GATE 2026 Mathematics Question Paper with Solutions

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General Instructions

(i) This booklet contains 65 questions, each provided with a complete, step-by-step solution.

(ii) It comprises 32 single-correct multiple-choice questions, 14 multiple-correct questions and 19 numerical / integer-type questions.

(iii) These questions follow the GATE Mathematics examination pattern.

(iv) Questions carry 1 or 2 marks. MCQs have negative marking (1/3 or 2/3 of the marks); MSQ and numerical-answer (NAT) questions carry no negative marking.

(v) GATE is a 3-hour paper of 65 questions (100 marks).

(vi) A virtual scientific calculator is provided; physical calculators are not allowed.

(vii) Attempt each question on your own before reviewing the given solution.

(viii) For multiple-correct questions, all correct options must be selected to earn full marks.

(ix) For numerical questions, report the answer rounded exactly as asked.

1. Suresh said, "I did it yesterday." Which one of the following options is the correct form of this sentence in indirect speech?

(A) Suresh said that I did it yesterday.

(B) Suresh says I did it yesterday.

(C) Suresh says that he did it the day before.

(D) Suresh said that he had done it the day before.

Correct Answer: (D) Suresh said that he had done it the day before.

Solution:

This question checks the rules for changing direct speech into indirect (reported) speech, specifically what happens to the tense of the verb and to time words like "yesterday" when the reporting verb is in the past tense.

The original sentence is: Suresh said, "I did it yesterday." The reporting verb here is "said," which is in the simple past tense. When the reporting verb is in the past tense, two changes are needed: the tense inside the quote must shift one step further into the past, and any time word tied to the speaker's "now" must change to match the new point of view.

1. **Suresh said that I did it yesterday.**: This keeps the pronoun "I" and the time word "yesterday" exactly as they were in direct speech. In reported speech the pronoun must change to match who is being talked about (he), and "yesterday" cannot stay the same because the day of reporting is not the same as the day of the action. This option is wrong.
2. **Suresh says I did it yesterday.**: This changes "said" to "says," turning the reporting verb into the present tense. But the question gives us "said," which is past tense, so the reporting verb cannot change to present. It also leaves the pronoun and the time word unchanged. This option is wrong.
3. **Suresh says that he did it the day before.**: This option correctly changes the time word to "the day before" and the pronoun to "he," but it still uses "says" for the reporting verb instead of "said." Since the original reporting verb is past tense, this option is wrong.
4. **Suresh said that he had done it the day before.**: This keeps "said" as the reporting verb, changes "I" to "he" since we

are now talking about Suresh in the third person, changes "yesterday" to "the day before" since the day of speaking is not the day of reporting, and shifts "did" (simple past) one step back to "had done" (past perfect), which is the standard backshift rule when the reporting verb is in the past tense. This option follows every rule correctly.

The correct option is "Suresh said that he had done it the day before," because it applies the backshift of tense from simple past to past perfect, changes the pronoun to third person, and updates the time word "yesterday" to "the day before."

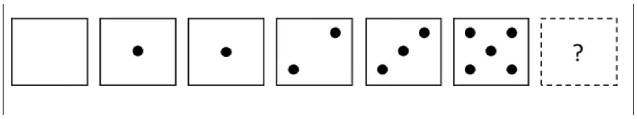
Let's summarize:

- When the reporting verb is in the past tense (said), the tense inside the reported clause shifts one step back: simple past becomes past perfect.
- Time and place words tied to the moment of speaking change in reported speech: "yesterday" becomes "the day before," "today" becomes "that day," "tomorrow" becomes "the next day."
- Pronouns change to match the new speaker's point of view: "I" becomes "he" when Suresh is being talked about.

So the correct indirect speech version is "Suresh said that he had done it the day before," which is option (D).

Quick Tip: When the reporting verb is in the past tense (said), shift the tense back one step (did becomes had done) and change time words like yesterday to the day before.

2. To continue the sequence of tiles shown, the tile indicated by the question mark should be:



(The tiles in the sequence contain, in order, 0 dots, 1 dot, 1 dot, 2 dots, 3 dots, and 5 dots, followed by a tile marked with a question mark.)

(A) A tile showing 4 dots arranged in a 2x2 square.

(B) A tile showing 6 dots.

(C) A tile showing 8 dots arranged in a ring.

(D) A tile showing 9 dots arranged in a 3x3 square.

Correct Answer: (C) A tile showing 8 dots arranged in a ring.

Solution:

Step 1: Count the dots on each tile in order.
Write down how many dots appear on each tile from left to right: the first tile is blank (0 dots), the second has 1 dot, the third has 1 dot, the fourth has 2 dots, the fifth has 3 dots, and the sixth has 5 dots.
So the sequence of dot counts is: 0, 1, 1, 2, 3, 5.

Step 2: Find the rule linking one term to the next.
Check if each term is the sum of the two terms before it: $0 + 1 = 1$ (matches the third term), $1 + 1 = 2$ (matches the fourth term), $1 + 2 = 3$ (matches the fifth term), $2 + 3 = 5$ (matches the sixth term).
Every term equals the sum of the two terms right before it, so this is the Fibonacci pattern.

Step 3: Apply the rule to find the next term.
The next tile's dot count should equal the sum of the fifth and sixth

terms: 3 + 5 = 8.
So the missing tile must show 8 dots.

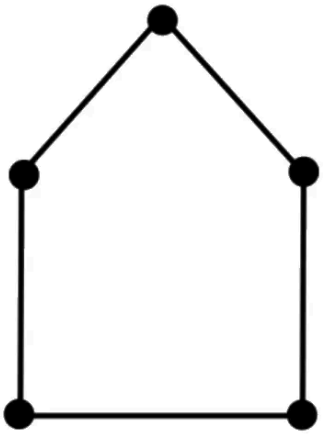
Step 4: Match this count to the given options.
Option (A) shows 4 dots, option (B) shows 6 dots, option (C) shows 8 dots arranged in a ring, and option (D) shows 9 dots.
Only option (C) has exactly 8 dots, so it is the tile that continues the sequence correctly. Options (A), (B), and (D) show 4, 6, and 9 dots, none of which fit the required count of 8.

Final Answer:
The dot counts follow the Fibonacci rule 0, 1, 1, 2, 3, 5, 8, so the missing tile has 8 dots.

8 dots, option (C)

Quick Tip: Count the dots on each tile: 0, 1, 1, 2, 3, 5. This is the Fibonacci pattern, so the next tile needs 3 + 5 = 8 dots.

3. Consider an art gallery whose walkways are shown as lines in the diagram. A black dot represents a junction of two walkways. A guard may be placed at a junction to watch over the walkways that join at that junction. The minimum number of guards needed to watch all the walkways is _____.



(The diagram shows five junctions joined in a closed loop shaped like a pentagon: a top junction connects down to an upper-left junction and an upper-right junction, the upper-left junction connects down to a lower-left junction, the upper-right junction connects down to a lower-right junction, and the lower-left and lower-right junctions connect to each other. This gives 5 junctions and 5 walkways in total.)

(A) 2

(B) 3

(C) 4

(D) 5

Correct Answer: (B) 3

Solution:

Step 1: Turn the picture into a graph.

Label the five junctions in the diagram: the top point as A, the upper-

left corner as UL, the upper-right corner as UR, the lower-left corner as LL, and the lower-right corner as LR.
The walkways (lines) join A to UL, A to UR, UL to LL, UR to LR, and LL to LR. This gives 5 walkways in total, connecting the 5 junctions in one closed loop shaped like a pentagon.

Step 2: Understand what placing a guard does.
A guard placed at a junction can watch every walkway that touches that junction, but not walkways that do not touch it.
We need to pick the smallest possible set of junctions so that every one of the 5 walkways has at least one of its two ends covered by a guard.

Step 3: Check whether 2 guards are enough.
Each junction touches exactly 2 walkways, so 2 guards can cover at most 4 walkway-ends between them if they do not share a walkway, and even fewer distinct walkways if they do.
Try guards at A and LL: this covers A-UL, A-UR, UL-LL, and LL-LR, but leaves UR-LR unwatched.
Try guards at UL and LR: this covers A-UL, UL-LL, UR-LR, and LL-LR, but leaves A-UR unwatched.
Every choice of 2 junctions leaves at least one walkway with neither end covered, because 5 walkways cannot all be covered by only 4 walkway-ends. So 2 guards are never enough.

Step 4: Check that 3 guards are enough.
Place guards at A, LL, and LR.
A covers A-UL and A-UR.
LL covers UL-LL and LL-LR.
LR covers LL-LR and UR-LR.
Together, all 5 walkways (A-UL, A-UR, UL-LL, UR-LR, LL-LR) have

at least one guarded end, so all walkways are watched.

Final Answer:
2 guards are never enough, and 3 guards are enough when placed correctly, so the minimum number of guards needed is 3.

$$\boxed{3}$$

Quick Tip: Model the junctions and walkways as a 5-sided loop (pentagon). This needs a minimum vertex cover, which for a 5-cycle is 3.

4. The 2nd of June is a Thursday in a certain year. Which day of the week is the 3rd of July in that year?

(A) Thursday

(B) Friday

(C) Saturday

(D) Sunday

Correct Answer: (D) Sunday

Solution:

Step 1: Find the number of days between 2nd June and 3rd July.
June has 30 days, so from the 2nd of June to the 2nd of July is exactly 30 days.
Adding one more day to reach the 3rd of July gives $30 + 1 = 31$ days in total from the 2nd of June to the 3rd of July.

Step 2: Convert the day gap into full weeks and extra days.

A week has 7 days, so divide 31 by 7: 31 = 7 x 4 + 3.
This means 31 days is exactly 4 full weeks plus 3 extra days.

Step 3: Use the extra days to shift the weekday.
Whole weeks do not change the day of the week, since the same weekday repeats every 7 days. So only the leftover 3 days matter.
Starting from Thursday (2nd June) and moving forward 3 days:
Thursday + 1 day = Friday, Friday + 1 day = Saturday, Saturday + 1 day = Sunday.

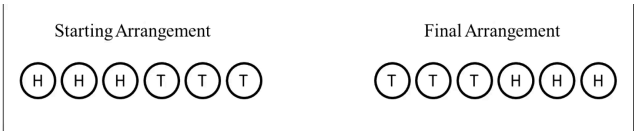
Step 4: State the day for 3rd July.
After moving 3 days forward from Thursday, we land on Sunday.

Final Answer:
The 3rd of July falls on a Sunday.

Sunday

Quick Tip: Count the days from 2nd June to 3rd July (31 days), find the remainder after dividing by 7 (3), and move that many weekdays forward from Thursday.

5. A coin with heads facing up is shown as H and a coin with tails facing up is shown as T. Six coins are placed in the Starting Arrangement, as shown in the figure below. A "step" is defined as interchanging a pair of adjacent coins without flipping them.



Starting Arrangement: H H H T T T
Final Arrangement: T T T H H H
The minimum number of steps needed to go from the Starting Arrangement to the Final Arrangement, as shown in the figure, is _____.

(A) 3

(B) 6

(C) 9

(D) 12

Correct Answer: (C) 9

Solution:

Step 1: Understand what one step can fix.
A step swaps two coins that are sitting next to each other, without flipping either one. A useful step always swaps a coin that is out of place with its neighbour, moving it one place closer to where it needs to end up.

Step 2: Compare the two arrangements.
Starting Arrangement: H H H T T T (positions 1 to 6).
Final Arrangement: T T T H H H.

Every H needs to end up to the right of every T, the exact opposite of how they start.

Step 3: Count how many H-T pairs are "wrongly ordered" to begin with.
In the Starting Arrangement, all 3 H coins sit to the left of all 3 T coins. In the Final Arrangement, every H must sit to the right of every T. So every one of the 3 H coins must cross over every one of the 3 T coins at least once.
This gives $3 \times 3 = 9$ required H-T crossings.

Step 4: Relate crossings to steps.
Each step swaps exactly one adjacent pair. A step that swaps an H past a T fixes exactly one H-T crossing and never needs to be reversed, since coins of the same kind (H with H, or T with T) never need to change places with each other; only H-T pairs need to trade places. So the minimum number of steps equals the number of H-T crossings needed, which is 9.

Final Answer:
The minimum number of steps needed is

$\boxed{9}$

Quick Tip: Each of the 3 heads must move past each of the 3 tails at least once; count how many such crossings are needed in total.

6. Exacerbate : Mitigate :: _____
Choose the option with the correct pair of words to fill the blank.

(A) Aggravate : Alleviate

(B) Alleviate : Precipitate

(C) Aggravate : Precipitate

(D) Emancipate : Exonerate

Correct Answer: (A) Aggravate : Alleviate

Solution:

This question tests word pairs that share the same relationship as the given pair. "Exacerbate" means to make a problem or situation worse, and "Mitigate" means to make a problem or situation less severe. So the first word in the blank must mean "to worsen" and the second word must mean "to lessen", the same worsen-to-lessen link seen in the stem pair.

1. **Aggravate : Alleviate**: "Aggravate" means to make a situation worse, the same sense as "exacerbate". "Alleviate" means to make suffering or a problem less severe, the same sense as "mitigate". This pair carries the exact worsen-to-lessen relationship of the stem.
2. **Alleviate : Precipitate**: "Alleviate" means to lessen, so it already breaks the pattern in the first slot, since the first word of the stem pair means to worsen, not to lessen. "Precipitate" means to bring on an event suddenly, which has nothing to do with severity either.
3. **Aggravate : Precipitate**: "Aggravate" fits the worsen sense correctly, but "Precipitate" means to trigger an event suddenly, not to lessen a problem. The second word does not carry the "mitigate" sense, so the pair breaks the relationship.
4. **Emancipate : Exonerate**: "Emancipate" means to free from restriction, and "Exonerate" means to clear from blame. Both

words describe freeing someone from a negative state, so this is a synonym pair, not a worsen-to-lessen antonym pair like the stem.

Only option (A) keeps both halves of the relationship intact: the first word means "to worsen" and the second word means "to lessen", matching "Exacerbate : Mitigate" exactly.

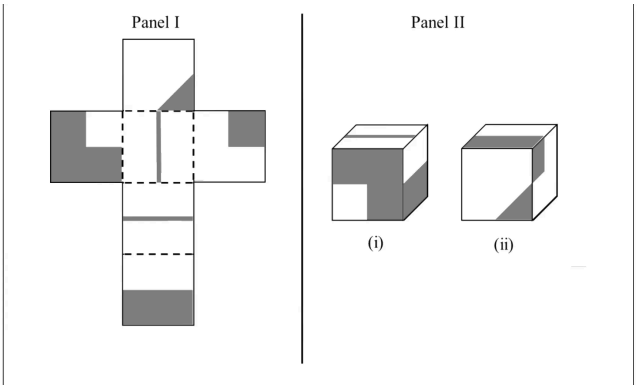
Let's summarize:

- Exacerbate and Mitigate form a worsen-to-lessen antonym pair.
- Aggravate and Alleviate form the same worsen-to-lessen antonym pair, so it is the correct match.
- The other three options each break the relationship in at least one half of the pair.

So the correct pair is Aggravate : Alleviate, option (A).

Quick Tip: Work out what relationship "worsen" and "lessen" describes, then look for the option where both words carry that same sense.

7. A paper shown in Panel I is folded along the dashed lines to construct a cube. The shaded regions shown in Panel I appear on the outer surface of the cube. Referring to the cubes shown in Panel II, which one of the options is correct?



(Panel I shows a flat cross-shaped net of six squares: a vertical strip of four squares one above another, with one extra square attached to the left of the second square in that strip and another extra square attached to its right, all joined along dashed fold lines. Some squares carry grey shaded regions and some are left plain white. Panel II shows two separate solid cubes, labelled (i) and (ii), each with grey shaded regions on some of their visible faces.)

(A) Only (i) can correspond to the unfolded cube in Panel I.

(B) Only (ii) can correspond to the unfolded cube in Panel I.

(C) Both (i) and (ii) can correspond to the unfolded cube in Panel I.

(D) Neither (i) nor (ii) can correspond to the unfolded cube in Panel I.

Correct Answer: (B) Only (ii) can correspond to the unfolded cube in Panel I.

Solution:

Step 1: Understanding the Question.

We are given a flat net of six squares (Panel I) that folds up into a cube, with certain regions shaded grey. We must decide which of the

two solid cubes drawn in Panel II, (i) or (ii), could actually be the folded result, based on where the grey regions land and how they sit next to each other.

Step 2: Key Approach.

Label one square of the net as the front face, since it stays fixed while the rest fold around it. The squares directly above, below, left, and right of the front face become the top, bottom, left, and right faces of the cube, and the square at the far end of the strip becomes the back face, opposite the front.

Once every square is assigned to a face of the cube, two facts must both be checked in each candidate cube: which faces carry the grey shading, and how the shaded region on one face lines up with the shaded region on its neighboring face along the edge they share, since folding fixes both the position and the rotation of every face at once.

Step 3: Detailed Explanation.

Assigning the net's six squares this way shows that the fully grey square becomes the back face, the square with the grey triangular corner becomes the top face, the mostly grey square with the white notch becomes the left face, and the square with the small grey corner patch becomes the right face, while the front and bottom faces stay plain white.

Because folding is rigid, the grey triangle on the top face and the grey patch on the right face must meet the shared top-right edge of the cube in one specific relative rotation, and the same is true for every other pair of neighboring shaded faces.

Checking cube (i): the grey regions shown on its visible faces meet each other at the wrong edge and in the wrong relative rotation compared to what the net requires, so cube (i) cannot be the folded result of this net.

Checking cube (ii): the grey regions shown on its visible faces meet at the same edges and in the same relative rotation that folding the net produces, so cube (ii) is consistent with the net.

Step 4: Final Answer.
Since only cube (ii) keeps the shaded faces in the positions and orientations that folding Panel I actually produces, only (ii) can correspond to the unfolded cube in Panel I.

Quick Tip: Trace which net squares become which cube faces (front, back, top, bottom, left, right) and check that shaded neighbors stay adjacent in the same orientation after folding.

8. In a population, patients who have high cholesterol also have high blood-pressure (BP). Some patients with high BP also have diabetes. There are no patients who have both high cholesterol and diabetes. Furthermore,
1. the total number of patients with at least one of these conditions is 75,
2. the number of patients with high cholesterol is 10,
3. the number of patients with high BP is 45, and
4. the number of patients with only high BP and no other conditions is 20.
Then the number of patients who have both diabetes and high BP is

_____

(A) 0

(B) 15

(C) 20

(D) 10

Correct Answer: (B) 15

Solution:

Step 1: Set up what each group means.

Let C be patients with high cholesterol, B be patients with high BP, and D be patients with diabetes. Every patient with high cholesterol also has high BP, so C sits completely inside B. No patient has both high cholesterol and diabetes, so C and D do not overlap at all.

Step 2: Break high BP patients into three parts.

Since C sits entirely inside B, and C never overlaps D, the 45 patients with high BP split into exactly three non-overlapping groups: patients with only high BP and nothing else (given as 20), patients with high BP and high cholesterol (all of C, since C sits inside B, so this group is 10), and patients with high BP and diabetes, the group we need, call it x.

$$|B| = (\text{only BP}) + |C| + x$$

$$45 = 20 + 10 + x$$

Step 3: Solve for x.

$$x = 45 - 20 - 10 = 15$$

If x were 0, 10, or 20 instead, the three parts of B would add up to 30, 40, or 50, not 45, so those values do not fit the given high BP count and are ruled out.

Step 4: Check against the total of 75.

Since every cholesterol patient already has high BP, patients with at least one condition are just the union of B and D. Patients with diabetes only (no BP) come to $75 - 45 = 30$, so $|D| = 15 + 30 = 45$, and $|B \cup D| = 45 + 45 - 15 = 75$, which matches the given total and confirms the answer is consistent.

Final Answer:
The number of patients with both diabetes and high BP is 15.

$$\boxed{15}$$

Quick Tip: Split the 45 high BP patients into three non-overlapping parts: only BP, BP with cholesterol, and BP with diabetes, then use the numbers you already have to find the missing part.

9. Four people P, Q, R, and S, of different ages, make the following observations.
P - I am younger than S.
Q - I am neither the youngest nor the oldest.
R - P is older than me.
Based on these observations, the youngest person is _____.

(A) P

(B) Q

(C) R

(D) S

Correct Answer: (C) R

Solution:

Step 1: Write each statement as an age comparison.
P says "I am younger than S", so P is younger than S.
R says "P is older than me", so P is older than R, which is the same as saying R is younger than P.
Putting these two together gives one chain: R is younger than P, and P is younger than S.

Step 2: Use Q's statement to place Q in the chain.
Q says Q is neither the youngest nor the oldest of all four. If Q were younger than R, Q would become the youngest of all four, which breaks Q's own statement, so Q must be older than R. If Q were older than S, Q would become the oldest of all four, again breaking Q's statement, so Q must be younger than S. This gives R is younger than Q, and Q is younger than S.

Step 3: Combine both chains.
From Step 1: R is younger than P, and P is younger than S. From Step 2: R is younger than Q, and Q is younger than S. In both chains, R sits below both P and Q, and S sits above both of them. So no matter whether P is younger than Q or Q is younger than P, R stays younger than both, and S stays older than both.

Step 4: Identify the youngest and rule out the rest.
P cannot be the youngest, since R's own statement makes R younger than P. Q cannot be the youngest, since Q's statement rules that out directly. S cannot be the youngest, since P's statement makes P younger than S, and R is younger still. R is younger than P (Step 1), younger than Q (Step 2), and therefore younger than S too, so R is younger than all three others, which makes R the youngest.

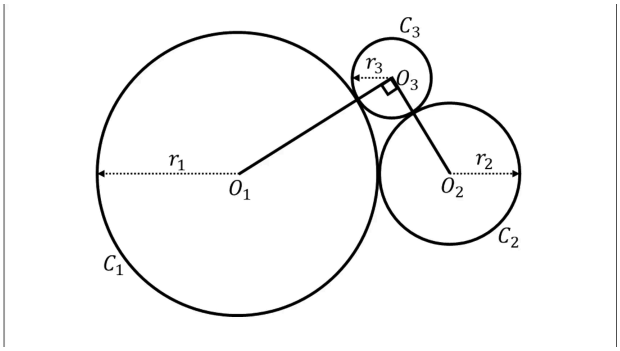
Final Answer:

The youngest person is R.

$$\boxed{R}$$

Quick Tip: Turn each statement into a "younger than" relation first, then check which single person is younger than all three of the others without breaking any statement.

10. Circles C_1, C_2, and C_3, with centers O_1, O_2, and O_3, and radii r_1, r_2, and r_3, respectively, touch each other as shown in the following figure.



Given $r_1 = 2$ cm, $r_2 = 1$ cm and the angle $\angle O_1O_3O_2$ is 90°, $r_3 =$ _____ cm.

(A) $\frac{1}{2}\left(-3+\sqrt{17}\right)$

(B) $\frac{1}{2}\left(3+\sqrt{17}\right)$

(C) $\frac{1}{2}\left(-2+\sqrt{17}\right)$

(D) $\frac{1}{2}\left(-3+2\sqrt{17}\right)$

Correct Answer: (A) $\frac{1}{2}\left(-3+\sqrt{17}\right)$

Solution:

Step 1: Use the tangency condition to write the three side lengths.

Since C_1 and C_2 touch each other from outside, the distance between their centers equals the sum of their radii: $O_1O_2 = r_1 + r_2$. The same rule applies to the other two pairs, since all three circles touch each other externally:

$$O_1O_2 = r_1 + r_2, \quad O_1O_3 = r_1 + r_3, \quad O_2O_3 = r_2 + r_3$$

Step 2: Apply the right angle at O_3.

Triangle $O_1O_2O_3$ has a right angle at O_3, so by the Pythagorean theorem, the side opposite the right angle, O_1O_2, satisfies:

$$O_1O_2^2 = O_1O_3^2 + O_2O_3^2$$

Step 3: Substitute the known values.

With $r_1 = 2$ and $r_2 = 1$:

$$O_1O_2 = 2 + 1 = 3, \quad O_1O_3 = 2 + r_3, \quad O_2O_3 = 1 + r_3$$

$$3^2 = (2 + r_3)^2 + (1 + r_3)^2$$

$$9 = 4 + 4r_3 + r_3^2 + 1 + 2r_3 + r_3^2$$

$$9 = 2r_3^2 + 6r_3 + 5$$

Step 4: Solve the quadratic equation.

$$2r_3^2 + 6r_3 - 4 = 0$$

$$r_3^2 + 3r_3 - 2 = 0$$

Using the quadratic formula:

$$r_3 = \frac{-3 \pm \sqrt{9+8}}{2} = \frac{-3 \pm \sqrt{17}}{2}$$

A radius must be positive, and the figure shows C_3 as clearly the smallest of the three circles, so r_3 must be smaller than both $r_1 = 2$ and $r_2 = 1$. Only one root fits both conditions:

$$r_3 = \frac{-3 + \sqrt{17}}{2} \approx 0.56 \text{ cm}$$

The other root, $\frac{-3-\sqrt{17}}{2}$, is negative and cannot be a radius. Option (B), $\frac{1}{2}(3 + \sqrt{17}) \approx 3.56$ cm, is bigger than r_1 itself, which does not match a circle drawn as the smallest of the three. Options (C) and (D) do not solve the quadratic equation above at all.

Final Answer:

$r_3 = \frac{1}{2}\left(-3 + \sqrt{17}\right)$ cm.

$$\boxed{r_3 = \frac{1}{2}\left(-3 + \sqrt{17}\right)}$$

Quick Tip: Use the fact that touching circles have their centers separated by the sum of their radii, then apply the Pythagorean theorem to the right angle at O_3.

11. Let $M_3(\mathbb{R})$ be the vector space of all 3×3 real matrices over $\mathbb{R}$ under usual matrix addition and scalar multiplication. Let $T_3(\mathbb{R})$ be the set of all 3×3 real upper triangular matrices. Which one of the following is TRUE?

(A) The quotient space $M_3(\mathbb{R})/T_3(\mathbb{R})$ is isomorphic to the vector space of all 3×3 real skew symmetric matrices over $\mathbb{R}$ under usual matrix addition and scalar multiplication.

(B) The quotient space $M_3(\mathbb{R})/T_3(\mathbb{R})$ is isomorphic to the vector space of all 3×3 real symmetric matrices over $\mathbb{R}$ under usual matrix addition and scalar multiplication.

(C) The quotient space $M_3(\mathbb{R})/T_3(\mathbb{R})$ is isomorphic to the vector space of all 3×3 real lower triangular matrices over $\mathbb{R}$ under usual matrix addition and scalar multiplication.

(D) The quotient space $M_3(\mathbb{R})/T_3(\mathbb{R})$ is isomorphic to the vector space of all 3×3 real matrices with trace zero over $\mathbb{R}$ under usual matrix addition and scalar multiplication.

Correct Answer: (A) The quotient space $M_3(\mathbb{R})/T_3(\mathbb{R})$ is isomorphic to the vector space of all 3×3 real skew symmetric matrices over $\mathbb{R}$ under usual matrix addition and scalar multiplication.

Solution:

Step 1: Understanding the Question.

We are told $M_3(\mathbb{R})$ is the vector space of all 3×3 real matrices, and $T_3(\mathbb{R})$ is the subspace of all 3×3 real upper triangular matrices (this includes the diagonal entries). We need to find which named vector

space the quotient $M_3(\mathbb{R})/T_3(\mathbb{R})$ is isomorphic to.

Step 2: Key Formula or Approach.

For a subspace W of a finite dimensional vector space V, the quotient space V/W has dimension $\dim(V/W) = \dim(V) - \dim(W)$. Also, two finite dimensional real vector spaces are isomorphic if and only if they have the same dimension, so once we know the dimension of the quotient we can match it to the option with the same dimension.

Step 3: Detailed Explanation.

The space $M_3(\mathbb{R})$ of all 3×3 real matrices has dimension 9, since each of the 9 entries can be chosen freely.

An upper triangular 3×3 matrix has entries only on and above the diagonal, giving $3 + 2 + 1 = 6$ free entries, so $\dim T_3(\mathbb{R}) = 6$.

So the quotient has dimension

$$\dim\left(M_3(\mathbb{R})/T_3(\mathbb{R})\right) = 9 - 6 = 3$$

Now check the dimension of each candidate space. Skew symmetric 3×3 matrices satisfy $A^T = -A$, which forces the diagonal to be zero and leaves only the 3 entries strictly above the diagonal free (each one fixes its mirror entry with a minus sign), so their dimension is 3. Symmetric matrices have 3 free diagonal entries plus 3 free off diagonal entries, so dimension 6. Lower triangular matrices, counted the same way as upper triangular, also have dimension 6. Matrices with trace zero remove only one linear condition from the 9-dimensional space, so their dimension is 8.

Only the skew symmetric matrices have dimension 3, matching the quotient space. There is even a concrete isomorphism: every coset $A + T_3(\mathbb{R})$ has a unique strictly lower triangular representative L (found

by using elements of $T_3(\mathbb{R})$ to cancel the diagonal and upper part of A), and the map $L \mapsto L - L^T$ carries this 3-dimensional space of strictly lower triangular matrices bijectively and linearly onto the skew symmetric matrices.

Step 4: Final Answer.
Since the quotient space has dimension 3, matching only the skew symmetric matrices, with an explicit linear isomorphism between the two, the quotient space $M_3(\mathbb{R})/T_3(\mathbb{R})$ is isomorphic to the vector space of 3×3 real skew symmetric matrices.

$$\boxed{M_3(\mathbb{R})/T_3(\mathbb{R}) \cong \{\text{skew symmetric } 3 \times 3 \text{ matrices}\}}$$

Quick Tip: Compare dim(M3(R)) minus dim(T3(R)) with the dimension of each candidate matrix space; only one option's dimension can match.

12. Let I be the integral defined as follows:

$$I = \int_0^1 \int_0^{\sqrt{y}} dx\,dy + \int_1^2 \int_{\sqrt{y-1}}^1 dx\,dy$$

If the order of integration is changed, then which one of the following is the correct expression for I?

(A)

$$\int_0^{1/2}\int_{-x}^{x} dy\,dx + \int_{1/2}^{1}\int_{-x^2}^{-x^2+1} dy\,dx$$

(B)

$$\int_0^{1}\int_{x^2}^{x^2+1} dy\,dx$$

(C)

$$\int_0^{1/2}\int_{x^2}^{x^2+1} dy\,dx + \int_{1/2}^{1}\int_{\sqrt{x}}^{\sqrt{x+1}} dy\,dx$$

(D)

$$\int_0^{1}\int_{x^2}^{x^2-1} dy\,dx$$

Correct Answer: (B)

$$\int_0^{1}\int_{x^2}^{x^2+1} dy\,dx$$

Solution:

Step 1: Understanding the Question.

We are given I as the sum of two double integrals in the order $dx\,dy$, and we need to sketch the region each one covers, then rewrite the same region in the order $dy\,dx$.

Step 2: Key Formula or Approach.

For a region described by $0 \le x \le \sqrt{y}$, squaring the inequality gives $x^2 \le y$, which describes the same region using a curve in x. We use this trick on both pieces, then check where the two pieces join so we can

write the combined region as a single integral in x first.

Step 3: Detailed Explanation.

First piece: $0 \le y \le 1, 0 \le x \le \sqrt{y}$. Squaring $x \le \sqrt{y}$ gives $x^2 \le y$. Since $y \le 1$ and $x \le \sqrt{y} \le 1$, this region is exactly $x^2 \le y \le 1$ for $0 \le x \le 1$.

Second piece: $1 \le y \le 2, \sqrt{y-1} \le x \le 1$. Squaring $\sqrt{y-1} \le x$ gives $y - 1 \le x^2$, i.e. $y \le x^2 + 1$. Combined with $y \ge 1$, this region is exactly $1 \le y \le x^2 + 1$ for $0 \le x \le 1$ (since $x^2 + 1 \le 2$ whenever $x \le 1$).

The two pieces share the boundary $y = 1$, so together they form one continuous region: for every x from 0 to 1, y runs from x^2 up to $x^2 + 1$, with no need to split the range of x. So

$$I = \int_0^1 \int_{x^2}^{x^2+1} dy\, dx$$

which is option (B). As a check, compute the value both ways. Directly, $\int_0^1 \sqrt{y}\, dy = \frac{2}{3}$ and $\int_1^2 \left(1 - \sqrt{y-1}\right) dy = \frac{1}{3}$, giving $I = \frac{2}{3} + \frac{1}{3} = 1$. From option (B), $\int_0^1 \left[(x^2 + 1) - x^2\right] dx = \int_0^1 1\, dx = 1$, the same value.

Option (A) uses the curves $y = -x$, $y = x$ and $y = -x^2$, $y = -x^2 + 1$, which never appear in the original region description; it evaluates to $\frac{3}{4}$, not 1. Option (C) splits at $x = \frac{1}{2}$ and brings in the curves $y = \sqrt{x}$, $y = \sqrt{x+1}$, also not part of the original region; it evaluates to an irrational number, not 1. Option (D) has upper limit $x^2 - 1$ below the lower limit x^2, an impossible negative height for a genuine region, and it evaluates to -1, the wrong sign.

Step 4: Final Answer.

The single merged region $x^2 \le y \le x^2 + 1, 0 \le x \le 1$, reproduces both the shape and the value of I.

$$\boxed{I = \int_0^1 \int_{x^2}^{x^2+1} dy\,dx = 1}$$

Quick Tip: Sketch both original regions in the xy-plane and see exactly where they join along the line y = 1.

13. Let C denote the Cantor set and $f : [0,1] \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} x^{2026} & \text{for } x \in C \\ \cos(\pi x) & \text{for } x \in \left[0, \frac{1}{2}\right] \setminus C \\ \sin(\pi x) & \text{for } x \in \left[\frac{1}{2}, 1\right] \setminus C \end{cases}$$

The value of the Lebesgue integral of $f(x)$ over the interval $[0,1]$ is equal to

(A) $\frac{2}{\pi}$

(B) $\frac{1}{\pi}$

(C) $\frac{3}{\pi}$

(D) 0

Correct Answer: (A) $\frac{2}{\pi}$

Solution:

Step 1: Understanding the Question.

The function f is defined piecewise, using one rule on the Cantor set C, one rule on $[0, \frac{1}{2}] \setminus C$, and one rule on $[\frac{1}{2}, 1] \setminus C$. We need the Lebesgue integral of f over $[0,1]$.

Step 2: Key Formula or Approach.

The Cantor set C has Lebesgue measure zero. A key fact about the Lebesgue integral is that a bounded function integrated over a set of measure zero always gives 0, and changing a function's values on a measure zero set never changes its integral over a larger set.

Step 3: Detailed Explanation.

Since $m(C) = 0$, the contribution of the branch x^{2026} on C to the integral is

$$\int_C x^{2026}\,dx = 0$$

no matter how the values x^{2026} behave on C.

Also, since C has measure zero, removing it from $[0, \frac{1}{2}]$ or from $[\frac{1}{2}, 1]$ does not change the measure of those pieces, so the Lebesgue integral over $[0, \frac{1}{2}] \setminus C$ equals the ordinary integral of $\cos(\pi x)$ over the full interval $[0, \frac{1}{2}]$, and similarly for $\sin(\pi x)$ over $[\frac{1}{2}, 1]$. So

$$\int_0^1 f(x)\,dx = \int_0^{1/2} \cos(\pi x)\,dx + \int_{1/2}^1 \sin(\pi x)\,dx$$

Compute the first piece:

$$\int_0^{1/2} \cos(\pi x)\,dx = \left[\frac{\sin(\pi x)}{\pi}\right]_0^{1/2} = \frac{\sin(\pi/2) - \sin 0}{\pi} = \frac{1}{\pi}$$

Compute the second piece:

$$\int_{1/2}^1 \sin(\pi x)\,dx = \left[-\frac{\cos(\pi x)}{\pi}\right]_{1/2}^1 = -\frac{\cos\pi}{\pi} + \frac{\cos(\pi/2)}{\pi} = \frac{1}{\pi} + 0 = \frac{1}{\pi}$$

Adding the two pieces gives $\frac{1}{\pi} + \frac{1}{\pi} = \frac{2}{\pi}$.
Option (B), $\frac{1}{\pi}$, only counts one of the two pieces. Option (C), $\frac{3}{\pi}$, adds an extra $\frac{1}{\pi}$ that has no source in the problem. Option (D), 0, would only hold if the two pieces canceled, but both pieces are positive and add up instead.

Step 4: Final Answer.
The Lebesgue integral of f over $[0, 1]$ equals $\frac{2}{\pi}$.

$$\boxed{\frac{2}{\pi}}$$

Quick Tip: The Cantor set has Lebesgue measure zero, so its branch of f never affects the value of the integral.

14. Let G be a group of order 595. Which one of the following is TRUE?

(A) G cannot have a proper normal subgroup.

(B) G must have a proper normal subgroup.

(C) G cannot have an element of order 17.

(D) The number of Sylow 5-subgroups is 17.

Correct Answer: (B) G must have a proper normal subgroup.

Solution:

Step 1: Understanding the Question.
We are given a group G of order 595 and need to decide which statement about its subgroups must be true. The first thing to do is factor 595 into primes.

Step 2: Key Formula or Approach.

We use Sylow's theorems. If n_p is the number of Sylow p-subgroups of a group of order n, then n_p divides n/p^a (where p^a is the highest power of p dividing n), and $n_p \equiv 1 \pmod{p}$. If $n_p = 1$, that unique Sylow p-subgroup is normal in G. We also use Cauchy's theorem: if a prime p divides $|G|$, then G has an element of order p.

Step 3: Detailed Explanation.

Factor $595 = 5 \times 7 \times 17$, three distinct primes, each to the first power. Look at the Sylow 5-subgroups. Here n_5 divides $7 \times 17 = 119$ and $n_5 \equiv 1 \pmod{5}$. The divisors of 119 are $1, 7, 17, 119$. Checking each mod 5: $1 \equiv 1$, $7 \equiv 2$, $17 \equiv 2$, $119 \equiv 4$. Only 1 satisfies the congruence, so $n_5 = 1$ is forced.

A unique Sylow 5-subgroup is automatically normal, and it has order 5, a proper subgroup of G since $5 < 595$. So G always has a proper normal subgroup, no matter which group of order 595 we picked. This makes option (B) TRUE and option (A) false.

Since 17 divides $|G| = 595$, Cauchy's theorem guarantees G has an element of order 17, so option (C) is false.

For option (D), check n_{17}: it divides $5 \times 7 = 35$ and satisfies $n_{17} \equiv 1 \pmod{17}$. The divisors of 35 are $1, 5, 7, 35$; mod 17 these are $1, 5, 7, 1$, so $n_{17} \in \{1, 35\}$. Also 17 itself does not even divide 35, so n_{17} can never equal 17. And separately, we already showed $n_5 = 1$, not 17. So option (D) is false either way it is read.

Step 4: Final Answer.

The unique, hence normal, Sylow 5-subgroup of order 5 is always a proper normal subgroup of G.

Option B: G must have a proper normal subgroup

Quick Tip: Factor 595 = 5 x 7 x 17 and check which Sylow subgroup count is forced to equal 1 by the congruence condition.

15. Let $u(x,t)$ be the solution of the initial value problem for the heat equation on the real line:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \qquad t > 0, \qquad k \in \mathbb{R}$$

with the initial condition

$$u(x,0) = e^{-a|x|}, \qquad a > 0.$$

If $\hat{u}(w,t) = \int_{-\infty}^{\infty} u(x,t)\, e^{iwx}\, dx$ is the Fourier transform of $u(x,t)$ with respect to x, then $\hat{u}(w,t)$ is equal to

(A)

$$\frac{2a}{a^2+w^2}\,e^{-kw^2t}$$

(B)

$$\frac{a}{a^2+w^2}\,e^{-kw^2t}$$

(C)

$$\frac{2a}{a^2+w^2}\,e^{-ka^2t}$$

(D)

$$\frac{1}{\sqrt{4\pi kt}}\,e^{\frac{-w^2}{4kt}}$$

Correct Answer: (A)

$$\frac{2a}{a^2+w^2}\,e^{-kw^2t}$$

Solution:

Step 1: Understanding the Question.

We are given the heat equation and an initial condition, and need the Fourier transform of the solution $u(x,t)$ with respect to x, using the convention $\hat{u}(w,t) = \int u(x,t)e^{iwx}\,dx$.

Step 2: Key Formula or Approach.

With this convention, taking the Fourier transform turns ∂_x into multiplication by $-iw$, so ∂_x^2 becomes multiplication by $(-iw)^2 = -w^2$. Taking the transform of both sides of the heat equation in x turns the PDE into an ordinary differential equation in t for each fixed w. We also need the Fourier transform of the initial data $e^{-a|x|}$.

Step 3: Detailed Explanation.

Applying the Fourier transform in x to both sides of $u_t = ku_{xx}$ gives

$$\frac{\partial \hat{u}}{\partial t} = k(-w^2)\hat{u} = -kw^2\hat{u}$$

This is a simple first order linear ODE in t (with w fixed as a parameter), whose solution is

$$\hat{u}(w,t) = \hat{u}(w,0)\,e^{-kw^2t}$$

Now find $\hat{u}(w,0)$, the transform of $u(x,0) = e^{-a|x|}$. Split the integral at $x = 0$:

$$\hat{u}(w,0) = \int_0^\infty e^{-ax}e^{iwx}\,dx + \int_{-\infty}^0 e^{ax}e^{iwx}\,dx = \frac{1}{a-iw} + \frac{1}{a+iw} = \frac{2a}{a^2+w^2}$$

Combining the two results,

$$\hat{u}(w,t) = \frac{2a}{a^2+w^2}\,e^{-kw^2t}$$

which is option (A). Option (B) is missing the factor of 2 that comes from adding the two halves of the integral for the initial data. Option (C) puts a^2 in the time-decay exponent instead of w^2, but the time decay must depend on w (the Fourier variable dual to x), not on a (which only measures how fast the initial data itself decays). Option (D) is the transform of the heat kernel alone, the solution for a delta function initial condition; it has no dependence on a at all, so it cannot be correct once the actual initial data $e^{-a|x|}$ is used.

Step 4: Final Answer.

The Fourier transform of the solution is the transform of the initial data multiplied by the time-decay factor from solving the transformed heat equation.

$$\boxed{\hat{u}(w,t) = \frac{2a}{a^2 + w^2} e^{-kw^2 t}}$$

Quick Tip: Transform the heat equation in x first to get a simple ODE in t, then separately transform the initial condition e^{-a|x|}.

16. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$f(x, y) = (e^x \cos y,\ e^x \sin y)$$

Which one of the following is TRUE?

(A) f is one-to-one.

(B) The Jacobian of f is negative.

(C) f is locally invertible.

(D) f is invertible on $\mathbb{R}^2$.

Correct Answer: (C) f is locally invertible.

Solution:

Step 1: Write the map in terms of the Jacobian.

The map is $f(x, y) = (e^x \cos y,\ e^x \sin y)$. Call the two component functions $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. To check invertibility we first find the Jacobian matrix of partial derivatives.

Step 2: Compute the Jacobian matrix and its determinant.

$$u_x = e^x \cos y, \quad u_y = -e^x \sin y, \quad v_x = e^x \sin y, \quad v_y = e^x \cos y$$

So the Jacobian determinant is

$$J = u_x v_y - u_y v_x = e^{2x} \cos^2 y + e^{2x} \sin^2 y = e^{2x}$$

Since $e^{2x} > 0$ for every real x, the Jacobian is never zero and never negative. This already rules out option (B).

Step 3: Use the Inverse Function Theorem.

The Inverse Function Theorem says that if the Jacobian determinant of a continuously differentiable map is nonzero at a point, the map is locally invertible near that point, meaning it has a smooth local inverse on some small neighborhood.

Here $J = e^{2x} \neq 0$ at every point of $\mathbb{R}^2$, so f is locally invertible at every point. This makes option (C) true.

Step 4: Check global injectivity, which is why A and D fail.

Being locally invertible everywhere does not mean f is globally one-to-one. Notice that $f(x, y)$ depends on y only through $\cos y$ and $\sin y$, and both repeat every 2π.

So $f(0, 0) = (e^0 \cos 0,\ e^0 \sin 0) = (1, 0)$ and $f(0, 2\pi)$ $= (e^0 \cos 2\pi,\ e^0 \sin 2\pi) = (1, 0)$ as well.

Two different points, $(0, 0)$ and $(0, 2\pi)$, give the same output, so f is not one-to-one. This rules out option (A). Since f fails to be injective on all of $\mathbb{R}^2$, it cannot be invertible on $\mathbb{R}^2$ either, which rules out option (D).

Final Answer:

The Jacobian of f is nonzero everywhere, so by the Inverse Function Theorem, f is locally invertible at every point of $\mathbb{R}^2$, even though it is not globally one-to-one.

$$\boxed{\text{(C) } f \text{ is locally invertible.}}$$

Quick Tip: Compute the Jacobian determinant of f, then compare local invertibility with global injectivity using the periodicity of sine and cosine in y.

17. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = |z|^2 - 5\bar{z} + 2$.
Which one of the following is TRUE?

(A) f is differentiable at $z = 5i$.

(B) f is differentiable at $z = 5$.

(C) f is differentiable at $z = -5i$.

(D) f is differentiable at $z = -5$.

Correct Answer: (B) f is differentiable at $z = 5$.

Solution:

Step 1: Write f in terms of real and imaginary parts.

Let $z = x + iy$ with x, y real. Then $|z|^2 = x^2 + y^2$ and $\bar{z} = x - iy$.

$$f(z) = |z|^2 - 5\bar{z} + 2 = (x^2 + y^2) - 5(x - iy) + 2 = (x^2 + y^2 - 5x + 2) + i(5y)$$

So if we write $f = u + iv$, then $u(x, y) = x^2 + y^2 - 5x + 2$ and $v(x, y)$

$= 5y$.

Step 2: Recall the Cauchy-Riemann equations.
A function $f = u + iv$ is complex differentiable at a point exactly where its partial derivatives are continuous there and satisfy the Cauchy-Riemann equations:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

Since u and v here are polynomials in x, y, their partials are continuous everywhere, so we only need to check where the two equations hold.

Step 3: Compute the partial derivatives and solve.

$$u_x = 2x - 5, \quad u_y = 2y, \quad v_x = 0, \quad v_y = 5$$

Setting $u_x = v_y$ gives $2x - 5 = 5$, so $x = 5$.
Setting $u_y = -v_x$ gives $2y = 0$, so $y = 0$.
Both Cauchy-Riemann equations hold only at the single point $(x, y) = (5, 0)$, which is $z = 5$.

Step 4: Check each option against this point.
Option (A) asks about $z = 5i$, that is $(x, y) = (0, 5)$: here $x = 0 \neq 5$, so the equations fail; not differentiable.
Option (C) asks about $z = -5i$, that is $(0, -5)$: again $x = 0 \neq 5$; not differentiable.
Option (D) asks about $z = -5$, that is $(-5, 0)$: here $x = -5 \neq 5$; not differentiable.
Option (B) asks about $z = 5$, that is $(5, 0)$: this is exactly the point we

found, so f is differentiable there.

Final Answer:

f is complex differentiable only at the single point $z = 5$, matching option (B).

$$\boxed{z = 5}$$

Quick Tip: Split f into real and imaginary parts and solve the Cauchy-Riemann equations for x and y.

18. Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ denote the unit disc in the complex plane $\mathbb{C}$. Let $f : D \to D$ be an analytic function which satisfies $f(0) = 0$. Then which one of the following is a possible value of $f'(0)$?

(A) $\frac{5}{2}i$

(B) $\frac{i}{10}$

(C) $\frac{3}{2}$

(D) $-\frac{5}{2}i$

Correct Answer: (B) $\frac{i}{10}$

Solution:

Step 1: Identify the tool needed.

We are told $D = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disc, and $f : D \to D$ is analytic with $f(0) = 0$. This is exactly the setup for the Schwarz Lemma.

Step 2: State the Schwarz Lemma.

The Schwarz Lemma says that if $f : D \to D$ is analytic and $f(0) = 0$, then for every $z \in D$,

$$|f(z)| \leq |z|$$

and in particular, letting $z \to 0$, the derivative at the origin satisfies

$$|f'(0)| \leq 1$$

Step 3: Apply this bound to each option.

Since $f'(0)$ must satisfy $|f'(0)| \leq 1$, any option whose magnitude exceeds 1 cannot be a possible value of $f'(0)$. Check the magnitude of each option:

$$\left|\frac{5}{2}i\right| = 2.5, \quad \left|\frac{i}{10}\right| = 0.1, \quad \left|\frac{3}{2}\right| = 1.5, \quad \left|-\frac{5}{2}i\right| = 2.5$$

Options (A), (C), and (D) all have magnitude greater than 1, so none of them can be $f'(0)$ for an analytic self-map of the disc fixing 0. Option (B), $\frac{i}{10}$, has magnitude 0.1, which is well within the allowed bound of 1.

Step 4: Confirm it is actually achievable.

Take the simple linear map $f(z) = \frac{i}{10}z$. This sends D into D since $|f(z)| = \frac{1}{10}|z| < |z| < 1$ for $z \in D$, it is analytic since it is a polynomial, and $f(0) = 0$ with $f'(0) = \frac{i}{10}$. So this value is genuinely attainable, not just consistent with the bound.

Final Answer:

Only $\frac{i}{10}$ satisfies $|f'(0)| \leq 1$ from the Schwarz Lemma, and it is realized by $f(z) = \frac{i}{10}z$.

$$\boxed{f'(0) = \frac{i}{10}}$$

Quick Tip: Apply the Schwarz Lemma to bound the magnitude of f'(0) for an analytic self-map of the disc that fixes 0.

19. Let X be the space of all continuously differentiable real valued functions on $[0, 1]$. Define the following norms on X:

$$p_1(x) = \sup\{|x(t)| : t \in [0, 1]\}$$

$$p_2(x) = \sup\left\{\left|\frac{d}{dt}x(t)\right| : t \in [0, 1]\right\}$$

and $p_3(x) = p_1(x) + p_2(x)$.
Which one of the following is TRUE?

(A) (X, p_1) is a Banach space.

(B) (X, p_2) is a Banach space.

(C) (X, p_3) is not a Banach space.

(D) (X, p_3) is a Banach space.

Correct Answer: (D) (X, p_3) is a Banach space.

Solution:

Step 1: Understand the space and the three norms.

X is the set of all continuously differentiable real functions on $[0,1]$, often written $C^1[0,1]$. The three quantities given are

$$p_1(x) = \sup_{t\in[0,1]} |x(t)|, \qquad p_2(x) = \sup_{t\in[0,1]} |x'(t)|, \qquad p_3(x) = p_1(x) + p_2(x)$$

We need to decide which of the four statements about completeness, meaning being a Banach space, a complete normed space, is true.

Step 2: Test p1 for completeness.

p_1 is just the usual sup norm, applied only to functions that happen to be C^1. Take a sequence of smooth functions x_n that converges uniformly to a function like $g(t) = |t - \frac{1}{2}|$, which is continuous but has a sharp corner at $t = \frac{1}{2}$ and so is not differentiable there.

This sequence is Cauchy in p_1, since it converges uniformly, but its limit g does not belong to X, because g is not continuously differentiable. So this Cauchy sequence has no limit inside X under p_1. This means (X, p_1) is not complete, so option (A) is false.

Step 3: Test p2, and notice it is not even a genuine norm.

A norm must be zero only for the zero function. Check p_2 on a nonzero constant function, say $x(t) = 5$ for all t. Its derivative is $x'(t) = 0$, so

$$p_2(x) = \sup_t |x'(t)| = 0$$

even though x is not the zero function. So p_2 fails the positive-definiteness property of a norm; it is only a seminorm. Since (X, p_2) is not even a genuine normed space, it cannot be called a Banach space in the standard sense. This rules out option (B).

Step 4: Test p3, the combined norm.

$p_3(x) = p_1(x) + p_2(x)$ is the standard C^1 norm: it is zero only when both $\sup|x| = 0$ and $\sup|x'| = 0$, which forces $x \equiv 0$. So unlike p_2 alone, p_3 is a genuine norm.

Now take any sequence x_n that is Cauchy in p_3. This means both x_n and x_n' are uniformly Cauchy, so both converge uniformly: $x_n \to g$ and $x_n' \to h$ for some continuous g, h.

A standard real-analysis fact says that if $x_n \to g$ uniformly and $x_n' \to h$ uniformly, then g is differentiable with $g' = h$. So g is continuously differentiable, meaning $g \in X$, and $x_n \to g$ in the p_3 norm since both parts converge uniformly.

So every p_3-Cauchy sequence converges to a limit inside X. This means (X, p_3) is complete, so it is a Banach space. This makes option (D) true and rules out option (C).

Final Answer:

Only the combined norm $p_3 = p_1 + p_2$, which controls both the function and its derivative uniformly, makes $X = C^1[0,1]$ complete.

$$\boxed{\text{(D) } (X, p_3) \text{ is a Banach space.}}$$

Quick Tip: Check whether each candidate is complete, and first check whether p2 alone is even a valid norm on constant functions.

20. Let X be a set with at least two elements. Let d_1, d_2 and d_3 be metrics on X.

Which one of the following is NOT a metric on X?

(A) $d(x,y) := \min(d_1(x,y), 3)$ for all $x, y \in X$.

(B) $d(x,y) := \max(d_2(x,y), 3)$ for all $x, y \in X$.

(C) $d(x,y) := \dfrac{10\, d_3(x,y)}{1 + d_3(x,y)}$ for all $x, y \in X$.

(D) $d(x,y) := \dfrac{1}{3}(d_1(x,y) + d_2(x,y) + d_3(x,y))$ for all $x, y \in X$.

Correct Answer: (B) $d(x,y) := \max(d_2(x,y), 3)$ for all $x, y \in X$.

Solution:

Step 1: Recall the axioms a metric must satisfy.

A function $d : X \times X \to [0, \infty)$ is a metric if, for all $x, y, z \in X$:

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$, symmetry;

(iii) $d(x,z) \le d(x,y) + d(y,z)$, the triangle inequality.

We are told d_1, d_2, d_3 are already metrics on X, and we must find which combination built from them fails one of these three properties.

Step 2: Check option (B) at x = y first, since that is the fastest way to break it.

Option (B) defines $d(x,y) = \max(d_2(x,y), 3)$. Take $x = y$; this is allowed since the definition is stated "for all $x, y \in X$", which includes $x = y$.

Since d_2 is a metric, $d_2(x,x) = 0$. So

$$d(x,x) = \max(d_2(x,x), 3) = \max(0, 3) = 3$$

But a metric must give $d(x,x) = 0$, and here we get 3 instead. So option (B) violates axiom (i) for every single point $x \in X$, and is therefore not a metric.

Step 3: Confirm the other three options do satisfy all the axioms.

For option (A), $d(x,y) = \min(d_1(x,y), 3)$: at $x = y$, $\min(0,3) = 0$, so axiom (i) holds there; symmetry is immediate since d_1 is symmetric; and it is a standard fact that capping a metric with $\min(d, c)$ for a constant $c > 0$ keeps the triangle inequality, since $\min(a,c) \le \min(b,c) + \min(e,c)$ whenever $a \le b + e$, which can be checked by splitting into cases on whether b, e exceed c. So (A) is a genuine metric.

For option (C), $d(x,y) = \dfrac{10\, d_3(x,y)}{1 + d_3(x,y)}$: this is the standard bounded-metric construction. Since $t \mapsto \dfrac{10t}{1+t}$ is increasing and satisfies $\dfrac{10(a+b)}{1+a+b} \le \dfrac{10a}{1+a} + \dfrac{10b}{1+b}$ for $a, b \ge 0$, a routine algebraic check, applying it to d_3, which already satisfies the triangle inequality, keeps the triangle inequality; it clearly vanishes only when $d_3(x,y) = 0$, meaning $x = y$; so (C) is a genuine metric.

For option (D), $d(x,y) = \frac{1}{3}(d_1(x,y) + d_2(x,y) + d_3(x,y))$: a positive combination, an average, of metrics is again a metric, since each axiom, being zero exactly on the diagonal, symmetry, and the triangle inequality, is preserved under addition and positive scaling. So (D) is a genuine metric.

Final Answer:

Option (B) is the only one that fails, because forcing a floor of 3 with $\max(\cdot, 3)$ makes $d(x,x) = 3$ instead of 0.

$$\boxed{\text{(B) } d(x,y) := \max(d_2(x,y), 3) \text{ is NOT a metric.}}$$

Quick Tip: Test each candidate distance function at x = y first and see whether it still comes out to zero.

21. Let P be a 3×3 real symmetric positive definite matrix. Which of the following statements is/are TRUE?

(A) Q^TPQ is positive definite for all nonzero $Q \in \mathbb{R}^{3\times3}$.

(B) Q^TPQ is positive semidefinite for all $Q \in \mathbb{R}^{3\times3}$.

(C) Q^TPQ is positive definite if $Q \in \mathbb{R}^{3\times3}$ is nonsingular.

(D) Q^TPQ is not positive semidefinite for some $Q \in \mathbb{R}^{3\times3}$.

Correct Answer: (B) Q^TPQ is positive semidefinite for all $Q \in \mathbb{R}^{3\times3}$. ; (C) Q^TPQ is positive definite if $Q \in \mathbb{R}^{3\times3}$ is nonsingular.

Solution:

Step 1: Set up the link between the two matrices.

A real symmetric matrix P is positive definite when $v^TPv > 0$ for every nonzero vector $v \in \mathbb{R}^3$, and positive semidefinite when $v^TPv \geq 0$ for every vector v, where equality is allowed for some nonzero v.

For any real 3×3 matrix Q and any vector v, we get

$$v^T(Q^TPQ)v = (Qv)^TP(Qv)$$

So the sign of the quadratic form of Q^TPQ at v depends on where Qv lands and what P does there.

Step 2: Test option (A).

Pick a nonzero but singular Q, for example $Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. This Q is nonzero, but it sends the nonzero vector $v = (0, 0, 1)$ to 0. Then

$$v^T(Q^TPQ)v = (Qv)^TP(Qv) = 0^TP0 = 0$$

So Q^TPQ is not positive definite for this choice of Q, even though Q is nonzero. Option (A) claims this holds for every nonzero Q, so one failing case is enough to make (A) FALSE.

Step 3: Test option (C).
Now suppose Q is nonsingular. Then $Qv = 0$ can only happen when $v = 0$, because a nonsingular matrix has trivial null space. So for every nonzero v, the vector Qv is nonzero, and since P is positive definite,

$$(Qv)^TP(Qv) > 0$$

This gives $v^T(Q^TPQ)v > 0$ for every nonzero v, which is exactly the definition of Q^TPQ being positive definite. Option (C) is TRUE.

Step 4: Test options (B) and (D) together.
For any real Q, singular or not, and for any vector v, the vector Qv is just some vector in $\mathbb{R}^3$ (possibly zero). Since P is positive definite, $w^TPw \geq 0$ for every vector w, including $w = Qv$. So

$$v^T(Q^TPQ)v = (Qv)^TP(Qv) \geq 0 \quad \text{for every } v$$

This holds without any restriction on Q, so Q^TPQ is always positive semidefinite. That makes option (D), which says Q^TPQ fails to be positive semidefinite for some Q, FALSE, and it makes option (B) TRUE.

Final Answer:
Q^TPQ is always positive semidefinite (option B), and it is positive

definite precisely when Q is nonsingular (option C). Options (A) and (D) fail.

$$\boxed{\text{(B) and (C)}}$$

Quick Tip: Write $v^{\top}(Q^{\top}PQ)v = (Qv)^{\top}P(Qv)$ and check what happens to Qv when Q is singular versus nonsingular.

22. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined as follows:

$$f(x,y) = \begin{cases} \dfrac{x\sqrt{x^2+y^2}}{|x|} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Which of the following statements is/are TRUE?

(A) $f(x,y)$ is continuous at $(0,0)$.

(B) For any $u \in \mathbb{R}^2$, the directional derivative of f at $(0,0)$ in the direction of u exists.

(C) $\dfrac{\partial f}{\partial x}(0,0) = 1 = \dfrac{\partial f}{\partial y}(0,0)$.

(D) $f(x,y)$ is differentiable at $(0,0)$.

Correct Answer: (A) $f(x,y)$ is continuous at $(0,0)$. ; (B) For any $u \in \mathbb{R}^2$, the directional derivative of f at $(0,0)$ in the direction of u exists.

Solution:

Step 1: Rewrite the function in a simpler form.

For $x \neq 0$, $\dfrac{x}{|x|}$ equals 1 when $x > 0$ and -1 when $x < 0$; call this $\text{sign}(x)$. So for $x \neq 0$,

$$f(x,y) = \text{sign}(x)\sqrt{x^2+y^2}$$

and $f(0,y) = 0$ for every y. This form makes it much easier to check each property at the origin.

Step 2: Check continuity at $(0,0)$, option (A).
For every (x,y), whether x is zero or not, we have

$$|f(x,y)| \le \sqrt{x^2+y^2}$$

because $|\text{sign}(x)| = 1$ when $x \neq 0$, and $f(0,y) = 0$ matches the bound too. As $(x,y) \to (0,0)$ along any path, the right side $\sqrt{x^2+y^2} \to 0$, so by the squeeze theorem $f(x,y) \to 0 = f(0,0)$. This holds along every path, so f is continuous at $(0,0)$. Option (A) is TRUE.

Step 3: Check the directional derivative for a general direction, option (B).
Take any unit vector $u = (u_1, u_2)$ and look at $g(t) = f(tu_1, tu_2)$ for small $t > 0$.
If $u_1 \neq 0$: $g(t) = \text{sign}(tu_1)\sqrt{t^2u_1^2 + t^2u_2^2} = \text{sign}(u_1) \cdot t\sqrt{u_1^2+u_2^2}$
$= \text{sign}(u_1) \cdot t$ (since u is a unit vector). So

$$\lim_{t \to 0^+} \frac{g(t) - g(0)}{t} = \text{sign}(u_1)$$

which is a finite number. If $u_1 = 0$ (direction along the y-axis), then $g(t) = f(0, tu_2) = 0$ for every t, so the derivative is 0. Either way, the directional derivative exists and is finite for every direction u. Option (B) is TRUE.

Step 4: Check the partial derivatives, option (C).

Using Step 3 with $u = (1, 0)$: $\frac{\partial f}{\partial x}(0, 0) = \text{sign}(1) = 1$.

Using Step 3 with $u = (0, 1)$: since $u_1 = 0$, $\frac{\partial f}{\partial y}(0, 0) = 0$.

So $\frac{\partial f}{\partial x}(0, 0) = 1$ but $\frac{\partial f}{\partial y}(0, 0) = 0$, not 1. Option (C) claims both equal 1, so (C) is FALSE.

Step 5: Check differentiability, option (D).

If f were differentiable at $(0, 0)$, the directional derivative in direction $u = (u_1, u_2)$ would have to equal the linear expression $\nabla f(0, 0) \cdot u = 1 \cdot u_1 + 0 \cdot u_2 = u_1$. But Step 3 showed the actual directional derivative is $\text{sign}(u_1)$, which is $+1$ or -1 and does not vary smoothly with u_1 the way a linear function does. For instance, take $u = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$: the linear guess gives $\frac{1}{\sqrt{2}} \approx 0.707$, but the true directional derivative is $\text{sign}(u_1) = 1$. Since these do not match, f is not differentiable at $(0, 0)$. Option (D) is FALSE.

Final Answer:

f is continuous at the origin and every directional derivative there exists, so (A) and (B) hold. But the two partial derivatives are not both 1, and f is not differentiable, so (C) and (D) fail.

$$\boxed{\text{(A) and (B)}}$$

Quick Tip: Rewrite $f(x, y) = \text{sign}(x)\sqrt{x^2 + y^2}$ for $x \neq 0$ and check whether this bound goes to zero along every path, and whether the directional derivative is linear in direction.

23. Let $\mathbb{Z}_{11}$ be the ring of integers modulo 11. Let $\mathbb{Z}_{11}[x]$ be the ring of all polynomials with coefficients in $\mathbb{Z}_{11}$. Which of the following statements is/are TRUE?

(A) The polynomial $x^2 + x + 4$ is irreducible over $\mathbb{Z}_{11}$.

(B) $\mathbb{Z}_{11}[x]/\langle x^2 + x + 4\rangle$ is a field.

(C) The ideal $\langle x^2 + x + 4\rangle$ is maximal.

(D) The ideal $\langle x^2 + x + 4\rangle$ is not a prime ideal.

Correct Answer: (A) The polynomial $x^2 + x + 4$ is irreducible over $\mathbb{Z}_{11}$. ; (B) $\mathbb{Z}_{11}[x]/\langle x^2 + x + 4\rangle$ is a field. ; (C) The ideal $\langle x^2 + x + 4\rangle$ is maximal.

Solution:

Step 1: Check whether $x^2 + x + 4$ has a root in $\mathbb{Z}_{11}$.

A degree 2 polynomial over a field is irreducible exactly when it has no root in that field (if it had a root r, it would factor as $(x - r)$ times a linear factor). So test every element of $\mathbb{Z}_{11} = \{0, 1, 2, \dots, 10\}$ in $x^2 + x + 4$:

$0 \to 4,\ 1 \to 6,\ 2 \to 10,\ 3 \to 5,\ 4 \to 2,\ 5 \to 1,\ 6 \to 2,\ 7 \to 5,\ 8 \to 10,\ 9 \to 6,\ 10 \to 4$ (all values taken mod 11).

None of these equal 0, so $x^2 + x + 4$ has no root in $\mathbb{Z}_{11}$, which means it is irreducible over $\mathbb{Z}_{11}$. Option (A) is TRUE.

Step 2: Use the fact that $\mathbb{Z}_{11}[x]$ is a principal ideal domain (PID).

Since $\mathbb{Z}_{11}$ is a field, $\mathbb{Z}_{11}[x]$ behaves like the integers do for prime numbers: every ideal is generated by a single polynomial, and an ideal generated by an irreducible polynomial is a maximal ideal. Since Step 1 showed $x^2 + x + 4$ is irreducible, the ideal $\langle x^2 + x + 4\rangle$ is maximal. Option (C) is TRUE.

Step 3: Use the maximal ideal to conclude the quotient is a field.

For a commutative ring R with unity, a basic theorem says an ideal M is maximal if and only if the quotient ring R/M is a field. Since Step 2 showed $\langle x^2 + x + 4 \rangle$ is maximal in $\mathbb{Z}_{11}[x]$, the quotient $\mathbb{Z}_{11}[x]/\langle x^2 + x + 4 \rangle$ must be a field. Option (B) is TRUE. In fact this field has $11^2 = 121$ elements, since every element can be written uniquely as $a + bx$ with $a, b \in \mathbb{Z}_{11}$.

Step 4: Check option (D).

Another basic fact is that every maximal ideal in a commutative ring with unity is also a prime ideal (a field, which R/M is, has no zero divisors, and that is exactly what makes M prime). Since we showed $\langle x^2 + x + 4 \rangle$ is maximal, it is automatically prime. So the claim in (D), that this ideal is not prime, is FALSE.

Final Answer:

$x^2 + x + 4$ is irreducible over $\mathbb{Z}_{11}$, so the ideal it generates is maximal and the quotient ring is a field; that same ideal is therefore also prime, making (D) false.

$$\boxed{\text{(A), (B) and (C)}}$$

Quick Tip: Check whether $x^2 + x + 4$ has a root in $\mathbb{Z}_{11}$; no root means irreducible, and in a PID that makes the ideal maximal and the quotient a field.

24. Consider $\mathbb{R}$ with the topology

$$\tau = \{A \subseteq \mathbb{R} : A^c \text{ is finite}\} \cup \{\phi\}.$$

Which of the following statements is/are TRUE?

(A) $(\mathbb{R}, \tau)$ is a Hausdorff space.

(B) Any finite subset of $(\mathbb{R}, \tau)$ is closed.

(C) $(\mathbb{R}, \tau)$ is compact.

(D) $(\mathbb{R}, \tau)$ is connected.

Correct Answer: (B) Any finite subset of $(\mathbb{R}, \tau)$ is closed. ; (C) $(\mathbb{R}, \tau)$ is compact. ; (D) $(\mathbb{R}, \tau)$ is connected.

Solution:

Step 1: Recognize the topology.
This τ is the standard cofinite (finite complement) topology on $\mathbb{R}$: the open sets are exactly the empty set and every subset whose complement is finite. Since $\mathbb{R}$ is infinite, this topology behaves very differently from the usual topology, and it is a classic example used to separate topological properties from each other.

Step 2: Check Hausdorff, option (A).
Take any two nonempty open sets $U, V \in \tau$. Their complements U^c and V^c are both finite. If U and V were disjoint, then $V \subseteq U^c$, which would force V to be finite. But V is a nonempty open set, so V^c is finite too, and a set with a finite complement inside an infinite space $\mathbb{R}$ cannot itself be finite (a finite set union its finite complement can't cover all of $\mathbb{R}$). This contradiction shows any two nonempty open sets must intersect, so we can never separate two distinct points with disjoint open neighborhoods. $(\mathbb{R}, \tau)$ is NOT Hausdorff. Option (A) is

FALSE.

Step 3: Check that finite sets are closed, option (B).

A set F is closed exactly when its complement F^c is open. If F is finite, then F^c has complement F, which is finite by assumption, so $F^c \in \tau$ directly by the definition of τ. So every finite set is closed. Option (B) is TRUE.

Step 4: Check compactness, option (C).

Let $\{U_i\}_{i\in I}$ be any open cover of $\mathbb{R}$. Since the cover is nonempty and its union is all of $\mathbb{R}$, pick one index i_0 with U_{i_0} nonempty. Then $U_{i_0}^c = \{p_1, \dots, p_n\}$ is a finite set of points not covered by U_{i_0}. Each p_j belongs to $\mathbb{R}$, so each p_j lies in some U_{i_j} from the cover. Then $\{U_{i_0}, U_{i_1}, \dots, U_{i_n}\}$ is a finite subcollection that still covers all of $\mathbb{R}$. Since every open cover has a finite subcover, $(\mathbb{R}, \tau)$ is compact. Option (C) is TRUE.

Step 5: Check connectedness, option (D).

Suppose, for contradiction, $\mathbb{R} = U \cup V$ with U, V disjoint, nonempty, and open. Since $U \cap V = \emptyset$, $V \subseteq U^c$, and U^c is finite (as U is open and nonempty), so V is finite. But V is also open and nonempty, so V^c is finite too. Then $\mathbb{R} = V \cup V^c$ would be a union of two finite sets, making $\mathbb{R}$ finite, which is false. So no such split exists, and $(\mathbb{R}, \tau)$ is connected. Option (D) is TRUE.

Final Answer:

The space fails to be Hausdorff, but every finite set is closed, the space is compact, and the space is connected.

$$\boxed{\text{(B), (C) and (D)}}$$

Quick Tip: This is the cofinite topology: every nonempty open set has a finite complement. Think about whether two such sets can ever be disjoint.

25. Consider the following partial differential equation:

$$a\frac{\partial^2 f(x,y)}{\partial x^2} + b\frac{\partial^2 f(x,y)}{\partial y^2} = 8f(x,y)$$

where a and b are distinct positive real numbers.
The combination(s) of the values of the real parameters ξ and η for which $f(x,y) = e^{2\xi x+\eta y}$ is a solution of the given partial differential equation, is/are

(A) $\xi = \frac{1}{\sqrt{a}}, \eta = \frac{2}{\sqrt{b}}$

(B) $\xi = 0, \eta = 0$

(C) $\xi = \frac{1}{\sqrt{2a}}, \eta = \frac{1}{\sqrt{2b}}$

(D) $\xi = 0, \eta = 2\sqrt{\frac{2}{b}}$

Correct Answer: (A) $\xi = \frac{1}{\sqrt{a}}, \eta = \frac{2}{\sqrt{b}}$; (D) $\xi = 0, \eta = 2\sqrt{\frac{2}{b}}$

Solution:

Step 1: Differentiate the trial solution.
We are given $f(x,y) = e^{2\xi x+\eta y}$. Differentiating with respect to x twice:

$$\frac{\partial f}{\partial x} = 2\xi e^{2\xi x+\eta y}, \qquad \frac{\partial^2 f}{\partial x^2} = 4\xi^2 e^{2\xi x+\eta y} = 4\xi^2 f$$

Differentiating with respect to y twice:

$$\frac{\partial f}{\partial y} = \eta e^{2\xi x+\eta y}, \qquad \frac{\partial^2 f}{\partial y^2} = \eta^2 e^{2\xi x+\eta y} = \eta^2 f$$

Step 2: Substitute into the PDE.

The PDE is $a\frac{\partial^2 f}{\partial x^2} + b\frac{\partial^2 f}{\partial y^2} = 8f$. Plugging in Step 1:

$$a(4\xi^2 f) + b(\eta^2 f) = 8f$$

Since $f = e^{2\xi x+\eta y}$ is never zero, we can divide both sides by f, leaving the condition

$$4a\xi^2 + b\eta^2 = 8$$

Any pair (ξ, η) that satisfies this equation makes f a genuine solution of the PDE; any pair that does not satisfy it fails.

Step 3: Test option (A): $\xi = \frac{1}{\sqrt{a}}, \eta = \frac{2}{\sqrt{b}}$.

$$4a\xi^2 + b\eta^2 = 4a \cdot \frac{1}{a} + b \cdot \frac{4}{b} = 4 + 4 = 8$$

This matches, so option (A) is TRUE.

Step 4: Test option (B): $\xi = 0, \eta = 0$.

$$4a(0)^2 + b(0)^2 = 0 \neq 8$$

This does not match, so option (B) is FALSE.

Step 5: Test option (C): $\xi = \frac{1}{\sqrt{2a}}, \eta = \frac{1}{\sqrt{2b}}$.

$$4a\xi^2 + b\eta^2 = 4a \cdot \frac{1}{2a} + b \cdot \frac{1}{2b} = 2 + \frac{1}{2} = \frac{5}{2}$$

Since $\frac{5}{2} \neq 8$, option (C) is FALSE.

Step 6: Test option (D): $\xi = 0, \eta = 2\sqrt{\frac{2}{b}}$.

$$4a\xi^2 + b\eta^2 = 4a(0)^2 + b\left(4 \cdot \frac{2}{b}\right) = 0 + 8 = 8$$

This matches, so option (D) is TRUE.

Final Answer:

Only the pairs in (A) and (D) satisfy the required condition $4a\xi^2 + b\eta^2 = 8$.

$$\boxed{\text{(A) and (D)}}$$

Quick Tip: Differentiate $f = e^{2\xi x + \eta y}$ twice in x and y, substitute into the PDE, and check which pairs satisfy $4a\xi^2 + b\eta^2 = 8$.

26.

Consider the following assignment problem where X, Y, Z are tasks, P, Q, R are agents and the cost matrix is given by:

	X	Y	Z
P	4	2	8
Q	2	3	7
R	3	1	6

Which of the following statements is/are TRUE for an optimal assignment?

(A) The cost is 9.

(B) Agent P is assigned to task Y.

(C) Agent Q is assigned to task X.

(D) Agent R is assigned to task Y.

Correct Answer: (B) Agent P is assigned to task Y. ; (C) Agent Q is assigned to task X.

Solution:

Step 1: Write down the cost matrix.

We are given agents P, Q, R and tasks X, Y, Z with the cost matrix:

P: X = 4, Y = 2, Z = 8.

Q: X = 2, Y = 3, Z = 7.

R: X = 3, Y = 1, Z = 6.

We need to find the assignment of one task to each agent that gives the minimum total cost.

Step 2: Do a row reduction.

Subtract the smallest entry in each row from every entry in that row.

Row P has minimum 2, so the new row is 2, 0, 6.
Row Q has minimum 2, so the new row is 0, 1, 5.
Row R has minimum 1, so the new row is 2, 0, 5.

Step 3: Do a column reduction.
Now look at each column of the row-reduced matrix.
Column X has entries 2, 0, 2, so its minimum is 0 (no change needed).
Column Y has entries 0, 1, 0, so its minimum is 0 (no change needed).
Column Z has entries 6, 5, 5, so its minimum is 5. Subtract 5 from column Z.
The matrix becomes: P: 2, 0, 1; Q: 0, 1, 0; R: 2, 0, 0.

Step 4: Pick an assignment using only the zero entries.
We need one zero in each row and each column, with no two zeros sharing a row or column.
P has a zero only at column Y, so P must go to Y.
Since Y is now used, Q must use its other zero, at column X.
Since X and Y are now used, R must go to the remaining column Z (which is also a zero for R).
This gives the assignment P to Y, Q to X, R to Z, and since we found a complete zero assignment, this is optimal.

Step 5: Compute the real cost and check each option.
Using the original cost matrix: cost = (P to Y) 2 + (Q to X) 2 + (R to Z) 6 = 10.
Option (A) says the cost is 9, but we found 10, so (A) is false.
Option (B) says agent P is assigned to task Y, which matches our optimal assignment, so (B) is true.
Option (C) says agent Q is assigned to task X, which also matches, so (C) is true.

Option (D) says agent R is assigned to task Y, but R is assigned to Z, so (D) is false.

Final Answer:
The optimal assignment is P to Y, Q to X, R to Z with total cost 10, so statements (B) and (C) are true.

$$\boxed{\text{(B) and (C)}}$$

Quick Tip: Try every one to one assignment of agents to tasks and pick the one with the smallest total cost.

27.

Consider the following differential equation:

$$2x(x-2)^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + (x-2)y = 0$$

Which of the following statements is/are TRUE?

(A) The point $x = 2$ is not a singular point.

(B) The point $x = 2$ is a singular point.

(C) The point $x = 2$ is a regular singular point.

(D) The point $x = 2$ is not a regular singular point.

Correct Answer: (B) The point $x = 2$ is a singular point. ; (D) The point $x = 2$ is not a regular singular point.

Solution:

Step 1: Write the equation in standard form.
The equation is $2x(x-2)^2y'' + 3xy' + (x-2)y = 0$.

Divide every term by the coefficient of y'', which is $2x(x-2)^2$, to get the standard form $y'' + P(x)y' + Q(x)y = 0$.

$$P(x) = \frac{3x}{2x(x-2)^2} = \frac{3}{2(x-2)^2}$$

$$Q(x) = \frac{x-2}{2x(x-2)^2} = \frac{1}{2x(x-2)}$$

Step 2: Check whether $x = 2$ is a singular point.

A point x_0 is an ordinary point only if $P(x)$ and $Q(x)$ are both analytic (finite and well behaved) there.

Here $P(x)$ has $(x-2)^2$ in the denominator and $Q(x)$ has $(x-2)$ in the denominator, so both blow up at $x = 2$.

This means $x = 2$ is not an ordinary point, so it is a singular point. So option (A) is false and option (B) is true.

Step 3: Check whether it is a regular singular point.

A singular point x_0 is a regular singular point only if $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ are both analytic (finite) at x_0.

Compute $(x-2)P(x)$:

$$(x-2)P(x) = (x-2) \cdot \frac{3}{2(x-2)^2} = \frac{3}{2(x-2)}$$

As $x \to 2$, this expression still blows up (goes to infinity), so it is not analytic at $x = 2$.

Step 4: Conclude the type of singular point.

Since $(x-2)P(x)$ fails to be analytic at $x = 2$, the point $x = 2$ does not

satisfy the condition for a regular singular point.
So $x = 2$ is an irregular singular point, meaning option (C) is false and option (D) is true.

Final Answer:
The point $x = 2$ is a singular point but not a regular one, so statements (B) and (D) are true.

$$\boxed{\text{(B) and (D)}}$$

Quick Tip: Write the equation as y'' + P(x)y' + Q(x)y = 0 and check if (x-2)P(x) stays finite at x=2.

28.

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation which reflects every vector in $\mathbb{R}^3$ through a two-dimensional subspace of $\mathbb{R}^3$. Let $P \in \mathbb{R}^{3\times3}$ be the matrix representation of T using the basis

$$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.$$

Then the value of $2 \times \text{trace}(P) - 3 \times \text{determinant}(P)$ is equal to _____. (answer in integer)

Correct Answer: 5

Solution:

Step 1: Understand what a reflection through a plane does.
A two-dimensional subspace of $\mathbb{R}^3$ is a plane through the origin.

Reflecting a vector through this plane means: any vector already lying in the plane is left unchanged, and any vector perpendicular to the plane gets flipped to its negative.
Every vector in $\mathbb{R}^3$ can be split into a part lying in the plane and a part perpendicular to it, and T acts on each part separately.

Step 2: Find the eigenvalues of T.
Since T leaves every vector in the 2-dimensional plane unchanged, the plane is an eigenspace with eigenvalue 1, and it is 2-dimensional, so the eigenvalue 1 is repeated twice.
The direction perpendicular to the plane is 1-dimensional, and T sends it to its negative, so this direction is an eigenspace with eigenvalue -1.
So the three eigenvalues of T (counted with multiplicity) are $1, 1, -1$.

Step 3: Use eigenvalues to get trace and determinant.
The trace of a matrix equals the sum of its eigenvalues, and this does not depend on which basis we use to write the matrix.

$$\text{trace}(P) = 1 + 1 + (-1) = 1$$

The determinant of a matrix equals the product of its eigenvalues, and this also does not depend on the basis.

$$\text{determinant}(P) = 1 \times 1 \times (-1) = -1$$

Because trace and determinant are basis-independent, using the standard basis given in the question does not change these values; P is simply the matrix of T itself.

Step 4: Compute the required expression.

$$2 \times \text{trace}(P) - 3 \times \text{determinant}(P) = 2(1) - 3(-1) = 2 + 3 = 5$$

Final Answer:

The value of $2 \times \text{trace}(P) - 3 \times \text{determinant}(P)$ is 5, no matter which particular plane is chosen as the subspace.

$$\boxed{5}$$

Quick Tip: A reflection through a plane in R3 has eigenvalues 1, 1, -1; use these to get the trace and determinant directly.

29.

Let $x^3 - x + 1 \in \mathbb{Z}_3[x]$, where $\mathbb{Z}_3[x]$ is the ring of all polynomials with coefficients in $\mathbb{Z}_3$. Then the degree of the field extension

$$\mathbb{Z}_3[x]/\langle x^3 - x + 1\rangle$$

of $\mathbb{Z}_3$ is equal to _____. (answer in integer)

Correct Answer: 3

Solution:

Step 1: Understand the setup.

We are working with $\mathbb{Z}_3 = \{0, 1, 2\}$, the integers modulo 3, and looking at the polynomial ring $\mathbb{Z}_3[x]$ over it.

We take the quotient of this ring by the ideal generated by $f(x) = x^3$

$-x+1$, written $\mathbb{Z}_3[x]/\langle f(x)\rangle$.
A standard fact from field theory says: if $f(x)$ is irreducible over $\mathbb{Z}_3$, then this quotient is a field, and its degree as an extension of $\mathbb{Z}_3$ equals the degree of $f(x)$.

Step 2: Check whether $f(x)$ is irreducible over $\mathbb{Z}_3$.
Since $f(x)$ has degree 3, it fails to be irreducible only if it has a root in $\mathbb{Z}_3$ (any factorization of a cubic must include at least one linear factor).
So we test all three elements of $\mathbb{Z}_3$:
$f(0) = 0^3 - 0 + 1 = 1$, which is not 0 in $\mathbb{Z}_3$.
$f(1) = 1^3 - 1 + 1 = 1$, which is not 0 in $\mathbb{Z}_3$.
$f(2) = 2^3 - 2 + 1 = 8 - 2 + 1 = 7 \equiv 1 \pmod 3$, which is not 0 in $\mathbb{Z}_3$.

Step 3: Conclude irreducibility.
Since none of $0, 1, 2$ is a root of $f(x)$ in $\mathbb{Z}_3$, $f(x)$ has no linear factor, so it cannot be factored into lower degree polynomials over $\mathbb{Z}_3$. This means $f(x)$ is irreducible over $\mathbb{Z}_3$.

Step 4: Find the degree of the extension.
Because $f(x)$ is an irreducible polynomial of degree 3, the quotient ring $\mathbb{Z}_3[x]/\langle f(x)\rangle$ is a field, and as a vector space over $\mathbb{Z}_3$ it has dimension equal to the degree of $f(x)$, which is 3. This dimension is exactly the degree of the field extension.

Final Answer:
The degree of the field extension is 3.

$\boxed{3}$

Quick Tip: Check if x^3-x+1 has a root in Z3; if not it is irreducible, and the extension degree equals its degree.

30.

Consider the differential equation $\frac{dy}{dx} = x + y$ with the initial condition $y(0) = 1$. Using the modified Euler's method, the second approximation to $y(h)$, where $h = 0.05$ (step size), is equal to _____. (rounded off to TWO decimal places)

Correct Answer: 1.05

Solution:

Step 1: Set up the known values.

We have $f(x, y) = x + y$, starting point $x_0 = 0$, $y_0 = 1$, and step size $h = 0.05$.

We want to estimate y at $x_1 = x_0 + h = 0.05$.

Step 2: Find the first (predictor) estimate.

The modified Euler's method first uses the plain Euler formula to get a rough starting guess:

$$y_1^{(0)} = y_0 + h\, f(x_0, y_0)$$

Here $f(x_0, y_0) = 0 + 1 = 1$, so

$$y_1^{(0)} = 1 + 0.05 \times 1 = 1.05$$

Step 3: Apply the corrector formula once (first approximation).

The corrector formula improves the guess by averaging the slope at the start and the estimated slope at the end:

$$y_1^{(k+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(k)})\right]$$

Using $y_1^{(0)} = 1.05$:

$$f(x_1, y_1^{(0)}) = 0.05 + 1.05 = 1.10$$

$$y_1^{(1)} = 1 + \frac{0.05}{2}(1 + 1.10) = 1 + 0.025 \times 2.10 = 1.0525$$

This $y_1^{(1)} = 1.0525$ is the first approximation from the corrector.

Step 4: Apply the corrector formula a second time (second approximation).

We repeat the same corrector step, now using the improved value $y_1^{(1)} = 1.0525$:

$$f(x_1, y_1^{(1)}) = 0.05 + 1.0525 = 1.1025$$

$$y_1^{(2)} = 1 + \frac{0.05}{2}(1 + 1.1025) = 1 + 0.025 \times 2.1025 = 1.0525625$$

This $y_1^{(2)} = 1.0525625$ is the second approximation.

Final Answer:

Rounding 1.0525625 to two decimal places gives 1.05.

1.05

Quick Tip: First find the predictor y1 = y0 + h f(x0,y0), then apply the corrector formula twice to get the second approximation.

31. Let

$$\alpha = \lim_{n\to\infty} \int_0^1 \frac{n^2 + (\sin e^x)^n}{7n^2 + x^8}\, dx.$$

The value of 14α is equal to _____. (Answer in integer)

Correct Answer: 2

Solution:

Step 1: Look at the two parts of the integrand separately.

The integrand is $\frac{n^2 + (\sin e^x)^n}{7n^2 + x^8}$ on $x \in [0,1]$. As $n \to \infty$, the term n^2 grows without bound, while $(\sin e^x)^n$ stays bounded because $|\sin e^x| \le 1$ for every real x, so $|(\sin e^x)^n| \le 1$ for every n.

Step 2: Divide numerator and denominator by n^2.

Write the integrand as

$$\frac{n^2 + (\sin e^x)^n}{7n^2 + x^8} = \frac{1 + \frac{(\sin e^x)^n}{n^2}}{7 + \frac{x^8}{n^2}}.$$

For each fixed $x \in [0,1]$, the term $\frac{(\sin e^x)^n}{n^2}$ goes to 0 as $n \to \infty$ because the numerator stays between -1 and 1 while the denominator

n^2 grows without bound. In the same way $\frac{x^8}{n^2} \to 0$ since $x^8 \le 1$ on $[0,1]$.

Step 3: Find the pointwise limit of the integrand.

So for every $x \in [0,1]$,

$$\lim_{n\to\infty} \frac{n^2 + (\sin e^x)^n}{7n^2 + x^8} = \frac{1+0}{7+0} = \frac{1}{7}.$$

The whole sequence of functions converges pointwise to the constant $\frac{1}{7}$ on $[0,1]$.

Step 4: Check that we can move the limit inside the integral.

For $n \ge 1$, the numerator $n^2 + (\sin e^x)^n$ is at most $n^2 + 1$, and the denominator $7n^2 + x^8$ is at least $7n^2$, so the integrand is bounded by $\frac{n^2+1}{7n^2} \le \frac{2}{7}$, a constant that does not depend on x. Since this bound is an integrable function on the finite interval $[0,1]$, the Dominated Convergence Theorem lets us swap the limit and the integral.

Step 5: Compute α and then 14α.

$$\alpha = \lim_{n\to\infty} \int_0^1 \frac{n^2 + (\sin e^x)^n}{7n^2 + x^8}\, dx = \int_0^1 \frac{1}{7}\, dx = \frac{1}{7}[x]_0^1 = \frac{1}{7}.$$

So

$$14\alpha = 14 \times \frac{1}{7} = 2.$$

Final Answer:

The value of 14α is 2.

$$\boxed{14\alpha = 2}$$

Quick Tip: Bound $(\sin e^x)^n$ between -1 and 1, divide numerator and denominator by n^2, then take the limit before integrating.

32. Let $X = (\mathbb{R}^3, \|\cdot\|_1)$, where $\left\|\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right\|_1 = |x| + |y| + |z|$, and let $T : X \to X$ be the linear transformation defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 2 & -2 \\ 1 & 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The operator norm of T is equal to _____. (Answer in integer)

Correct Answer: 8

Solution:

Step 1: Recall what the operator norm means here.

For a linear map $T : X \to X$ where $X = (\mathbb{R}^3, \|\cdot\|_1)$, the operator norm is defined as

$$\|T\| = \sup_{\|v\|_1=1} \|Tv\|_1.$$

The matrix of T is $A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 2 & -2 \\ 1 & 3 & -3 \end{pmatrix}$.

Step 2: Get an upper bound for $\|Tv\|_1$ in terms of the columns of A.

Write $v = (x, y, z)$ with $\|v\|_1 = |x| + |y| + |z| = 1$. Then $Tv = xA_1 + yA_2 + zA_3$, where A_1, A_2, A_3 are the three columns of A. By the triangle inequality,

$$\|Tv\|_1 \leq |x|\,\|A_1\|_1 + |y|\,\|A_2\|_1 + |z|\,\|A_3\|_1.$$

If M is the largest of $\|A_1\|_1, \|A_2\|_1, \|A_3\|_1$, the right side is at most $M(|x| + |y| + |z|) = M$. So $\|T\| \leq M$, where M is the largest column sum of absolute values.

Step 3: Compute the three column sums.

Column 1: $|1| + |2| + |1| = 4$.

Column 2: $|1| + |2| + |3| = 6$.

Column 3: $|3| + |-2| + |-3| = 8$.

The largest of these is $M = 8$, from column 3.

Step 4: Show this bound is actually attained.

Take $v = e_3 = (0, 0, 1)$, which has $\|v\|_1 = 1$. Then $Tv = A_3 = (3, -2, -3)$, and

$$\|Tv\|_1 = |3| + |-2| + |-3| = 8.$$

So the bound $M = 8$ from Step 2 is reached at $v = e_3$, which means the supremum defining $\|T\|$ equals 8.

Final Answer:

The operator norm of T is 8.

$$\boxed{\|T\| = 8}$$

Quick Tip: For a linear map on $(\mathbb{R}^3, \|\cdot\|_1)$, the operator norm equals the largest sum of absolute values found in any single column of its matrix.

33. Consider the following Linear Programming Problem (LPP):

$$\text{maximize } z = 10x_1 + 20x_2$$

subject to

$$x_1 \leq 36, \quad x_2 \geq 42, \quad x_1 + x_2 \geq 48, \quad 5x_1 + x_2 \leq 150, \quad x_1, x_2 \geq 0.$$

The maximum value of z in the above LPP is equal to _____. (Answer in integer)

Correct Answer: 3000

Solution:

Step 1: Identify the constraints and understand the region.

We must maximize $z = 10x_1 + 20x_2$ subject to $x_1 \leq 36$, $x_2 \geq 42$, $x_1 + x_2 \geq 48$, $5x_1 + x_2 \leq 150$, and $x_1, x_2 \geq 0$. Since $x_2 \geq 42$ already, the constraint $x_1 + x_2 \geq 48$ only bites when x_2 is close to 42; for example at $x_2 = 42$ we need $x_1 \geq 6$, but once $x_2 \geq 48$ it holds for any $x_1 \geq 0$.

Step 2: Find the corner points of the feasible region.

The maximum of a linear objective over a region bounded by straight lines always occurs at a corner point, so we find where the boundary lines cross.

Line $x_2 = 42$ meets line $x_1 + x_2 = 48$ at $x_1 = 48 - 42 = 6$, giving $(6, 42)$.

Line $x_2 = 42$ meets line $5x_1 + x_2 = 150$ at $5x_1 = 150 - 42 = 108$, so $x_1 = 21.6$, giving $(21.6, 42)$.

Line $5x_1 + x_2 = 150$ meets the line $x_1 = 0$ at $x_2 = 150$, giving $(0, 150)$.

Line $x_1 + x_2 = 48$ meets the line $x_1 = 0$ at $x_2 = 48$, giving $(0, 48)$.

Checking each point against all five constraints confirms these four points are the corners of the feasible region, and $x_1 \leq 36$ never actually cuts this region since x_1 never exceeds 21.6 inside it.

Step 3: Evaluate z at every corner.

$$z(6, 42) = 10(6) + 20(42) = 60 + 840 = 900.$$

$$z(21.6, 42) = 10(21.6) + 20(42) = 216 + 840 = 1056.$$

$$z(0, 150) = 10(0) + 20(150) = 0 + 3000 = 3000.$$

$$z(0, 48) = 10(0) + 20(48) = 0 + 960 = 960.$$

Step 4: Pick the largest value.

Comparing 900, 1056, 3000 and 960, the largest is 3000, reached at $(x_1, x_2) = (0, 150)$, and this point satisfies every constraint.

Final Answer:

The maximum value of z is 3000.

$$\boxed{z_{max} = 3000}$$

Quick Tip: Find the corner points of the feasible region formed by the given constraints and evaluate z at each corner; the largest value is the answer.

34. The number of distinct topologies on the set $\{1, 2, 3\}$ consisting of exactly four elements is equal to _____. (Answer in integer)

Correct Answer: 9

Solution:

Step 1: Understand what is being counted.
A topology on $\{1, 2, 3\}$ is a collection of subsets of $\{1, 2, 3\}$ that contains the empty set $\emptyset$ and the whole set $X = \{1, 2, 3\}$, and is closed under taking unions and intersections of its members. We want topologies with exactly four sets in them. Since $\emptyset$ and X are always two of those four sets, we need exactly two more distinct sets P and Q, each a proper nonempty subset of X, such that $P \cup Q$ and $P \cap Q$ also lie inside $\{\emptyset, X, P, Q\}$.

Step 2: List the possible proper nonempty subsets.
X has three singletons, $\{1\}, \{2\}, \{3\}$, and three 2-element sets, $\{1, 2\}, \{1, 3\}, \{2, 3\}$. We must choose P and Q from these six sets.

Step 3: Rule out picking two singletons.
If $P = \{a\}$ and $Q = \{b\}$ with $a \neq b$, then $P \cup Q = \{a, b\}$, a two-element set not equal to $\emptyset$, X, P or Q. Closure under union fails, so no topology of this type exists.

Step 4: Rule out picking two 2-element sets.
If P and Q are two different 2-element sets, their intersection is a single element, the one common to both, which is not equal to $\emptyset$, X, P or Q. Closure under intersection fails, so no topology of this type exists either.

Step 5: Study the mixed case, one singleton and one 2-element set.
Let $P = \{a\}$ and let Q be a 2-element set. Two things can happen:
(i) Q contains a, so $Q = \{a, b\}$ for the other element b. Then $P \cup Q = Q$ and $P \cap Q = P$, both already in the collection, so closure holds automatically. For each choice of a (3 ways) there are 2 choices of b, giving $3 \times 2 = 6$ topologies of this chain type, $\{\emptyset, \{a\}, \{a, b\}, X\}$.
(ii) Q does not contain a, so $Q = X \setminus \{a\}$, the complement of P. Then P

$\cup Q = X$ and $P \cap Q = \emptyset$, both already in the collection, so closure holds. For each of the 3 choices of a, there is exactly 1 such Q, giving 3 topologies of this partition type, $\{\emptyset, \{a\}, X \setminus \{a\}, X\}$.

Step 6: Add up all valid topologies.

$$6 \text{ (chain type)} + 3 \text{ (partition type)} = 9.$$

Final Answer:

There are 9 distinct topologies on $\{1, 2, 3\}$ with exactly four elements.

$$\boxed{9}$$

Quick Tip: Every topology must contain $\emptyset$ and X; check which pairs of the remaining proper subsets keep the collection closed under union and intersection.

35. For $a_n \in \mathbb{C}$, $n = 0, 1, 2, \ldots$, the power series

$$\sum_{n=0}^{\infty} a_n (z-2)^n$$

converges at $z = 5$ and diverges at $z = -1$. Then the radius of convergence of this power series is equal to _____. (Answer in integer)

Correct Answer: 3

Solution:

Step 1: Recall the basic fact about radius of convergence.

For a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ with radius of convergence R, the

series converges for every z with $|z - z_0| < R$, and it diverges for every z with $|z - z_0| > R$. Here the center is $z_0 = 2$.

Step 2: Use the convergence information at $z = 5$.
The distance from the center to $z = 5$ is

$$|5 - 2| = 3.$$

Since the series converges at this point, this point cannot lie outside the disc of convergence, so

$$R \geq 3.$$

Step 3: Use the divergence information at $z = -1$.
The distance from the center to $z = -1$ is

$$|-1 - 2| = |-3| = 3.$$

Since the series diverges at this point, this point cannot lie inside the disc of convergence, so

$$R \leq 3.$$

Step 4: Combine both bounds.
From Step 2, $R \geq 3$; from Step 3, $R \leq 3$. The only value that satisfies both at once is

$$R = 3.$$

Final Answer:
The radius of convergence of the power series is 3.

$$\boxed{R = 3}$$

Quick Tip: Both given points are the same distance from the center of the series, so the radius of convergence must equal that common distance.

36. The solution $y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$ of the initial value problem

$$\frac{dy}{dx} = \begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix} y, \qquad y(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

is equal to

(A) $3\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-x}$

(B) $-3\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-x}$

(C) $3\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{x} + 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-x}$

(D) $3\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-x}$

Correct Answer: (D) $3\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-x}$

Solution:

Step 1: Find the eigenvalues of the coefficient matrix.

The system is $y' = Ay$ with $A = \begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix}$. The trace of A is $-3 + 3 = 0$ and the determinant is $(-3)(3) - (4)(-2) = -9 + 8 = -1$.

The characteristic equation is $\lambda^2 - (\text{trace})\lambda + \det = 0$, so $\lambda^2 - 1 = 0$, which gives $\lambda = 1$ and $\lambda = -1$.

Step 2: Find the eigenvector for each eigenvalue.

For $\lambda = 1$: solve $(A - I)v = 0$, that is $\begin{pmatrix} -4 & 4 \\ -2 & 2 \end{pmatrix} v = 0$. This gives $v_1 = v_2$, so take $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda = -1$: solve $(A + I)v = 0$, that is $\begin{pmatrix} -2 & 4 \\ -2 & 4 \end{pmatrix} v = 0$. This gives $v_1 = 2v_2$, so take $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Step 3: Write the general solution and use the initial condition.

Since the eigenvalues are real and distinct, the general solution is $y(x) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{x} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{-x}$.

At $x = 0$: $c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, which gives $c_1 + 2c_2 = 1$ and $c_1 + c_2 = 2$. Subtracting the second equation from the first gives $c_2 = -1$, and then $c_1 = 2 - c_2 = 3$.

Step 4: Check the wrong options.

Option (A) uses $c_1 = 3, c_2 = 1$, which gives $y(0) = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$, not $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, so it is wrong.

Option (B) uses $c_1 = -3, c_2 = 1$, which gives $y(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$, the negative of the required vector, so it is wrong.

Option (C) uses $c_1 = 3, c_2 = 2$, which gives $y(0) = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$, not $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, so it is also wrong.

Final Answer:

With $c_1 = 3$ and $c_2 = -1$, the solution is $y(x) = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{-x}$.

$$\boxed{y(x) = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{-x}}$$

Quick Tip: Find the eigenvalues and eigenvectors of the coefficient matrix, then fix the constants using the given initial vector $y(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

37. Let $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{pmatrix}$. Define an inner product on $\mathbb{R}^3$ with respect to P as

$$\langle x, y \rangle_P = x^{\top} P y, \quad \text{for all } x, y \in \mathbb{R}^3.$$

Consider the subspace $V = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$. Which one of the following sets is an orthonormal basis of V with respect to $\langle x, y \rangle_P$?

(A) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

(B) $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

(C) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{5} \\ \frac{1}{5} \end{pmatrix} \right\}$

(D) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right\}$

Correct Answer: (C) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{5} \\ \frac{1}{5} \end{pmatrix} \right\}$

Solution:

Step 1: Understand the weighted inner product.

Here $P = \text{diag}(1, 9, 16)$, so for vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, the inner product is $\langle x, y \rangle_P = x_1 y_1 + 9 x_2 y_2 + 16 x_3 y_3$. A set is orthonormal for this inner product if each vector has $\langle v, v \rangle_P = 1$ and any two distinct vectors satisfy $\langle v, w \rangle_P = 0$.

Step 2: Normalize the first spanning vector.

Take $v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$. Then $\langle v_1, v_1\rangle_P = 1(1)(1) = 1$, so v_1 is already a unit vector under this inner product. Set $e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$.

Step 3: Apply Gram-Schmidt to the second spanning vector.

Take $v_2 = \begin{pmatrix}1\\1\\1\end{pmatrix}$. Its component along e_1 is $\langle v_2, e_1\rangle_P = 1(1)(1) + 9(1)(0) + 16(1)(0) = 1$.

Subtract this component: $w_2 = v_2 - 1 \cdot e_1 = \begin{pmatrix}0\\1\\1\end{pmatrix}$.

Find its P-norm: $\langle w_2, w_2\rangle_P = 1(0)^2 + 9(1)^2 + 16(1)^2 = 9 + 16 = 25$, so $\|w_2\|_P = 5$.

Normalizing, $e_2 = \frac{w_2}{5} = \begin{pmatrix}0\\\frac{1}{5}\\\frac{1}{5}\end{pmatrix}$.

Step 4: Check the wrong options.

Option (A) uses w_2 without dividing by its norm 5, and $\langle w_2, w_2\rangle_P = 25 \neq 1$, so it is not normalized and is wrong.

Option (B) replaces e_1 with $\begin{pmatrix}0\\1\\0\end{pmatrix}$, which has $\langle \cdot, \cdot\rangle_P = 9 \neq 1$, so it fails to be a unit vector and is wrong.

Option (D) divides the second vector by $\sqrt{2}$ instead of 5, using ordinary Euclidean length instead of the P-weighted length, giving $\langle \cdot, \cdot\rangle_P = 9(\frac{1}{2}) + 16(\frac{1}{2}) = 12.5 \neq 1$, so it is also wrong.

Final Answer:

The orthonormal basis with respect to $\langle x, y \rangle_P$ is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{5} \\ \frac{1}{5} \end{pmatrix} \right\}$.

$$\boxed{\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{5} \\ \frac{1}{5} \end{pmatrix} \right\}}$$

Quick Tip: Work out $\langle v, v \rangle_P = v_1^2 + 9v_2^2 + 16v_3^2$ for each candidate vector and check it equals 1, then check the cross term is 0.

38. The value of the contour integral

$$\int_\Gamma \frac{(1-z)(3\cos z + 5\sin z)}{1 - z^{101}}\, dz,$$

where $\Gamma = \{z \in \mathbb{C} : |z| = \frac{7}{9}\}$, oriented in the counter-clockwise direction, is equal to

(A) $2\pi i$

(B) $-2\pi i$

(C) 0

(D) $4\pi i$

Correct Answer: (C) 0

Solution:

Step 1: Simplify the integrand using a factorization.

Recall the identity $1 - z^n = (1-z)(1 + z + z^2 + \cdots + z^{n-1})$. With n

$= 101$, this gives $1 - z^{101} = (1 - z)(1 + z + z^2 + \cdots + z^{100})$.
So the integrand becomes

$$\frac{(1 - z)(3\cos z + 5\sin z)}{(1 - z)(1 + z + \cdots + z^{100})} = \frac{3\cos z + 5\sin z}{1 + z + z^2 + \cdots + z^{100}}$$

after cancelling the common factor $(1 - z)$; this cancellation turns the apparent singularity at $z = 1$ into a removable one.

Step 2: Locate the singularities of the simplified integrand.
The numerator $3\cos z + 5\sin z$ is entire. The denominator $1 + z + z^2 + \cdots + z^{100}$ is zero exactly at the 101th roots of unity other than $z = 1$, because $(1 - z)$ times it equals $1 - z^{101}$. Every such root satisfies $|z| = 1$.

Step 3: Compare with the contour.
The contour Γ is the circle $|z| = \frac{7}{9}$, and $\frac{7}{9} < 1$. Since all 100 zeros of the denominator lie exactly on $|z| = 1$, none of them lie inside or on Γ. The removable point $z = 1$ is also outside Γ, since $|1| = 1 > \frac{7}{9}$.
So the simplified integrand is analytic on and inside Γ, with no singular points enclosed.

Step 4: Apply Cauchy's theorem and check the wrong options.
By the Cauchy-Goursat theorem, the integral of a function that is analytic everywhere inside and on a closed contour is zero. Options (A), (B) and (D) would all require some singularity to be enclosed by Γ so that a residue contributes a nonzero multiple of πi, but there is no such singularity, so they are wrong.

Final Answer:
The contour integral equals 0.

$\boxed{0}$

Quick Tip: Factor $1 - z^{101}$ as $(1 - z)(1 + z + \cdots + z^{100})$ and check whether any zero of the simplified denominator lies inside $|z| = \frac{7}{9}$.

39. Let $f(z) = \sum_{n=1}^{\infty} 5^{-n} \cos(nz)$. On which one of the following domains, does $f(z)$ represent an analytic function?

(A) $\{z \in \mathbb{C} : |\operatorname{Im} z| < \ln 5\}$

(B) $\{z \in \mathbb{C} : |\operatorname{Im} z| > \ln 5\}$

(C) $\{z \in \mathbb{C} : |\operatorname{Re} z| < \ln 5\}$

(D) $\{z \in \mathbb{C} : |\operatorname{Re} z| > \ln 5\}$

Correct Answer: (A) $\{z \in \mathbb{C} : |\operatorname{Im} z| < \ln 5\}$

Solution:

Step 1: Rewrite the cosine terms using complex exponentials.

Write $z = x + iy$, so $\operatorname{Im} z = y$. Using $\cos(nz) = \dfrac{e^{inz} + e^{-inz}}{2}$, the series becomes

$$f(z) = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{e^{iz}}{5} \right)^n + \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{e^{-iz}}{5} \right)^n$$

which is a sum of two geometric series in the ratios $w_1 = \dfrac{e^{iz}}{5}$ and $w_2 = \dfrac{e^{-iz}}{5}$.

Step 2: Find the modulus of each ratio.

Since $z = x + iy$, $e^{iz} = e^{ix-y} = e^{-y}e^{ix}$, so $|e^{iz}| = e^{-y}$, and $e^{-iz} = e^{-ix+y}$

$= e^{y}e^{-ix}$, so $|e^{-iz}| = e^{y}$.

So $|w_1| = \dfrac{e^{-y}}{5}$ and $|w_2| = \dfrac{e^{y}}{5}$.

Step 3: Apply the geometric series convergence condition.

A geometric series $\sum w^n$ converges, and sums to an analytic function, exactly when $|w| < 1$. Both $|w_1| < 1$ and $|w_2| < 1$ are needed together, since $f(z)$ is the sum of both series.

$|w_1| < 1$ means $e^{-y} < 5$, that is $y > -\ln 5$.

$|w_2| < 1$ means $e^{y} < 5$, that is $y < \ln 5$.

Together, $-\ln 5 < y < \ln 5$, which is exactly $|\operatorname{Im} z| < \ln 5$.

Step 4: Check the wrong options.

Option (B) reverses the inequality and describes the region where the series actually diverges, since there $e^{y} \geq 5$ or $e^{-y} \geq 5$, so it is wrong.

Options (C) and (D) depend on $\operatorname{Re} z = x$, but the modulus of $e^{\pm iz}$ does not depend on x at all, only on y, so the convergence of this series cannot depend on $\operatorname{Re} z$; both are wrong.

Final Answer:

The series converges absolutely, and $f(z)$ is analytic, exactly on the strip $|\operatorname{Im} z| < \ln 5$.

$$\boxed{\{z \in \mathbb{C} : |\operatorname{Im} z| < \ln 5\}}$$

Quick Tip: Split $\cos(nz)$ into $\dfrac{e^{inz} + e^{-inz}}{2}$ and check when both resulting geometric series converge.

40. Let $\delta(t)$ be the unit impulse function defined by $\delta(x - x_0) = 0$ when $x \neq x_0$ and $\int_{-\infty}^{\infty} \delta(x - x_0)\, dx = 1$.

Consider the following initial value problem

$$2\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2y = \delta(x - 5)$$

with $y(0) = 0$ and $\frac{dy}{dx}(0) = 0$. Which one of the following is TRUE?

(A) $y(10) = \frac{10}{\sqrt{15}} e^{-5/4} \sin\left(\frac{\sqrt{15}}{4}\right)$

(B) $y(10) = \frac{2}{\sqrt{15}} e^{-5/4} \sin\left(\frac{5\sqrt{15}}{4}\right)$

(C) $y(10) = \frac{10}{\sqrt{15}} e^{5/4} \sin\left(\frac{\sqrt{15}}{4}\right)$

(D) $y(10) = \frac{2}{\sqrt{15}} e^{5/4} \sin\left(\frac{5\sqrt{15}}{4}\right)$

Correct Answer: (B) $y(10) = \frac{2}{\sqrt{15}} e^{-5/4} \sin\left(\frac{5\sqrt{15}}{4}\right)$

Solution:

Step 1: Take the Laplace transform of the whole equation.

Using $\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$ and $\mathcal{L}\{y'\} = sY(s) - y(0)$, and since $y(0) = 0$ and $y'(0) = 0$, these reduce to $\mathcal{L}\{y''\} = s^2Y(s)$ and $\mathcal{L}\{y'\} = sY(s)$. Also $\mathcal{L}\{\delta(x - 5)\} = e^{-5s}$.

So the transformed equation is $2s^2Y(s) + sY(s) + 2Y(s) = e^{-5s}$, which gives

$$Y(s) = \frac{e^{-5s}}{2s^2 + s + 2}$$

Step 2: Rewrite the denominator by completing the square.

$$2s^2+s+2=2\left[\left(s+\frac{1}{4}\right)^2+\frac{15}{16}\right]$$

so $\dfrac{1}{2s^2+s+2}=\dfrac{1}{2}\cdot\dfrac{1}{(s+\frac{1}{4})^2+\frac{15}{16}}$.

Step 3: Invert the Laplace transform.

Using $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+a)^2+w^2}\right\}=\dfrac{1}{w}e^{-at}\sin(wt)$ with $a=\frac{1}{4}$ and $w=\dfrac{\sqrt{15}}{4}$, the inverse transform of $\dfrac{1}{2s^2+s+2}$ is

$$g(t)=\frac{1}{2}\cdot\frac{4}{\sqrt{15}}e^{-t/4}\sin\left(\frac{\sqrt{15}}{4}t\right)=\frac{2}{\sqrt{15}}e^{-t/4}\sin\left(\frac{\sqrt{15}}{4}t\right)$$

Because of the factor e^{-5s}, the time-shift property gives $y(x)=g(x-5)$ for $x>5$ (and $y(x)=0$ for $x<5$).

Step 4: Evaluate at $x=10$ and check the wrong options.

$$y(10)=g(5)=\frac{2}{\sqrt{15}}e^{-5/4}\sin\left(\frac{5\sqrt{15}}{4}\right)$$

Options (A) and (C) use $\dfrac{10}{\sqrt{15}}$ with argument $\dfrac{\sqrt{15}}{4}$, mixing up the coefficient and forgetting to shift the argument by the delay of 5, so they are wrong. Options (C) and (D) use a growing exponential $e^{5/4}$ instead of the decaying $e^{-5/4}$ that this damped system must have, so they are also wrong.

Final Answer:

$$\boxed{y(10)=\frac{2}{\sqrt{15}}e^{-5/4}\sin\left(\frac{5\sqrt{15}}{4}\right)}$$

Quick Tip: Take the Laplace transform of the ODE, use $\mathcal{L}\{\delta(x-5)\} = e^{-5s}$, and invert with the time-shift property.

41. Which one of the following statements is TRUE?

(A) The set of all 2×2 real matrices under usual matrix addition and multiplication is a Unique Factorization Domain.

(B) The set of all real valued functions under usual function addition and multiplication is a Unique Factorization Domain.

(C) The set of all polynomials with coefficients in $\mathbb{R}$ under usual polynomial addition and multiplication is a Unique Factorization Domain.

(D) The set of Riemann integrable functions in $[0, 1]$ under usual function addition and multiplication is a Unique Factorization Domain.

Correct Answer: (C) The set of all polynomials with coefficients in $\mathbb{R}$ under usual polynomial addition and multiplication is a Unique Factorization Domain.

Solution:

Step 1: Recall what a Unique Factorization Domain needs.
A Unique Factorization Domain (UFD) is first of all an integral domain: a commutative ring with identity that has no zero divisors, meaning $ab = 0$ forces $a = 0$ or $b = 0$. Only then do we ask whether every non zero, non unit element factors into irreducibles in a way that is unique up to order and units. So the first check on every option is whether the ring even has zero divisors.

Step 2: Rule out option (A), the 2×2 real matrices.
Matrix multiplication is not commutative, and this ring also has

plenty of zero divisors. Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Both are non zero, but AB is the zero matrix. A ring with zero divisors is not even an integral domain, so it can never be a UFD. Option (A) is false.

Step 3: Rule out options (B) and (D), the function rings.
Let f be the function that equals 1 on the first half of $[0, 1]$ and 0 on the second half, and let g equal 0 where f is 1 and 1 where f is 0. Both f and g are Riemann integrable and non zero, yet fg is the zero function on all of $[0, 1]$. The same split works for all real valued functions on any set with more than one point. So both the ring of all real valued functions and the ring of Riemann integrable functions on $[0, 1]$ have zero divisors and are not integral domains. Options (B) and (D) are false.

Step 4: Check option (C), the polynomial ring $\mathbb{R}[x]$.
$\mathbb{R}$ is a field, and whenever F is a field, the ring $F[x]$ is an integral domain, because the degree of a product of two non zero polynomials adds the degrees, so a product of non zero polynomials is never zero. $F[x]$ is in fact a Euclidean domain under the division algorithm for polynomials, every Euclidean domain is a Principal Ideal Domain, and every Principal Ideal Domain is a Unique Factorization Domain. So $\mathbb{R}[x]$ is a UFD, where every polynomial factors into irreducible polynomials uniquely up to order and unit (non zero constant) multiples. Option (C) is true.

Final Answer:
Only the polynomial ring $\mathbb{R}[x]$ is an integral domain among the four choices, and being a Euclidean domain over a field pushes it all the way to being a UFD.

Option (C)

Quick Tip: Check whether each ring is even an integral domain (no zero divisors) first. Matrices and function rings on more than one point always have zero divisors; only one candidate is a domain, and it is a well known type of UFD.

42. Let $\ell^\infty = \{x = (x_n)_{n\geq 1} \mid x_n \in \mathbb{R},\ \sup\{|x_n| : n = 1, 2, \ldots\} < \infty\}$ with the supremum norm. Let $T : \ell^\infty \to \ell^\infty$ be given by $T(x_1, x_2, x_3, \ldots) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right)$. Which one of the following is TRUE?

(A) T is bounded but not one-to-one.

(B) T is one-to-one but not bounded.

(C) T is bounded and the inverse (from the range of T) exists but not bounded.

(D) T is bounded and the inverse (from the range of T) is bounded.

Correct Answer: (C) T is bounded and the inverse (from the range of T) exists but not bounded.

Solution:

Step 1: Check that T is bounded.

For $x = (x_n) \in \ell^\infty$, the n-th entry of Tx is x_n/n. Since $n \geq 1$, $|x_n/n| \leq |x_n|$ for every n, so

$$\|Tx\|_\infty = \sup_n \left|\frac{x_n}{n}\right| \leq \sup_n |x_n| = \|x\|_\infty.$$

So T is a bounded linear operator with $\|T\| \leq 1$. Taking $x = (1, 0, 0, \ldots)$

gives $Tx = (1, 0, 0, \ldots)$ with equal norms, so in fact $\|T\| = 1$.

Step 2: Check that T is one-to-one.

Suppose $Tx = 0$. Then $x_n/n = 0$ for every n, and since $n \neq 0$, this forces $x_n = 0$ for every n. So only $x = 0$ maps to 0, meaning T is injective (one-to-one). This already rules out options (A) and (B): (A) says T is not one-to-one and (B) says T is not bounded, both false.

Step 3: Look at the inverse on the range of T.

Since T is injective, T^{-1} is well defined from the range of T back to ℓ^∞, by $(T^{-1}y)_n = ny_n$. We need to check if this inverse is bounded, meaning whether some constant C satisfies $\|T^{-1}y\|_\infty \leq C\|y\|_\infty$ for every y in the range of T.

Step 4: Build a sequence that breaks boundedness of the inverse.

For each N, let $y_n^{(N)} = 1/\sqrt{n}$ for $n = 1, \ldots, N$ and $y_n^{(N)} = 0$ for $n > N$. Then $\|y^{(N)}\|_\infty = 1$ for every N (the largest value is at $n = 1$). This $y^{(N)}$ is in the range of T, since its preimage $x_n^{(N)} = ny_n^{(N)} = \sqrt{n}$ for $n \leq N$ (and 0 after) is a genuinely bounded sequence, bounded by $\sqrt{N}$. But

$$\|T^{-1}y^{(N)}\|_\infty = \|x^{(N)}\|_\infty = \sqrt{N},$$

which grows without bound as $N \to \infty$, while $\|y^{(N)}\|_\infty$ stays fixed at 1. So no single constant works for every y in the range, and T^{-1} is not bounded.

Final Answer:

T is bounded and one-to-one, but its inverse on the range of T exists and is unbounded, matching option (C).

Option (C)

Quick Tip: Check boundedness using the multiplier 1/n, injectivity by solving Tx=0, then test the inverse (multiply back by n) on a sequence like y_n = 1/sqrt(n) truncated at N and let N grow.

43. Let $u(x,t)$ satisfy the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty,\ t > 0,\ c > 0$$

with

$$u(x,0) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\frac{\partial u}{\partial t}(x,0) = 0.$$

By using D'Alembert's formula, the maximum value of $u(0,t)$ for $t > 0$ is

(A) 1

(B) $\frac{1}{2}$

(C) $\frac{1}{4}$

(D) 0

Correct Answer: (A) 1

Solution:

Step 1: Write down D'Alembert's formula.

For the 1D wave equation $u_{tt} = c^2 u_{xx}$ on the whole real line with initial position $u(x,0) = f(x)$ and initial velocity $u_t(x,0) = g(x)$, D'Alembert's formula gives

$$u(x,t) = \frac{1}{2}\big[f(x-ct) + f(x+ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

Here $f(x) = 1$ if $|x| < 1$ and 0 otherwise, and $g(x) = 0$ for every x.

Step 2: Drop the velocity term.

Since $g \equiv 0$, the integral term vanishes, so

$$u(x,t) = \frac{1}{2}\big[f(x-ct) + f(x+ct)\big].$$

Step 3: Plug in x = 0.

At $x = 0$,

$$u(0,t) = \frac{1}{2}\big[f(-ct) + f(ct)\big].$$

f only depends on $|x|$ (it is 1 exactly when the argument is strictly between -1 and 1), so $f(-ct) = f(ct)$ for every t, and

$$u(0,t) = \frac{1}{2}\big[f(ct) + f(ct)\big] = f(ct).$$

A common slip is to assume one of $f(ct)$ or $f(-ct)$ could be 1 while the other is 0, giving an average like $1/2$; but since both terms are always equal here, that never happens at $x = 0$.

Step 4: Evaluate f(ct).

Since $c > 0$ and $t > 0$, $ct > 0$, so $f(ct) = 1$ exactly when $ct < 1$, that is when $0 < t < 1/c$, and $f(ct) = 0$ once $t \geq 1/c$. So

$$u(0,t) = \begin{cases} 1 & 0 < t < 1/c \\ 0 & t \geq 1/c. \end{cases}$$

Final Answer:

For every t between 0 and $1/c$, $u(0,t)$ equals 1, and it never exceeds that, so the maximum value of $u(0,t)$ for $t > 0$ is 1.

$$\boxed{1}$$

Quick Tip: Use D'Alembert's formula with zero initial velocity: u(0,t) = f(ct) since f is even, and f is 1 whenever ct is less than 1.

44. Consider the Laplace equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \qquad 0 < x < 1,\ 0 < y < 1,$$

with the boundary conditions

$$T(x,0) = x, \qquad T(0,y) = y$$

$$T(x,1) = 1 + x, \qquad T(1,y) = 1 + y.$$

Then the value of $T\left(\frac{1}{2}, \frac{1}{3}\right)$ is equal to

(A) $\frac{7}{6}$

(B) $\frac{5}{6}$

(C) $\frac{1}{6}$

(D) $\frac{1}{2}$

Correct Answer: (B) $\frac{5}{6}$

Solution:

Step 1: Try the simplest harmonic function that could fit.

The boundary values x, y, $1+x$, $1+y$ are all linear in x and y, so try $T(x,y) = x+y$, since any function of the form $ax+by+c$ automatically satisfies Laplace's equation, as its second derivatives in x and y are both zero.

Step 2: Check T(x,y) = x+y against every boundary condition.

On $y=0$: $T(x,0) = x+0 = x$, matching $T(x,0) = x$.

On $x=0$: $T(0,y) = 0+y = y$, matching $T(0,y) = y$.

On $y=1$: $T(x,1) = x+1 = 1+x$, matching $T(x,1) = 1+x$.

On $x=1$: $T(1,y) = 1+y$, matching $T(1,y) = 1+y$.

So $T(x,y) = x+y$ matches all four boundary edges exactly, not just at the corners.

Step 3: Use uniqueness of the Dirichlet problem.

The Laplace equation on a bounded region, together with prescribed values on the whole boundary, has exactly one solution (this is the Dirichlet problem uniqueness result). Since $T(x,y) = x+y$ is harmonic and matches the boundary data everywhere on the boundary of the unit square, it must be the solution to this problem; there is no other function to consider.

Step 4: Evaluate at the given point.

$$T\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}.$$

Final Answer:

$T(1/2, 1/3) = 5/6$. Options like $7/6$, $1/6$, or $1/2$ come from mis-adding the fractions or guessing a wrong form for T; the direct sum $x + y$ evaluated here gives exactly $5/6$.

$$\boxed{\frac{5}{6}}$$

Quick Tip: Try T(x,y) = x+y; it is harmonic and matches all four boundary conditions exactly, so by uniqueness of the Dirichlet problem it is the solution.

45. Let E_1 and E_2 be subsets of a normed linear space X, and

$$E_1 + E_2 = \{x + y : x \in E_1,\ y \in E_2\}$$

$$E_1 \times E_2 = \{(x, y) : x \in E_1,\ y \in E_2\}.$$

Which one of the following is NOT TRUE?

(A) If either of E_1 or E_2 is open, then $E_1 + E_2$ is open.

(B) If E_1 and E_2 are convex, then $E_1 + E_2$ is convex.

(C) If E_1 and E_2 are closed, then $E_1 + E_2$ is closed.

(D) If E_1 and E_2 are connected, then $E_1 \times E_2$ is connected.

Correct Answer: (C) If E_1 and E_2 are closed, then $E_1 + E_2$ is closed.

Solution:

Step 1: Check option (A), sum with one set open.

Suppose E_1 is open. Then $E_1 + E_2 = \bigcup_{y \in E_2} (E_1 + y)$, a union over all $y \in E_2$ of the translated sets $E_1 + y$. Translating an open set by a fixed

vector gives another open set, since translation is a continuous, invertible map on a normed space. A union of open sets is open, so $E_1 + E_2$ is open. Option (A) is TRUE.

Step 2: Check option (B), sum of convex sets.

Let $z_1 = x_1 + y_1$ and $z_2 = x_2 + y_2$ be in $E_1 + E_2$, with $x_1, x_2 \in E_1$ and $y_1, y_2 \in E_2$. For $0 \le \lambda \le 1$,

$$\lambda z_1 + (1-\lambda)z_2 = \big[\lambda x_1 + (1-\lambda)x_2\big] + \big[\lambda y_1 + (1-\lambda)y_2\big].$$

Since E_1 is convex, the first bracket is in E_1; since E_2 is convex, the second bracket is in E_2. So the whole point lies in $E_1 + E_2$, which shows $E_1 + E_2$ is convex. Option (B) is TRUE.

Step 3: Check option (D), product of connected sets.

It is a standard topology fact that the product of two connected spaces, with the product topology, is connected. Since E_1 and E_2 are connected, $E_1 \times E_2$ is connected. Option (D) is TRUE.

Step 4: Check option (C), sum of closed sets, using a counterexample.

Work in $X = \mathbb{R}^2$ with the usual norm. Let

$$E_1 = \{(x, 0) : x \in \mathbb{R}\}, \qquad E_2 = \left\{\left(x, \frac{1}{x}\right) : x > 0\right\}.$$

Both E_1 (the x-axis) and E_2 (one branch of a hyperbola) are closed subsets of $\mathbb{R}^2$. A point of $E_1 + E_2$ is $\left(x_1 + x_2, \frac{1}{x_2}\right)$ with $x_1 \in \mathbb{R}$, $x_2 > 0$. Since x_1 ranges over all reals, for any second coordinate $v = \frac{1}{x_2} > 0$ the first coordinate can take any real value, so $E_1 + E_2$ is exactly the open upper half plane $\{(u, v) : v > 0\}$, which does not contain its

boundary line $v = 0$ and so is not closed. Here two closed sets add up to a set that is not closed. Option (C) is FALSE, so this is the statement that is NOT TRUE.

Final Answer:
Options (A), (B), (D) are always true in a normed linear space, but (C) can fail, as the hyperbola-plus-line example shows.

Option (C)

Quick Tip: Check each claim directly: open-plus-anything and convex-plus-convex are provable from definitions, and connected times connected is a standard fact, but closed-plus-closed has a classic counterexample (a line and a hyperbola branch in the plane).

46. For a transportation problem, let c_{ij} denote the unit cost of the cell (i, j). The known unit costs are shown below (a dash marks a cell whose cost is not given).

	j = 1	**j = 2**	**j = 3**
i = 1	10	12	19
i = 2	11	13	-
i = 3	-	-	-

Let α_i and β_j, $i, j = 1, 2, 3$, be the simplex multipliers associated with a basis corresponding to this unit cost table, so that $\alpha_1 = x$, $\alpha_2 = x + 1$, $\beta_1 = y$ and $\beta_2 = y + 2$. The relative cost coefficient d_{ij} is the difference between the current solution and the new improved solution.
If $x = 4$, $c_{13} = 19$ and $\beta_3 = y + 5$, then which one of the following is TRUE?

(A) $y = 6$ and $d_{13} = 7$

(B) $y = 6$ and $d_{13} = 4$

(C) $y = 5$ and $d_{13} = 7$

(D) $y = 5$ and $d_{13} = 4$

Correct Answer: (B) $y = 6$ and $d_{13} = 4$

Solution:

Step 1: Understanding the Question.

In a transportation problem, the simplex multipliers (also called the u-v values) satisfy $\alpha_i + \beta_j = c_{ij}$ for every basic (occupied) cell, and give the relative cost coefficient $d_{ij} = c_{ij} - (\alpha_i + \beta_j)$ for every non-basic (empty) cell. The basic cells here are $(1,1), (1,2), (2,1), (2,2)$, since their costs $c_{11}, c_{12}, c_{21}, c_{22}$ are given directly.

Step 2: Set up the multiplier equations.

Since $(1,1)$ is a basic cell, $c_{11} = \alpha_1 + \beta_1$, so

$$10 = x + y$$

Check this against the other basic cells. For $(1,2)$: $c_{12} = \alpha_1 + \beta_2$ gives $12 = x + (y + 2) = (x + y) + 2 = 10 + 2 = 12$, which matches. For $(2,1)$: $c_{21} = \alpha_2 + \beta_1$ gives $11 = (x + 1) + y = (x + y) + 1 = 11$, which matches. For $(2,2)$: $c_{22} = \alpha_2 + \beta_2$ gives $13 = (x + 1) + (y + 2) = (x + y) + 3 = 13$, which also matches. So all four basic-cell equations reduce to the single relation $x + y = 10$.

Step 3: Solve for y.

We are given $x = 4$. Put this into $x + y = 10$:

$$y = 10 - x = 10 - 4 = 6$$

Step 4: Find beta_3 and compute d13.

We are given $\beta_3 = y + 5 = 6 + 5 = 11$. The cell $(1, 3)$ is non-basic, so its relative cost coefficient is

$$d_{13} = c_{13} - (\alpha_1 + \beta_3) = 19 - (4 + 11) = 19 - 15 = 4$$

Final Answer:

We get $y = 6$ and $d_{13} = 4$, which is option (B). The other options change either the value of y or of d_{13}: option (A) keeps $y = 6$ but wrongly has $d_{13} = 7$, and options (C), (D) both use $y = 5$, which contradicts $x + y = 10$ with $x = 4$.

$$\boxed{y = 6,\ d_{13} = 4}$$

Quick Tip: Use the basic-cell rule $c_{ij} = \alpha_i + \beta_j$ to first pin down $x + y$, then use $x = 4$ to get y, and finally use $d_{13} = c_{13} - (\alpha_1 + \beta_3)$ for the non-basic cell $(1, 3)$.

47. Let X and Y be topological spaces and $f : X \to Y$ be a continuous and bijective mapping. Which one of the following statements is TRUE?

(A) f is a homeomorphism if X and Y are compact.

(B) f is a homeomorphism if X is Hausdorff and Y is compact.

(C) f is a homeomorphism if X is compact and Y is Hausdorff.

(D) f is a homeomorphism if X and Y are Hausdorff.

Correct Answer: (C) f is a homeomorphism if X is compact and Y is Hausdorff.

Solution:

Step 1: Understanding the Question.

A continuous bijection $f : X \to Y$ is a homeomorphism only when its inverse $f^{-1} : Y \to X$ is also continuous. A continuous bijection need not automatically have a continuous inverse, so we need the extra condition on X and Y that forces f^{-1} to be continuous.

Step 2: Recall the key theorem.

f^{-1} is continuous exactly when f is a closed map, that is, f sends closed sets of X to closed sets of Y. The standard theorem is: if X is compact and Y is Hausdorff, then every continuous bijection $f : X \to Y$ is automatically a homeomorphism.

Step 3: Prove why compact X plus Hausdorff Y works.

Take any closed set $C \subseteq X$. Since X is compact, C is a closed subset of a compact space, so C itself is compact. A continuous image of a compact set is compact, so $f(C)$ is compact in Y. Since Y is Hausdorff, every compact subset of Y is closed. So $f(C)$ is closed in Y. This shows f sends closed sets to closed sets, so f is a closed map, so f^{-1} is continuous, and f is a homeomorphism. This matches option (C).

Step 4: Why the other combinations fail.

Option (D), both X and Y Hausdorff with no compactness anywhere, is not enough: take $X = [0, 2\pi)$ with the usual topology (Hausdorff but not compact) and Y the unit circle S^1 (Hausdorff), with $f(t) = (\cos t, \sin t)$. This f is continuous and bijective, but f^{-1} is not continuous at $(1, 0)$, since points on S^1 just below angle 0 come from t

near 2π, far from $t = 0$ in $[0, 2\pi)$. So f is not a homeomorphism, even though both spaces are Hausdorff, ruling out (D). The same example has Y compact and X Hausdorff but not compact, so it also rules out option (B), since (B) needs X compact, not just Hausdorff.
Option (A) drops the Hausdorff requirement on Y and keeps only compactness on both sides, which is also not enough, since the proof in Step 3 genuinely needs Y Hausdorff. For instance, the identity map from $[0, 1]$ with the usual topology to $[0, 1]$ with the cofinite topology (in which every infinite space is compact but not Hausdorff) is a continuous bijection between two compact spaces, yet it is not a homeomorphism, since the cofinite topology is not Hausdorff while the usual topology is. This rules out (A).

Final Answer:
The correct requirement is X compact and Y Hausdorff, which is option (C).

Quick Tip: Remember the standard theorem: a continuous bijection from a compact space onto a Hausdorff space is always a homeomorphism, because compactness of the domain plus Hausdorff-ness of the codomain force the map to be closed.

48. Let P and Q be 3×3 nonzero real matrices. Assume that there exists a 3×3 real nonsingular matrix S such that $S^{-1}PS$ and $S^{-1}QS$ are both upper triangular. Which of the following statements is/are TRUE?

(A) The matrix $PQ - QP$ is invertible.

(B) The matrix $PQ - QP$ is nilpotent.

(C) The matrix $I + PQ - QP$ is invertible, where I is the 3×3 identity matrix.

(D) If $PQ - QP$ is a nonzero matrix, then $PQ - QP$ is diagonalizable.

Correct Answer: (B) The matrix $PQ - QP$ is nilpotent. ; (C) The matrix $I + PQ - QP$ is invertible, where I is the 3×3 identity matrix.

Solution:

Step 1: Understanding the Question.

We are told P and Q can be brought to upper triangular form by the same similarity transform S, that is, $T_1 = S^{-1}PS$ and $T_2 = S^{-1}QS$ are both upper triangular 3×3 matrices. We need to work out what this forces about the commutator $PQ - QP$.

Step 2: Transform the commutator.

Since $T_1 = S^{-1}PS$ and $T_2 = S^{-1}QS$, we get $P = ST_1S^{-1}$ and $Q = ST_2S^{-1}$. So

$$S^{-1}(PQ - QP)S = T_1T_2 - T_2T_1$$

This means $PQ - QP$ is similar to the commutator $T_1T_2 - T_2T_1$ of two upper triangular matrices, so both matrices share the same eigenvalues and the same nilpotency behaviour.

Step 3: Show the commutator of two triangular matrices is strictly upper triangular.

The product of two upper triangular matrices is upper triangular, and the diagonal entry of the product only involves the diagonal entries of

the factors: $(T_1T_2)_{ii} = (T_1)_{ii}(T_2)_{ii}$, and likewise $(T_2T_1)_{ii} = (T_2)_{ii}(T_1)_{ii}$. Since ordinary numbers commute, these two diagonal entries are equal, so every diagonal entry of $T_1T_2 - T_2T_1$ is zero. A 3×3 upper triangular matrix with all zero diagonal entries is strictly upper triangular, and every strictly upper triangular 3×3 matrix N satisfies $N^3 = 0$.

Step 4: Conclude PQ - QP is nilpotent.

Since $T_1T_2 - T_2T_1$ is nilpotent and $PQ - QP$ is similar to it, $PQ - QP$ is nilpotent too (similarity preserves $M^k = 0$, because if $S^{-1}MS = N$ and $N^3 = 0$, then $M^3 = SN^3S^{-1} = 0$). So option (B) is TRUE.

Step 5: Use nilpotency to test options (A) and (C).

A nilpotent matrix has all eigenvalues equal to 0, so its determinant, the product of its eigenvalues, is 0. This means $PQ - QP$ can never be invertible, so option (A) is FALSE. On the other hand, the eigenvalues of $I + (PQ - QP)$ are $1 + 0 = 1$ for each of the three eigenvalues of $PQ - QP$, so

$$\det(I + PQ - QP) = 1 \times 1 \times 1 = 1 \neq 0$$

so $I + PQ - QP$ is always invertible. Option (C) is TRUE.

Step 6: Test option (D).

Suppose $PQ - QP$ is a nonzero matrix and, for contradiction, that it is diagonalizable. A diagonalizable matrix is similar to a diagonal matrix built from its own eigenvalues. Since all the eigenvalues of $PQ - QP$ are 0 (Step 5), that diagonal matrix would be the zero matrix, so $PQ - QP$ itself would have to be zero, contradicting the assumption that it is nonzero. So a nonzero nilpotent matrix is never diagonalizable, and

option (D) is FALSE.

Final Answer:

The true statements are (B) and (C).

$$\boxed{PQ - QP \text{ is nilpotent, and } I + PQ - QP \text{ is invertible}}$$

Quick Tip: Simultaneous triangularization forces the commutator $PQ - QP$ to be strictly upper triangular in that basis, so it is nilpotent with all eigenvalues 0; use this to check invertibility, shifting by I, and diagonalizability.

49. Let $S = \bigcup_{n=1}^{\infty} \left\{(x, y) \in \mathbb{R}^2 : (x - n)^2 + y^2 = \frac{1}{n^2}\right\}$ with the usual topology. Which of the following statements is/are TRUE?

(A) S is not compact.

(B) S is compact.

(C) S is connected.

(D) S is not connected.

Correct Answer: (A) S is not compact. ; (D) S is not connected.

Solution:

Step 1: Understanding the Question.

S is the union of infinitely many circles C_n, one for each $n = 1, 2, 3, \ldots$, where C_n is centred at $(n, 0)$ with radius $\frac{1}{n}$. We need to check whether S is compact and whether it is connected, as a subspace of $\mathbb{R}^2$ with the usual topology.

Step 2: Check boundedness (needed for compactness).

By the Heine-Borel theorem, a subset of $\mathbb{R}^2$ is compact if and only if it is closed and bounded. The centre of C_n is at $x = n$, so as n grows without bound, points of S appear arbitrarily far to the right along the x-axis (for instance, the point $(n + \frac{1}{n}, 0)$ lies on C_n for every n). So S contains points with arbitrarily large x-coordinate, which means S is unbounded.

Step 3: Conclude on compactness.

Since S is unbounded, it cannot be compact, regardless of whether it is closed. So option (A), "S is not compact," is TRUE, and option (B), "S is compact," is FALSE.

Step 4: Check whether adjacent circles meet.

Two circles C_m and C_n (say $m < n$) have centres a distance $n - m$ apart and radii $\frac{1}{m}$ and $\frac{1}{n}$. Two circles intersect exactly when the distance between centres is between the difference and the sum of their radii, that is

$$\left|\frac{1}{m} - \frac{1}{n}\right| \leq n - m \leq \frac{1}{m} + \frac{1}{n}$$

For $m = 1, n = 2$: distance $= 1$, and $\frac{1}{1} + \frac{1}{2} = 1.5 \geq 1$, so C_1 and C_2 do intersect (in fact at two points, near $x \approx 1.875$). But for any $m \geq 2$, $\frac{1}{m} + \frac{1}{m+1} \leq \frac{1}{2} + \frac{1}{3} = \frac{5}{6} < 1 =$ the distance between adjacent centres, so C_m and C_{m+1} do NOT meet once $m \geq 2$. Circles that are further apart ($n - m \geq 2$) are even less likely to meet, since the distance only grows while the radii only shrink, so no two non-adjacent circles ever intersect either.

Step 5: Count the connected pieces.

So C_1 and C_2 touch and fuse into one connected piece, while $C_3, C_4, C_5, \ldots$ each stand completely on their own, disjoint from every other circle in S. This gives infinitely many separate, non-touching pieces, so S splits into infinitely many disjoint nonempty pieces, and a space with more than one such piece is by definition not connected.

Step 6: Conclude on connectedness.
So option (D), "S is not connected," is TRUE, and option (C), "S is connected," is FALSE.

Final Answer:
S is neither compact nor connected.

$$\boxed{\text{(A) and (D)}}$$

Quick Tip: Check boundedness first (the circles march off to $x = \infty$, so S cannot be compact), then check which circles actually touch: only C_1 and C_2 intersect, every other circle stands alone, so S is not connected.

50. Let $f(z) = \dfrac{z}{1-z}$ and $g(z) = \dfrac{1+z}{1-z}$ be two Mobius transformations defined on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$. Consider the following statements:
S_1: $f(D) \subseteq g(D)$
S_2: $g(D) \subseteq f(D)$
Which of the following statements is/are CORRECT?

(A) S_1 is true.

(B) S_2 is true.

(C) S_2 is true and S_1 is false.

(D) Neither S_1 is true nor S_2 is true.

Correct Answer: (B) S_2 is true. ; (C) S_2 is true and S_1 is false.

Solution:

Step 1: Understanding the Question.

We need to find the actual image regions $f(D)$ and $g(D)$ under the two given Mobius maps, then compare them to check the two inclusions S_1 and S_2. A Mobius transformation always sends a disc or a half-plane to another disc or half-plane, so we look for which half-plane or disc each map produces.

Step 2: Find f(D).

Let $w = f(z) = \dfrac{z}{1-z}$. Solve for z in terms of w:

$$w(1-z) = z \implies w = z + wz = z(1+w) \implies z = \frac{w}{1+w}$$

The condition $|z| < 1$ becomes $\left|\dfrac{w}{1+w}\right| < 1$, that is $|w| < |w+1|$. This says w is closer to 0 than to -1. The set equidistant from 0 and -1 is the vertical line $\mathrm{Re}(w) = -\frac{1}{2}$, and the side closer to 0 is $\mathrm{Re}(w) > -\frac{1}{2}$. So

$$f(D) = \left\{w : \mathrm{Re}(w) > -\tfrac{1}{2}\right\}$$

Step 3: Find g(D).

Let $w = g(z) = \dfrac{1+z}{1-z}$. Solve for z:

$$w(1-z) = 1+z \implies w-1 = z+wz = z(1+w) \implies z = \frac{w-1}{w+1}$$

The condition $|z| < 1$ becomes $|w-1| < |w+1|$, meaning w is closer to 1 than to -1. The set equidistant from 1 and -1 is the imaginary axis $\mathrm{Re}(w) = 0$, and the side closer to 1 is $\mathrm{Re}(w) > 0$. So

$$g(D) = \{w : \mathrm{Re}(w) > 0\}$$

Step 4: Compare the two half-planes.

$f(D)$ is the half-plane $\mathrm{Re}(w) > -\frac{1}{2}$, and $g(D)$ is the smaller half-plane $\mathrm{Re}(w) > 0$. Every point with $\mathrm{Re}(w) > 0$ also satisfies $\mathrm{Re}(w) > -\frac{1}{2}$, so

$$g(D) \subseteq f(D)$$

This means S_2 is TRUE.

Step 5: Check S1.

Is $f(D) \subseteq g(D)$? Take $w = -\frac{1}{4}$, which satisfies $\mathrm{Re}(w) = -\frac{1}{4} > -\frac{1}{2}$, so it lies in $f(D)$. But $\mathrm{Re}(w) = -\frac{1}{4}$ is not greater than 0, so this point is NOT in $g(D)$. This single point already shows $f(D)$ is not contained in $g(D)$, so S_1 is FALSE.

Step 6: Match to the options.

Since S_2 is true, option (B), "S_2 is true," is TRUE. Since S_2 is true and S_1 is false, option (C) is also TRUE (it is a more detailed, equally correct restatement). Option (A), "S_1 is true," is FALSE, and option

(D), "neither is true," is FALSE since S_2 does hold.

Final Answer:

$g(D) \subseteq f(D)$ but $f(D) \not\subseteq g(D)$, so S_2 is true and S_1 is false.

$$\boxed{\text{(B) and (C)}}$$

Quick Tip: Both maps send $z = 1$ to infinity, so the unit circle maps to a vertical line in each case; find that line for f and g using $z = 0$ and $z = -1$, then compare the two half-planes.

51. Let α and β be the roots of the indicial equation obtained in the method of finding the Frobenius series solution to the differential equation:

$$2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (1 - x^2)y = 0.$$

Which of the following statements is/are TRUE?

(A) $\alpha \neq \beta$

(B) $\alpha - \beta \in \mathbb{Z}$

(C) $\alpha - \beta \notin \mathbb{Z}$

(D) $\alpha = \beta$

Correct Answer: (A) $\alpha \neq \beta$; (C) $\alpha - \beta \notin \mathbb{Z}$

Solution:

Step 1: Write the equation near the singular point.

The given equation is $2x^2y'' - xy' + (1 - x^2)y = 0$, where $y'' = \frac{d^2y}{dx^2}$

and $y' = \dfrac{dy}{dx}$.

Here $x = 0$ is a regular singular point of the equation, so we look for a Frobenius series solution of the form $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ with $a_0 \neq 0$.

Step 2: Substitute the lowest power term to get the indicial equation.

Put $y = x^r$ and keep only the terms that come from the lowest power of x, since these fix the indicial equation.

$y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$.

So $2x^2y'' = 2r(r-1)x^r$, $-xy' = -rx^r$, and the leading part of $(1 - x^2)y$ is $1 \cdot x^r$ (the $-x^2y$ term shifts the power up by 2 and does not enter the indicial equation).

Adding the coefficients of x^r and setting the sum to zero gives the indicial equation:

$$2r(r-1) - r + 1 = 0$$

Step 3: Solve the indicial equation.

Expand:

$$2r^2 - 2r - r + 1 = 0 \implies 2r^2 - 3r + 1 = 0$$

Factor:

$$(2r - 1)(r - 1) = 0$$

So the roots are $r = 1$ and $r = \dfrac{1}{2}$.

Take $\alpha = 1$ and $\beta = \dfrac{1}{2}$ (which root we call α and which we call β does not change the size of the difference).

Step 4: Check each option.

(A) $\alpha \neq \beta$: since $1 \neq \dfrac{1}{2}$, this is TRUE.

(B) $\alpha - \beta \in \mathbb{Z}$: $\alpha - \beta = 1 - \frac{1}{2} = \frac{1}{2}$, which is not an integer, so this is FALSE.

(C) $\alpha - \beta \notin \mathbb{Z}$: since $\frac{1}{2}$ is not an integer, this is TRUE.

(D) $\alpha = \beta$: the two roots are different, so this is FALSE.

Final Answer:

The roots of the indicial equation are 1 and $\frac{1}{2}$, so they are distinct and their difference is not a whole number.

$$\boxed{\text{(A) and (C) are TRUE}}$$

Quick Tip: Substitute y = x^r into only the lowest-power terms of the equation to get a quadratic in r, then solve it and compare the two roots.

52. Let $\mathbb{Q}[x]$ be the ring of all polynomials with coefficients in $\mathbb{Q}$ under the usual polynomial addition and multiplication. Let $T : \mathbb{Q}[x] \to \mathbb{Q}[x]$ be defined by

$$T(p(x)) = p(x^2), \quad \text{for all } p(x) \in \mathbb{Q}[x].$$

Which of the following statements is/are TRUE?

(A) T is a ring homomorphism.

(B) T is one-to-one.

(C) T is onto.

(D) T is a ring isomorphism.

Correct Answer: (A) T is a ring homomorphism. ; (B) T is one-to-one.

Solution:

Step 1: Check whether T preserves addition and multiplication.

A map $T : R \to R$ between rings is a ring homomorphism if $T(p+q) = T(p) + T(q)$ and $T(pq) = T(p)T(q)$ for all p, q, and it also sends the identity 1 to 1.

Here T is the substitution map that replaces x with x^2 in a polynomial. For any two polynomials $p(x), q(x) \in \mathbb{Q}[x]$:

$$T(p(x) + q(x)) = (p+q)(x^2) = p(x^2) + q(x^2) = T(p(x)) + T(q(x))$$

$$T(p(x)q(x)) = (pq)(x^2) = p(x^2)\,q(x^2) = T(p(x))\,T(q(x))$$

Also $T(1) = 1(x^2) = 1$, so the identity is preserved too.

So T is a ring homomorphism, this is the general fact that substituting a fixed element (here x^2) for the variable in a polynomial ring always gives a ring homomorphism. Option (A) is TRUE.

Step 2: Check whether T is one-to-one.

A ring homomorphism is one-to-one exactly when its kernel is just the zero element.

Suppose $T(p(x)) = 0$, that is $p(x^2) = 0$ as a polynomial. If $p(x)$ is a nonzero polynomial of degree n with leading coefficient $a_n \neq 0$, then $p(x^2)$ is a nonzero polynomial of degree $2n$ with the same leading coefficient a_n, so it cannot be the zero polynomial.

So $p(x^2) = 0$ forces $p(x) = 0$, which means the kernel of T is $\{0\}$, so T is one-to-one. Option (B) is TRUE.

Step 3: Check whether T is onto.

Every polynomial in the image of T has the form $p(x^2) = a_0 + a_1x^2 + a_2x^4 + \cdots$, which only has even powers of x.

Take the polynomial $q(x) = x$, which has an odd power with a nonzero coefficient. There is no $p(x) \in \mathbb{Q}[x]$ with $p(x^2) = x$, because substituting x^2 for x can never produce an odd-degree term.

So x is not in the image of T, which means T is not onto. Option (C) is FALSE.

Step 4: Decide about the isomorphism and give the final answer.
A ring isomorphism must be both one-to-one and onto. Since T is not onto, it cannot be a ring isomorphism, even though it is an injective ring homomorphism. Option (D) is FALSE.
So among the four statements, only (A) and (B) hold.

T is a ring homomorphism and one-to-one, but not onto and not an isomorphism

Quick Tip: Check T(p+q)=T(p)+T(q) and T(pq)=T(p)T(q) for the homomorphism property, then look at which polynomials (odd or even degree terms) can actually appear as p(x^2).

53. Let X be any normed linear space and X' be the dual space of X. Which of the following statements is/are TRUE?

(A) If X is separable and X' is non-separable, then X must be reflexive.

(B) If X is separable and X' is non-separable, then X cannot be reflexive.

(C) If X is reflexive, then X' is reflexive.

(D) If X' is separable, then X is separable.

Correct Answer: (B) If X is separable and X' is non-separable, then X cannot be reflexive. ; (C) If X is reflexive, then X' is reflexive. ; (D) If X' is separable, then X is separable.

Solution:

Step 1: Recall the basic theorem linking separability of a space and its dual.

A standard result in functional analysis says: if the dual space X' of a normed linear space X is separable (has a countable dense subset), then X itself is separable.

This gives statement (D) directly: "If X' is separable, then X is separable" is TRUE.

The converse of this fact does not hold in general, a separable space can have a non-separable dual, the classic example being $X = \ell^1$, which is separable, while its dual $X' = \ell^\infty$ is not separable.

Step 2: Recall the reflexivity duality theorem.

Another standard theorem says a normed linear space X is reflexive if and only if its dual X' is reflexive.

So if X is reflexive, its dual X' must also be reflexive. This gives statement (C): "If X is reflexive, then X' is reflexive" is TRUE.

Step 3: Use these two facts together to test statements (A) and (B).

Suppose X is separable and, for the sake of argument, that X is reflexive.

Since X is reflexive, X'' (the dual of X') is isometrically the same as X, which is separable, so X'' is separable too.

Applying the Step 1 theorem to the space X' in place of X: since the dual of X', namely X'', is separable, X' itself must be separable.

So if X is separable and reflexive, then X' must be separable. Taking the contrapositive: if X is separable and X' is NOT separable, then X cannot be reflexive.

This matches statement (B), so (B) is TRUE, and it directly rules out statement (A), which claims the opposite ("X must be reflexive"), so (A) is FALSE.

The classical example $X = \ell^1$ (separable) with $X' = \ell^\infty$ (not

separable) confirms this: ℓ^1 is well known to NOT be reflexive.

Step 4: Final answer.
Statements (B), (C), and (D) are TRUE, and statement (A) is FALSE.

$$\boxed{\text{(B), (C) and (D) are TRUE}}$$

Quick Tip: Use the theorem that a separable dual forces the space to be separable, plus the fact that reflexivity of a space and its dual always go together, then argue by contrapositive.

54. Let $(\mathbb{Z}_n, +)$ be the group of integers modulo n. Let $G = \mathbb{Z}_{30} \oplus \mathbb{Z}_{12}$ be the external direct product of $\mathbb{Z}_{30}$ and $\mathbb{Z}_{12}$. Then which of the following statements is/are TRUE?

(A) G is not a cyclic group.

(B) The order of the element $(14, 7)$ in G is 98.

(C) The order of the element $(14, 7)$ in G is 60.

(D) There is an element in G of order 360.

Correct Answer: (A) G is not a cyclic group. ; (C) The order of the element $(14, 7)$ in G is 60.

Solution:

Step 1: Check whether G is cyclic.
A basic fact about direct products of cyclic groups says $\mathbb{Z}_m \oplus \mathbb{Z}_n$ is cyclic if and only if $\gcd(m, n) = 1$.
Here $m = 30$ and $n = 12$, and $\gcd(30, 12) = 6 \neq 1$.
So $G = \mathbb{Z}_{30} \oplus \mathbb{Z}_{12}$ is NOT cyclic. Option (A) is TRUE.

Step 2: Recall how to find the order of an element in a direct sum.

For $G = \mathbb{Z}_m \oplus \mathbb{Z}_n$, the order of an element (a, b) is the least common multiple of the order of a in $\mathbb{Z}_m$ and the order of b in $\mathbb{Z}_n$. The order of a in $\mathbb{Z}_m$ is $\frac{m}{\gcd(a, m)}$.

Step 3: Compute the order of the element (14, 7).

Order of 14 in $\mathbb{Z}_{30}$: $\gcd(14, 30) = 2$, so its order is $\frac{30}{2} = 15$.

Order of 7 in $\mathbb{Z}_{12}$: $\gcd(7, 12) = 1$, so its order is $\frac{12}{1} = 12$.

So the order of $(14, 7)$ in G is $\text{lcm}(15, 12)$. Since $15 = 3 \times 5$ and $12 = 2^2 \times 3$, $\text{lcm}(15, 12) = 2^2 \times 3 \times 5 = 60$.

So the order of $(14, 7)$ is 60, not 98. Option (C) is TRUE and option (B) is FALSE.

Step 4: Check whether G has an element of order 360.

The order of every element of $G = \mathbb{Z}_{30} \oplus \mathbb{Z}_{12}$ must divide $\text{lcm}(30, 12) = 60$ (this number is called the exponent of the group), because the order of any (a, b) is $\text{lcm}(\text{ord}(a), \text{ord}(b))$, and $\text{ord}(a) \mid 30$, $\text{ord}(b) \mid 12$, so their lcm always divides $\text{lcm}(30, 12) = 60$.

Since $360 > 60$, no element of G can have order 360. Option (D) is FALSE.

Final Answer:

The true statements are that G is not cyclic and that the order of $(14, 7)$ is 60.

(A) and (C) are TRUE

Quick Tip: Use the rule that Z_m (+) Z_n is cyclic only when gcd(m,n)=1, and that the order of (a,b) is the lcm of the individual orders of a and b.

55. Let

$$J = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

Then the geometric multiplicity of the eigenvalue 2 of J is equal to _____. (Answer in integer)

Correct Answer: 2

Solution:

Step 1: Understand what "geometric multiplicity" means.

The geometric multiplicity of an eigenvalue λ of a matrix A is the dimension of its eigenspace, that is the dimension of the null space of $A - \lambda I$, or the number of linearly independent eigenvectors for that eigenvalue.

Step 2: Notice the block structure of J.

J is a 6×6 upper triangular matrix that splits into three 2×2 blocks along the diagonal, with all other entries zero:

$$J_1 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

Here J_1 and J_2 are both 2×2 Jordan blocks with eigenvalue 2, and J_3 is a 2×2 Jordan block with eigenvalue 3.

Step 3: Find the nullity of $J - 2I$ block by block.

For a single 2×2 Jordan block with eigenvalue 2, namely $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, subtracting $2I$ gives $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, which has rank 1 and nullity $2 - 1 = 1$.

So each of J_1 and J_2 contributes a nullity of 1 to the eigenspace of 2 (each block is one Jordan chain, giving exactly one independent eigenvector).

The block J_3 has eigenvalue 3, not 2, so $J_3 - 2I = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is invertible, contributing nullity 0 to the eigenspace of 2.

Step 4: Add up the contributions and give the final answer.

Since J is block diagonal, the nullity of $J - 2I$ is the sum of the nullities of each block: $1 + 1 + 0 = 2$.

So the geometric multiplicity of the eigenvalue 2 is 2, coming from two separate 2×2 Jordan blocks, each giving one independent eigenvector.

$$\boxed{2}$$

Quick Tip: Write J - 2I and solve the homogeneous system directly, or note that J splits into 2x2 diagonal blocks and count how many of those blocks belong to eigenvalue 2.

56. Let $\alpha = \iint_S \vec{F} \cdot \hat{n}\, dS$, where $\vec{F} = (2x + 3z)\hat{i} + (xz - y)\hat{j} + (y^2 + 2z)\hat{k}$ and S is the sphere with centre at $(3, -1, 2)$ and radius 9. Here $\hat{n}$ is the unit normal drawn outward and $\hat{i}, \hat{j}, \hat{k}$ are unit vectors.

Then the value of $\frac{1}{36\pi}\alpha$ is equal to _____. (Answer in integer)

Correct Answer: 81

Solution:

Step 1: Pick the right tool.

S is a closed surface, a full sphere, and $\hat{n}$ points outward. Whenever a flux integral is taken over a closed surface, the Gauss divergence theorem turns it into a volume integral.

$$\alpha = \iint_S \vec{F} \cdot \hat{n}\, dS = \iiint_V (\nabla \cdot \vec{F})\, dV$$

where V is the solid ball enclosed by S.

Step 2: Find the divergence of $\vec{F}$.

$\vec{F} = (2x + 3z)\hat{i} + (xz - y)\hat{j} + (y^2 + 2z)\hat{k}$, so take the partial derivative of each component with respect to its own variable and add them.

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2x + 3z) + \frac{\partial}{\partial y}(xz - y) + \frac{\partial}{\partial z}(y^2 + 2z)$$

$$\nabla \cdot \vec{F} = 2 + (-1) + 2 = 3$$

The divergence is a constant, 3, the same at every point in space.

Step 3: Turn the volume integral into 3 times the volume.

Since $\nabla \cdot \vec{F} = 3$ everywhere, it can be pulled out of the integral.

$$\alpha = \iiint_V 3\,dV = 3\iiint_V dV = 3V$$

Here V is the volume of the ball of radius 9. The centre $(3, -1, 2)$ does not change the volume, only the radius matters.

Step 4: Compute the volume of the ball.

The volume of a ball of radius r is $\frac{4}{3}\pi r^3$. With $r = 9$,

$$V = \frac{4}{3}\pi(9)^3 = \frac{4}{3}\pi(729) = 972\pi$$

Step 5: Find α and then the required value.

$$\alpha = 3V = 3(972\pi) = 2916\pi$$

Now divide by 36π:

$$\frac{1}{36\pi}\alpha = \frac{2916\pi}{36\pi} = 81$$

Final Answer:

The value of $\frac{1}{36\pi}\alpha$ is 81.

$$\boxed{81}$$

Quick Tip: Use the divergence theorem: for a closed surface, the flux equals the divergence integrated over the enclosed volume, and this divergence turns out to be constant here.

57. Consider the problem of maximizing

$$z = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

subject to

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 1,$$

where $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$.

Then the maximum value of z is _____. (Answer in integer)

Correct Answer: 3

Solution:

Step 1: Recognise the form of the problem.

Write $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ and $x = (x_1, x_2, x_3)^T$. The problem is to maximize $z = x^T A x$ subject to $x^T x = 1$, that is, over all unit vectors in $\mathbb{R}^3$. A is symmetric.

Step 2: Use Lagrange multipliers.

Set $L = x^T Ax - \lambda(x^T x - 1)$. Taking the gradient with respect to x and setting it to zero gives

$$2Ax - 2\lambda x = 0 \implies Ax = \lambda x$$

So the critical points of z on the unit sphere are exactly the unit eigenvectors of A, and at such a point $z = x^T Ax = \lambda x^T x = \lambda$. This means the maximum value of z is the largest eigenvalue of A.

Step 3: Find the eigenvalues of A.

Solve $\det(A - \lambda I) = 0$.

$$\det\begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{pmatrix} = (2-\lambda)\left[(2-\lambda)^2 - 1\right] = 0$$

One root is $\lambda = 2$ directly. The other factor gives $(2-\lambda)^2 = 1$, so $2 - \lambda = \pm 1$, which means $\lambda = 1$ or $\lambda = 3$.

Step 4: Pick the largest eigenvalue.

The three eigenvalues of A are $1, 2, 3$. The largest is 3, so the maximum value of z on the unit sphere is 3.

Final Answer:

The maximum value of z is 3.

$$\boxed{3}$$

Quick Tip: The maximum of a quadratic form x^T A x over unit vectors equals the largest eigenvalue of A, attained at the matching unit eigenvector.

58. Let $P_3(\mathbb{R})$ be the vector space of all polynomials of degree at most three with real coefficients under usual polynomial addition and scalar multiplication. Let $T : P_3(\mathbb{R}) \to \mathbb{R}^2$ be the linear transformation defined as

$$T(p) = \big(p(1), p'(1)\big)$$

for all $p \in P_3(\mathbb{R})$, where p' is the derivative of p.

Then the nullity of T is equal to _____. (Answer in integer)

Correct Answer: 2

Solution:

Step 1: Note the dimension of the domain.

$P_3(\mathbb{R})$, the space of polynomials of degree at most 3, has basis $\{1, x, x^2, x^3\}$, so $\dim P_3(\mathbb{R}) = 4$. The codomain is $\mathbb{R}^2$, which has dimension 2.

Step 2: Recall the rank-nullity theorem.

For a linear map $T : V \to W$ with V finite dimensional,

$$\dim V = \text{rank}(T) + \text{nullity}(T)$$

Here $\dim V = 4$, so $\text{nullity}(T) = 4 - \text{rank}(T)$. To use this, the rank of T is needed.

Step 3: Show that T is onto $\mathbb{R}^2$.

Take any target pair $(a, b) \in \mathbb{R}^2$. Choose the polynomial $p(x) = a + b(x - 1)$, which has degree at most 1, so it certainly lies in $P_3(\mathbb{R})$. Then

$$p(1) = a + b(1 - 1) = a, \qquad p'(x) = b \implies p'(1) = b$$

So $T(p) = (a, b)$, and since (a, b) was arbitrary, T is surjective. This means $\text{rank}(T) = \dim \mathbb{R}^2 = 2$.

Step 4: Apply rank-nullity.

$$\text{nullity}(T) = \dim P_3(\mathbb{R}) - \text{rank}(T) = 4 - 2 = 2$$

Final Answer:
The nullity of T is 2.

$$\boxed{2}$$

Quick Tip: Use rank-nullity: find the rank of T first by checking whether it is onto, then nullity equals dim(domain) minus rank.

59. The number of zeros of the complex polynomial $z^6 + 5z^3 + 4z + 11$ in the annulus $\{z \in \mathbb{C} : 1 < |z| < 3\}$ is equal to _____. (Answer in integer)

Correct Answer: 6

Solution:

Step 1: Plan the count using Rouche's theorem.
Let $f(z) = z^6 + 5z^3 + 4z + 11$. To count zeros in the annulus $1 < |z| < 3$, first count zeros inside $|z| < 3$, then count zeros inside $|z| < 1$, and subtract. Rouche's theorem says that if $|g(z)| < |h(z)|$ on a circle, then $h + g$ and h have the same number of zeros inside that circle.

Step 2: Count zeros inside $|z| = 3$.

Split $f(z) = z^6 + (5z^3 + 4z + 11)$, taking $h(z) = z^6$ as the dominant piece. On $|z| = 3$:

$$|z^6| = 3^6 = 729$$

$$|5z^3 + 4z + 11| \leq 5|z|^3 + 4|z| + 11 = 5(27) + 4(3) + 11 = 135 + 12 + 11 = 158$$

Since $158 < 729$, Rouche's theorem applies with $h(z) = z^6$ and $g(z) = 5z^3 + 4z + 11$, so f has the same number of zeros inside $|z| < 3$ as z^6 does, which is 6 (all at the origin, counted with multiplicity). So f has 6 zeros inside $|z| < 3$, which accounts for all 6 roots of this degree-6 polynomial.

Step 3: Count zeros inside $|z| = 1$.

Now split $f(z) = 11 + (z^6 + 5z^3 + 4z)$, taking the constant $h(z) = 11$ as dominant this time. On $|z| = 1$:

$$|z^6 + 5z^3 + 4z| \leq |z|^6 + 5|z|^3 + 4|z| = 1 + 5 + 4 = 10$$

Since $10 < 11 = |h(z)|$, Rouche's theorem again applies, so f has the same number of zeros inside $|z| < 1$ as the constant function 11 does. A nonzero constant has 0 zeros, so f has 0 zeros inside $|z| < 1$.

Step 4: Subtract to get the count in the annulus.

Zeros in $1 < |z| < 3$ equals (zeros inside $|z| < 3$) minus (zeros inside $|z| \leq 1$), which is $6 - 0 = 6$. Neither bound above was tight ($158 < 729$ and $10 < 11$), so no zero lies exactly on $|z| = 1$ or $|z| = 3$, and the count of 6 zeros strictly inside the annulus is exact.

Final Answer:

The number of zeros of $z^6 + 5z^3 + 4z + 11$ in the annulus $1 < |z| < 3$ is 6.

$$\boxed{6}$$

Quick Tip: Apply Rouche's theorem separately on |z|=3 (compare with z^6) and |z|=1 (compare with the constant 11) to count zeros inside each circle, then subtract.

60. Let $L^2[0, \pi]$ denote the space of all real valued Lebesgue square integrable functions on $[0, \pi]$. Let $T : L^2[0, \pi] \to L^2[0, \pi]$ be defined as follows:

$$T(f(x)) = \sin x \int_0^{\pi} f(t) \cos t \, dt + \cos x \int_0^{\pi} f(t) \sin t \, dt$$

Then the value of $\frac{4}{\pi}\|T\|$ is equal to _____. (Answer in integer)

Correct Answer: 2

Solution:

Step 1: Rewrite T using inner product notation.

Working in $L^2[0, \pi]$ with inner product $\langle f, g \rangle = \int_0^{\pi} f(t)g(t) \, dt$, the operator can be written as

$$T(f) = \langle f, \cos t \rangle \sin x + \langle f, \sin t \rangle \cos x$$

So T sends every function f to some combination of just $\sin x$ and $\cos x$. This means T is a rank-2 operator, its entire range is the 2-dimensional space spanned by $\sin x$ and $\cos x$.

Step 2: Check that $\sin x$ and $\cos x$ are orthogonal on $[0, \pi]$.

$$\langle \cos t, \sin t \rangle = \int_0^{\pi} \cos t \sin t \, dt = \frac{1}{2} \int_0^{\pi} \sin 2t \, dt = \frac{1}{2} \left[-\frac{\cos 2t}{2} \right]_0^{\pi} = \frac{1}{4}(-1 + 1) = 0$$

So $\cos t$ and $\sin t$ are orthogonal in $L^2[0, \pi]$. Also their norms match:

$$\| \cos t \|^2 = \int_0^{\pi} \cos^2 t \, dt = \frac{\pi}{2}, \qquad \| \sin t \|^2 = \int_0^{\pi} \sin^2 t \, dt = \frac{\pi}{2}$$

So $\| \cos t \| = \| \sin t \| = \sqrt{\pi/2}$. Call this common value c.

Step 3: Build an orthonormal pair and rewrite T.
Let $e_1 = \cos t / c$ and $e_2 = \sin t / c$, unit vectors. Any $f \in L^2[0, \pi]$ can be split as

$$f = \alpha e_1 + \beta e_2 + g, \qquad \alpha = \langle f, e_1 \rangle, \ \beta = \langle f, e_2 \rangle$$

where g is orthogonal to both e_1 and e_2. Then $\langle f, \cos t \rangle = c\alpha$ and $\langle f, \sin t \rangle = c\beta$, so

$$T(f) = c\alpha \sin x + c\beta \cos x = c\alpha(c\, e_2) + c\beta(c\, e_1) = c^2(\beta e_1 + \alpha e_2)$$

Step 4: Compute the operator norm.

Since e_1, e_2 are orthonormal, $\|T(f)\|^2 = c^4(\alpha^2 + \beta^2)$, while $\|f\|^2 = \alpha^2 + \beta^2 + \|g\|^2 \geq \alpha^2 + \beta^2$. So

$$\frac{\|T(f)\|}{\|f\|} \leq \frac{c^2\sqrt{\alpha^2 + \beta^2}}{\sqrt{\alpha^2 + \beta^2}} = c^2$$

with equality when $g = 0$, which is achievable by taking f as a pure combination of $\cos t, \sin t$. So

$$\|T\| = c^2 = \left(\sqrt{\frac{\pi}{2}}\right)^2 = \frac{\pi}{2}$$

Step 5: Compute the required value.

$$\frac{4}{\pi}\|T\| = \frac{4}{\pi} \cdot \frac{\pi}{2} = 2$$

Final Answer:

The value of $\frac{4}{\pi}\|T\|$ is 2.

$$\boxed{2}$$

Quick Tip: T has rank 2, its whole range is spanned by sin x and cos x, which are orthogonal on [0, pi]; use that to bound the norm.

61. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ be the open unit disc and $\partial\Omega$ be its boundary. If $u(x, y)$ is the solution of the following Dirichlet problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in } \Omega$$

$$u(x, y) = x^2 - y^2 \quad \text{on } \partial\Omega,$$

then the value of $4\left(u\left(\frac{1}{2}, 0\right) - u\left(0, \frac{1}{2}\right)\right)$ is _____.

Correct Answer: 2

Solution:

Step 1: Understanding the Question.

We need to solve the Laplace equation $u_{xx} + u_{yy} = 0$ inside the open unit disc Ω, with the boundary value $u = x^2 - y^2$ fixed on the circle $\partial\Omega$. This is a Dirichlet problem. Once we know u, we plug in the two given points and combine the values.

Step 2: Check whether the boundary function is already harmonic.

A function is harmonic when it satisfies Laplace's equation everywhere, not just on the boundary. Take $v(x, y) = x^2 - y^2$ and find its second derivatives:

$$\frac{\partial^2 v}{\partial x^2} = 2, \qquad \frac{\partial^2 v}{\partial y^2} = -2$$

Adding them gives $2 + (-2) = 0$, so $v(x, y) = x^2 - y^2$ is harmonic on the whole plane, not just on the circle.

Step 3: Use uniqueness of the Dirichlet problem.

The Dirichlet problem on a bounded region has exactly one harmonic solution for a given continuous boundary value. Here $v(x, y) = x^2 - y^2$ is harmonic inside Ω and it also equals the prescribed boundary data $x^2 - y^2$ on $\partial\Omega$. Since the solution is unique, we get

$$u(x, y) = x^2 - y^2 \quad \text{for all } (x, y) \in \Omega$$

There is no need to build the Poisson integral separately, since we already found a harmonic function that matches the boundary condition exactly.

Step 4: Evaluate at the two given points.
At $\left(\frac{1}{2}, 0\right)$:

$$u\left(\frac{1}{2}, 0\right) = \left(\frac{1}{2}\right)^2 - 0^2 = \frac{1}{4}$$

At $\left(0, \frac{1}{2}\right)$:

$$u\left(0, \frac{1}{2}\right) = 0^2 - \left(\frac{1}{2}\right)^2 = -\frac{1}{4}$$

Final Answer:

$$4\left(u\left(\frac{1}{2}, 0\right) - u\left(0, \frac{1}{2}\right)\right) = 4\left(\frac{1}{4} - \left(-\frac{1}{4}\right)\right) = 4 \times \frac{1}{2} = 2$$

$$\boxed{2}$$

Quick Tip: Check if the boundary function is already harmonic in the whole disc. If it is, it must equal the unique solution itself.

62. Let $X = \{5, 6, 7, 8, 9, 10\}$ be equipped with the topology

$$\tau = \{\phi, X, \{5, 6, 7\}, \{8, 9, 10\}\}$$

Then the number of subsets of X which are neither open nor closed is _____.

Correct Answer: 60

Solution:

Step 1: Understanding the Question.

The set X has 6 elements, so it has $2^6 = 64$ subsets in total. A subset is called open if it belongs to τ, and closed if its complement (in X) belongs to τ. We need to count the subsets that fail both tests, that is, subsets that are neither open nor closed.

Step 2: List the open sets.

By definition, the open sets are exactly the members of τ:

$$\phi, \quad X, \quad \{5, 6, 7\}, \quad \{8, 9, 10\}$$

That gives 4 open sets.

Step 3: Find the closed sets by taking complements.

A set is closed when X minus that set lies in τ. Take the complement of each open set:

$$X \setminus \phi = X, \qquad X \setminus X = \phi$$

$$X \setminus \{5,6,7\} = \{8,9,10\}, \qquad X \setminus \{8,9,10\} = \{5,6,7\}$$

All four of these complements, $X, \phi, \{8,9,10\}, \{5,6,7\}$, are themselves already in τ, so every open set here is also closed. The closed sets are the same four sets: $\phi, X, \{5,6,7\}, \{8,9,10\}$.

Step 4: Count the sets that are open or closed.
Because the list of closed sets is identical to the list of open sets, the union of open sets and closed sets still has only 4 sets. No new set gets added by including the closed ones.

Final Answer:
Subsets that are neither open nor closed equal total subsets minus open-or-closed subsets:

$$64 - 4 = 60$$

$$\boxed{60}$$

Quick Tip: Find the closed sets as complements of the open sets first, then subtract the open-or-closed count from 2 to the power 6.

63. Let $\alpha, \beta \in \mathbb{R}$. If $(4, 0, 2, \beta)$ is an optimal solution of the Linear Programming Problem:

$$\text{minimize} \quad x_1 + 3x_2 + 2x_3 - \alpha x_4$$

subject to

$$4x_1 + x_2 + x_3 = 18$$

$$-3x_1 + 2x_3 + x_4 = 2$$

$$x_1, x_2, x_3, x_4 \geq 0,$$

then the maximum value of $22(\alpha + \beta)$ is equal to _____.

Correct Answer: 206

Solution:

Step 1: Understanding the Question.

We are told the point $(x_1, x_2, x_3, x_4) = (4, 0, 2, \beta)$ is optimal for this linear program. We first pin down β using the constraints, then use the optimality condition to pin down α, and finally compute $22(\alpha + \beta)$.

Step 2: Find β from the constraints.

Substitute $x_1 = 4, x_2 = 0, x_3 = 2$ into the first constraint as a check:

$$4(4) + 0 + 2 = 18$$

This holds true, confirming the point is feasible in the first constraint (it does not fix β). Use the second constraint to solve for $\beta = x_4$:

$$-3(4) + 2(2) + \beta = 2 \implies -12 + 4 + \beta = 2 \implies \beta = 10$$

Step 3: Key Formula or Approach.

For a linear program of the form minimize $c^T x$ subject to $Ax = b, x \geq 0$, a feasible point x^* is optimal exactly when there exist multipliers y_1, y_2 (one per equality constraint) such that every variable's reduced cost is non negative,

$$c_j - (\text{column}_j \cdot y) \geq 0$$

and this reduced cost is exactly zero for every variable that is strictly positive in x^* (complementary slackness). Here $x_1 = 4, x_3 = 2, x_4 = \beta = 10$ are all strictly positive, so their reduced costs must vanish.

Step 4: Set up and solve the reduced cost equations.

The columns of the constraint matrix are $x_1 : (4, -3),\ x_2 : (1, 0),\ x_3 : (1, 2),\ x_4 : (0, 1)$, with costs $c_1 = 1,\ c_2 = 3,\ c_3 = 2,\ c_4 = -\alpha$. Setting the reduced costs of x_1 and x_3 to zero:

$$1 - (4y_1 - 3y_2) = 0 \implies 4y_1 - 3y_2 = 1$$

$$2 - (y_1 + 2y_2) = 0 \implies y_1 + 2y_2 = 2$$

From the second equation, $y_1 = 2 - 2y_2$. Substitute into the first:

$$4(2-2y_2)-3y_2=1 \implies 8-8y_2-3y_2=1 \implies 11y_2=7 \implies y_2=\frac{7}{11}$$

Then $y_1=2-2\left(\frac{7}{11}\right)=2-\frac{14}{11}=\frac{8}{11}$.

Step 5: Use the x_4 reduced cost to find α.

$$-\alpha-y_2=0 \implies \alpha=-y_2=-\frac{7}{11}$$

As a check, the reduced cost of x_2 (which is zero in x^*, so it only needs to be non negative) is $3-y_1=3-\frac{8}{11}=\frac{25}{11}\geq 0$, so all the optimality conditions hold together.

Final Answer:

$$\alpha+\beta=-\frac{7}{11}+10=\frac{103}{11}$$

$$22(\alpha+\beta)=22\times\frac{103}{11}=2\times 103=206$$

$$\boxed{206}$$

Quick Tip: Find beta from the second constraint directly, then apply complementary slackness (reduced cost equal to zero on the positive variables) to pin down alpha.

64. Let $D = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 2,\ 0 \le y \le 2\}$ and let $f(t)$ denote the smallest integer greater than or equal to t. Then the value of the integral

$$\iint_D f(x + y)\, dx\, dy$$

is _____.

Correct Answer: 10

Solution:

Step 1: Understanding the Question.

Here $f(t) = \lceil t \rceil$, the ceiling function, the smallest integer that is greater than or equal to t. We need to integrate $\lceil x + y \rceil$ over the square $D = [0, 2] \times [0, 2]$. Since $x + y$ ranges from 0 to 4 on this square, $\lceil x + y \rceil$ only takes the whole number values $1, 2, 3, 4$ (the single point where $x + y = 0$ has zero area, so it does not affect the integral).

Step 2: Key Formula or Approach.

Split the square into four regions based on the value of $s = x + y$: the region where $0 < s \le 1$ contributes value 1, where $1 < s \le 2$ contributes value 2, where $2 < s \le 3$ contributes value 3, and where $3 < s \le 4$ contributes value 4. The integral becomes

$$\iint_D \lceil x + y \rceil\, dx\, dy = \sum_{k=1}^{4} k \cdot \text{Area}(k - 1 < x + y \le k)$$

So we just need the area of the square lying between consecutive lines $x + y = k - 1$ and $x + y = k$.

Step 3: Compute the area under each line $x + y \le s$.

Let $A(s)$ be the area of the part of the square where $x + y \le s$. For $0 \le s \le 2$, this region is a right triangle with both legs of length s, so $A(s) = \frac{s^2}{2}$. For $2 \le s \le 4$, the region is the whole square minus a small triangle near the far corner $(2, 2)$ with legs of length $4 - s$, so $A(s) = 4 - \frac{(4-s)^2}{2}$. Using these:

$$A(0) = 0, \quad A(1) = \frac{1}{2}, \quad A(2) = 2, \quad A(3) = 4 - \frac{1}{2} = \frac{7}{2}, \quad A(4) = 4$$

Step 4: Take the differences (the area of each band).

$$A(1) - A(0) = \frac{1}{2}, \quad A(2) - A(1) = \frac{3}{2}, \quad A(3) - A(2) = \frac{3}{2}, \quad A(4) - A(3) = \frac{1}{2}$$

These four numbers add up to 4, the total area of the square, which is a good check.

Step 5: Weight each band by its ceiling value and add.

$$1\left(\frac{1}{2}\right) + 2\left(\frac{3}{2}\right) + 3\left(\frac{3}{2}\right) + 4\left(\frac{1}{2}\right) = 0.5 + 3 + 4.5 + 2 = 10$$

Final Answer:

$$\boxed{10}$$

Quick Tip: Split the square by the level lines x+y=1, 2, 3 and weight each strip's area by the ceiling value on it.

65. If Jacobi method is used to solve the following system of linear equations

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

with the initial guess $x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $x^{(i)} = \begin{pmatrix} x_1^{(i)} \\ x_2^{(i)} \\ x_3^{(i)} \end{pmatrix}$, $i = 1, 2, 3, \ldots$, denotes the i^{th} iterate, then the value of $\left| x_1^{(2)} + x_2^{(2)} + x_3^{(2)} \right|$ is equal to _____.

Correct Answer: 4

Solution:

Step 1: Understanding the Question.

The Jacobi method solves $Ax = b$ by updating each variable using only the OLD values of the other variables from the previous iterate, never the values just computed in the same round. We need to run this for two rounds starting from $x^{(0)} = (0, 0, 0)$ and then add up the three components of $x^{(2)}$.

Step 2: Key Formula or Approach.

Write out the three equations of the system:

$$x_1 + 2x_2 + x_3 = 2, \qquad 2x_2 + 2x_3 = 2, \qquad x_1 + x_2 + x_3 = 2$$

Solve each equation for the variable sitting on its own diagonal position, using old iterate values on the right:

$$x_1^{(k+1)} = 2 - 2x_2^{(k)} - x_3^{(k)}$$

$$x_2^{(k+1)} = \frac{2 - 2x_3^{(k)}}{2} = 1 - x_3^{(k)}$$

$$x_3^{(k+1)} = 2 - x_1^{(k)} - x_2^{(k)}$$

Step 3: Compute the first iterate $x^{(1)}$.

Using $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$:

$$x_1^{(1)} = 2 - 0 - 0 = 2$$

$$x_2^{(1)} = 1 - 0 = 1$$

$$x_3^{(1)} = 2 - 0 - 0 = 2$$

So $x^{(1)} = (2, 1, 2)$.

Step 4: Compute the second iterate $x^{(2)}$.

Now use $x_1^{(1)} = 2, x_2^{(1)} = 1, x_3^{(1)} = 2$ on the right side of each formula:

$$x_1^{(2)} = 2 - 2(1) - 2 = 2 - 2 - 2 = -2$$

$$x_2^{(2)} = 1 - 2 = -1$$

$$x_3^{(2)} = 2 - 2 - 1 = -1$$

So $x^{(2)} = (-2, -1, -1)$.

Final Answer:

$$x_1^{(2)} + x_2^{(2)} + x_3^{(2)} = -2 - 1 - 1 = -4$$

$$|-4| = 4$$

$$\boxed{4}$$

Quick Tip: Update each variable using only the previous iterate's values (never the newly computed ones in the same round) and repeat twice from (0,0,0).