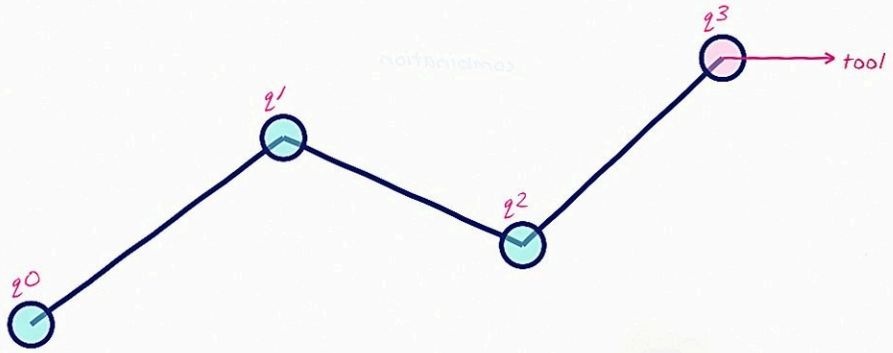
GATE Robotics Revision Notes

Manipulator Kinematics Jacobians and Singularities

SYLLABUS MAP

Purpose :-

Manipulator kinematics maps joint variables to end-effector pose and velocity. Forward models, inverse solutions, Jacobians and singularities form the geometric core of robot analysis.



Coverage route :-

- → robot joints
- → kinematic chains
- → denavit hartenberg parameters
- → forward kinematics
- → planar two link arm

How to revise :-

Read the physical idea first. Draw the model, write symbols, select the governing relation and finish with a limiting-case check.

ROBOT JOINTS

Definition :-

A robot joint permits a specified relative motion between two links.

Concept :-

Revolute joints contribute angles, while prismatic joints contribute translations.

$$q = [q_1 \dots q_n]^T$$

GATE check :-

Count independent joint variables, not visible mechanical parts.

Question route :-

Mark the known state and the required output. Use the model assumptions before substituting any number.

→ draw the frame, graph or free-body model

→ keep signs, units and coordinates consistent

Answer check :-

Test one simple or limiting case. The result must match the physical trend and remain inside the stated assumptions.

→ reject options with impossible units or direction

Connection forward :-

Next topic: KINEMATIC CHAINS.

Serial robots have broad workspaces; parallel robots can be stiff but introduce loop constraints.

KINEMATIC CHAINS

Definition :-

A serial chain connects links in one path, while a parallel chain has multiple paths.

Concept :-

Serial robots have broad workspaces; parallel robots can be stiff but introduce loop constraints.

$$\text{pose} = f(q)$$

GATE check :-

Closed chains require constraint equations in addition to joint transforms.

Question route :-

Mark the known state and the required output. Use the model assumptions before substituting any number.

→ draw the frame, graph or free-body model

→ keep signs, units and coordinates consistent

Answer check :-

Test one simple or limiting case. The result must match the physical trend and remain inside the stated assumptions.

→ reject options with impossible units or direction

Connection forward :-

Next topic: DENAVIT HARTENBERG PARAMETERS.

A consistent frame assignment turns a long geometry problem into a repeatable transform product.

DENAVIT HARTENBERG PARAMETERS

Definition :-

DH notation represents each link transform using four geometric parameters.

Concept :-

A consistent frame assignment turns a long geometry problem into a repeatable transform product.

$$A_i = Rotz(\theta)\ Transz(d)\ Transx(a)\ Rotx(\alpha)$$

GATE check :-

Do not mix standard and modified DH conventions.

Question route :-

Mark the known state and the required output. Use the model assumptions before substituting any number.

→ draw the frame, graph or free-body model

→ keep signs, units and coordinates consistent

Answer check :-

Test one simple or limiting case. The result must match the physical trend and remain inside the stated assumptions.

→ reject options with impossible units or direction

Connection forward :-

Next topic: FORWARD KINEMATICS.

Multiply transforms from base to tool in physical chain order.

FORWARD KINEMATICS

Definition :-

Forward kinematics computes end-effector pose from known joint variables.

Concept :-

Multiply transforms from base to tool in physical chain order.

$$T_{0n}(q) = \text{product}(A_i(q_i))$$

GATE check :-

Check the zero configuration against the actual robot drawing.

Question route :-

Mark the known state and the required output. Use the model assumptions before substituting any number.

→ draw the frame, graph or free-body model
→ keep signs, units and coordinates consistent

Answer check :-

Test one simple or limiting case. The result must match the physical trend and remain inside the stated assumptions.

→ reject options with impossible units or direction

Connection forward :-

Next topic: PLANAR TWO LINK ARM.

Its geometry is the basic test case for forward and inverse kinematics.

PLANAR TWO LINK ARM

Definition :-

A two-link planar arm reaches a point by adding two rotated link vectors.

Concept :-

Its geometry is the basic test case for forward and inverse kinematics.

$$x = l_1 \cos q_1 + l_2 \cos(q_1 + q_2)$$

GATE check :-

The y equation uses sine with the same cumulative angles.

Question route :-

Mark the known state and the required output. Use the model assumptions before substituting any number.

→ draw the frame, graph or free-body model

→ keep signs, units and coordinates consistent

Answer check :-

Test one simple or limiting case. The result must match the physical trend and remain inside the stated assumptions.

→ reject options with impossible units or direction

Connection forward :-

Next topic: WORKSPACE.

Link lengths, joint limits, collisions and orientation requirements shrink the usable workspace.

WORKSPACE

Definition :-

Workspace is the set of poses or positions reachable by the end effector.

Concept :-

Link lengths, joint limits, collisions and orientation requirements shrink the usable workspace.

$$l1 - l2 \leq r \leq l1 + l2$$

GATE check :-

Reachability of position does not guarantee reachability of every orientation.

Question route :-

Mark the known state and the required output. Use the model assumptions before substituting any number.

→ draw the frame, graph or free-body model

→ keep signs, units and coordinates consistent

Answer check :-

Test one simple or limiting case. The result must match the physical trend and remain inside the stated assumptions.

→ reject options with impossible units or direction

Connection forward :-

Next topic: INVERSE KINEMATICS.

Solutions may be multiple, absent or continuous depending on geometry and task dimension.

INVERSE KINEMATICS

Definition :-

Inverse kinematics finds joint values that produce a desired pose.

Concept :-

Solutions may be multiple, absent or continuous depending on geometry and task dimension.

$$q = f^{-1}(x)$$

GATE check :-

Verify every candidate with forward kinematics and joint limits.

Question route :-

Mark the known state and the required output. Use the model assumptions before substituting any number.

→ draw the frame, graph or free-body model
→ keep signs, units and coordinates consistent

Answer check :-

Test one simple or limiting case. The result must match the physical trend and remain inside the stated assumptions.

→ reject options with impossible units or direction

Connection forward :-

Next topic: GEOMETRIC IK.

It gives closed-form solutions when the mechanism has suitable geometry.

GEOMETRIC IK

Definition :-

Geometric inverse kinematics uses triangles, projections and wrist-center construction.

Concept :-

It gives closed-form solutions when the mechanism has suitable geometry.

$$\cos q2 = (r2 - l1^2 - l2^2)/(2\, l1\, l2)$$

GATE check :-

Both signs of sine q2 give elbow-up and elbow-down branches.

Question route :-

Mark the known state and the required output. Use the model assumptions before substituting any number.

→ draw the frame, graph or free-body model

→ keep signs, units and coordinates consistent

Answer check :-

Test one simple or limiting case. The result must match the physical trend and remain inside the stated assumptions.

→ reject options with impossible units or direction

Connection forward :-

Next topic: NUMERICAL IK.

It works for general robots but depends on initialization, step size and conditioning.

NUMERICAL IK

Definition :-

Numerical IK iteratively reduces pose error using local derivatives.

Concept :-

It works for general robots but depends on initialization, step size and conditioning.

$$q_next = q + J + e$$

GATE check :-

Large steps can jump branches or violate joint limits.

Question route :-

Mark the known state and the required output. Use the model assumptions before substituting any number.

→ draw the frame, graph or free-body model

→ keep signs, units and coordinates consistent

Answer check :-

Test one simple or limiting case. The result must match the physical trend and remain inside the stated assumptions.

→ reject options with impossible units or direction

Connection forward :-

Next topic: VELOCITY KINEMATICS.

Each Jacobian column is the motion caused by one unit joint rate.

VELOCITY KINEMATICS

Definition :-

Velocity kinematics relates joint rates to end-effector linear and angular velocity.

Concept :-

Each Jacobian column is the motion caused by one unit joint rate.

$$\dot{x} = J(q)\,\dot{q}$$

GATE check :-

Match the twist ordering and expression frame.

Question route :-

Mark the known state and the required output. Use the model assumptions before substituting any number.

→ draw the frame, graph or free-body model

→ keep signs, units and coordinates consistent

Answer check :-

Test one simple or limiting case. The result must match the physical trend and remain inside the stated assumptions.

→ reject options with impossible units or direction

Connection forward :-

Next topic: GEOMETRIC JACOBIAN.

A revolute column contains an axis cross moment arm plus the axis itself.

GEOMETRIC JACOBIAN

Definition :-

The geometric Jacobian is built from joint axes and position vectors.

Concept :-

A revolute column contains an axis cross moment arm plus the axis itself.

$$Jv_i = z_i \times (p_e - p_i)$$

GATE check :-

A prismatic column has linear axis z_i and zero angular part.

Question route :-

Mark the known state and the required output. Use the model assumptions before substituting any number.

→ draw the frame, graph or free-body model
→ keep signs, units and coordinates consistent

Answer check :-

Test one simple or limiting case. The result must match the physical trend and remain inside the stated assumptions.

→ reject options with impossible units or direction

Connection forward :-

Next topic: ANALYTIC JACOBIAN.

Euler-angle rates are not identical to angular velocity and require a conversion matrix.

ANALYTIC JACOBIAN

Definition :-

The analytic Jacobian maps joint rates to derivatives of chosen pose parameters.

Concept :-

Euler-angle rates are not identical to angular velocity and require a conversion matrix.

$$\dot{x}_{param} = J_a \dot{q}$$

GATE check :-

Analytic Jacobians inherit parameterization singularities.

Question route :-

Mark the known state and the required output. Use the model assumptions before substituting any number.

→ draw the frame, graph or free-body model

→ keep signs, units and coordinates consistent

Answer check :-

Test one simple or limiting case. The result must match the physical trend and remain inside the stated assumptions.

→ reject options with impossible units or direction

Connection forward :-

Next topic: STATIC FORCE MAPPING.

This follows from equality of virtual work at joints and task space.

STATIC FORCE MAPPING

Definition :-

The Jacobian transpose maps an end-effector wrench to equivalent joint torques.

Concept :-

This follows from equality of virtual work at joints and task space.

$$\mathcal{T} = J^T F$$

GATE check :-

A wrench and Jacobian must be expressed at compatible points and frames.

Question route :-

Mark the known state and the required output. Use the model assumptions before substituting any number.

→ draw the frame, graph or free-body model
→ keep signs, units and coordinates consistent

Answer check :-

Test one simple or limiting case. The result must match the physical trend and remain inside the stated assumptions.

→ reject options with impossible units or direction

Connection forward :-

Next topic: KINEMATIC SINGULARITY.

Some task motion becomes impossible while certain joint rates or forces grow very large.

KINEMATIC SINGULARITY

Definition :-

A singularity occurs when the Jacobian loses rank.

Concept :-

Some task motion becomes impossible while certain joint rates or forces grow very large.

$$\text{rank } J < \text{task dimension}$$

GATE check :-

A small determinant is not a safe test for a non-square Jacobian.

Question route :-

Mark the known state and the required output. Use the model assumptions before substituting any number.

→ draw the frame, graph or free-body model
→ keep signs, units and coordinates consistent

Answer check :-

Test one simple or limiting case. The result must match the physical trend and remain inside the stated assumptions.

→ reject options with impossible units or direction

Connection forward :-

Next topic: MANIPULABILITY.

Singular values of the Jacobian give the principal velocity directions.

MANIPULABILITY

Definition :-

Manipulability measures how freely a robot can generate task velocity near a configuration.

Concept :-

Singular values of the Jacobian give the principal velocity directions.

$$w = \text{root}(\det(J\ J\ T))$$

GATE check :-

Manipulability approaches zero at a full task-space singularity.

Question route :-

Mark the known state and the required output. Use the model assumptions before substituting any number.

→ draw the frame, graph or free-body model

→ keep signs, units and coordinates consistent

Answer check :-

Test one simple or limiting case. The result must match the physical trend and remain inside the stated assumptions.

→ reject options with impossible units or direction

Connection forward :-

Next topic: PSEUDOINVERSE.

It handles non-square Jacobians and redundant manipulators.

PSEUDOINVERSE

Definition :-

The Moore-Penrose pseudoinverse gives a minimum-norm least-squares solution.

Concept :-

It handles non-square Jacobians and redundant manipulators.

$$q_dot = J^{+} \times x_dot$$

GATE check :-

Near singularity, small singular values amplify noise and rates.

Question route :-

Mark the known state and the required output. Use the model assumptions before substituting any number.

→ draw the frame, graph or free-body model

→ keep signs, units and coordinates consistent

Answer check :-

Test one simple or limiting case. The result must match the physical trend and remain inside the stated assumptions.

→ reject options with impossible units or direction

Connection forward :-

Next topic: DAMPED LEAST SQUARES.

It trades exact tracking for bounded joint rates and numerical robustness.

DAMPED LEAST SQUARES

Definition :-

Damped least squares regularizes the inverse near a singular Jacobian.

Concept :-

It trades exact tracking for bounded joint rates and numerical robustness.

$$J^{\#} = J^{T}(J J^{T} + \lambda^{2} I)^{-1}$$

GATE check :-

Too much damping creates avoidable tracking error.

Question route :-

Mark the known state and the required output. Use the model assumptions before substituting any number.

→ draw the frame, graph or free-body model

→ keep signs, units and coordinates consistent

Answer check :-

Test one simple or limiting case. The result must match the physical trend and remain inside the stated assumptions.

→ reject options with impossible units or direction

Connection forward :-

Next topic: REDUNDANCY AND NULL SPACE.

Null-space motion changes posture without changing the primary task velocity.

REDUNDANCY AND NULL SPACE

Definition :-

A redundant robot has more joint variables than the task requires.

Concept :-

Null-space motion changes posture without changing the primary task velocity.

$$q_dot = J^{+} x_dot + (I - J^{+}J)z$$

GATE check :-

The secondary command must be projected into the true null space.

Question route :-

Mark the known state and the required output. Use the model assumptions before substituting any number.

→ draw the frame, graph or free-body model

→ keep signs, units and coordinates consistent

Answer check :-

Test one simple or limiting case. The result must match the physical trend and remain inside the stated assumptions.

→ reject options with impossible units or direction

Connection forward :-

Next topic: ROBOT JOINTS.

Revolute joints contribute angles, while prismatic joints contribute translations.

QUICK FORMULA MAP

Core relations :-

Robot Joints : $q = [q1 \dots qn]T$

Kinematic Chains : $pose = f(q)$

Denavit Hartenberg Parameters : $A_i = Rotz(\ominus)\ Transz(d)\ Transx(a)\ Rotx(\bigcirc)$

Forward Kinematics : $T_On(q) = product(A_i(q_i))$

Planar Two Link Arm : $x = l1 \cos q1 + l2 \cos(q1+q2)$

Workspace : $l1-l2 \leq r \leq l1+l2$

Inverse Kinematics : $q = f^{-1}(x)$

Geometric IK : $\cos q2 = (r2-l1^2-l2^2)/(2\ l1\ l2)$

Numerical IK : $q_next = q + J^{+} e$

Velocity Kinematics : $x_dot = J(q)\ q_dot$

Geometric Jacobian : $Jv_i = z_i \times (p_e-p_i)$

Recall ladder :-

→ state the model and coordinate frame

→ write symbols before numbers

→ test units and limiting behaviour

GATE TIPS & TRAPS

High-return checks :-

Count independent joint variables, not visible mechanical parts.

Closed chains require constraint equations in addition to joint transforms.

Do not mix standard and modified DH conventions.

Check the zero configuration against the actual robot drawing.

The y equation uses sine with the same cumulative angles.

Common mistakes :-

Large steps can jump branches or violate joint limits.

Match the twist ordering and expression frame.

A prismatic column has linear axis z_i and zero angular part.

Analytic Jacobians inherit parameterization singularities.

Final recall :-

Sketch first, state the frame, keep one sign convention and verify whether the answer is physically reachable.

Rapid recall checks :-

STATIC FORCE MAPPING: A wrench and Jacobian must be expressed at compatible points and frames.

KINEMATIC SINGULARITY: A small determinant is not a safe test for a non-square Jacobian.

GATE PROBLEM DRILL 1

Focus pair :-

Connect robot joints with workspace. Many robotics questions hide this connection inside a diagram or data table.

$$q = [q1 \dots qn]T$$

Question route :-

Translate the wording into a state, input and output. Decide whether the relation is geometric, dynamic, probabilistic or algorithmic.

- → label every frame, axis and time index
- → separate measured data from estimated state
- → apply constraints before solving
- → use the simplest valid model first

$$11-12 <= r <= 11+12$$

Fast elimination :-

Count independent joint variables, not visible mechanical parts. Reachability of position does not guarantee reachability of every orientation.

Answer check :-

Verify dimensions, range, sign and limiting behaviour. For a matrix answer, also check its shape and frame order.

GATE PROBLEM DRILL 2

Focus pair :-

Connect denavit hartenberg parameters with geometric ik. Many robotics questions hide this connection inside a diagram or data table.

$$A_i = Rotz(\theta)\ Transz(d)\ Transx(a)\ Rotx(\alpha)$$

Question route :-

Translate the wording into a state, input and output. Decide whether the relation is geometric, dynamic, probabilistic or algorithmic.

- → label every frame, axis and time index
- → separate measured data from estimated state
- → apply constraints before solving
- → use the simplest valid model first

$$\cos q2 = (r2 - l1^2 - l2^2)/(2\ l1\ l2)$$

Fast elimination :-

Do not mix standard and modified DH conventions. Both signs of sine q2 give elbow-up and elbow-down branches.

Answer check :-

Verify dimensions, range, sign and limiting behaviour. For a matrix answer, also check its shape and frame order.

GATE PROBLEM DRILL 3

Focus pair :-

Connect planar two link arm with velocity kinematics. Many robotics questions hide this connection inside a diagram or data table.

$$x = l_1 \cos q_1 + l_2 \cos(q_1+q_2)$$

Question route :-

Translate the wording into a state, input and output. Decide whether the relation is geometric, dynamic, probabilistic or algorithmic.

- → label every frame, axis and time index
- → separate measured data from estimated state
- → apply constraints before solving
- → use the simplest valid model first

$$\dot{x} = J(q)\,\dot{q}$$

Fast elimination :-

The y equation uses sine with the same cumulative angles. Match the twist ordering and expression frame.

Answer check :-

Verify dimensions, range, sign and limiting behaviour. For a matrix answer, also check its shape and frame order.

GATE PROBLEM DRILL 4

Focus pair :-

Connect inverse kinematics with analytic jacobian. Many robotics questions hide this connection inside a diagram or data table.

$$q = f^{-1}(x)$$

Question route :-

Translate the wording into a state, input and output. Decide whether the relation is geometric, dynamic, probabilistic or algorithmic.

- → label every frame, axis and time index
- → separate measured data from estimated state
- → apply constraints before solving
- → use the simplest valid model first

$$\dot{x}_{param} = J_a \dot{q}$$

Fast elimination :-

Verify every candidate with forward kinematics and joint limits. Analytic Jacobians inherit parameterization singularities.

Answer check :-

Verify dimensions, range, sign and limiting behaviour. For a matrix answer, also check its shape and frame order.

GATE PROBLEM DRILL 5

Focus pair :-

Connect numerical ik with kinematic singularity. Many robotics questions hide this connection inside a diagram or data table.

$$q_next = q + J + e$$

Question route :-

Translate the wording into a state, input and output. Decide whether the relation is geometric, dynamic, probabilistic or algorithmic.

- → label every frame, axis and time index
- → separate measured data from estimated state
- → apply constraints before solving
- → use the simplest valid model first

$$\text{rank } J < \text{task dimension}$$

Fast elimination :-

Large steps can jump branches or violate joint limits. A small determinant is not a safe test for a non-square Jacobian.

Answer check :-

Verify dimensions, range, sign and limiting behaviour. For a matrix answer, also check its shape and frame order.

GATE PROBLEM DRILL 6

Focus pair :-

Connect geometric jacobian with pseudoinverse. Many robotics questions hide this connection inside a diagram or data table.

```
Jv_i = z_i cross (p_e-p_i)
```

Question route :-

Translate the wording into a state, input and output. •Decide whether the relation is geometric, dynamic, probabilistic or algorithmic.

→ label every frame, axis and time index
→ separate measured data from estimated state
→ apply constraints before solving
→ use the simplest valid model first

```
q_dot = J  + x_dot
```

Fast elimination :-

A prismatic column has linear axis z_i and zero angular part. Near singularity, small singular values amplify noise and rates.

Answer check :-

Verify dimensions, range, sign and limiting behaviour. For a matrix answer, also check its shape and frame order.

GATE PROBLEM DRILL 7

Focus pair :-

Connect static force mapping with redundancy and null space. Many robotics questions hide this connection inside a diagram or data table.

$$\mathcal{J} = J\ T\ F$$

Question route :-

Translate the wording into a state, input and output. Decide whether the relation is geometric, dynamic, probabilistic or algorithmic.

- → label every frame, axis and time index
- → separate measured data from estimated state
- → apply constraints before solving
- → use the simplest valid model first

$$q_dot = J\ +\ x_dot + (I - J\ +J)z$$

Fast elimination :-

A wrench and Jacobian must be expressed at compatible points and frames. The secondary command must be projected into the true null space.

Answer check :-

Verify dimensions, range, sign and limiting behaviour. For a matrix answer, also check its shape and frame order.