GATE 2026 Mathematics Question Paper with Solutions

ZOLLEGE

General Instructions

(i) This booklet contains 65 questions, each provided with a complete, step-by-step solution.

(ii) It comprises 32 single-correct multiple-choice questions, 14 multiple-correct questions and 19 numerical / integer-type questions.

(iii) These questions follow the GATE Mathematics examination pattern.

(iv) Questions carry 1 or 2 marks. MCQs have negative marking (1/3 or 2/3 of the marks); MSQ and numerical-answer (NAT) questions carry no negative marking.

(v) GATE is a 3-hour paper of 65 questions (100 marks).

(vi) A virtual scientific calculator is provided; physical calculators are not allowed.

(vii) Attempt each question on your own before reviewing the given solution.

(viii) For multiple-correct questions, all correct options must be selected to earn full marks.

(ix) For numerical questions, report the answer rounded exactly as asked.

1. Suresh said, "I did it yesterday." Which one of the following options is the correct form of this sentence in indirect speech?

(A) Suresh said that I did it yesterday.

(B) Suresh says I did it yesterday.

(C) Suresh says that he did it the day before.

(D) Suresh said that he had done it the day before.

Correct Answer: (D) Suresh said that he had done it the day before.

SOLUTION:

Step 1: Understanding the Concept.
This is a direct-to-indirect speech conversion question. Whenever the reporting verb (the verb that introduces the quote, like "said") is in the past tense, three things change together: the verb tense inside the quote, the pronouns, and any time expression tied to the moment of speaking.

Step 2: Key Rule to Apply.
Because "said" is past tense, the verb inside the quote must move one tense further back: simple past ("did") becomes past perfect ("had done"). At the same time, "I" (the speaker, Suresh) becomes "he" since we are now reporting his words from outside, and "yesterday" becomes "the day before" because the day of reporting is not the same as the day the action happened.

Step 3: Detailed Explanation.
Apply all three changes to the original sentence "I did it yesterday":
Pronoun change: "I" becomes "he."
Tense change: "did" (simple past) becomes "had done" (past perfect).
Time change: "yesterday" becomes "the day before."
Putting these together gives: "Suresh said that he had done it the day before."
Checking the other options against these three rules: "Suresh said that I did it yesterday" fails the pronoun and time changes. "Suresh says I did it yesterday" and "Suresh says that he did it the day before" both wrongly keep the reporting verb as "says" instead of "said," so the tense inside cannot stay in simple past either.

Step 4: Final Answer.
Only the sentence that shifts the pronoun, the tense, and the time

word together is grammatically correct: "Suresh said that he had done it the day before," which is option (D).

Quick Tip: When the reporting verb is in the past tense (said), shift the tense back one step (did becomes had done) and change time words like yesterday to the day before.

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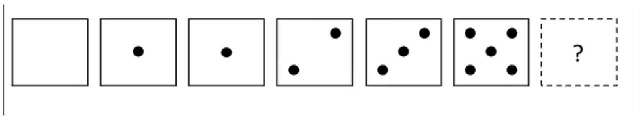
2. To continue the sequence of tiles shown, the tile indicated by the question mark should be:



(The tiles in the sequence contain, in order, 0 dots, 1 dot, 1 dot, 2 dots, 3 dots, and 5 dots, followed by a tile marked with a question mark.)

(A) A tile showing 4 dots arranged in a 2x2 square.

(B) A tile showing 6 dots.

(C) A tile showing 8 dots arranged in a ring.

(D) A tile showing 9 dots arranged in a 3x3 square.

Correct Answer: (C) A tile showing 8 dots arranged in a ring.

SOLUTION:

This question is a Fibonacci-style number pattern hidden inside pictures instead of numbers. Each tile carries a certain number of dots, and the job is to work out what number of dots the seventh tile should carry.

List the dot counts of the six visible tiles: 0, 1, 1, 2, 3, 5. A well known number pattern called the Fibonacci sequence starts

$0, 1, 1, 2, 3, 5, 8, 13, \ldots$, where from the third term onward, each number is the sum of the two numbers just before it. Comparing our six dot counts to this pattern shows they match term by term: $0, 1, 1, 2, 3, 5$ is exactly the start of the Fibonacci sequence.

Since the seventh Fibonacci number (right after $0, 1, 1, 2, 3, 5$) is $5 + 3 = 8$, the missing tile in the question must carry 8 dots.

1. **Option (A), 4 dots**: 4 does not equal $5 + 3$, so this tile does not continue the pattern.
2. **Option (B), 6 dots**: 6 also does not match the required sum of 8, so this is not the right tile either.
3. **Option (C), 8 dots arranged in a ring**: 8 matches $5 + 3$ exactly, so this is the tile that correctly continues the sequence.
4. **Option (D), 9 dots**: 9 is one more than the required 8, so this tile also fails to fit the pattern.

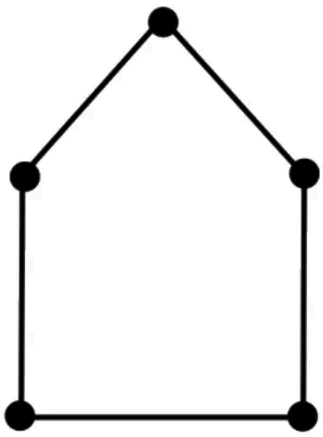
Let's summarize:

- The dot counts 0, 1, 1, 2, 3, 5 follow the Fibonacci rule, where each term is the sum of the two terms before it.
- The next term after 3 and 5 must be $3 + 5 = 8$.
- Only the tile with 8 dots, option (C), matches this required count.

So the tile indicated by the question mark must show 8 dots, confirming option (C) as correct.

Quick Tip: Count the dots on each tile: 0, 1, 1, 2, 3, 5. This is the Fibonacci pattern, so the next tile needs 3 + 5 = 8 dots.

3. Consider an art gallery whose walkways are shown as lines in the diagram. A black dot represents a junction of two walkways. A guard may be placed at a junction to watch over the walkways that join at that junction. The minimum number of guards needed to watch all the walkways is ________.



(The diagram shows five junctions joined in a closed loop shaped like a pentagon: a top junction connects down to an upper-left junction and an upper-right junction, the upper-left junction connects down to a lower-left junction, the upper-right junction connects down to a lower-right junction, and the lower-left and lower-right junctions connect to each other. This gives 5 junctions and 5 walkways in total.)

(A) 2

(B) 3

(C) 4

(D) 5

Correct Answer: (B) 3

SOLUTION:

The diagram is a closed loop of 5 junctions and 5 walkways, shaped like a pentagon: a top junction connects to two upper corners, each upper corner connects down to a lower corner, and the two lower corners connect to each other. This kind of closed loop with an odd number of junctions has a known minimum guard count, which can be worked out directly by testing each answer choice.

1. **2 guards**: With only 2 junctions chosen, at most 4 walkway-ends can be covered (2 walkways touch each junction), but there are 5 walkways in the loop. Since $2 \times 2 = 4$, which is less than 5, at least one walkway is always left with no guard at either end. So 2 guards can never watch every walkway.
2. **3 guards**: Choosing the top junction and the two lower corners covers every walkway: the top junction watches the two walkways running down from it, and each lower corner watches the walkway coming down to it plus the walkway joining the two lower corners. All 5 walkways end up watched, using only 3 guards.
3. **4 guards**: This works too (any 4 out of 5 junctions will cover all walkways, since only one junction is left unguarded), but it uses one more guard than necessary, so it is not the minimum.
4. **5 guards**: Placing a guard at every junction obviously watches every walkway, but this is far more than needed, so it is not the minimum either.

Since 2 guards are proven impossible and 3 guards are shown to work with an explicit placement, the smallest possible number of guards is 3.

Let's summarize:

- With 5 walkways and each junction touching only 2 of them, 2 guards can cover at most 4 walkway-ends, so 2 is not enough.
- Placing guards at the top junction and both lower corners covers all 5 walkways at once.
- 4 or 5 guards would also work but use more guards than the minimum required.

So the minimum number of guards needed to watch all the walkways is 3, which is option (B).

Quick Tip: Model the junctions and walkways as a 5-sided loop (pentagon). This needs a minimum vertex cover, which for a 5-cycle is 3.

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4. The 2nd of June is a Thursday in a certain year. Which day of the week is the 3rd of July in that year?

(A) Thursday

(B) Friday

(C) Saturday

(D) Sunday

Correct Answer: (D) Sunday

SOLUTION:

This question can be solved by numbering the days of the week and using simple modular arithmetic instead of counting day by day.

Assign each weekday a number starting from Thursday as 0: Thursday = 0, Friday = 1, Saturday = 2, Sunday = 3, Monday = 4, Tuesday = 5, Wednesday = 6. Since 2nd June is Thursday, it corresponds to number 0.

Next, count the number of days from 2nd June to 3rd July. June has 30 days, so the 2nd of June to the 2nd of July spans 30 days, and one more day brings us to the 3rd of July, giving $30 + 1 = 31$ days in total.

To find the weekday number for the 3rd of July, take the remainder when 31 is divided by 7, since the weekday pattern repeats every 7 days: $31 = 4 \times 7 + 3$, so the remainder is 3.

Add this remainder to the starting weekday number: $0 + 3 = 3$. Looking back at the numbering, weekday number 3 corresponds to Sunday.

1. **Thursday**: this is weekday number 0, which would only be correct if the day gap were a multiple of 7, but 31 is not a multiple of 7.
2. **Friday**: this is weekday number 1, which would need a remainder of 1, not 3.
3. **Saturday**: this is weekday number 2, which would need a remainder of 2, not 3.
4. **Sunday**: this is weekday number 3, which matches the remainder found above exactly.

Let's summarize:

- There are 31 days between 2nd June and 3rd July.
- Dividing 31 by 7 leaves a remainder of 3, so the weekday shifts forward by 3 places.
- Three places after Thursday is Sunday.

So the 3rd of July falls on a Sunday, which is option (D).

Quick Tip: Count the days from 2nd June to 3rd July (31 days), find the remainder after dividing by 7 (3), and move that many weekdays forward from Thursday.

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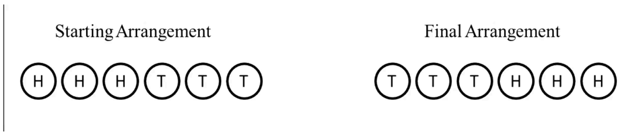
5. A coin with heads facing up is shown as H and a coin with tails facing up is shown as T. Six coins are placed in the Starting Arrangement, as shown in the figure below. A "step" is defined as interchanging a pair of adjacent coins without flipping them.



Starting Arrangement: H H H T T T
Final Arrangement: T T T H H H
The minimum number of steps needed to go from the Starting Arrangement to the Final Arrangement, as shown in the figure, is __________.

(A) 3

(B) 6

(C) 9

(D) 12

Correct Answer: (C) 9

SOLUTION:

Step 1: Track the target position of each coin type.
There are 3 H coins, starting at positions 1, 2 and 3, and 3 T coins, starting at positions 4, 5 and 6. In the Final Arrangement, all 3 T coins

must occupy positions 1, 2 and 3, and all 3 H coins must occupy positions 4, 5 and 6.

Step 2: Work out how far each H coin must move past T coins.
Take the H coin at position 1: to reach the H-block on the right side of the row, it must move past all 3 T coins that currently sit to its right, since every T coin needs to end up to its left. That is 3 required adjacent swaps for this one H coin.
The same reasoning applies to the H coin at position 2 and the H coin at position 3: each of them must also cross past all 3 T coins, since the coins keep their internal order (an H never needs to swap past another H, and a T never needs to swap past another T).

Step 3: Add up the moves for all three H coins.
Each of the 3 H coins needs 3 adjacent swaps against T coins, and these swaps never overlap or cancel out.
Total swaps: $3 + 3 + 3 = 9$.

Step 4: Confirm this is the minimum, not just one way of doing it.
Since every H coin genuinely must end up to the right of every T coin, and an adjacent swap can only fix one H-T pair at a time, no sequence of swaps can do the job in fewer than 9 steps.

Final Answer:

$$\boxed{9}$$

Quick Tip: Each of the 3 heads must move past each of the 3 tails at least once; count how many such crossings are needed in total.

6. Exacerbate : Mitigate :: ___________

Choose the option with the correct pair of words to fill the blank.

(A) Aggravate : Alleviate

(B) Alleviate : Precipitate

(C) Aggravate : Precipitate

(D) Emancipate : Exonerate

Correct Answer: (A) Aggravate : Alleviate

SOLUTION:

Understanding the Question:

We need a pair of words that fits into the blank so it has the same link as "Exacerbate : Mitigate".

Key Formula or Approach:

Find what "Exacerbate" and "Mitigate" mean on their own, write down the link between them in plain words, then test each option against that same link.

"Exacerbate" = to make something worse.

"Mitigate" = to make something less severe.

So the link is: word 1 means "worsen", word 2 means "lessen".

Detailed Explanation:

Option (A) Aggravate : Alleviate. "Aggravate" = worsen, "Alleviate" = lessen. This keeps the same link.

Option (B) Alleviate : Precipitate. "Alleviate" = lessen, which is already the wrong sense for word 1 (should be worsen). Fails at the first word itself.

Option (C) Aggravate : Precipitate. Word 1 is fine ("Aggravate" = worsen), but "Precipitate" means to bring on an event fast, not to

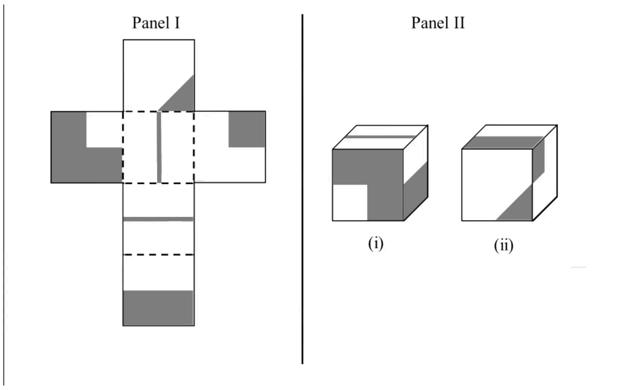
lessen anything. Fails at the second word.
Option (D) Emancipate : Exonerate. Both words mean "to free or clear from something", so this pair means nearly the same thing on both sides, not a worsen and lessen pair at all.
Since only option (A) keeps both halves of the worsen-to-lessen link, it is the one that matches.

Final Answer:
The correct pair is Aggravate : Alleviate.

$$\boxed{\text{Aggravate : Alleviate}}$$

Quick Tip: Work out what relationship "worsen" and "lessen" describes, then look for the option where both words carry that same sense.

7. A paper shown in Panel I is folded along the dashed lines to construct a cube. The shaded regions shown in Panel I appear on the outer surface of the cube. Referring to the cubes shown in Panel II, which one of the options is correct?



(Panel I shows a flat cross-shaped net of six squares: a vertical strip of four squares one above another, with one extra square attached to the left of the second square in that strip and another extra square attached to its right, all joined along dashed fold lines. Some squares carry grey shaded regions and some are left plain white. Panel II shows two separate solid cubes, labelled (i) and (ii), each with grey shaded regions on some of their visible faces.)

(A) Only (i) can correspond to the unfolded cube in Panel I.

(B) Only (ii) can correspond to the unfolded cube in Panel I.

(C) Both (i) and (ii) can correspond to the unfolded cube in Panel I.

(D) Neither (i) nor (ii) can correspond to the unfolded cube in Panel I.

Correct Answer: (B) Only (ii) can correspond to the unfolded cube in Panel I.

SOLUTION:

A quick way to check a cube-folding question like this is to pick two shaded faces that share an edge in the flat net, and then check every candidate cube to see if those same two faces still share an edge in the same relative direction after folding. If the shared edge or the rotation does not match, that candidate is impossible, no matter how correct it looks at first glance.

In the net, the square carrying the grey diagonal triangle sits right next to the square carrying the small grey corner patch, joined along one fold line, and the fully grey square sits at the far end of the strip, which becomes the face directly opposite the front face once folded. The mostly grey square with the white notch sits on the opposite side of the strip from the small-patch square, so it becomes the left face while the small-patch square becomes the right face.

1. **Cube (i)**: The shaded faces visible on this cube do not sit next to each other the way the net demands. Folding a flat net fixes both which faces touch and how their patterns line up along that shared edge, and cube (i) breaks this requirement, so it cannot be the result of folding Panel I.
2. **Cube (ii)**: The shaded faces visible on this cube touch each other along the same edge, and in the same rotation, that the net produces when folded. Every shaded region lands on the correct face in the correct orientation, so cube (ii) is a valid folded form of Panel I.

Since cube (i) fails the adjacency and orientation check while cube (ii) passes it, only cube (ii) can be the actual folded cube.

Let's summarize:

- Folding a net fixes not just which face goes where, but also how each face is rotated relative to its neighbors.
- Two shaded faces that are neighbors in the flat net must stay neighbors, in the same relative rotation, after folding.

- Cube (i) violates this, cube (ii) satisfies it, so only (ii) is a valid fold of the net in Panel I.

So the correct choice is that only (ii) can correspond to the unfolded cube in Panel I, which is option (B).

Quick Tip: Trace which net squares become which cube faces (front, back, top, bottom, left, right) and check that shaded neighbors stay adjacent in the same orientation after folding.

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8. In a population, patients who have high cholesterol also have high blood-pressure (BP). Some patients with high BP also have diabetes. There are no patients who have both high cholesterol and diabetes. Furthermore,
1. the total number of patients with at least one of these conditions is 75,
2. the number of patients with high cholesterol is 10,
3. the number of patients with high BP is 45, and
4. the number of patients with only high BP and no other conditions is 20.
Then the number of patients who have both diabetes and high BP is

____

(A) 0

(B) 15

(C) 20

(D) 10

Correct Answer: (B) 15

SOLUTION:

This question is a set-overlap problem, so let's place the numbers directly onto a Venn diagram in words instead of writing the equation first.

1. High cholesterol patients are all inside the high BP group, and high cholesterol never overlaps diabetes. So on the diagram, the cholesterol circle sits fully inside the BP circle, in a region that does not touch the diabetes circle at all. That region holds 10 patients.
2. The BP circle also has a region marked "only BP" (no cholesterol, no diabetes), which holds 20 patients.
3. The rest of the BP circle, the part that overlaps diabetes, is what the question calls "both diabetes and high BP". Since the BP circle totals 45, and 10 (cholesterol overlap) and 20 (only BP) are already placed inside it, the remaining part of the BP circle is $45 - 10 - 20 = 15$.
4. As a check, add up everyone with at least one condition: only BP (20) plus BP-and-cholesterol (10) plus BP-and-diabetes (15) plus diabetes-only. Since the grand total is 75, diabetes-only must be $75 - 20 - 10 - 15 = 30$, a sensible positive number, so the diagram is consistent.

Let's summarize:

- The high BP circle of 45 splits into three parts: only BP (20), BP with cholesterol (10), and BP with diabetes (the unknown part).
- Subtracting the known parts from 45 gives the BP-with-diabetes part as 15.
- The overall total of 75 checks out once diabetes-only works out to 30, confirming the split is correct.

So the number of patients with both diabetes and high BP is 15.

Quick Tip: Split the 45 high BP patients into three non-overlapping parts: only BP, BP with cholesterol, and BP with diabetes, then use the numbers you already have to find the missing part.

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9. Four people P, Q, R, and S, of different ages, make the following observations.
P - I am younger than S.
Q - I am neither the youngest nor the oldest.
R - P is older than me.
Based on these observations, the youngest person is _______.

(A) P

(B) Q

(C) R

(D) S

Correct Answer: (C) R

SOLUTION:

Instead of chaining the comparisons, let's test each option directly against the three statements and see which one survives.

1. **Testing P as youngest**: R says "P is older than me", which directly means P is older than R. If P were the youngest of all four, P would have to be younger than R too, but that contradicts R's own statement. So P cannot be the youngest.

2. **Testing Q as youngest**: Q says "I am neither the youngest nor the oldest". If Q were the youngest, Q's own statement would be false. So Q cannot be the youngest.
3. **Testing S as youngest**: P says "I am younger than S", which means P is younger than S. If S were the youngest of all four, S would have to be younger than P too, but that contradicts P's own statement. So S cannot be the youngest.
4. **Testing R as youngest**: We need R younger than P, younger than Q, and younger than S. R younger than P is given directly by R's own statement. R younger than S follows since R is younger than P and P is younger than S. R younger than Q holds too, because if R were older than Q, Q would become the overall youngest, which breaks Q's statement, so Q must be older than R.

Only the fourth case, R as the youngest, avoids every contradiction, so R must be the youngest person.

Let's summarize:

- R's own statement rules out P as the youngest.
- Q's own statement rules out Q as the youngest.
- P's statement rules out S as the youngest.
- Only R is left once the other three are ruled out, and checking R directly confirms it works.

So the youngest person is R.

Quick Tip: Turn each statement into a "younger than" relation first, then check which single person is younger than all three of the others without breaking any statement.

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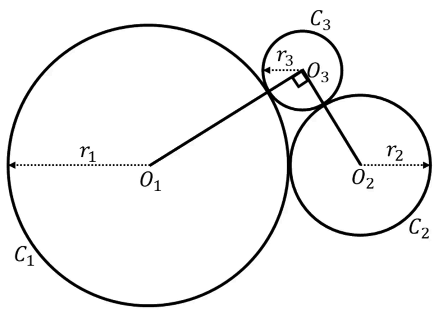
10. Circles C_1, C_2, and C_3, with centers O_1, O_2, and O_3, and radii r_1, r_2, and r_3, respectively, touch each other as shown in the following figure.



Given $r_1 = 2$ cm, $r_2 = 1$ cm and the angle $\angle O_1O_3O_2$ is 90°, $r_3 =$ ______ cm.

(A) $\frac{1}{2}\left(-3+\sqrt{17}\right)$

(B) $\frac{1}{2}\left(3+\sqrt{17}\right)$

(C) $\frac{1}{2}\left(-2+\sqrt{17}\right)$

(D) $\frac{1}{2}\left(-3+2\sqrt{17}\right)$

Correct Answer: (A) $\frac{1}{2}\left(-3+\sqrt{17}\right)$

SOLUTION:

Here is a second way to reach the same value, using coordinates instead of plugging straight into the Pythagorean theorem.

1. Place O_3 at the origin, with the two radii to O_1 and O_2 running along the two coordinate axes, since the angle between them is 90°. So O_1 sits at $(-(r_1 + r_3), 0)$, that is $(-(2 + r_3), 0)$, and O_2 sits at $(0, -(r_2 + r_3))$, that is $(0, -(1 + r_3))$, using the tangency rule that touching circles have their centers separated by the sum of the two radii.

2. The distance from O_1 to O_2 must equal $r_1 + r_2 = 3$, because C_1 and C_2 also touch each other externally. Using the distance formula between O_1 and O_2: $\sqrt{(2 + r_3)^2 + (1 + r_3)^2} = 3$.
3. Square both sides: $(2 + r_3)^2 + (1 + r_3)^2 = 9$, which expands to $4 + 4r_3 + r_3^2 + 1 + 2r_3 + r_3^2 = 9$, so $2r_3^2 + 6r_3 - 4 = 0$, or $r_3^2 + 3r_3 - 2 = 0$.
4. Solve with the quadratic formula: $r_3 = \frac{-3 \pm \sqrt{17}}{2}$. A radius cannot be negative, and the figure shows C_3 as clearly the smallest circle, so take the root that gives a small positive value: $r_3 = \frac{-3 + \sqrt{17}}{2} \approx 0.56$ cm.

Let's summarize:

- Placing O_3 at the right angle and the other two centers along the axes turns the tangency conditions into a distance formula equation.
- Squaring and simplifying gives the same quadratic $r_3^2 + 3r_3 - 2 = 0$ as the direct Pythagorean route.
- Only the positive root, about 0.56 cm, matches a circle smaller than both C_1 and C_2, as shown in the figure.

So $r_3 = \frac{1}{2}\left(-3 + \sqrt{17}\right)$ cm.

$$\boxed{r_3 = \frac{1}{2}\left(-3 + \sqrt{17}\right) \text{ cm}}$$

Quick Tip: Use the fact that touching circles have their centers separated by the sum of their radii, then apply the Pythagorean theorem to the right angle at O_3.

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11. Let $M_3(\mathbb{R})$ be the vector space of all 3×3 real matrices over $\mathbb{R}$ under usual matrix addition and scalar multiplication. Let $T_3(\mathbb{R})$ be the set of all 3×3 real upper triangular matrices. Which one of the following is TRUE?

(A) The quotient space $M_3(\mathbb{R})/T_3(\mathbb{R})$ is isomorphic to the vector space of all 3×3 real skew symmetric matrices over $\mathbb{R}$ under usual matrix addition and scalar multiplication.

(B) The quotient space $M_3(\mathbb{R})/T_3(\mathbb{R})$ is isomorphic to the vector space of all 3×3 real symmetric matrices over $\mathbb{R}$ under usual matrix addition and scalar multiplication.

(C) The quotient space $M_3(\mathbb{R})/T_3(\mathbb{R})$ is isomorphic to the vector space of all 3×3 real lower triangular matrices over $\mathbb{R}$ under usual matrix addition and scalar multiplication.

(D) The quotient space $M_3(\mathbb{R})/T_3(\mathbb{R})$ is isomorphic to the vector space of all 3×3 real matrices with trace zero over $\mathbb{R}$ under usual matrix addition and scalar multiplication.

Correct Answer: (A) The quotient space $M_3(\mathbb{R})/T_3(\mathbb{R})$ is isomorphic to the vector space of all 3×3 real skew symmetric matrices over $\mathbb{R}$ under usual matrix addition and scalar multiplication.

SOLUTION:

Step 1: Understanding the Concept.
Instead of just subtracting dimensions, let's build an actual basis of the quotient space $M_3(\mathbb{R})/T_3(\mathbb{R})$ by extending a basis of $T_3(\mathbb{R})$ to a basis of $M_3(\mathbb{R})$, and match it with a basis of one of the four candidate spaces.

Step 2: Key Formula or Approach.
Write E_{ij} for the matrix with a 1 in row i, column j, and 0 everywhere else. A basis of $T_3(\mathbb{R})$ (upper triangular, diagonal included) is

$\{E_{11}, E_{12}, E_{13}, E_{22}, E_{23}, E_{33}\}$, six matrices in total. Adding the three strictly lower entries E_{21}, E_{31}, E_{32} completes a basis of all nine matrices in $M_3(\mathbb{R})$.

Step 3: Detailed Explanation.

In the quotient space, the cosets of the six basis matrices of $T_3(\mathbb{R})$ all become the zero vector, since each already lies inside $T_3(\mathbb{R})$. So the quotient space is spanned by the remaining three cosets $[E_{21}]$, $[E_{31}]$, $[E_{32}]$, and these three are linearly independent, since no nonzero combination of E_{21}, E_{31}, E_{32} can land inside $T_3(\mathbb{R})$ (which has no entries below the diagonal). This gives $\dim(M_3(\mathbb{R})/T_3(\mathbb{R})) = 3$ directly. Now define a linear map φ from this quotient space to the skew symmetric matrices by sending $[E_{21}] \mapsto E_{21} - E_{12}$, $[E_{31}] \mapsto E_{31} - E_{13}$, and $[E_{32}] \mapsto E_{32} - E_{23}$, extended linearly. Each of $E_{21} - E_{12}$, $E_{31} - E_{13}$, $E_{32} - E_{23}$ is skew symmetric, since its transpose is its own negative, and these three matrices form a basis of the 3-dimensional space of skew symmetric matrices. So φ sends a basis of the quotient to a basis of the skew symmetric matrices, which makes φ a linear isomorphism. Checking the other options the same way: symmetric matrices need a 6-element basis (3 diagonal plus 3 off diagonal free entries), full lower triangular matrices also need a 6-element basis, and trace zero matrices need an 8-element basis (only one condition dropped from nine). None of these can be put in one to one linear correspondence with our 3-element quotient basis $\{[E_{21}], [E_{31}], [E_{32}]\}$.

Step 4: Final Answer.

The quotient space has a basis of exactly three cosets, and this matches a basis of the skew symmetric matrices through an explicit linear bijection.

$$\boxed{M_3(\mathbb{R})/T_3(\mathbb{R}) \cong \{\text{skew symmetric } 3 \times 3 \text{ matrices}\}}$$

Quick Tip: Compare dim(M3(R)) minus dim(T3(R)) with the dimension of each candidate matrix space; only one option's dimension can match.

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12. Let I be the integral defined as follows:

$$I = \int_0^1 \int_0^{\sqrt{y}} dx\, dy \;+\; \int_1^2 \int_{\sqrt{y-1}}^1 dx\, dy$$

If the order of integration is changed, then which one of the following is the correct expression for I?

(A)

$$\int_0^{1/2} \int_{-x}^{x} dy\, dx \;+\; \int_{1/2}^1 \int_{-x^2}^{-x^2+1} dy\, dx$$

(B)

$$\int_0^1 \int_{x^2}^{x^2+1} dy\, dx$$

(C)

$$\int_0^{1/2} \int_{x^2}^{x^2+1} dy\, dx \;+\; \int_{1/2}^1 \int_{\sqrt{x}}^{\sqrt{x+1}} dy\, dx$$

(D)

$$\int_0^1 \int_{x^2}^{x^2-1} dy\, dx$$

Correct Answer: (B)

$$\int_0^1 \int_{x^2}^{x^2+1} dy\,dx$$

SOLUTION:

This question asks us to redraw the region of integration for I and read off the new order $dy\,dx$. Let's identify the exact region traced by the two given pieces and check each option by matching both the region description and the numeric value of I.

The first piece $\int_0^1 \int_0^{\sqrt{y}} dx\,dy$ covers all points with $0 \le x \le \sqrt{y}$ and $0 \le y \le 1$; squaring the bound $x \le \sqrt{y}$ gives $x^2 \le y$, so this piece is the region between the parabola $y = x^2$ and the line $y = 1$, for $0 \le x \le 1$.

The second piece $\int_1^2 \int_{\sqrt{y-1}}^1 dx\,dy$ covers points with $\sqrt{y-1} \le x \le 1$ and $1 \le y \le 2$; squaring gives $y - 1 \le x^2$, i.e. $y \le x^2 + 1$, so this piece is the region between $y = 1$ and $y = x^2 + 1$, again for $0 \le x \le 1$.

The two pieces meet exactly at $y = 1$, so together they form one continuous region: for every x from 0 to 1, y runs from x^2 up to $x^2 + 1$. No split at $x = \frac{1}{2}$ is needed.

1. **Option A**: uses bounds $-x$ to x and $-x^2$ to $-x^2 + 1$, a completely different pair of curves that does not match either piece above. Direct evaluation gives $\frac{3}{4}$, not the true value of I.
2. **Option B**: $\int_0^1 \int_{x^2}^{x^2+1} dy\,dx$, exactly the single merged region found above, with no artificial split needed.
3. **Option C**: splits the region at $x = \frac{1}{2}$ and uses $y = \sqrt{x}$ and $y = \sqrt{x+1}$ in the second piece, curves that never appeared in the original description. It evaluates to an irrational number, not matching I.

4. **Option D**: $\int_0^1 \int_{x^2}^{x^2-1} dy\,dx$ has an upper limit $x^2 - 1$ smaller than the lower limit x^2, an impossible negative height for a real region, and it evaluates to -1, the wrong sign entirely.

Computing the original integral directly confirms the check: $\int_0^1 \sqrt{y}\,dy = \frac{2}{3}$ and $\int_1^2 \left(1 - \sqrt{y-1}\right) dy = \frac{1}{3}$, so $I = \frac{2}{3} + \frac{1}{3} = 1$. Evaluating option B the same way, $\int_0^1 \left[(x^2+1) - x^2\right] dx = \int_0^1 1\,dx = 1$, the same value.

Let's summarize:

- The two given pieces are the regions between $y = x^2$ and $y = 1$, and between $y = 1$ and $y = x^2 + 1$, which join into one region $x^2 \le y \le x^2 + 1$ for $0 \le x \le 1$.
- Only option B reproduces this region and the correct value $I = 1$; the other three either use the wrong bounding curves or an impossible negative height.

So the correct expression after changing the order of integration is $\int_0^1 \int_{x^2}^{x^2+1} dy\,dx$, option B.

$$\boxed{I = \int_0^1 \int_{x^2}^{x^2+1} dy\,dx}$$

Quick Tip: Sketch both original regions in the xy-plane and see exactly where they join along the line y = 1.

13. Let C denote the Cantor set and $f : [0,1] \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} x^{2026} & \text{for } x \in C \\ \cos(\pi x) & \text{for } x \in \left[0, \frac{1}{2}\right] \setminus C \\ \sin(\pi x) & \text{for } x \in \left[\frac{1}{2}, 1\right] \setminus C \end{cases}$$

The value of the Lebesgue integral of $f(x)$ over the interval $[0,1]$ is equal to

(A) $\dfrac{2}{\pi}$

(B) $\dfrac{1}{\pi}$

(C) $\dfrac{3}{\pi}$

(D) 0

Correct Answer: (A) $\dfrac{2}{\pi}$

SOLUTION:

The key idea here is that the Cantor set C has Lebesgue measure zero, so let's recompute the measure of C from scratch and then see why the values of f on C never affect the integral.

The Cantor set is built by repeatedly removing the open middle third of each remaining interval. At step 1 we remove a piece of length $\frac{1}{3}$, at step 2 we remove 2 pieces of length $\frac{1}{9}$ each, and at step n we remove 2^{n-1} pieces of length $\frac{1}{3^n}$. The total length removed is the geometric series $\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{3} \cdot \dfrac{1}{1 - \frac{2}{3}} = 1$. Since we started with an interval of length 1 and removed a total length of 1, the Cantor set has Lebesgue measure 0.

Because $m(C) = 0$, no matter how large x^{2026} gets on C, the piece $\int_C x^{2026}\, dx = 0$ (a bounded function integrated over a measure zero set is always 0). Also, removing the measure zero set C from $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$ does not change the Lebesgue integral over those pieces, so we may integrate $\cos(\pi x)$ and $\sin(\pi x)$ over the full sub-intervals as ordinary integrals.

1. $\frac{2}{\pi}$: matches $\int_0^{1/2} \cos(\pi x)\, dx + \int_{1/2}^{1} \sin(\pi x)\, dx = \frac{1}{\pi} + \frac{1}{\pi}$.
2. $\frac{1}{\pi}$: this is only one of the two pieces, so it undercounts the total by half.
3. $\frac{3}{\pi}$: this would need an extra $\frac{1}{\pi}$ that has no source in the problem, an overcount.
4. 0: this would only be right if the two pieces canceled out, but both $\int_0^{1/2} \cos(\pi x)\, dx$ and $\int_{1/2}^{1} \sin(\pi x)\, dx$ are positive, so they add instead of cancel.

Now compute the two pieces directly. First, $\int_0^{1/2} \cos(\pi x)\, dx$ $= \left[\dfrac{\sin(\pi x)}{\pi}\right]_0^{1/2} = \dfrac{\sin(\pi/2) - \sin 0}{\pi} = \dfrac{1}{\pi}$. Second, $\int_{1/2}^{1} \sin(\pi x)\, dx$ $= \left[-\dfrac{\cos(\pi x)}{\pi}\right]_{1/2}^{1} = -\dfrac{\cos \pi}{\pi} + \dfrac{\cos(\pi/2)}{\pi} = \dfrac{1}{\pi} + 0 = \dfrac{1}{\pi}$.

Let's summarize:

- The Cantor set has measure zero, so the x^{2026} branch of f never contributes to the integral.
- The integral reduces to $\int_0^{1/2} \cos(\pi x)\, dx + \int_{1/2}^{1} \sin(\pi x)\, dx = \frac{1}{\pi} + \frac{1}{\pi}$ $= \frac{2}{\pi}$.

So the value of the Lebesgue integral of f over $[0, 1]$ is $\frac{2}{\pi}$.

$$\boxed{\frac{2}{\pi}}$$

Quick Tip: The Cantor set has Lebesgue measure zero, so its branch of f never affects the value of the integral.

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14. Let G be a group of order 595. Which one of the following is TRUE?

(A) G cannot have a proper normal subgroup.

(B) G must have a proper normal subgroup.

(C) G cannot have an element of order 17.

(D) The number of Sylow 5-subgroups is 17.

Correct Answer: (B) G must have a proper normal subgroup.

SOLUTION:

Step 1: Understanding the Concept.
Here is a different way to see that G must have a proper normal subgroup: instead of pinning down n_5 directly by congruence, count elements and show that the Sylow 5-subgroup and the Sylow 7-subgroup cannot both fail to be normal at the same time.

Step 2: Key Formula or Approach.
If a Sylow p-subgroup is not normal, then $n_p > 1$, and since every Sylow p-subgroup here has prime order p, any two distinct ones intersect only in the identity. So all n_p of them together contribute $n_p(p-1)$ distinct non-identity elements of order p to G. If two different primes both give $n_p > 1$, we can add up these counts and compare with $|G|$.

Step 3: Detailed Explanation.
As before, $595 = 5 \times 7 \times 17$. For the Sylow 5-subgroups, n_5 divides 119 and is $\equiv 1 \pmod 5$; among the divisors $1, 7, 17, 119$ of 119, only 1 and 119 can possibly satisfy this (checking mod 5 gives $1, 2, 2, 4$, so really

only 1 works, but suppose for the sake of this alternate check that n_5 could be as large as 119). For the Sylow 7-subgroups, n_7 divides 85 and is $\equiv 1 \pmod{7}$; the divisors of 85 are $1, 5, 17, 85$, and mod 7 these are $1, 5, 3, 1$, so $n_7 \in \{1, 85\}$.

Suppose, for contradiction, that both $n_5 > 1$ and $n_7 > 1$, so $n_5 = 119$ and $n_7 = 85$. Then the number of elements of order 5 is $119 \times (5 - 1)$ $= 476$, and the number of elements of order 7 is $85 \times (7 - 1) = 510$. These elements are all distinct from each other and from the identity, so just these two counts already add up to $476 + 510 = 986$, which is more than $|G| = 595$. That is impossible.

So n_5 and n_7 cannot both be greater than 1 at the same time; at least one of the Sylow 5-subgroup or the Sylow 7-subgroup must be the unique one of its kind, and a unique Sylow subgroup is always normal. Either way, G has a normal subgroup of order 5 or 7, and both are proper subgroups of a group of order 595.

Step 4: Final Answer.

Counting elements rules out having both the Sylow 5-subgroup and the Sylow 7-subgroup non-normal at once, so G is forced to have a proper normal subgroup.

$$\boxed{\text{Option B: } G \text{ must have a proper normal subgroup}}$$

Quick Tip: Factor 595 = 5 x 7 x 17 and check which Sylow subgroup count is forced to equal 1 by the congruence condition.

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15. Let $u(x,t)$ be the solution of the initial value problem for the heat equation on the real line:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \qquad t > 0, \qquad k \in \mathbb{R}$$

with the initial condition

$$u(x,0) = e^{-a|x|}, \qquad a > 0.$$

If $\hat{u}(w,t) = \displaystyle\int_{-\infty}^{\infty} u(x,t)\, e^{iwx}\, dx$ is the Fourier transform of $u(x,t)$ with respect to x, then $\hat{u}(w,t)$ is equal to

(A)

$$\frac{2a}{a^2+w^2}\, e^{-kw^2 t}$$

(B)

$$\frac{a}{a^2+w^2}\, e^{-kw^2 t}$$

(C)

$$\frac{2a}{a^2+w^2}\, e^{-ka^2 t}$$

(D)

$$\frac{1}{\sqrt{4\pi k t}}\, e^{\frac{-w^2}{4kt}}$$

Correct Answer: (A)

$$\frac{2a}{a^2+w^2}\, e^{-kw^2 t}$$

SOLUTION:

Here is a different way to reach the same transform, by first writing the solution in real space as a convolution and then using the convolution theorem, instead of transforming the PDE directly.

The general solution of the heat equation $u_t = ku_{xx}$ on the whole real line, with initial data $u(x, 0) = g(x)$, is the convolution of g with the heat kernel:

$$u(x,t) = \int_{-\infty}^{\infty} g(y)\,K(x-y,t)\,dy, \qquad K(x,t) = \frac{1}{\sqrt{4\pi kt}}\,e^{-x^2/(4kt)}$$

Taking the Fourier transform of a convolution turns it into a product of the two individual transforms, so

$$\hat{u}(w,t) = \hat{g}(w)\cdot\hat{K}(w,t)$$

The Fourier transform of the heat kernel $K(x, t)$, with the convention $\hat{f}(w) = \int f(x)e^{iwx}dx$, works out to $\hat{K}(w,t) = e^{-kw^2t}$; this is the standard fact that a Gaussian in x transforms into a decaying exponential in w.

Now find $\hat{g}(w)$, the transform of the initial data $g(x) = e^{-a|x|}$. Split the integral at $x = 0$:

$$\hat{g}(w) = \int_0^{\infty} e^{-ax}e^{iwx}dx + \int_{-\infty}^{0} e^{ax}e^{iwx}dx = \frac{1}{a-iw} + \frac{1}{a+iw} = \frac{2a}{a^2+w^2}$$

Multiplying the two pieces together:

$$\hat{u}(w,t) = \frac{2a}{a^2+w^2}\cdot e^{-kw^2t}$$

Let's check this against the other options. Option B is missing the factor of 2 that comes from adding the two halves of the integral for g. Option C puts a^2 in the exponent instead of w^2, but the exponential decay in time must depend on w (the Fourier variable dual to x), not on a (which only measures how fast the initial condition itself decays). Option D is just $\hat{K}(w,t)$ alone, the transform for a point source (delta function) initial condition; it has no a in it at all, so it cannot be the right answer once the actual initial data $e^{-a|x|}$ is folded in.

Let's summarize:

- $u(x,t)$ is the convolution of the initial data with the heat kernel, so its transform is the product of the two transforms.
- $\hat{g}(w) = \dfrac{2a}{a^2+w^2}$ and $\hat{K}(w,t) = e^{-kw^2t}$, giving $\hat{u}(w,t) = \dfrac{2a}{a^2+w^2}e^{-kw^2t}$.

So the correct expression is $\hat{u}(w,t) = \dfrac{2a}{a^2+w^2}e^{-kw^2t}$.

$$\boxed{\hat{u}(w,t) = \frac{2a}{a^2+w^2}\,e^{-kw^2t}}$$

Quick Tip: Transform the heat equation in x first to get a simple ODE in t, then separately transform the initial condition e^{-a|x|}.

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16. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$f(x,y) = (e^x \cos y,\; e^x \sin y)$$

Which one of the following is TRUE?

(A) f is one-to-one.

(B) The Jacobian of f is negative.

(C) f is locally invertible.

(D) f is invertible on $\mathbb{R}^2$.

Correct Answer: (C) f is locally invertible.

SOLUTION:

Look at this map through complex numbers instead of raw partial derivatives. Write $z = x + iy$. Then $e^z = e^x(\cos y + i \sin y) = e^x \cos y + i\, e^x \sin y$, so the given $f(x, y) = (e^x \cos y, e^x \sin y)$ is exactly the real and imaginary parts of the complex exponential e^z.

1. **(A) f is one-to-one**: Since e^z has period $2\pi i$ in z, the points $z = 0$ and $z = 2\pi i$, that is $(x, y) = (0, 0)$ and $(x, y) = (0, 2\pi)$, both give $e^z = 1$. Two distinct inputs give the same output, so f is not injective. This option is false.
2. **(B) The Jacobian of f is negative**: A holomorphic function like e^z has Jacobian determinant equal to $|e^z|^2 = e^{2x}$, which is always strictly positive, never negative. This option is false.
3. **(C) f is locally invertible**: The complex derivative of e^z is e^z itself, and $e^z \neq 0$ for every z, so the map is a local diffeomorphism, meaning it is locally one-to-one with a smooth local inverse, at every point. This matches exactly what "locally invertible" means. This option is true.
4. **(D) f is invertible on R2**: Global invertibility needs global injectivity, which is already ruled out in option (A) using the $2\pi i$ periodicity. So f cannot be invertible on all of $\mathbb{R}^2$. This option is false.

So the only correct statement is that f is locally invertible everywhere, even though the periodicity of e^z stops it from being globally one-to-one. This is a classic example showing that a nonzero Jacobian only guarantees a local inverse, not a global one.

Let's summarize:

- The Jacobian determinant of f works out to e^{2x}, which never vanishes and is always positive.
- A nonzero Jacobian at a point gives local invertibility there by the Inverse Function Theorem, but says nothing about injectivity on the whole plane.
- The 2π periodicity in y, equivalently $2\pi i$ periodicity of e^z, is what breaks global injectivity, so options (A) and (D) both fail.

Hence the correct choice is that f is locally invertible.

$$\boxed{\text{(C) } f \text{ is locally invertible everywhere on } \mathbb{R}^2}$$

Quick Tip: Compute the Jacobian determinant of f, then compare local invertibility with global injectivity using the periodicity of sine and cosine in y.

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17. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = |z|^2 - 5\bar{z} + 2$.
Which one of the following is TRUE?

(A) f is differentiable at $z = 5i$.

(B) f is differentiable at $z = 5$.

(C) f is differentiable at $z = -5i$.

(D) f is differentiable at $z = -5$.

Correct Answer: (B) f is differentiable at $z = 5$.

SOLUTION:

Here is a quicker route using Wirtinger derivatives instead of splitting into real and imaginary parts by hand. Recall $|z|^2 = z\bar{z}$, so we can write

$f(z) = z\bar{z} - 5\bar{z} + 2$ directly as a function of z and $\bar{z}$ treated as independent variables.

For a function built from z and $\bar{z}$, complex differentiability at a point, in the sense tested by the Cauchy-Riemann equations, happens exactly where $\frac{\partial f}{\partial \bar{z}} = 0$ at that point, as long as f is smooth as a function of x, y, which is true here since it is a polynomial. Compute this Wirtinger derivative, treating z as a constant while differentiating with respect to $\bar{z}$:

$$\frac{\partial f}{\partial \bar{z}} = z - 5$$

So f is differentiable exactly where $z - 5 = 0$, that is, only at the single point $z = 5$. Now check each option against this:

1. **(A) z = 5i**: Here $z - 5 = 5i - 5 \neq 0$, so f is not differentiable here.
2. **(B) z = 5**: Here $z - 5 = 0$, so this is exactly the point of differentiability.
3. **(C) z = -5i**: Here $z - 5 = -5i - 5 \neq 0$, not differentiable.
4. **(D) z = -5**: Here $z - 5 = -10 \neq 0$, not differentiable.

Only $z = 5$ makes the Wirtinger derivative $\partial f / \partial \bar{z}$ vanish, so f is complex differentiable at that single point and nowhere else nearby.

Let's summarize:

- Writing f using z and $\bar{z}$ turns the Cauchy-Riemann test into the single condition $\partial f / \partial \bar{z} = 0$.
- Here that condition reduces to the linear equation $z - 5 = 0$, giving one isolated point of differentiability.
- That point is $z = 5$, matching option (B) and confirming the same answer found using the standard real-variable Cauchy-Riemann equations.

$$\boxed{z = 5}$$

Quick Tip: Split f into real and imaginary parts and solve the Cauchy-Riemann equations for x and y.

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18. Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ denote the unit disc in the complex plane $\mathbb{C}$. Let $f : D \to D$ be an analytic function which satisfies $f(0) = 0$. Then which one of the following is a possible value of $f'(0)$?

(A) $\frac{5}{2}i$

(B) $\frac{i}{10}$

(C) $\frac{3}{2}$

(D) $-\frac{5}{2}i$

Correct Answer: (B) $\frac{i}{10}$

SOLUTION:

Instead of just quoting the Schwarz Lemma, let's build the bound on $f'(0)$ from the maximum modulus principle directly, then use it to eliminate the wrong options.

Since $f(0) = 0$ and f is analytic on D, we can write $f(z) = z\,g(z)$ for some function g that is analytic on all of D, which comes from factoring out the zero of f at the origin using the power series of f. Note that $g(0) = f'(0)$, since the power series of f starts as $f(z) = f'(0)z + \cdots$, so dividing by z and setting $z = 0$ gives $g(0) = f'(0)$.

Now fix any radius r with $0 < r < 1$. On the circle $|z| = r$, we have $|f(z)| < 1$ since f maps into D, so

$$|g(z)| = \frac{|f(z)|}{|z|} < \frac{1}{r} \quad \text{on } |z| = r$$

By the maximum modulus principle, this bound also holds inside the circle, so $|g(0)| \le 1/r$. Letting $r \to 1$ gives $|g(0)| \le 1$, that is,

$$|f'(0)| \le 1$$

This is the same conclusion as the Schwarz Lemma, just derived directly. Now check the four options against $|f'(0)| \le 1$:

1. **(A)** $\frac{5}{2}i$: magnitude 2.5, too big, ruled out.
2. **(B)** $\frac{i}{10}$: magnitude 0.1, satisfies the bound comfortably.
3. **(C)** $\frac{3}{2}$: magnitude 1.5, too big, ruled out.
4. **(D)** $-\frac{5}{2}i$: magnitude 2.5, too big, ruled out.

Only option (B) is small enough in magnitude to be a valid derivative value. A concrete example that actually achieves it is the linear map $f(z) = \frac{i}{10}z$, which clearly maps D into D, fixes 0, and has $f'(0) = \frac{i}{10}$.

Let's summarize:

- Writing $f(z) = zg(z)$ and applying the maximum modulus principle to g gives $|f'(0)| \le 1$, the same bound the Schwarz Lemma provides.
- Three of the four options have magnitude bigger than 1 and are immediately ruled out.
- Only $\frac{i}{10}$ satisfies the bound, and it is realized by a genuine example, $f(z) = \frac{i}{10}z$.

$$\boxed{f'(0) = \frac{i}{10}}$$

Quick Tip: Apply the Schwarz Lemma to bound the magnitude of f'(0) for an analytic self-map of the disc that fixes 0.

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19. Let X be the space of all continuously differentiable real valued functions on $[0, 1]$. Define the following norms on X:

$$p_1(x) = \sup\{|x(t)| : t \in [0, 1]\}$$

$$p_2(x) = \sup\left\{\left|\frac{d}{dt}x(t)\right| : t \in [0, 1]\right\}$$

and $p_3(x) = p_1(x) + p_2(x)$.
Which one of the following is TRUE?

(A) (X, p_1) is a Banach space.

(B) (X, p_2) is a Banach space.

(C) (X, p_3) is not a Banach space.

(D) (X, p_3) is a Banach space.

Correct Answer: (D) (X, p_3) is a Banach space.

SOLUTION:

This question is really about testing completeness of $C^1[0, 1]$ under three different ways of measuring size. Let's go option by option with concrete examples first, then confirm with a general argument for p_3.

1. **(A) (X, p1) is a Banach space**: Pick a sequence of smooth functions $x_n(t)$ that converges uniformly to the tent function $g(t) = |t - 1/2|$, for example by smoothing the corner more and more

sharply as n grows. Each x_n is in X and the sequence is Cauchy under the sup norm p_1, since it converges uniformly. But g is not differentiable at $t = 1/2$, so $g \notin X$. A Cauchy sequence with no limit inside X means (X, p_1) is not complete. This option is false.

2. **(B) (X, p2) is a Banach space**: Before even asking about completeness, check whether p_2 qualifies as a norm at all. Take the nonzero constant function $x(t) = 1$. Its derivative is 0 everywhere, so $p_2(x) = \sup|x'(t)| = 0$, even though $x \neq 0$. A norm must be zero only for the zero vector, so p_2 fails this basic requirement and is only a seminorm. Since (X, p_2) is not a genuine normed space to begin with, calling it a Banach space is not correct. This option is false.
3. **(C) (X, p3) is not a Banach space**: The next point shows (X, p_3) is in fact complete, so this option is false.
4. **(D) (X, p3) is a Banach space**: Let (x_n) be Cauchy in $p_3 = p_1 + p_2$. Since $p_1(x_n - x_m) \to 0$ and $p_2(x_n - x_m) \to 0$ as $n, m \to \infty$, both x_n and its derivative x_n' are uniformly Cauchy sequences of continuous functions on $[0, 1]$. By completeness of the space of continuous functions with the sup norm, $x_n \to$ g uniformly and $x_n' \to h$ uniformly for some continuous g and h. A standard theorem on uniform convergence of derivatives then tells us g is differentiable and $g' = h$, so $g \in C^1[0, 1] = X$. Since both $x_n \to g$ and $x_n' \to g'$ uniformly, $p_3(x_n - g) \to 0$, so $x_n \to g$ in the p_3 norm. Every Cauchy sequence converges inside X, so (X, p_3) is complete, that is, a Banach space. This option is true.

Let's summarize:

- p_1 alone is not enough because the uniform limit of C^1 functions need not be differentiable.

- p_2 alone is not even a valid norm, since it vanishes on every nonzero constant function.
- Only $p_3 = p_1 + p_2$, which tracks the function and its derivative together, is both a genuine norm and complete.

$$\boxed{\text{(D) } (X, p_3) \text{ is a Banach space}}$$

Quick Tip: Check whether each candidate is complete, and first check whether p2 alone is even a valid norm on constant functions.

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20. Let X be a set with at least two elements. Let d_1, d_2 and d_3 be metrics on X.
Which one of the following is NOT a metric on X?

(A) $d(x, y) := \min(d_1(x, y), 3)$ for all $x, y \in X$.

(B) $d(x, y) := \max(d_2(x, y), 3)$ for all $x, y \in X$.

(C) $d(x, y) := \dfrac{10\, d_3(x, y)}{1 + d_3(x, y)}$ for all $x, y \in X$.

(D) $d(x, y) := \dfrac{1}{3}(d_1(x, y) + d_2(x, y) + d_3(x, y))$ for all $x, y \in X$.

Correct Answer: (B) $d(x, y) := \max(d_2(x, y), 3)$ for all $x, y \in X$.

SOLUTION:

There is a general pattern worth knowing here: if d is a metric and $c > 0$ is a constant, then $\min(d, c)$ is always again a metric, since it just caps the distance at c, but $\max(d, c)$ is never a metric, because it forces every point to be at distance at least c from itself. Let's use this idea to spot the wrong option quickly, then verify the rest.

1. **(A) min(d1(x,y), 3)**: At $x = y$, this gives $\min(0, 3) = 0$, so the identity property survives. Symmetry is inherited directly from d_1. For the triangle inequality, if $a = d_1(x, z) \leq b + e = d_1(x, y) + d_1(y, z)$, then capping each side at 3 still respects the inequality, checking the cases where b, e are above or below 3 shows $\min(a, 3) \leq \min(b, 3) + \min(e, 3)$ always. So this is a genuine metric.
2. **(B) max(d2(x,y), 3)**: Set $x = y$. Since $d_2(x, x) = 0$, as d_2 is a metric, we get $d(x, x) = \max(0, 3) = 3 \neq 0$. Every point ends up at distance 3 from itself, which directly breaks the requirement that $d(x, x) = 0$. This fails to be a metric, and the failure does not depend on which particular metric d_2 we started with; it happens purely because of the $\max(\cdot, 3)$ construction.
3. **(C) 10 d3(x,y) / (1 + d3(x,y))**: This is the classic bounded-metric transform. It is 0 exactly when $d_3(x, y) = 0$, that is when $x = y$, it is symmetric since d_3 is, and because $t \mapsto \frac{10t}{1+t}$ is increasing and subadditive on $t \geq 0$, applying it to the triangle inequality for d_3 preserves the triangle inequality for d. So this is a genuine metric.
4. **(D) (1/3)(d1(x,y)+d2(x,y)+d3(x,y))**: A positive-weighted sum of metrics is again a metric: it vanishes only when all three vanish, which happens only at $x = y$ since each d_i is already a metric, it is symmetric, and the triangle inequality follows by adding the three individual triangle inequalities and dividing by 3. So this is a genuine metric.

Only option (B) breaks down, and it breaks down specifically at the diagonal case $x = y$, giving a nonzero self-distance of 3.

Let's summarize:

- Capping a metric from above with $\min(d, c)$ always keeps it a metric.

- Flooring a metric from below with $\max(d, c)$ for $c > 0$ always destroys the metric property, since it makes $d(x, x) = c \neq 0$.
- The bounded transform $\frac{10d_3}{1+d_3}$ and the average $\frac{1}{3}(d_1 + d_2 + d_3)$ both preserve every metric axiom.

$$\boxed{\text{(B) } \max(d_2(x, y), 3) \text{ is NOT a metric}}$$

Quick Tip: Test each candidate distance function at x = y first and see whether it still comes out to zero.

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21. Let P be a 3×3 real symmetric positive definite matrix. Which of the following statements is/are TRUE?

(A) $Q^\top PQ$ is positive definite for all nonzero $Q \in \mathbb{R}^{3\times3}$.

(B) $Q^\top PQ$ is positive semidefinite for all $Q \in \mathbb{R}^{3\times3}$.

(C) $Q^\top PQ$ is positive definite if $Q \in \mathbb{R}^{3\times3}$ is nonsingular.

(D) $Q^\top PQ$ is not positive semidefinite for some $Q \in \mathbb{R}^{3\times3}$.

Correct Answer: (B) $Q^\top PQ$ is positive semidefinite for all $Q \in \mathbb{R}^{3\times3}$. ; (C) $Q^\top PQ$ is positive definite if $Q \in \mathbb{R}^{3\times3}$ is nonsingular.

SOLUTION:

A clean way to answer this is to write the positive definite matrix P as $P = R^\top R$ for some invertible 3×3 matrix R (this is always possible for a symmetric positive definite matrix, for instance through a Cholesky decomposition). Then for any real matrix Q,

$$Q^\top PQ = Q^\top R^\top RQ = (RQ)^\top (RQ)$$

Any matrix of the form $M^\top M$ satisfies $v^\top M^\top M v = \|Mv\|^2 \geq 0$ for every vector v, so it is automatically positive semidefinite, no matter what M is. Setting $M = RQ$, this immediately tells us $Q^\top PQ$ is positive semidefinite for every choice of Q, with no extra condition needed.

1. **(A)**: If Q is nonzero but singular, then RQ is also singular (multiplying by the invertible R does not fix a rank deficiency), so there is a nonzero v with $(RQ)v = 0$, giving $v^\top(Q^\top PQ)v = \|(RQ)v\|^2 = 0$. This breaks strict positivity, so (A) is false.
2. **(B)**: True, shown directly above using $Q^\top PQ = (RQ)^\top(RQ)$, which holds for every Q including the zero matrix and every singular matrix.
3. **(C)**: If Q is nonsingular, then RQ is a product of two invertible matrices, so RQ is invertible too, and $(RQ)v = 0$ only when $v = 0$. So $v^\top(Q^\top PQ)v = \|(RQ)v\|^2 > 0$ for every nonzero v, meaning $Q^\top PQ$ is positive definite. (C) is true.
4. **(D)**: This says $Q^\top PQ$ fails to be positive semidefinite for some Q. Since we showed it equals $(RQ)^\top(RQ)$ for every Q, and that form is always positive semidefinite, (D) is false.

So the true statements are (B) and (C): writing P as $R^\top R$ turns the whole problem into a statement about $M^\top M$ matrices, which are semidefinite by construction and become strictly definite exactly when Q is invertible.

Let's summarize:

- $Q^\top PQ$ can be rewritten as $(RQ)^\top(RQ)$, a Gram-matrix form that is always positive semidefinite.
- Invertibility of Q is exactly the extra condition that upgrades semidefinite to definite.

The correct options are (B) and (C).

$$\boxed{\text{(B) and (C)}}$$

Quick Tip: Write $v^\top(Q^\top PQ)v = (Qv)^\top P(Qv)$ and check what happens to Qv when Q is singular versus nonsingular.

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22. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined as follows:

$$f(x,y) = \begin{cases} \dfrac{x\sqrt{x^2+y^2}}{|x|} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Which of the following statements is/are TRUE?

(A) $f(x,y)$ is continuous at $(0,0)$.

(B) For any $u \in \mathbb{R}^2$, the directional derivative of f at $(0,0)$ in the direction of u exists.

(C) $\dfrac{\partial f}{\partial x}(0,0) = 1 = \dfrac{\partial f}{\partial y}(0,0)$.

(D) $f(x,y)$ is differentiable at $(0,0)$.

Correct Answer: (A) $f(x,y)$ is continuous at $(0,0)$. ; (B) For any $u \in \mathbb{R}^2$, the directional derivative of f at $(0,0)$ in the direction of u exists.

SOLUTION:

A neat way to handle this problem is to switch to polar coordinates near the origin: write $x = r\cos\theta$, $y = r\sin\theta$ with $r \geq 0$. For $x \neq 0$ (that is, $\cos\theta \neq 0$),

$$f(r,\theta) = \frac{x\sqrt{x^2+y^2}}{|x|} = \frac{r\cos\theta \cdot r}{r|\cos\theta|} = r \cdot \text{sign}(\cos\theta)$$

and $f = 0$ when $\cos\theta = 0$ (the y-axis, where $x = 0$ by definition). So near the origin, f always looks like r times a bounded factor of size at most 1.

1. **(A) Continuity at** $(0,0)$: True. Since $|f(r,\theta)| \le r$ for every angle θ, letting $r \to 0$ forces $f \to 0$, which matches $f(0,0) = 0$. The angle plays no role in the limit, so the limit is the same from every direction.
2. **(B) Directional derivative exists for every** u: True. Along a ray at fixed angle θ, $f(r,\theta) = r \cdot \operatorname{sign}(\cos\theta)$ is linear in r, so its derivative in r at $r = 0$ is just the constant $\operatorname{sign}(\cos\theta)$, a finite number for every angle (and 0 when $\cos\theta = 0$).
3. **(C)** $\frac{\partial f}{\partial x}(0,0) = 1 = \frac{\partial f}{\partial y}(0,0)$: False. Taking $\theta = 0$ (the positive x-direction) gives $\operatorname{sign}(\cos 0) = 1$, so $\partial f/\partial x(0,0) = 1$. But taking $\theta = \pi/2$ (the y-direction) gives $\cos\theta = 0$, which falls in the $x = 0$ branch where $f \equiv 0$, so $\partial f/\partial y(0,0) = 0$, not 1.
4. **(D) Differentiable at** $(0,0)$: False. A differentiable function must have its directional derivative equal to $\cos\theta \cdot f_x(0,0) + \sin\theta \cdot f_y(0,0) = \cos\theta$, a smooth function of θ. But the actual directional derivative here is $\operatorname{sign}(\cos\theta)$, which jumps between -1 and 1 and does not match $\cos\theta$ except at $\theta = 0, \pi$. This mismatch rules out differentiability.

So only (A) and (B) hold up.

Let's summarize:

- In polar form f collapses to r times a sign that depends only on the angle, which is enough to guarantee continuity and existence of every directional derivative.
- That same sign factor is not a smooth (linear) function of direction, which is exactly why the two partials are unequal and

why differentiability fails.

The correct options are (A) and (B).

$$\boxed{\text{(A) and (B)}}$$

Quick Tip: Rewrite $f(x, y) = \text{sign}(x)\sqrt{x^2 + y^2}$ for $x \neq 0$ and check whether this bound goes to zero along every path, and whether the directional derivative is linear in direction.

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23. Let $\mathbb{Z}_{11}$ be the ring of integers modulo 11. Let $\mathbb{Z}_{11}[x]$ be the ring of all polynomials with coefficients in $\mathbb{Z}_{11}$. Which of the following statements is/are TRUE?

(A) The polynomial $x^2 + x + 4$ is irreducible over $\mathbb{Z}_{11}$.

(B) $\mathbb{Z}_{11}[x]/\langle x^2 + x + 4\rangle$ is a field.

(C) The ideal $\langle x^2 + x + 4\rangle$ is maximal.

(D) The ideal $\langle x^2 + x + 4\rangle$ is not a prime ideal.

Correct Answer: (A) The polynomial $x^2 + x + 4$ is irreducible over $\mathbb{Z}_{11}$. ; (B) $\mathbb{Z}_{11}[x]/\langle x^2 + x + 4\rangle$ is a field. ; (C) The ideal $\langle x^2 + x + 4\rangle$ is maximal.

SOLUTION:

Instead of testing every element of $\mathbb{Z}_{11}$ one by one, we can use the quadratic formula idea. For a quadratic $x^2 + x + 4$ over a field where 2 is invertible (true for $\mathbb{Z}_{11}$ since 11 is odd), a root exists exactly when the discriminant $D = 1^2 - 4(1)(4) = 1 - 16 = -15$ is a perfect square in that field. Reducing mod 11, $D \equiv -15 + 22 = 7 \pmod{11}$.

Euler's criterion says 7 is a square mod 11 exactly when $7^{(11-1)/2} = 7^5 \equiv 1 \pmod{11}$, and a non-square when $7^5 \equiv -1 \pmod{11}$. Computing step by step: $7^2 = 49 \equiv 5$, $7^4 \equiv 5^2 = 25 \equiv 3$, and $7^5 \equiv 3 \times 7 = 21 \equiv 10 \equiv -1 \pmod{11}$. So 7 is not a square mod 11, which means the discriminant has no square root in $\mathbb{Z}_{11}$, and the quadratic $x^2 + x + 4$ has no root there.

1. **(A) Irreducible over $\mathbb{Z}_{11}$**: True. No root in $\mathbb{Z}_{11}$ means a degree 2 polynomial cannot split into two linear factors over $\mathbb{Z}_{11}$, so it is irreducible.
2. **(B) $\mathbb{Z}_{11}[x]/\langle x^2 + x + 4\rangle$ is a field**: True. $\mathbb{Z}_{11}[x]$ is a principal ideal domain because $\mathbb{Z}_{11}$ is a field, and quotienting by the ideal generated by an irreducible element in a PID always gives a field.
3. **(C) $\langle x^2 + x + 4\rangle$ is maximal**: True, this is really the same fact as (B) stated differently: an ideal is maximal exactly when the quotient by it is a field.
4. **(D) $\langle x^2 + x + 4\rangle$ is not a prime ideal**: False. Every maximal ideal is also prime (a field has no zero divisors), so this ideal is in fact a prime ideal, and (D) is wrong to claim otherwise.

So the true statements are (A), (B), and (C).

Let's summarize:

- Checking the discriminant against quadratic residues is a shortcut for irreducibility that avoids testing all 11 elements directly.
- In a PID, irreducible generator, maximal ideal, and field quotient are three ways of saying the same thing, and maximal always implies prime.

The correct options are (A), (B), and (C).

$\boxed{\text{(A), (B) and (C)}}$

Quick Tip: Check whether $x^2 + x + 4$ has a root in $\mathbb{Z}_{11}$; no root means irreducible, and in a PID that makes the ideal maximal and the quotient a field.

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24. Consider $\mathbb{R}$ with the topology

$$\tau = \{A \subseteq \mathbb{R} : A^c \text{ is finite}\} \cup \{\phi\}.$$

Which of the following statements is/are TRUE?

(A) $(\mathbb{R}, \tau)$ is a Hausdorff space.

(B) Any finite subset of $(\mathbb{R}, \tau)$ is closed.

(C) $(\mathbb{R}, \tau)$ is compact.

(D) $(\mathbb{R}, \tau)$ is connected.

Correct Answer: (B) Any finite subset of $(\mathbb{R}, \tau)$ is closed. ; (C) $(\mathbb{R}, \tau)$ is compact. ; (D) $(\mathbb{R}, \tau)$ is connected.

SOLUTION:

The topology $\tau = \{A \subseteq \mathbb{R} : A^c \text{ is finite}\} \cup \{\phi\}$ is the well known cofinite topology. Its single defining feature, that every nonempty open set misses only finitely many points of $\mathbb{R}$, drives every property below.

1. **(A) Hausdorff**: False. If two nonempty open sets U and V were disjoint, each would have to sit inside the complement of the other. Since complements are finite, both U and V would have to be finite. But a nonempty open set already has a finite

complement in an infinite space $\mathbb{R}$, so it cannot itself be finite as well (that would make $\mathbb{R}$ finite). So no two nonempty open sets can ever be disjoint, and the Hausdorff separation property fails completely.

2. **(B) Finite subsets are closed**: True. Take any finite set F. Its complement F^c has complement equal to F itself, which is finite by assumption, so F^c satisfies the exact rule that makes a set open in τ. Hence F^c is open, which by definition makes F closed.
3. **(C) Compact**: True. Given any open cover of $\mathbb{R}$, choose one nonempty set U from it. Only finitely many points of $\mathbb{R}$ lie outside U, and each of those points is covered by at least one more set from the cover. Adding those finitely many sets to U produces a finite subcover, so every open cover reduces to a finite one.
4. **(D) Connected**: True. A space is disconnected only if it splits into two disjoint nonempty open sets. Here, if $\mathbb{R} = U \sqcup V$ with both open and nonempty, then V sits inside the finite set U^c, so V is finite; but V being open and nonempty also forces V^c finite, and then $\mathbb{R} = V \cup V^c$ would be a union of two finite sets, which is impossible since $\mathbb{R}$ is infinite. So no such split can exist.

Every open set except the empty set is almost all of $\mathbb{R}$, missing only a finite chunk. That single idea is what breaks Hausdorff separation, yet guarantees closedness of finite sets, compactness, and connectedness all at once.

Let's summarize:

- Two nonempty open sets in this topology can never be disjoint, so Hausdorff fails.
- Finiteness of complements is exactly what makes finite sets closed, covers reducible to finite subcovers, and the whole space impossible to split into two open pieces.

The correct options are (B), (C), and (D).

$$\boxed{\text{(B), (C) and (D)}}$$

Quick Tip: This is the cofinite topology: every nonempty open set has a finite complement. Think about whether two such sets can ever be disjoint.

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25. Consider the following partial differential equation:

$$a\frac{\partial^2 f(x,y)}{\partial x^2} + b\frac{\partial^2 f(x,y)}{\partial y^2} = 8f(x,y)$$

where a and b are distinct positive real numbers.
The combination(s) of the values of the real parameters ξ and η for which $f(x,y) = e^{2\xi x+\eta y}$ is a solution of the given partial differential equation, is/are

(A) $\xi = \dfrac{1}{\sqrt{a}},\ \eta = \dfrac{2}{\sqrt{b}}$

(B) $\xi = 0,\ \eta = 0$

(C) $\xi = \dfrac{1}{\sqrt{2a}},\ \eta = \dfrac{1}{\sqrt{2b}}$

(D) $\xi = 0,\ \eta = 2\sqrt{\dfrac{2}{b}}$

Correct Answer: (A) $\xi = \dfrac{1}{\sqrt{a}},\ \eta = \dfrac{2}{\sqrt{b}}$; (D) $\xi = 0,\ \eta = 2\sqrt{\dfrac{2}{b}}$

SOLUTION:

Rather than deriving one general algebraic condition first, we can plug each candidate pair (ξ, η) straight into the original equation af_{xx}

$+ bf_{yy} = 8f$ and check the arithmetic directly, using $f_{xx} = 4\xi^2 f$ and $f_{yy} = \eta^2 f$ from differentiating $f = e^{2\xi x+\eta y}$ twice.

1. **(A) $\xi = 1/\sqrt{a}$, $\eta = 2/\sqrt{b}$**: Here $\xi^2 = 1/a$ and $\eta^2 = 4/b$. The left side becomes $a \cdot 4 \cdot (1/a)f + b \cdot (4/b)f = 4f + 4f = 8f$, which equals the right side exactly. This pair works.
2. **(B) $\xi = 0$, $\eta = 0$**: Both second derivative terms vanish since $\xi = \eta = 0$, so the left side is 0, but the right side is $8f$, which is never zero for real x, y (an exponential is always positive). This pair fails.
3. **(C) $\xi = 1/\sqrt{2a}$, $\eta = 1/\sqrt{2b}$**: Here $\xi^2 = 1/(2a)$ and $\eta^2 = 1/(2b)$. The left side is $a \cdot 4 \cdot \frac{1}{2a} f + b \cdot \frac{1}{2b} f = 2f + 0.5f = 2.5f$, which is not equal to $8f$. This pair fails.
4. **(D) $\xi = 0$, $\eta = 2\sqrt{2/b}$**: Here $\xi = 0$ kills the x-term entirely, and $\eta^2 = 4 \cdot (2/b) = 8/b$. The left side is $0 + b \cdot (8/b)f = 8f$, matching the right side exactly. This pair works.

Since a and b are distinct positive numbers, none of these divisions run into trouble (we never divide by zero), so the arithmetic above holds for every valid choice of a, b.

Let's summarize:

- The pair in (A) makes both terms on the left equal to $4f$ each, adding to $8f$.
- The pair in (D) sends the x-term to zero and makes the y-term alone equal to $8f$.
- (B) gives 0 and (C) gives $2.5f$, neither of which reaches $8f$.

The correct options are (A) and (D).

$$\boxed{\text{(A) and (D)}}$$

Quick Tip: Differentiate $f = e^{2\xi x+\eta y}$ twice in x and y, substitute into the PDE, and check which pairs satisfy $4a\xi^2 + b\eta^2 = 8$.

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26.

Consider the following assignment problem where X, Y, Z are tasks, P, Q, R are agents and the cost matrix is given by:

	X	Y	Z
P	4	2	8
Q	2	3	7
R	3	1	6

Which of the following statements is/are TRUE for an optimal assignment?

(A) The cost is 9.

(B) Agent P is assigned to task Y.

(C) Agent Q is assigned to task X.

(D) Agent R is assigned to task Y.

Correct Answer: (B) Agent P is assigned to task Y. ; (C) Agent Q is assigned to task X.

SOLUTION:

Instead of using the row and column reduction method, we can just list every possible way to assign the three agents to the three tasks and compare the total costs directly, since there are only a few of them.

1. **Understanding the approach:** With 3 agents and 3 tasks, there are $3! = 6$ possible one to one assignments. We calculate the total cost of each one directly from the matrix and compare them.
2. **P-X, Q-Y, R-Z:** $\text{cost} = 4 + 3 + 6 = 13$.
3. **P-X, Q-Z, R-Y:** $\text{cost} = 4 + 7 + 1 = 12$.
4. **P-Y, Q-X, R-Z:** $\text{cost} = 2 + 2 + 6 = 10$.
5. **P-Y, Q-Z, R-X:** $\text{cost} = 2 + 7 + 3 = 12$.
6. **P-Z, Q-X, R-Y:** $\text{cost} = 8 + 2 + 1 = 11$.
7. **P-Z, Q-Y, R-X:** $\text{cost} = 8 + 3 + 3 = 14$.

The smallest total among these six values is 10, coming from the single assignment P-Y, Q-X, R-Z. Since it is the only assignment with cost 10, it is the unique optimal assignment.

Let's summarize:

- Checking the statements against this optimal assignment: the cost is 10, not 9, so (A) is false.
- Agent P does go to Y, so (B) is true. Agent Q does go to X, so (C) is true. Agent R goes to Z, not Y, so (D) is false.

So the true statements are (B) and (C).

$$\boxed{(B) \text{ and } (C)}$$

Quick Tip: Try every one to one assignment of agents to tasks and pick the one with the smallest total cost.

27.

Consider the following differential equation:

$$2x(x-2)^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+(x-2)y=0$$

Which of the following statements is/are TRUE?

(A) The point $x = 2$ is not a singular point.

(B) The point $x = 2$ is a singular point.

(C) The point $x = 2$ is a regular singular point.

(D) The point $x = 2$ is not a regular singular point.

Correct Answer: (B) The point $x = 2$ is a singular point. ; (D) The point $x = 2$ is not a regular singular point.

SOLUTION:

Instead of computing limits of $P(x)$ and $Q(x)$ directly, we can classify the point $x = 2$ by counting the order of the zero in each coefficient of the equation $a(x)y'' + b(x)y' + c(x)y = 0$, where $a(x) = 2x(x-2)^2$, $b(x) = 3x$, and $c(x) = x - 2$.

1. **Order of zero in $a(x)$ at $x = 2$:** $a(x) = 2x(x-2)^2$ has the factor $(x-2)^2$, so $a(x)$ vanishes to order 2 at $x = 2$. This confirms $x = 2$ is a singular point, since the coefficient of the highest derivative vanishes there. So statement (A) is false and statement (B) is true.
2. **Order of zero in $b(x)$ at $x = 2$:** $b(x) = 3x$ gives $b(2) = 6$, which is nonzero, so $b(x)$ has order 0 (no zero) at $x = 2$.
3. **Order of zero in $c(x)$ at $x = 2$:** $c(x) = x - 2$ vanishes to order 1 at $x = 2$.
4. **Regular singular point test:** A singular point is regular only if the order of zero of $a(x)$ exceeds that of $b(x)$ by at most 1, and

exceeds that of $c(x)$ by at most 2. Here the order of $a(x)$ exceeds the order of $b(x)$ by $2 - 0 = 2$, which is more than the allowed 1, so this test fails, meaning $x = 2$ is an irregular singular point.

Since the excess order condition on $b(x)$ is violated, $x = 2$ cannot be a regular singular point, confirming statement (D) is true and statement (C) is false.

Let's summarize:

- $x = 2$ makes the leading coefficient $a(x)$ vanish, so it is a singular point.
- The order of vanishing of $a(x)$ is too high compared to $b(x)$, so the singular point is irregular, not regular.

So the true statements are (B) and (D).

$$\boxed{(B) \text{ and } (D)}$$

Quick Tip: Write the equation as y'' + P(x)y' + Q(x)y = 0 and check if (x-2)P(x) stays finite at x=2.

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28.

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation which reflects every vector in $\mathbb{R}^3$ through a two-dimensional subspace of $\mathbb{R}^3$. Let $P \in \mathbb{R}^{3\times 3}$ be the matrix representation of T using the basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Then the value of $2 \times \text{trace}(P) - 3 \times \text{determinant}(P)$ is equal to ________. (answer in integer)

Correct Answer: 5

SOLUTION:

Instead of arguing through eigenvalues in general, we can just pick one concrete two-dimensional subspace, write down the actual matrix of T for it, and compute trace and determinant directly. Since the answer does not depend on which plane is chosen, any valid choice will do.

1. **Choose a plane:** Take the two-dimensional subspace to be the xy-plane, that is, all vectors of the form $(x, y, 0)$.
2. **Write down what T does:** Reflecting through the xy-plane keeps the x and y coordinates the same and flips the sign of the z coordinate, since z is measured perpendicular to this plane. So $T(x, y, z) = (x, y, -z)$.
3. **Write the matrix P:** Using the standard basis $(1, 0, 0), (0, 1, 0), (0, 0, 1)$, this transformation has the matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

4. **Compute trace and determinant:** The trace is the sum of the diagonal entries, $1 + 1 + (-1) = 1$. The determinant of this diagonal matrix is the product of the diagonal entries, $1 \times 1 \times (-1) = -1$.

Now substitute these into the required expression: $2 \times \text{trace}(P) - 3 \times \text{determinant}(P) = 2(1) - 3(-1) = 2 + 3 = 5$.

Let's summarize:

- Reflection through any plane in $\mathbb{R}^3$ always has trace 1 and determinant -1, since two directions stay fixed and one direction flips sign.
- The specific plane chosen does not matter; only the type of transformation matters.

So the required value is 5 for any choice of plane.

$$\boxed{5}$$

Quick Tip: A reflection through a plane in R3 has eigenvalues 1, 1, -1; use these to get the trace and determinant directly.

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29.

Let $x^3 - x + 1 \in \mathbb{Z}_3[x]$, where $\mathbb{Z}_3[x]$ is the ring of all polynomials with coefficients in $\mathbb{Z}_3$. Then the degree of the field extension

$$\mathbb{Z}_3[x]/\langle x^3 - x + 1\rangle$$

of $\mathbb{Z}_3$ is equal to ________. (answer in integer)

Correct Answer: 3

SOLUTION:

There is another way to get the same answer, by directly counting the elements of the quotient ring instead of quoting the rule that the degree equals the degree of the irreducible polynomial.

1. **Confirm $f(x) = x^3 - x + 1$ has no root in $\mathbb{Z}_3$:** checking $x = 0, 1, 2$ gives $f(0) = 1$, $f(1) = 1$, $f(2) = 7 \equiv 1 \pmod 3$. None of these is 0, so $f(x)$ has no linear factor over $\mathbb{Z}_3$, and being a cubic with no root means it cannot be broken into a product of a linear and a quadratic factor, so it is irreducible.
2. **Describe elements of the quotient ring:** When we divide any polynomial in $\mathbb{Z}_3[x]$ by $f(x)$, the remainder always has degree less than 3, so it can be written as $a_0 + a_1x + a_2x^2$ with each $a_i \in \{0, 1, 2\}$. Every element of $\mathbb{Z}_3[x]/\langle f(x)\rangle$ is uniquely represented this way.
3. **Count the elements:** There are 3 choices for each of a_0, a_1, a_2, giving $3 \times 3 \times 3 = 27$ elements in total. Since $f(x)$ is irreducible, this quotient ring is a field with 27 elements.
4. **Relate the size to the degree:** A finite field that is an extension of $\mathbb{Z}_3$ with 3^n elements has degree n over $\mathbb{Z}_3$, because it is a vector space over $\mathbb{Z}_3$ with n basis elements $(1, x, x^2, \dots, x^{n-1})$. Here the field has $27 = 3^3$ elements, so $n = 3$.

Let's summarize:

- $f(x) = x^3 - x + 1$ has no root in $\mathbb{Z}_3$, so it is irreducible.
- The quotient field has $3^3 = 27$ elements, matching a degree-3 extension of $\mathbb{Z}_3$.

So the degree of the field extension is 3.

$\boxed{3}$

Quick Tip: Check if x^3-x+1 has a root in Z3; if not it is irreducible, and the extension degree equals its degree.

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30.

Consider the differential equation $\frac{dy}{dx} = x + y$ with the initial condition $y(0) = 1$. Using the modified Euler's method, the second approximation to $y(h)$, where $h = 0.05$ (step size), is equal to ________. (rounded off to TWO decimal places)

Correct Answer: 1.05

SOLUTION:

The modified Euler's method comes from approximating the integral form of the differential equation using the trapezoidal rule, and then repeating the correction step to refine the estimate. We can lay out the whole calculation in a short table instead of a step by step narrative.

1. **Why the formula looks the way it does:** Integrating $\frac{dy}{dx} = f(x, y)$ from x_0 to $x_1 = x_0 + h$ gives $y(x_1) = y_0 + \int_{x_0}^{x_1} f(x, y)\, dx$. Approximating this integral by the trapezoidal rule (averaging the slope at the two ends) gives $y_1 \approx y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$. Since y_1 appears on both sides, we solve it by repeated substitution, starting from a first guess.
2. **Round 0 (initial guess by plain Euler):** $y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.05(0 + 1) = 1.05$.

3. **Round 1 (first correction):** $f(x_1, y_1^{(0)}) = 0.05 + 1.05 = 1.10$, so $y_1^{(1)} = 1 + \frac{0.05}{2}(1 + 1.10) = 1 + 0.025(2.10) = 1.0525$.
4. **Round 2 (second correction):** $f(x_1, y_1^{(1)}) = 0.05 + 1.0525$ $= 1.1025$, so $y_1^{(2)} = 1 + \frac{0.05}{2}(1 + 1.1025) = 1 + 0.025(2.1025)$ $= 1.0525625$.

The value $y_1^{(2)} = 1.0525625$ is the second approximation asked for in the question.

Let's summarize:

- The predictor gives a rough first guess using the plain Euler formula.
- Each pass of the corrector formula (based on the trapezoidal rule) refines this guess, and the second approximation means the corrector has been applied twice.

Rounding 1.0525625 to two decimal places gives 1.05.

$$\boxed{1.05}$$

Quick Tip: First find the predictor y1 = y0 + h f(x0,y0), then apply the corrector formula twice to get the second approximation.

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31. Let

$$\alpha = \lim_{n\to\infty} \int_0^1 \frac{n^2 + (\sin e^x)^n}{7n^2 + x^8}\, dx.$$

The value of 14α is equal to ________. (Answer in integer)

Correct Answer: 2

SOLUTION:

Step 1: Bound the numerator between two simple expressions.

Since $-1 \le \sin e^x \le 1$ for every real x, we get $-1 \le (\sin e^x)^n \le 1$ for every positive integer n, regardless of what e^x actually equals on $[0, 1]$. Adding n^2 to all three parts gives

$$n^2 - 1 \le n^2 + (\sin e^x)^n \le n^2 + 1.$$

Step 2: Bound the whole integrand using this.

Dividing through by the positive quantity $7n^2 + x^8$,

$$\frac{n^2 - 1}{7n^2 + x^8} \le \frac{n^2 + (\sin e^x)^n}{7n^2 + x^8} \le \frac{n^2 + 1}{7n^2 + x^8}, \qquad x \in [0, 1].$$

Step 3: Integrate all three parts over $[0, 1]$.

Since $0 \le x^8 \le 1$ on $[0, 1]$, we can further simplify the outer bounds by replacing x^8 with its extreme values 0 and 1:

$$\frac{n^2 - 1}{7n^2 + 1} \le \frac{n^2 + (\sin e^x)^n}{7n^2 + x^8} \le \frac{n^2 + 1}{7n^2}, \qquad x \in [0, 1].$$

Integrating each part over x from 0 to 1 (the outer bounds are constants, so their integrals equal themselves):

$$\frac{n^2 - 1}{7n^2 + 1} \le \int_0^1 \frac{n^2 + (\sin e^x)^n}{7n^2 + x^8}\, dx \le \frac{n^2 + 1}{7n^2}.$$

Step 4: Take $n \to \infty$ on both outer bounds.

$$\lim_{n \to \infty} \frac{n^2 - 1}{7n^2 + 1} = \frac{1}{7}, \qquad \lim_{n \to \infty} \frac{n^2 + 1}{7n^2} = \frac{1}{7}.$$

Both outer sequences converge to $\frac{1}{7}$, so by the Sandwich (Squeeze) Theorem, the middle sequence, which is exactly the integral defining α, also converges to $\frac{1}{7}$. This gives the same value as before, but reaches it purely through inequalities and the Sandwich Theorem, without invoking the Dominated Convergence Theorem at all.

Step 5: State α and compute 14α.

$$\alpha = \frac{1}{7} \implies 14\alpha = 14 \times \frac{1}{7} = 2.$$

$$\boxed{14\alpha = 2}$$

Quick Tip: Bound $(\sin e^x)^n$ between -1 and 1, divide numerator and denominator by n^2, then take the limit before integrating.

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32. Let $X = (\mathbb{R}^3, \|\cdot\|_1)$, where $\left\|\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right\|_1 = |x| + |y| + |z|$, and let $T : X \to X$ be the linear transformation defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 2 & -2 \\ 1 & 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The operator norm of T is equal to ________. (Answer in integer)

Correct Answer: 8

SOLUTION:

Step 1: Use the shape of the unit ball in $\|\cdot\|_1$.

The closed unit ball of $(\mathbb{R}^3, \|\cdot\|_1)$ is the octahedron with corners at $\pm e_1, \pm e_2, \pm e_3$, that is, the six points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$. Every point v with $\|v\|_1 \leq 1$ is a convex combination of these six corners.

Step 2: Use convexity to reduce the search to the corners.

The function $v \mapsto \|Tv\|_1$ is convex, since it is the composition of the linear map T with the convex norm $\|\cdot\|_1$. A convex function on a convex set attains its maximum at an extreme point, so the maximum of $\|Tv\|_1$ over the unit ball occurs at one of the six corners. Also $\|T(-e_i)\|_1 = \|Te_i\|_1$, so it is enough to check just e_1, e_2, e_3.

Step 3: Evaluate T at each of the three basis vectors.

The matrix of T is $A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 2 & -2 \\ 1 & 3 & -3 \end{pmatrix}$, and Te_j is the j-th column of A.

$Te_1 = (1, 2, 1)$, so $\|Te_1\|_1 = 1 + 2 + 1 = 4$.

$Te_2 = (1, 2, 3)$, so $\|Te_2\|_1 = 1 + 2 + 3 = 6$.

$Te_3 = (3, -2, -3)$, so $\|Te_3\|_1 = 3 + 2 + 3 = 8$.

Step 4: Pick the largest value.

Among $4, 6, 8$, the largest is 8, attained at $v = e_3$. Since the maximum of $\|Tv\|_1$ over the whole unit ball must be one of these three numbers by Step 2, the operator norm of T is exactly 8.

$$\boxed{\|T\| = 8}$$

Quick Tip: For a linear map on $(\mathbb{R}^3, \|\cdot\|_1)$, the operator norm equals the largest sum of absolute values found in any single column of its matrix.

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33. Consider the following Linear Programming Problem (LPP):

$$\text{maximize } z = 10x_1 + 20x_2$$

subject to

$$x_1 \le 36, \quad x_2 \ge 42, \quad x_1 + x_2 \ge 48, \quad 5x_1 + x_2 \le 150, \quad x_1, x_2 \ge 0.$$

The maximum value of z in the above LPP is equal to ________. (Answer in integer)

Correct Answer: 3000

SOLUTION:

Step 1: Compare how efficiently x_1 and x_2 use the tightest resource.
Look at the constraint that ties both variables together: $5x_1 + x_2 \le 150$. Think of 150 as a budget, where increasing x_1 by 1 unit uses up 5 units of that budget, while increasing x_2 by 1 unit uses up only 1 unit.
Increasing x_1 by 1 unit costs 5 units of budget and adds 10 to z, a return of $10/5 = 2$ per unit of budget used.
Increasing x_2 by 1 unit costs 1 unit of budget and adds 20 to z, a return of $20/1 = 20$ per unit of budget used.

Step 2: Decide which variable to push as far as possible.
Since x_2 gives a much better return per unit of the shared budget (20 against 2), the maximum of z is reached by keeping x_1 as small as the constraints allow, that is $x_1 = 0$, and making x_2 as large as the budget constraint permits.

Step 3: Set $x_1 = 0$ and find the largest allowed x_2.
With $x_1 = 0$, the constraint $5x_1 + x_2 \le 150$ becomes $x_2 \le 150$, so the largest possible value is $x_2 = 150$.

Step 4: Check this point against every other constraint.

At $(x_1, x_2) = (0, 150)$: $x_1 = 0 \leq 36$ holds, $x_2 = 150 \geq 42$ holds, $x_1 + x_2 = 150 \geq 48$ holds, and $x_1, x_2 \geq 0$ holds. Every constraint is satisfied, so $(0, 150)$ is feasible, and by the return-per-budget argument in Step 2, no other feasible point can give a larger z.

Step 5: Compute the maximum value of z.

$$z = 10(0) + 20(150) = 3000.$$

$$\boxed{z_{max} = 3000}$$

Quick Tip: Find the corner points of the feasible region formed by the given constraints and evaluate z at each corner; the largest value is the answer.

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34. The number of distinct topologies on the set $\{1, 2, 3\}$ consisting of exactly four elements is equal to ________. (Answer in integer)

Correct Answer: 9

SOLUTION:

Step 1: Set up the count as choosing 2 extra sets out of 6 candidates.

Any topology on $\{1, 2, 3\}$ with exactly four members must contain $\emptyset$ and $X = \{1, 2, 3\}$, plus two more distinct proper nonempty subsets. There are exactly 6 proper nonempty subsets of X: the three singletons $\{1\}, \{2\}, \{3\}$ and the three pairs $\{1, 2\}, \{1, 3\}, \{2, 3\}$. Choosing 2 of these 6 to go with $\emptyset$ and X can be done in $\binom{6}{2} = 15$ ways, and each way must be checked for closure under union and intersection.

Step 2: Check the 3 ways of choosing two singletons.
For any two singletons, say $\{1\}$ and $\{2\}$, the union $\{1\} \cup \{2\} = \{1, 2\}$ is missing from the collection $\{\emptyset, \{1\}, \{2\}, X\}$. All 3 such choices fail the same way, so none of them give a valid topology.

Step 3: Check the 3 ways of choosing two pairs.
For any two of the 2-element sets, say $\{1, 2\}$ and $\{1, 3\}$, the intersection $\{1, 2\} \cap \{1, 3\} = \{1\}$ is missing from the collection $\{\emptyset, \{1, 2\}, \{1, 3\}, X\}$. All 3 such choices fail the same way, so none of them work either.

Step 4: Check the remaining 9 mixed choices, one singleton and one pair.
There are $3 \times 3 = 9$ ways to pick one singleton and one pair. Splitting these by whether the singleton sits inside the pair: if it does, for example $\{1\}$ with $\{1, 2\}$, the union is $\{1, 2\}$ (already present) and the intersection is $\{1\}$ (already present), so this is valid; there are 6 such combinations. If the singleton is disjoint from the pair, for example $\{1\}$ with $\{2, 3\}$, the union is X (already present) and the intersection is $\emptyset$ (already present), so this is valid too; there are 3 such combinations. All 9 mixed choices pass the closure check.

Step 5: Total the valid combinations.
Out of the 15 possible choices, 0 come from two singletons, 0 from two pairs, and 9 from the mixed case, giving a total of 9 topologies with exactly four elements.

$$\boxed{9}$$

Quick Tip: Every topology must contain $\emptyset$ and X; check which pairs of the remaining proper subsets keep the collection closed under union and intersection.

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35. For $a_n \in \mathbb{C}$, $n = 0, 1, 2, \ldots$, the power series

$$\sum_{n=0}^{\infty} a_n (z-2)^n$$

converges at $z = 5$ and diverges at $z = -1$. Then the radius of convergence of this power series is equal to ________. (Answer in integer)

Correct Answer: 3

SOLUTION:

Step 1: Note that both given points are the same distance from the center.

The power series is centered at $z_0 = 2$ since it is written in powers of $(z - 2)$. The point $z = 5$ is at distance $|5 - 2| = 3$ from the center, and the point $z = -1$ is at distance $|-1-2| = 3$ from the center as well. So both points lie on the same circle $|z - 2| = 3$, but one gives convergence and the other gives divergence.

Step 2: Suppose, for contradiction, that $R < 3$.

If R were less than 3, then $z = 5$, at distance exactly $3 > R$ from the center, would lie strictly outside the disc of convergence $|z - 2| < R$. A power series always diverges strictly outside its disc of convergence, which contradicts the fact that the series converges at $z = 5$. So $R < 3$ is impossible.

Step 3: Suppose, for contradiction, that $R > 3$.

If R were greater than 3, then $z = -1$, at distance exactly $3 < R$ from the center, would lie strictly inside the disc of convergence $|z - 2| < R$. A power series always converges strictly inside its disc of convergence, which contradicts the fact that the series diverges at $z = -1$. So $R > 3$ is impossible too.

Step 4: Conclude the only possible value.

Since $R < 3$ and $R > 3$ are both impossible, the only value left is $R = 3$.

$$\boxed{R = 3}$$

Quick Tip: Both given points are the same distance from the center of the series, so the radius of convergence must equal that common distance.

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36. The solution $y(x) = \binom{y_1(x)}{y_2(x)}$ of the initial value problem

$$\frac{dy}{dx} = \begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix} y, \qquad y(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

is equal to

(A) $3\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{-x}$

(B) $-3\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{-x}$

(C) $3\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{x} + 2\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{-x}$

(D) $3\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{-x}$

Correct Answer: (D) $3\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{-x}$

SOLUTION:

A different way to solve this system is to eliminate one variable and get a single second order equation for y_1, instead of working with eigenvectors directly.

The system is $y_1' = -3y_1 + 4y_2$ and $y_2' = -2y_1 + 3y_2$, with $y_1(0) = 1$, $y_2(0) = 2$.

From the first equation, $y_2 = \dfrac{y_1' + 3y_1}{4}$. Differentiating this gives y_2' $= \dfrac{y_1'' + 3y_1'}{4}$.

Substitute both expressions into the second equation:

$$\frac{y_1'' + 3y_1'}{4} = -2y_1 + 3\left(\frac{y_1' + 3y_1}{4}\right)$$

Multiply through by 4: $y_1'' + 3y_1' = -8y_1 + 3y_1' + 9y_1 = 3y_1' + y_1$. The $3y_1'$ terms cancel, leaving $y_1'' = y_1$, so $y_1'' - y_1 = 0$.

The characteristic roots are $r = \pm 1$, so $y_1(x) = Ae^x + Be^{-x}$ for constants A, B.

Using $y_1(0) = 1$ gives $A + B = 1$. Differentiating, $y_1'(x) = Ae^x - Be^{-x}$, and from the first equation $y_1'(0) = -3y_1(0) + 4y_2(0) = -3(1) + 4(2)$ $= 5$, so $A - B = 5$.

Solving these two equations, $2A = 6$, so $A = 3$ and $B = -2$. This gives $y_1(x) = 3e^x - 2e^{-x}$.

Now find y_2 using $y_2 = \dfrac{y_1' + 3y_1}{4}$. Here $y_1' = 3e^x + 2e^{-x}$, so $y_1' + 3y_1$ $= 3e^x + 2e^{-x} + 9e^x - 6e^{-x} = 12e^x - 4e^{-x}$, and dividing by 4 gives $y_2(x)$ $= 3e^x - e^{-x}$.

Writing y_1 and y_2 together as a vector, $y(x) = 3\binom{1}{1}e^x - \binom{2}{1}e^{-x}$, which matches the same coefficients found by the eigenvector method, confirming option (D) is correct.

$$\boxed{y(x) = 3\begin{pmatrix}1\\1\end{pmatrix}e^{x} - \begin{pmatrix}2\\1\end{pmatrix}e^{-x}}$$

Quick Tip: Find the eigenvalues and eigenvectors of the coefficient matrix, then fix the constants using the given initial vector $y(0) = \binom{1}{2}$.

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37. Let $P = \begin{pmatrix}1 & 0 & 0\\0 & 9 & 0\\0 & 0 & 16\end{pmatrix}$. Define an inner product on $\mathbb{R}^3$ with respect to P as

$$\langle x, y\rangle_P = x^{\top}Py, \quad \text{for all } x, y \in \mathbb{R}^3.$$

Consider the subspace $V = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$. Which one of the following sets is an orthonormal basis of V with respect to $\langle x, y\rangle_P$?

(A) $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$

(B) $\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$

(C) $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\\frac{1}{5}\\\frac{1}{5}\end{pmatrix}\right\}$

(D) $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}\right\}$

Correct Answer: (C) $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\\frac{1}{5}\\\frac{1}{5}\end{pmatrix}\right\}$

SOLUTION:

Another way to solve this is to change coordinates so the weighted inner product $\langle x, y\rangle_P$ turns into the ordinary dot product, and then do plain Gram-Schmidt in the new coordinates.

Since $P = \mathrm{diag}(1, 9, 16)$, write $P = D^2$ where $D = \mathrm{diag}(1, 3, 4)$. Then for any x, y, $\langle x, y\rangle_P = x^\top D^2 y = (Dx) \cdot (Dy)$, the ordinary dot product of the scaled vectors $u = Dx$ and $v = Dy$.

Scale the two spanning vectors using D: $u_1 = Dv_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, and $u_2 = Dv_2 = \begin{pmatrix}1\\3\\4\end{pmatrix}$.

Now do ordinary Gram-Schmidt on u_1, u_2. The vector u_1 already has length 1, so $\hat{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$.

The projection of u_2 on $\hat{e}_1$ is $(u_2 \cdot \hat{e}_1)\hat{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, so the leftover part is $u_2 - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\3\\4\end{pmatrix}$, which has length $\sqrt{3^2 + 4^2} = 5$.

So $\hat{e}_2 = \begin{pmatrix}0\\\frac{3}{5}\\\frac{4}{5}\end{pmatrix}$.

Convert back to the original coordinates using $x = D^{-1}u$, with $D^{-1} = \operatorname{diag}(1, \frac{1}{3}, \frac{1}{4})$. This gives $e_1 = D^{-1}\hat{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ and $e_2 = D^{-1}\hat{e}_2 = \begin{pmatrix}0\\\frac{1}{5}\\\frac{1}{5}\end{pmatrix}$, exactly matching the basis found by direct Gram-Schmidt with the P-inner product.

Let's summarize:

- Scaling by $D = \operatorname{diag}(1, 3, 4)$ turns the weighted inner product into the plain dot product.
- Doing Gram-Schmidt in the scaled coordinates and then scaling back by D^{-1} gives the same orthonormal basis as working directly with P.

So the correct orthonormal basis is $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\\frac{1}{5}\\\frac{1}{5}\end{pmatrix}\right\}$, option (C).

$$\boxed{\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\\frac{1}{5}\\\frac{1}{5}\end{pmatrix}\right\}}$$

Quick Tip: Work out $\langle v, v\rangle_P = v_1^2 + 9v_2^2 + 16v_3^2$ for each candidate vector and check it equals 1, then check the cross term is 0.

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38. The value of the contour integral

$$\int_{\Gamma} \frac{(1-z)(3\cos z + 5\sin z)}{1-z^{101}}\, dz,$$

where $\Gamma = \left\{z \in \mathbb{C} : |z| = \frac{7}{9}\right\}$, oriented in the counter-clockwise direction, is equal to

(A) $2\pi i$

(B) $-2\pi i$

(C) 0

(D) $4\pi i$

Correct Answer: (C) 0

SOLUTION:

A different way to see this is to expand the integrand directly as a convergent power series inside the disk bounded by Γ, instead of factoring out roots of unity.

Since Γ has radius $\frac{7}{9} < 1$, every point z inside or on Γ satisfies $|z| \le \frac{7}{9} < 1$. For such z, the geometric series

$$\frac{1}{1-z^{101}} = \sum_{k=0}^{\infty} z^{101k}$$

converges, because $|z^{101}| = |z|^{101} < 1$.

The function $3\cos z + 5\sin z$ is entire, so it also has a convergent Taylor series for every z, in particular for $|z| \le \frac{7}{9}$.

Multiplying $(1-z)$, the entire function $3\cos z + 5\sin z$, and the convergent geometric series $\sum z^{101k}$ together gives a power series in z that converges for all $|z| < 1$. A function represented by a convergent power series on a disk is analytic throughout that disk.

So the whole integrand is analytic on the closed disk $|z| \le \frac{7}{9}$ enclosed by Γ, with no singularities inside at all.

Let's summarize:

- Writing $\frac{1}{1-z^{101}}$ as a geometric series shows the integrand is just an ordinary convergent power series inside $|z| < 1$.
- Cauchy's theorem says the integral of an analytic function around any closed contour lying in its domain of analyticity is zero.

Since Γ lies entirely inside the region of analyticity, the integral must be 0, option (C).

$$\boxed{0}$$

Quick Tip: Factor $1 - z^{101}$ as $(1-z)(1+z+\cdots+z^{100})$ and check whether any zero of the simplified denominator lies inside $|z| = \frac{7}{9}$.

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39. Let $f(z) = \sum_{n=1}^{\infty} 5^{-n} \cos(nz)$. On which one of the following domains, does $f(z)$ represent an analytic function?

(A) $\{z \in \mathbb{C} : |\operatorname{Im} z| < \ln 5\}$

(B) $\{z \in \mathbb{C} : |\operatorname{Im} z| > \ln 5\}$

(C) $\{z \in \mathbb{C} : |\operatorname{Re} z| < \ln 5\}$

(D) $\{z \in \mathbb{C} : |\operatorname{Re} z| > \ln 5\}$

Correct Answer: (A) $\{z \in \mathbb{C} : |\operatorname{Im} z| < \ln 5\}$

SOLUTION:

Another way to find the domain is to apply the root test to the size of each term directly, instead of splitting $\cos(nz)$ into two exponential geometric series.

Write $z = x + iy$. Using $\cos(nz) = \cos(nx)\cosh(ny) - i\sin(nx)\sinh(ny)$, the size of $\cos(nz)$ is bounded for large n by roughly $\frac{1}{2}e^{n|y|}$, since both $\cosh(ny)$ and $\sinh(ny)$ grow like $\frac{1}{2}e^{n|y|}$ as $n \to \infty$.

So the n-th term of the series has size

$$|5^{-n}\cos(nz)| \approx \frac{1}{2}\left(\frac{e^{|y|}}{5}\right)^n$$

The root test says a series $\sum a_n$ converges absolutely when $\lim_{n\to\infty} |a_n|^{1/n} < 1$. Here that limit works out to $\dfrac{e^{|y|}}{5}$.

Setting this below 1 gives $e^{|y|} < 5$, that is $|y| < \ln 5$, that is $|\operatorname{Im} z| < \ln 5$.

When $|y| > \ln 5$, the same limit is bigger than 1, so the series diverges there instead, confirming that region cannot be the domain of analyticity.

Let's summarize:

- The size of each term depends only on $y = \operatorname{Im} z$, never on $x = \operatorname{Re} z$, through the growth of $\cosh(ny)$ and $\sinh(ny)$.
- The root test gives the same strip $|\operatorname{Im} z| < \ln 5$ as the exponential-splitting method, confirming $f(z)$ is analytic there.

So the correct domain is $\{z \in \mathbb{C} : |\operatorname{Im} z| < \ln 5\}$, option (A).

$$\boxed{\{z \in \mathbb{C} : |\operatorname{Im} z| < \ln 5\}}$$

Quick Tip: Split $\cos(nz)$ into $\dfrac{e^{inz} + e^{-inz}}{2}$ and check when both resulting geometric series converge.

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40. Let $\delta(t)$ be the unit impulse function defined by $\delta(x-x_0)=0$ when $x \neq x_0$ and $\int_{-\infty}^{\infty} \delta(x-x_0)\,dx = 1$.

Consider the following initial value problem

$$2\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2y = \delta(x-5)$$

with $y(0)=0$ and $\frac{dy}{dx}(0)=0$. Which one of the following is TRUE?

(A) $y(10) = \frac{10}{\sqrt{15}} e^{-5/4} \sin\left(\frac{\sqrt{15}}{4}\right)$

(B) $y(10) = \frac{2}{\sqrt{15}} e^{-5/4} \sin\left(\frac{5\sqrt{15}}{4}\right)$

(C) $y(10) = \frac{10}{\sqrt{15}} e^{5/4} \sin\left(\frac{\sqrt{15}}{4}\right)$

(D) $y(10) = \frac{2}{\sqrt{15}} e^{5/4} \sin\left(\frac{5\sqrt{15}}{4}\right)$

Correct Answer: (B) $y(10) = \frac{2}{\sqrt{15}} e^{-5/4} \sin\left(\frac{5\sqrt{15}}{4}\right)$

SOLUTION:

A different way to solve this is to use the jump condition that an impulse creates in the derivative, instead of using the Laplace transform.

For $x < 5$, there is no forcing yet and the initial conditions are $y(0) = 0$, $y'(0) = 0$, so the unique solution of the homogeneous equation on that interval is simply $y(x) = 0$.

Integrate $2y'' + y' + 2y = \delta(x-5)$ across a tiny interval around $x = 5$, from $5 - \epsilon$ to $5 + \epsilon$, and let $\epsilon \to 0$. The terms $\int y'\, dx$ and $\int 2y\, dx$ shrink to 0 because y and y' stay bounded, but $\int 2y''\, dx = 2[y'(5^+) - y'(5^-)]$ must equal $\int \delta(x-5)\, dx = 1$.

Since $y'(5^-) = 0$ (from the zero solution before the impulse), this gives $y'(5^+) = \frac{1}{2}$. Also y stays continuous at $x = 5$, so $y(5^+) = y(5^-) = 0$.

For $x > 5$, let $\tau = x - 5$. The equation becomes homogeneous again, $2y'' + y' + 2y = 0$, with fresh initial data $y(0) = 0$ and $y'(0) = \frac{1}{2}$ in terms of τ.

The characteristic equation is $2r^2 + r + 2 = 0$, giving $r = \dfrac{-1 \pm \sqrt{1-16}}{4} = -\dfrac{1}{4} \pm i\dfrac{\sqrt{15}}{4}$.

So the general solution is $y(\tau) = e^{-\tau/4}\left[C_1 \cos\left(\dfrac{\sqrt{15}}{4}\tau\right) + C_2 \sin\left(\dfrac{\sqrt{15}}{4}\tau\right)\right]$.

Using $y(0) = 0$ gives $C_1 = 0$. Differentiating and using $y'(0) = \frac{1}{2}$ gives $C_2 \cdot \dfrac{\sqrt{15}}{4} = \dfrac{1}{2}$, so $C_2 = \dfrac{2}{\sqrt{15}}$.

So $y(\tau) = \dfrac{2}{\sqrt{15}} e^{-\tau/4} \sin\left(\dfrac{\sqrt{15}}{4}\tau\right)$, and at $x = 10$ we have $\tau = 5$, giving

$$y(10) = \frac{2}{\sqrt{15}} e^{-5/4} \sin\left(\frac{5\sqrt{15}}{4}\right)$$

Let's summarize:

- Before the impulse hits at $x = 5$, the response is zero because the system starts at rest.
- The impulse instantly changes the derivative y' by $\frac{1}{2}$ at $x = 5$, and after that the system just follows its natural damped oscillation.

This matches the Laplace transform result exactly, confirming option (B).

$$\boxed{y(10) = \frac{2}{\sqrt{15}} e^{-5/4} \sin\left(\frac{5\sqrt{15}}{4}\right)}$$

Quick Tip: Take the Laplace transform of the ODE, use $\mathcal{L}\{\delta(x-5)\} = e^{-5s}$, and invert with the time-shift property.

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41. Which one of the following statements is TRUE?

(A) The set of all 2×2 real matrices under usual matrix addition and multiplication is a Unique Factorization Domain.

(B) The set of all real valued functions under usual function addition and multiplication is a Unique Factorization Domain.

(C) The set of all polynomials with coefficients in $\mathbb{R}$ under usual polynomial addition and multiplication is a Unique Factorization Domain.

(D) The set of Riemann integrable functions in $[0, 1]$ under usual function addition and multiplication is a Unique Factorization Domain.

Correct Answer: (C) The set of all polynomials with coefficients in $\mathbb{R}$ under usual polynomial addition and multiplication is a Unique Factorization Domain.

SOLUTION:

This question is really about one property: a Unique Factorization Domain (UFD) must first be an integral domain, a commutative ring where no product of two non zero elements is zero. If we can find one pair of non zero elements whose product is zero, that ring is finished as a candidate, since it fails to even be a domain. Let's go through the four candidate rings one at a time.

1. **2 by 2 real matrices:** the matrices $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are both non zero, but $E_{11}E_{22}$ is the zero matrix. This ring has zero divisors, and is not even commutative, so it cannot be a UFD.
2. **Real valued functions:** pick f equal to 1 on one part of the domain and 0 elsewhere, and g equal to 0 where f is 1 and 1 where f is 0. Then f and g are both non zero but fg is the zero function everywhere. This ring has zero divisors.
3. **Polynomials $\mathbb{R}[x]$:** if $p(x)$ and $q(x)$ are non zero real polynomials of degrees m and n, the leading coefficient of $p(x)q(x)$ is the product of the two leading, non zero, real coefficients, which is non zero since $\mathbb{R}$ has no zero divisors. So $p(x)q(x)$ is never zero when both factors are non zero. This ring is an integral domain, and since $\mathbb{R}$ is a field, $\mathbb{R}[x]$ is a Euclidean domain, hence a PID, hence a UFD.
4. **Riemann integrable functions on $[0, 1]$:** the same trick as functions works, take f as the indicator of the first half interval and g as the indicator of the second half; both are Riemann integrable and non zero, but $fg = 0$ on all of $[0, 1]$. Zero divisors again.

Only $\mathbb{R}[x]$ survives the very first test of being an integral domain, and being a Euclidean domain over a field pushes it all the way to being a UFD. So the correct statement is option (C).

Let's summarize:

- A ring with zero divisors can never be a UFD, since UFD requires being an integral domain first.
- Matrix rings, and rings of real (or Riemann integrable) functions on more than one point, always have zero divisors built from split indicator-like elements.
- Polynomials over a field form a Euclidean domain, hence a PID, hence a UFD.

So the true statement is option (C): $\mathbb{R}[x]$ under usual addition and multiplication is a Unique Factorization Domain.

$$\boxed{\text{Option (C): } \mathbb{R}[x] \text{ is a UFD}}$$

Quick Tip: Check whether each ring is even an integral domain (no zero divisors) first. Matrices and function rings on more than one point always have zero divisors; only one candidate is a domain, and it is a well known type of UFD.

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42. Let $\ell^\infty = \{x = (x_n)_{n\geq 1} \mid x_n \in \mathbb{R},\ \sup\{|x_n| : n = 1, 2, \ldots\} < \infty\}$ with the supremum norm. Let $T : \ell^\infty \to \ell^\infty$ be given by $T(x_1, x_2, x_3, \ldots)$ $= \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right)$. Which one of the following is TRUE?

(A) T is bounded but not one-to-one.

(B) T is one-to-one but not bounded.

(C) T is bounded and the inverse (from the range of T) exists but not bounded.

(D) T is bounded and the inverse (from the range of T) is bounded.

Correct Answer: (C) T is bounded and the inverse (from the range of T) exists but not bounded.

SOLUTION:

T is a diagonal operator on ℓ^∞: it multiplies the n-th coordinate of x by the scalar $a_n = 1/n$. For a diagonal operator, three standard facts about the multipliers a_n settle everything.

1. **Boundedness of T:** a diagonal operator with multipliers a_n is bounded on ℓ^∞ exactly when (a_n) is a bounded sequence, and then $\|T\| = \sup_n |a_n|$. Here $a_n = 1/n \le 1$ for all $n \ge 1$, so T is bounded with $\|T\| = 1$, attained at $n = 1$.
2. **Injectivity of T:** a diagonal operator is one-to-one exactly when no multiplier is zero. Since $a_n = 1/n \neq 0$ for every n, T never sends a non zero coordinate to zero, so T is injective.
3. **Boundedness of the inverse:** on the range of T, the inverse is again diagonal, with multipliers $1/a_n = n$. A diagonal operator is bounded exactly when its multipliers form a bounded sequence. Here the multipliers are $n = 1, 2, 3, \ldots$, an unbounded sequence, so T^{-1} cannot be bounded.

A concrete check on point 3: take $y = (1, 1, \ldots, 1, 0, 0, \ldots)$ with exactly N ones, so $\|y\|_\infty = 1$. Its preimage under T has n-th entry $n \cdot 1 = n$ for $n \le N$, giving $x = (1, 2, 3, \ldots, N, 0, 0, \ldots)$ with $\|x\|_\infty = N$. The ratio $\|T^{-1}y\|_\infty / \|y\|_\infty = N$ can be made as large as we like by taking N larger, so no fixed constant bounds it, confirming T^{-1} is unbounded.

Let's summarize:

- Diagonal operator with multipliers $1/n$: bounded (multipliers bounded by 1) and one-to-one (no multiplier is zero).
- Its inverse has multipliers n, an unbounded sequence, so the inverse exists on the range of T but is not bounded.

So T is bounded, one-to-one, and its inverse on the range exists but is unbounded, which is option (C).

$$\boxed{\text{Option (C): bounded, injective, inverse exists but unbounded}}$$

Quick Tip: Check boundedness using the multiplier 1/n, injectivity by solving Tx=0, then test the inverse (multiply back by n) on a sequence like y_n = 1/sqrt(n) truncated at N and let N grow.

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43. Let $u(x,t)$ satisfy the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty,\ t > 0,\ c > 0$$

with

$$u(x,0) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\frac{\partial u}{\partial t}(x,0) = 0.$$

By using D'Alembert's formula, the maximum value of $u(0,t)$ for $t > 0$ is

(A) 1

(B) $\frac{1}{2}$

(C) $\frac{1}{4}$

(D) 0

Correct Answer: (A) 1

SOLUTION:

The initial displacement here is a bump of height 1 sitting between $x = -1$ and $x = 1$, starting from rest (zero initial velocity). D'Alembert's solution splits this bump into two waves, each of half the original height, one moving right at speed c and one moving left at speed c:

$$u(x,t) = \frac{1}{2}f(x - ct) + \frac{1}{2}f(x + ct)$$

where f is the indicator function that is 1 on $(-1, 1)$ and 0 elsewhere.

The first piece, $\frac{1}{2}f(x - ct)$, is the original bump shrunk to half height and shifted right by ct. The second piece, $\frac{1}{2}f(x + ct)$, is the same half height bump shifted left by ct. At $t = 0$ these two half height bumps sit exactly on top of each other over $(-1, 1)$, adding up to the full height 1, matching the given initial shape.

Now watch the point $x = 0$ as t increases from 0. The right moving half bump has support at $x \in (ct - 1, ct + 1)$, and the left moving half bump has support at $x \in (-ct - 1, -ct + 1)$. As long as 0 lies inside both supports, both terms contribute $\frac{1}{2}$ each at $x = 0$, giving $u(0,t) = \frac{1}{2} + \frac{1}{2} = 1$.

Check when 0 is inside the right moving support $(ct - 1, ct + 1)$: this needs $ct - 1 < 0 < ct + 1$, that is $-1 < ct < 1$, and since $ct > 0$ this means $ct < 1$, i.e. $t < 1/c$. By symmetry the left moving bump gives the same condition. So for every t with $0 < t < 1/c$, both half bumps still cover $x = 0$, and $u(0,t) = 1$ exactly. Once $t \geq 1/c$, both half bumps have moved past $x = 0$ and $u(0,t)$ drops to 0.

Let's summarize:

- D'Alembert's solution splits a zero velocity bump into two half height copies moving apart at speed c.
- At $x = 0$, both half height copies overlap for $0 < t < 1/c$, adding to the full value 1.

- The value never exceeds 1 at any time, so 1 is the maximum.

So the maximum value of $u(0,t)$ for $t > 0$ is 1.

$$\boxed{u(0,t)_{max} = 1}$$

Quick Tip: Use D'Alembert's formula with zero initial velocity: u(0,t) = f(ct) since f is even, and f is 1 whenever ct is less than 1.

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44. Consider the Laplace equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \qquad 0 < x < 1,\ 0 < y < 1,$$

with the boundary conditions

$$T(x,0) = x, \qquad T(0,y) = y$$

$$T(x,1) = 1 + x, \qquad T(1,y) = 1 + y.$$

Then the value of $T\left(\frac{1}{2}, \frac{1}{3}\right)$ is equal to

(A) $\frac{7}{6}$

(B) $\frac{5}{6}$

(C) $\frac{1}{6}$

(D) $\frac{1}{2}$

Correct Answer: (B) $\frac{5}{6}$

SOLUTION:

Instead of directly guessing the answer, build it by comparing T to the plane function $x + y$ using the maximum principle for harmonic functions.

Define $w(x, y) = T(x, y) - (x + y)$. Since $x + y$ is harmonic (linear functions always satisfy Laplace's equation, as their second partial derivatives are zero), and T is harmonic by the problem statement, the difference w is also harmonic on the square, being a difference of two harmonic functions.

Now check what w looks like on each edge of the square, using the given boundary values of T:

1. On $y = 0$: $w(x, 0) = T(x, 0) - (x + 0) = x - x = 0$.
2. On $x = 0$: $w(0, y) = T(0, y) - (0 + y) = y - y = 0$.
3. On $y = 1$: $w(x, 1) = T(x, 1) - (x + 1) = (1 + x) - (x + 1) = 0$.
4. On $x = 1$: $w(1, y) = T(1, y) - (1 + y) = (1 + y) - (1 + y) = 0$.

So w is harmonic on the whole square and equals 0 on all four edges of the boundary. By the maximum principle, a harmonic function attains both its maximum and minimum on the boundary of the region. Since w is 0 everywhere on the boundary, its maximum and minimum are both 0, which forces $w(x, y) = 0$ at every point inside the square too.

This means $T(x, y) - (x + y) = 0$ for all (x, y) in the square, so $T(x, y) = x + y$ everywhere.

Let's summarize:

- Writing $w = T - (x + y)$ turns the problem into checking a harmonic function with all zero boundary data.
- The maximum principle says such a w must be identically zero inside the region, not just on the boundary.
- So $T(x, y) = x + y$ exactly, and $T(1/2, 1/3) = 1/2 + 1/3 = 5/6$.

So the value of $T(1/2, 1/3)$ is 5/6.

$$\boxed{T(1/2, 1/3) = \frac{5}{6}}$$

Quick Tip: Try T(x,y) = x+y; it is harmonic and matches all four boundary conditions exactly, so by uniqueness of the Dirichlet problem it is the solution.

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45. Let E_1 and E_2 be subsets of a normed linear space X, and

$$E_1 + E_2 = \{x + y : x \in E_1,\ y \in E_2\}$$

$$E_1 \times E_2 = \{(x, y) : x \in E_1,\ y \in E_2\}.$$

Which one of the following is NOT TRUE?

(A) If either of E_1 or E_2 is open, then $E_1 + E_2$ is open.

(B) If E_1 and E_2 are convex, then $E_1 + E_2$ is convex.

(C) If E_1 and E_2 are closed, then $E_1 + E_2$ is closed.

(D) If E_1 and E_2 are connected, then $E_1 \times E_2$ is connected.

Correct Answer: (C) If E_1 and E_2 are closed, then $E_1 + E_2$ is closed.

SOLUTION:

To find which statement is NOT true, it helps to know that three of these four properties are standard preserved-property results for normed linear spaces, and one is a classic trap. Let's go through each option.

1. **(A) open plus anything gives an open sum:** if E_1 is open, then for each fixed $y \in E_2$ the shifted set $E_1 + y$ is open, because adding a constant vector is a continuous, invertible map on a normed space. Since $E_1 + E_2 = \bigcup_{y \in E_2}(E_1 + y)$ is a union of open sets, it is open. This holds true.
2. **(B) convex plus convex is convex:** for $z_1 = x_1 + y_1, z_2 = x_2 + y_2$ in $E_1 + E_2$ and $\lambda \in [0, 1]$, the point $\lambda z_1 + (1 - \lambda)z_2$ splits as $[\lambda x_1 + (1 - \lambda)x_2] + [\lambda y_1 + (1 - \lambda)y_2]$, a sum of a convex combination inside E_1 and one inside E_2, so it lands back in $E_1 + E_2$. This holds true.
3. **(D) connected times connected is connected:** this is a basic topology fact, the product of connected spaces with the product topology is always connected. This holds true.
4. **(C) closed plus closed is closed:** this is the one that can fail. Take $X = \mathbb{R}$, $A = \{0, 1, 2, 3, \ldots\}$ (the non negative integers, a closed discrete set) and $B = \{-n + \frac{1}{n} : n = 1, 2, 3, \ldots\}$ (also closed, since its points march off to $-\infty$ with no finite accumulation point). Taking the element n from A and $-n + \frac{1}{n}$ from B with matching index gives the sum $\frac{1}{n}$, which tends to 0 as $n \to \infty$ but is never equal to 0 for any finite n. So 0 is a limit point of $A + B$ that does not belong to $A + B$, meaning $A + B$ is not closed even though both A and B are closed.

So three of the four statements are guaranteed by general theorems, and only the closed-plus-closed claim can break down, as the counterexample above shows.

Let's summarize:

- Open-plus-anything, convex-plus-convex, and connected-times-connected are always safe, provable directly from definitions.

- Closed-plus-closed is the classic exception: two closed sets can add up to a non closed set, as the integers and the shifted-reciprocal sequence show.

So the statement that is NOT true is option (C).

$$\boxed{\text{Option (C) is NOT true}}$$

Quick Tip: Check each claim directly: open-plus-anything and convex-plus-convex are provable from definitions, and connected times connected is a standard fact, but closed-plus-closed has a classic counterexample (a line and a hyperbola branch in the plane).

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46. For a transportation problem, let c_{ij} denote the unit cost of the cell (i, j). The known unit costs are shown below (a dash marks a cell whose cost is not given).

	j = 1	**j = 2**	**j = 3**
i = 1	10	12	19
i = 2	11	13	-
i = 3	-	-	-

Let α_i and β_j, $i, j = 1, 2, 3$, be the simplex multipliers associated with a basis corresponding to this unit cost table, so that $\alpha_1 = x$, $\alpha_2 = x + 1$, $\beta_1 = y$ and $\beta_2 = y + 2$. The relative cost coefficient d_{ij} is the difference between the current solution and the new improved solution.
If $x = 4$, $c_{13} = 19$ and $\beta_3 = y + 5$, then which one of the following is TRUE?

(A) $y = 6$ and $d_{13} = 7$

(B) $y = 6$ and $d_{13} = 4$

(C) $y = 5$ and $d_{13} = 7$

(D) $y = 5$ and $d_{13} = 4$

Correct Answer: (B) $y = 6$ and $d_{13} = 4$

SOLUTION:

Step 1: Understanding the Concept.

This is a u-v method (MODI method) question. For every basic cell of a transportation table, the rule $c_{ij} = u_i + v_j$ holds, where u_i plays the role of α_i and v_j plays the role of β_j. For a non-basic cell, the reduced (relative) cost is $d_{ij} = c_{ij} - (u_i + v_j)$.

Step 2: Pick one basic-cell equation and plug in x directly.

Take the cell $(2, 2)$, whose cost is given as $c_{22} = 13$. Since it is basic, $c_{22} = \alpha_2 + \beta_2 = (x + 1) + (y + 2)$. With $x = 4$:

$$13 = (4 + 1) + (y + 2) = 5 + y + 2 = y + 7$$

so $y = 13 - 7 = 6$.

Step 3: Cross-check with a second basic cell.

Using cell $(1, 1)$: $c_{11} = \alpha_1 + \beta_1 = x + y = 4 + 6 = 10$, and indeed $c_{11} = 10$ as given, so the value $y = 6$ is consistent across the whole table, not just from one equation.

Step 4: Work out beta_3 and the reduced cost of the empty cell (1,3).

$\beta_3 = y + 5 = 6 + 5 = 11$. The cell $(1, 3)$ is empty (non-basic), so its reduced cost is

$$d_{13} = c_{13} - (\alpha_1 + \beta_3) = 19 - (4 + 11) = 4$$

A reduced cost of 4 (positive) means moving material into this cell would raise the total transportation cost by 4 per unit, confirming the current basis is not improved by bringing in that cell.

Step 5: Final Answer.

So $y = 6$ and $d_{13} = 4$, which is option (B); options with $y = 5$ break the relation $x + y = 10$, and the option with $d_{13} = 7$ uses the wrong value while combining with β_3 and c_{13}.

$$\boxed{y = 6,\ d_{13} = 4}$$

Quick Tip: Use the basic-cell rule $c_{ij} = \alpha_i + \beta_j$ to first pin down $x + y$, then use $x = 4$ to get y, and finally use $d_{13} = c_{13} - (\alpha_1 + \beta_3)$ for the non-basic cell $(1, 3)$.

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47. Let X and Y be topological spaces and $f : X \to Y$ be a continuous and bijective mapping. Which one of the following statements is TRUE?

(A) f is a homeomorphism if X and Y are compact.

(B) f is a homeomorphism if X is Hausdorff and Y is compact.

(C) f is a homeomorphism if X is compact and Y is Hausdorff.

(D) f is a homeomorphism if X and Y are Hausdorff.

Correct Answer: (C) f is a homeomorphism if X is compact and Y is Hausdorff.

SOLUTION:

This question tests when a continuous bijection between topological spaces is guaranteed to be a homeomorphism. A homeomorphism needs both f and f^{-1} continuous, and continuity of f^{-1} is the same as f being a closed map (closed sets go to closed sets). Let's go through each option.

1. **X and Y compact:** This is not enough on its own. Compactness of X lets us say a closed (hence compact) subset of X maps to a compact subset of Y, but without Y Hausdorff, a compact subset of Y need not be closed. So f need not be a closed map, and this option can fail.
2. **X Hausdorff and Y compact:** Hausdorff-ness of X says nothing about images of closed sets in Y, and compactness of Y alone does not help either, since we need compactness on the domain side. The classic map from $[0, 2\pi)$ onto the circle S^1 (continuous, bijective, X Hausdorff, Y compact) fails to be a homeomorphism, so this option is false.
3. **X compact and Y Hausdorff:** Take any closed set C in X. Since X is compact, C is compact. Continuous maps send compact sets to compact sets, so $f(C)$ is compact in Y. Since Y is Hausdorff, compact subsets of Hausdorff spaces are always closed, so $f(C)$ is closed. This makes f a closed map, so f^{-1} is continuous, and f is a homeomorphism. This option is always true.
4. **X and Y both Hausdorff:** Hausdorff-ness alone never forces compactness anywhere, so the closed-map argument has no compact set to work with. The same $[0, 2\pi) \to S^1$ map is a counterexample here too, since both spaces are Hausdorff but f is not a homeomorphism. This option is false.

Only the combination "domain compact, codomain Hausdorff" is strong enough to guarantee a continuous bijection is a homeomorphism, so the correct choice is option (C).

Let's summarize:

- A continuous bijection is a homeomorphism exactly when it is also a closed map.
- Compactness of the domain plus the Hausdorff property on the codomain is exactly the condition that forces every continuous bijection to be closed.
- Swapping which space is compact and which is Hausdorff, or dropping compactness altogether, breaks the argument, as the $[0, 2\pi) \to S^1$ example shows.

So the correct answer is option (C): f is a homeomorphism if X is compact and Y is Hausdorff.

Quick Tip: Remember the standard theorem: a continuous bijection from a compact space onto a Hausdorff space is always a homeomorphism, because compactness of the domain plus Hausdorff-ness of the codomain force the map to be closed.

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48. Let P and Q be 3×3 nonzero real matrices. Assume that there exists a 3×3 real nonsingular matrix S such that $S^{-1}PS$ and $S^{-1}QS$ are both upper triangular. Which of the following statements is/are TRUE?

(A) The matrix $PQ - QP$ is invertible.

(B) The matrix $PQ - QP$ is nilpotent.

(C) The matrix $I + PQ - QP$ is invertible, where I is the 3×3 identity matrix.

(D) If $PQ - QP$ is a nonzero matrix, then $PQ - QP$ is diagonalizable.

Correct Answer: (B) The matrix $PQ - QP$ is nilpotent. ; (C) The matrix $I + PQ - QP$ is invertible, where I is the 3×3 identity matrix.

SOLUTION:

Step 1: Understanding the Concept.

Simultaneous triangularization by the same S means P and Q can each be split, in the basis given by S, into a diagonal part and a strictly upper triangular part: $T_1 = D_1 + N_1$ and $T_2 = D_2 + N_2$, where D_1, D_2 are diagonal (carrying the eigenvalues of P and Q) and N_1, N_2 are strictly upper triangular.

Step 2: Expand the commutator using this split.

$$T_1T_2 - T_2T_1 = (D_1 + N_1)(D_2 + N_2) - (D_2 + N_2)(D_1 + N_1)$$

Expanding and grouping, this equals $[D_1, D_2] + [D_1, N_2] + [N_1, D_2] + [N_1, N_2]$, where $[A, B] = AB - BA$. Two diagonal matrices always commute, so $[D_1, D_2] = 0$.

Step 3: Track the remaining terms.

A diagonal matrix times a strictly upper triangular matrix, in either order, is still strictly upper triangular, and a commutator of two strictly upper triangular matrices is also strictly upper triangular. So every one of the three remaining terms is strictly upper triangular, and their sum $T_1T_2 - T_2T_1$ is strictly upper triangular as well.

Step 4: Nilpotency and its consequences.

Any 3×3 strictly upper triangular matrix N satisfies $N^3 = 0$, so it is nilpotent, with every eigenvalue equal to 0. Since $PQ - QP$ is similar to $T_1T_2 - T_2T_1$ (via the same S), it is nilpotent too, and all three of its eigenvalues are 0. This directly makes option (B) true.

Step 5: Check the determinant-based options.

Because the eigenvalues of $PQ - QP$ are all 0, $\det(PQ - QP) = 0 \times 0 \times 0 = 0$, so $PQ - QP$ is singular, not invertible, making option (A) false. Shifting by the identity shifts every eigenvalue up by 1, so the eigenvalues of $I + PQ - QP$ are $1, 1, 1$, giving $\det(I + PQ - QP) = 1 \neq 0$. So $I + PQ - QP$ is always invertible, making option (C) true.

Step 6: Check diagonalizability in option (D).

A nonzero matrix all of whose eigenvalues are 0 can be diagonalizable only if it is similar to the zero matrix, which forces the matrix itself to be zero. That contradicts the "nonzero" assumption in option (D), so a nonzero nilpotent matrix is never diagonalizable, and option (D) is false.

Step 7: Final Answer.

The statements that are always true are (B), $PQ - QP$ is nilpotent, and (C), $I + PQ - QP$ is invertible.

$$\boxed{\text{(B) and (C)}}$$

Quick Tip: Simultaneous triangularization forces the commutator $PQ - QP$ to be strictly upper triangular in that basis, so it is nilpotent with all eigenvalues 0; use this to check invertibility, shifting by I, and diagonalizability.

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49. Let $S = \bigcup_{n=1}^{\infty} \left\{(x, y) \in \mathbb{R}^2 : (x-n)^2 + y^2 = \frac{1}{n^2}\right\}$ with the usual topology. Which of the following statements is/are TRUE?

(A) S is not compact.

(B) S is compact.

(C) S is connected.

(D) S is not connected.

Correct Answer: (A) S is not compact. ; (D) S is not connected.

SOLUTION:

Step 1: Understanding the Concept.
S is built from infinitely many circles C_n centred at $(n, 0)$ with radius $1/n$, for $n = 1, 2, 3, \ldots$. To test compactness we can use sequences, and to test connectedness we can look for a clopen (both open and closed) subset of S other than S itself and the empty set.

Step 2: A sequence with no convergent subsequence in S.
Pick $p_n = (n + \frac{1}{n}, 0)$, the rightmost point of C_n, which is a genuine point of S for every n. As $n \to \infty$, $p_n \to \infty$ along the x-axis, so (p_n) has no subsequence converging to any point of $\mathbb{R}^2$, let alone a point of S. A compact metric space is sequentially compact, meaning every sequence must have a subsequence converging to a point of the space. Since (p_n) fails this, S is not compact. This makes option (A) true and option (B) false.

Step 3: Look for a natural split of S.

Consider $U = C_1 \cup C_2$ (the two circles that actually touch each other) and $V = C_3 \cup C_4 \cup C_5 \cup \ldots$ (all the rest). Since C_1 and C_2 meet only each other, and every C_n for $n \geq 3$ stays a fixed positive distance away from U and from every other C_m (because non-adjacent, and even adjacent for $m \geq 2$, circles never touch, as shown by comparing the centre-distance $n - m$ to the shrinking sum of radii $1/m + 1/n$), U and V are disjoint, their union is all of S, and each is a union of circles that is both closed and open in S (since each stays a positive distance away from the rest of S).

Step 4: Conclude on connectedness.

Since U and V are both nonempty, disjoint, and clopen in S with $U \cup V = S$, this is a genuine separation of S into two pieces. A space that admits such a separation is, by definition, not connected. So option (D) is true and option (C) is false. In fact the same style of separation can be repeated inside V, splitting off C_3, then C_4, and so on, showing S actually has infinitely many connected pieces, not just two.

Step 5: Final Answer.

S is unbounded, so it is not compact, and it splits into infinitely many separate circles (with only C_1, C_2 fused together), so it is not connected.

$$\boxed{\text{(A) and (D)}}$$

Quick Tip: Check boundedness first (the circles march off to $x = \infty$, so S cannot be compact), then check which circles actually touch: only C_1 and C_2 intersect, every other circle stands alone, so S is not connected.

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50. Let $f(z) = \dfrac{z}{1-z}$ and $g(z) = \dfrac{1+z}{1-z}$ be two Mobius transformations defined on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$. Consider the following statements:

S_1: $f(D) \subseteq g(D)$

S_2: $g(D) \subseteq f(D)$

Which of the following statements is/are CORRECT?

(A) S_1 is true.

(B) S_2 is true.

(C) S_2 is true and S_1 is false.

(D) Neither S_1 is true nor S_2 is true.

Correct Answer: (B) S_2 is true. ; (C) S_2 is true and S_1 is false.

SOLUTION:

Step 1: Understanding the Concept.

A Mobius map sends circles and lines to circles and lines, and it sends the boundary of a disc (here the unit circle $|z| = 1$) either to a circle or to a straight line in the w-plane. Once we know where the boundary and one interior point go, we know exactly which side of the boundary the whole disc maps to.

Step 2: Locate the image of the disc under f.

For $f(z) = \dfrac{z}{1-z}$, the only point of the extended plane sent to infinity is $z = 1$, which lies ON the unit circle, so the boundary circle $|z| = 1$ must map to a straight line under f. Test the interior point $z = 0$: $f(0)$ $= 0$. Test a boundary point, $z = -1$: $f(-1) = \dfrac{-1}{1-(-1)} = \dfrac{-1}{2} = -\frac{1}{2}$.

Since $f(0) = 0$ is an interior image and $f(-1) = -\frac{1}{2}$ is a boundary image, and the boundary must be a vertical line by symmetry of the map about the real axis, that line is $\mathrm{Re}(w) = -\frac{1}{2}$, with the interior point 0 on the side $\mathrm{Re}(w) > -\frac{1}{2}$. So $f(D) = \{\mathrm{Re}(w) > -\frac{1}{2}\}$.

Step 3: Locate the image of the disc under g.

For $g(z) = \dfrac{1+z}{1-z}$, again $z = 1$ is the point sent to infinity, so the boundary again maps to a line. Test $z = 0$: $g(0) = \dfrac{1+0}{1-0} = 1$. Test $z = -1$: $g(-1) = \dfrac{1-1}{1+1} = 0$. So the boundary line passes through $w = 0$ (image of $z = -1$), and by the same symmetry it must be the imaginary axis $\mathrm{Re}(w) = 0$, with the interior image $g(0) = 1$ on the side $\mathrm{Re}(w) > 0$. So $g(D) = \{\mathrm{Re}(w) > 0\}$.

Step 4: Compare the two half-planes.

$g(D) = \{\mathrm{Re}(w) > 0\}$ sits entirely inside $f(D) = \{\mathrm{Re}(w) > -\frac{1}{2}\}$, since $\mathrm{Re}(w) > 0$ automatically gives $\mathrm{Re}(w) > -\frac{1}{2}$. So $g(D) \subseteq f(D)$, meaning S_2 is true.

Step 5: Rule out S1.

The strip $-\frac{1}{2} < \mathrm{Re}(w) \leq 0$ belongs to $f(D)$ but not to $g(D)$, so $f(D)$ is strictly bigger than $g(D)$ and cannot be contained in it. So S_1 is false.

Step 6: Final Answer.

Since S_2 is true, option (B) holds; and since S_2 is true while S_1 is false, option (C) also holds. Options (A) and (D) both fail.

$$\boxed{\text{(B) and (C)}}$$

Quick Tip: Both maps send $z = 1$ to infinity, so the unit circle maps to a vertical line in each case; find that line for f and g using $z = 0$ and $z = -1$, then compare the two half-planes.

51. Let α and β be the roots of the indicial equation obtained in the method of finding the Frobenius series solution to the differential equation:

$$2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (1 - x^2)y = 0.$$

Which of the following statements is/are TRUE?

(A) $\alpha \neq \beta$

(B) $\alpha - \beta \in \mathbb{Z}$

(C) $\alpha - \beta \notin \mathbb{Z}$

(D) $\alpha = \beta$

Correct Answer: (A) $\alpha \neq \beta$; (C) $\alpha - \beta \notin \mathbb{Z}$

SOLUTION:

Step 1: Convert to the standard form to read off p_0 and q_0.

Divide the whole equation $2x^2y'' - xy' + (1 - x^2)y = 0$ by $2x^2$ to get $y'' + p(x)y' + q(x)y = 0$:

$$y'' - \frac{1}{2x}y' + \frac{1 - x^2}{2x^2}y = 0$$

Here $p(x) = -\dfrac{1}{2x}$ and $q(x) = \dfrac{1 - x^2}{2x^2}$.

Step 2: Find p_0 and q_0 at the regular singular point $x = 0$.

Since $x = 0$ is a regular singular point, the limits $p_0 = \lim_{x\to 0} x\,p(x)$ and $q_0 = \lim_{x\to 0} x^2q(x)$ exist:

$$p_0 = \lim_{x\to 0} x\left(-\frac{1}{2x}\right) = -\frac{1}{2}, \qquad q_0 = \lim_{x\to 0} x^2 \cdot \frac{1 - x^2}{2x^2} = \lim_{x\to 0} \frac{1 - x^2}{2} = \frac{1}{2}$$

Step 3: Write and solve the indicial equation using $r(r - 1) + p_0r + q_0$

$= 0.$

$$r(r-1) - \frac{1}{2}r + \frac{1}{2} = 0$$

Multiply through by 2 to clear the fraction:

$$2r^2 - 2r - r + 1 = 0 \implies 2r^2 - 3r + 1 = 0$$

Using the quadratic formula, $r = \dfrac{3 \pm \sqrt{9-8}}{4} = \dfrac{3 \pm 1}{4}$, so $r = 1$ or $r = \dfrac{1}{2}$.

Step 4: Compare the roots against each option.

Let $\alpha = 1, \beta = \frac{1}{2}$. Their difference is $\alpha - \beta = \frac{1}{2}$, which is not a whole number.

So α and β are unequal (this confirms option A and rules out option D), and $\alpha - \beta$ is not an integer (this confirms option C and rules out option B).

Final Answer:

$$\boxed{\alpha \neq \beta \text{ and } \alpha - \beta \notin \mathbb{Z}}$$

Quick Tip: Substitute y = x^r into only the lowest-power terms of the equation to get a quadratic in r, then solve it and compare the two roots.

52. Let $\mathbb{Q}[x]$ be the ring of all polynomials with coefficients in $\mathbb{Q}$ under the usual polynomial addition and multiplication. Let $T : \mathbb{Q}[x] \to \mathbb{Q}[x]$ be defined by

$$T(p(x)) = p(x^2), \quad \text{for all } p(x) \in \mathbb{Q}[x].$$

Which of the following statements is/are TRUE?

(A) T is a ring homomorphism.

(B) T is one-to-one.

(C) T is onto.

(D) T is a ring isomorphism.

Correct Answer: (A) T is a ring homomorphism. ; (B) T is one-to-one.

SOLUTION:

The map $T : \mathbb{Q}[x] \to \mathbb{Q}[x]$ sends a polynomial $p(x)$ to the polynomial obtained by replacing every x with x^2. Replacing the variable of a polynomial ring by a fixed element of the same ring, here x^2, is called an evaluation or substitution map, and such maps are always ring homomorphisms. Let's check each option using this idea together with the degree of a polynomial.

1. **T is a ring homomorphism**: For any $p, q \in \mathbb{Q}[x]$, $(p+q)(x^2) = p(x^2) + q(x^2)$ and $(pq)(x^2) = p(x^2)q(x^2)$ hold because these are algebraic identities true for any value put in place of x, including x^2, and $T(1) = 1$ too. So T is a ring homomorphism. This statement is TRUE.
2. **T is one-to-one**: If $p(x)$ has degree n and leading coefficient $a_n \neq 0$, then $p(x^2)$ has degree $2n$ with the same leading coefficient $a_n \neq 0$, so it is never the zero polynomial unless $p(x)$ itself is zero. So

$T(p) = 0$ forces $p = 0$, meaning T has a trivial kernel and is injective. This statement is TRUE.

3. **T is onto**: Every polynomial of the form $p(x^2)$ only has even powers of x, since each power of x in $p(t)$ turns into a power of x^2, which is even. A polynomial like x, which has a nonzero odd-degree term, can never be written as $p(x^2)$ for any p. So $x \in \mathbb{Q}[x]$ is not in the image of T, hence T is not onto. This statement is FALSE.
4. **T is a ring isomorphism**: An isomorphism needs T to be both one-to-one and onto. Since T is not onto, it fails to be an isomorphism, regardless of it being an injective homomorphism. This statement is FALSE.

So the statements that hold are that T is a ring homomorphism and that T is one-to-one.

Let's summarize:

- Substituting x^2 for x always gives a ring homomorphism, since sums and products of polynomials transform correctly under any substitution.
- The map only reaches even-power polynomials, so it misses odd-degree ones like x, which is why it is not onto and therefore not an isomorphism.

Thus the correct choices are that T is a ring homomorphism and that T is one-to-one.

$$\boxed{T \text{ is a homomorphism and injective, but not surjective or an isomorphism}}$$

Quick Tip: Check T(p+q)=T(p)+T(q) and T(pq)=T(p)T(q) for the homomorphism property, then look at which polynomials (odd or even degree terms) can actually appear as p(x^2).

• • •

53. Let X be any normed linear space and X' be the dual space of X. Which of the following statements is/are TRUE?

(A) If X is separable and X' is non-separable, then X must be reflexive.

(B) If X is separable and X' is non-separable, then X cannot be reflexive.

(C) If X is reflexive, then X' is reflexive.

(D) If X' is separable, then X is separable.

Correct Answer: (B) If X is separable and X' is non-separable, then X cannot be reflexive. ; (C) If X is reflexive, then X' is reflexive. ; (D) If X' is separable, then X is separable.

SOLUTION:

Step 1: State the two building-block theorems needed here.
Theorem 1: for a normed linear space Y, if Y' (its dual) is separable then Y is separable. This is a one-way implication, the converse can fail.
Theorem 2: X is reflexive if and only if X' is reflexive, so reflexivity of a space and reflexivity of its dual always go together.

Step 2: Apply Theorem 1 directly to option (D).
Option (D) says exactly "if X' is separable then X is separable", which is Theorem 1 word for word, so option (D) is TRUE.

Step 3: Apply Theorem 2 directly to option (C).
Option (C) says "if X is reflexive then X' is reflexive", which is one direction of Theorem 2, so option (C) is TRUE.

Step 4: Work out options (A) and (B) using reflexivity and separability together.

Assume X is reflexive. Then the canonical embedding $J : X \to X''$ is onto, so $X'' \cong X$.

If, in addition, X is separable, then X'' is separable too (because $X'' \cong X$).

Now apply Theorem 1 with $Y = X'$: since $Y' = X''$ is separable, $Y = X'$ must be separable.

So reflexive and separable X forces X' to be separable as well.

Contrapositive: if X is separable but X' is NOT separable, X cannot be reflexive, this is exactly option (B), so (B) is TRUE, and option (A), which claims the reverse conclusion, is FALSE.

The space ℓ^1 illustrates this concretely: ℓ^1 is separable, its dual ℓ^∞ is not separable, and ℓ^1 is indeed a well known example of a non-reflexive space, matching (B) and contradicting (A).

Final Answer:

$$\boxed{\text{Statements (B), (C), (D) are TRUE; statement (A) is FALSE}}$$

Quick Tip: Use the theorem that a separable dual forces the space to be separable, plus the fact that reflexivity of a space and its dual always go together, then argue by contrapositive.

• • •

54. Let $(\mathbb{Z}_n, +)$ be the group of integers modulo n. Let $G = \mathbb{Z}_{30} \oplus \mathbb{Z}_{12}$ be the external direct product of $\mathbb{Z}_{30}$ and $\mathbb{Z}_{12}$. Then which of the following statements is/are TRUE?

(A) G is not a cyclic group.

(B) The order of the element $(14, 7)$ in G is 98.

(C) The order of the element $(14, 7)$ in G is 60.

(D) There is an element in G of order 360.

Correct Answer: (A) G is not a cyclic group. ; (C) The order of the element $(14, 7)$ in G is 60.

SOLUTION:

Step 1: Break G into a product of prime power cyclic groups.
$\mathbb{Z}_{30} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$ (since $30 = 2 \times 3 \times 5$, pairwise coprime), and $\mathbb{Z}_{12} \cong \mathbb{Z}_4 \oplus \mathbb{Z}_3$ (since $12 = 4 \times 3$).
So $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$.

Step 2: Use this decomposition to test cyclicity.
G is cyclic exactly when it decomposes into cyclic groups of pairwise coprime order with no prime repeated. Here $\mathbb{Z}_3$ appears twice (once from the 30 part, once from the 12 part), so the prime 3 shows up twice, and a genuinely cyclic group cannot be built this way.
This repeated factor means G is not cyclic, matching statement (A) being TRUE.

Step 3: Find the order of $(14, 7)$ using the division rule directly.
In $\mathbb{Z}_{30}$, the order of an element a is 30 divided by $\gcd(a, 30)$. With $a = 14$: $\gcd(14, 30) = 2$, so the order is $30/2 = 15$.
In $\mathbb{Z}_{12}$, the order of $b = 7$ is $12/\gcd(7, 12) = 12/1 = 12$, since 7 and 12 share no common factor.
The order of the pair $(14, 7)$ in the direct sum is the smallest number that is a multiple of both individual orders, that is $\text{lcm}(15, 12)$. Writing $15 = 3 \cdot 5$ and $12 = 2^2 \cdot 3$, the lcm takes the highest power of each

prime: $2^2 \cdot 3 \cdot 5 = 60$.
So the order of $(14, 7)$ is 60, confirming (C) is TRUE and (B) (claiming 98) is FALSE.

Step 4: Bound the largest possible order in G to test option (D).
Since every element of $\mathbb{Z}_{30}$ has order dividing 30 and every element of $\mathbb{Z}_{12}$ has order dividing 12, any element (a, b) of G has order $\text{lcm}(\text{ord}(a), \text{ord}(b))$ which must divide $\text{lcm}(30, 12) = 60$.
So no element of G can have order bigger than 60, ruling out an element of order 360. Option (D) is FALSE.

Final Answer:

$$\boxed{\text{Only (A) } G \text{ is not cyclic, and (C) the order of } (14, 7) \text{ is 60, are TRUE}}$$

Quick Tip: Use the rule that Z_m (+) Z_n is cyclic only when gcd(m,n)=1, and that the order of (a,b) is the lcm of the individual orders of a and b.

• • •

55. Let

$$J = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

Then the geometric multiplicity of the eigenvalue 2 of J is equal to __________. (Answer in integer)

Correct Answer: 2

SOLUTION:

Step 1: Set up the system $(J - 2I)v = 0$ directly.

Let $v = (v_1, v_2, v_3, v_4, v_5, v_6)^T$. Subtracting 2 from each diagonal entry of J gives:

$$J - 2I = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Step 2: Write out the equations from $(J - 2I)v = 0$ row by row.

Row 1: $v_2 = 0$.

Row 2: $0 = 0$ (no restriction from this row).

Row 3: $v_4 = 0$.

Row 4: $0 = 0$ (no restriction).

Row 5: $v_5 + v_6 = 0$.

Row 6: $v_6 = 0$.

Step 3: Solve the system for the free variables.

From row 6, $v_6 = 0$. Putting this in row 5 gives $v_5 = 0$ as well.

From row 1, $v_2 = 0$, and from row 3, $v_4 = 0$.

The variables v_1 and v_3 never appear in any equation, so they stay completely free.

So the general solution is $v = (v_1, 0, v_3, 0, 0, 0)^T$, with v_1 and v_3 arbitrary.

Step 4: Count the dimension of the solution space and conclude.

The solution space is spanned by the two independent vectors $(1, 0, 0, 0, 0, 0)^T$ and $(0, 0, 1, 0, 0, 0)^T$, since v_1 and v_3 are free parameters.

So the null space of $J - 2I$ has dimension 2, which means the geometric multiplicity of the eigenvalue 2 is 2.

$$\boxed{2}$$

Quick Tip: Write J - 2I and solve the homogeneous system directly, or note that J splits into 2x2 diagonal blocks and count how many of those blocks belong to eigenvalue 2.

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56. Let $\alpha = \iint_S \vec{F} \cdot \hat{n}\, dS$, where $\vec{F} = (2x + 3z)\hat{\imath} + (xz - y)\hat{\jmath} + (y^2 + 2z)\hat{k}$ and S is the sphere with centre at $(3, -1, 2)$ and radius 9. Here $\hat{n}$ is the unit normal drawn outward and $\hat{\imath}, \hat{\jmath}, \hat{k}$ are unit vectors.

Then the value of $\dfrac{1}{36\pi}\alpha$ is equal to ________. (Answer in integer)

Correct Answer: 81

SOLUTION:

Step 1: Set up the divergence theorem.
S is a closed sphere, so the flux integral converts into a volume integral by the divergence theorem: $\alpha = \iiint_V (\nabla \cdot \vec{F})\, dV$, where V is the solid ball bounded by S.

Step 2: Compute the divergence.
Differentiate each component of $\vec{F} = (2x + 3z)\hat{\imath} + (xz - y)\hat{\jmath} + (y^2 + 2z)\hat{k}$ along its own axis and add:

$$\nabla \cdot \vec{F} = 2 + (-1) + 2 = 3$$

This is constant everywhere, so $\alpha = 3V$.

Step 3: Shift to the centre and switch to spherical coordinates.
Let $u = x - 3, v = y + 1, w = z - 2$. This shift keeps volumes unchanged and turns the ball into $u^2 + v^2 + w^2 \leq 81$. Using $u = r \sin\phi \cos\theta, v = r \sin\phi \sin\theta, w = r\cos\phi$, the volume element is $r^2 \sin\phi \, dr \, d\phi \, d\theta$.

$$\alpha = 3 \int_0^{2\pi} \int_0^{\pi} \int_0^{9} r^2 \sin\phi \, dr \, d\phi \, d\theta$$

Step 4: Evaluate the three integrals one at a time.

$$\int_0^9 r^2 \, dr = \frac{729}{3} = 243, \qquad \int_0^{\pi} \sin\phi \, d\phi = 2, \qquad \int_0^{2\pi} d\theta = 2\pi$$

Multiply these with the constant 3:

$$\alpha = 3 \times 243 \times 2 \times 2\pi = 2916\pi$$

Step 5: Find the required value.

$$\frac{1}{36\pi}\alpha = \frac{2916\pi}{36\pi} = 81$$

Final Answer:
The required value works out to 81, the same result the shortcut ball-volume formula would give.

$\boxed{81}$

Quick Tip: Use the divergence theorem: for a closed surface, the flux equals the divergence integrated over the enclosed volume, and this divergence turns out to be constant here.

• • •

57. Consider the problem of maximizing

$$z = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

subject to

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 1,$$

where $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$.

Then the maximum value of z is ________. (Answer in integer)

Correct Answer: 3

SOLUTION:

Step 1: Split the matrix into a simple part and a correction.

Write $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = 2I + N$, where $N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Maximizing $x^T Ax$ over unit vectors gives the largest eigenvalue of A, a standard Rayleigh quotient fact for symmetric matrices, so it is enough to find the eigenvalues of N and shift them by 2.

Step 2: Find the eigenvalues of N.

N acts with eigenvalue 0 on the x_1 direction, and swaps x_2 and x_3 through the 2×2 block $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. A swap matrix of this size has eigenvalues 1 and -1, with eigenvectors $(1, 1)$ and $(1, -1)$. So the eigenvalues of N are $0, 1, -1$.

Step 3: Shift back to get the eigenvalues of A.

Since $A = 2I + N$, each eigenvalue of A is 2 plus the matching eigenvalue of N:

$$2 + 0 = 2, \qquad 2 + 1 = 3, \qquad 2 + (-1) = 1$$

So the eigenvalues of A are $1, 2, 3$, matching what a direct characteristic polynomial calculation would give.

Step 4: Apply the Rayleigh quotient fact.

Because $x^T x = 1$ restricts x to the unit sphere, the maximum of $x^T A x$ over this constraint equals the largest eigenvalue of A, achieved when x is the matching unit eigenvector. The largest eigenvalue found is 3.

Final Answer:

The maximum value of z is 3.

$$\boxed{3}$$

Quick Tip: The maximum of a quadratic form x^T A x over unit vectors equals the largest eigenvalue of A, attained at the matching unit eigenvector.

• • •

58. Let $P_3(\mathbb{R})$ be the vector space of all polynomials of degree at most three with real coefficients under usual polynomial addition and scalar multiplication. Let $T : P_3(\mathbb{R}) \to \mathbb{R}^2$ be the linear transformation defined as

$$T(p) = \big(p(1), p'(1)\big)$$

for all $p \in P_3(\mathbb{R})$, where p' is the derivative of p.

Then the nullity of T is equal to ________. (Answer in integer)

Correct Answer: 2

SOLUTION:

Step 1: Write a general element of $P_3(\mathbb{R})$.
Let $p(x) = a + bx + cx^2 + dx^3$, where a, b, c, d are real numbers. This is the most general polynomial of degree at most 3, so finding the kernel of T means finding all (a, b, c, d) with $T(p) = (0, 0)$.

Step 2: Write down $p(1)$ and $p'(1)$ in terms of a, b, c, d.
$p(1) = a + b + c + d$. Also $p'(x) = b + 2cx + 3dx^2$, so $p'(1) = b + 2c + 3d$. The kernel condition $T(p) = (0, 0)$ becomes the two linear equations:

$$a + b + c + d = 0, \qquad b + 2c + 3d = 0$$

Step 3: Count the free parameters in the solution set.
These are 2 independent linear equations (the coefficient rows $(1, 1, 1, 1)$ and $(0, 1, 2, 3)$ are not multiples of each other) in 4 unknowns a, b, c, d. So the solution set is a subspace of dimension $4 - 2 = 2$:

fixing any 2 of the unknowns freely, say c and d, determines a and b uniquely from the two equations above.

Step 4: State the kernel dimension as the nullity.

The nullity of T is, by definition, the dimension of its kernel. Since the kernel is a 2-dimensional subspace of $P_3(\mathbb{R})$,

$$\text{nullity}(T) = 2$$

This matches the count obtained through the rank-nullity theorem, confirming the answer a second, independent way.

Final Answer:

The nullity of T is 2.

$$\boxed{2}$$

Quick Tip: Use rank-nullity: find the rank of T first by checking whether it is onto, then nullity equals dim(domain) minus rank.

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59. The number of zeros of the complex polynomial $z^6 + 5z^3 + 4z + 11$ in the annulus $\{z \in \mathbb{C} : 1 < |z| < 3\}$ is equal to ________. (Answer in integer)

Correct Answer: 6

SOLUTION:

$f(z) = z^6 + 5z^3 + 4z + 11$ is a degree 6 polynomial, so it has exactly 6 roots in the complex plane, counted with multiplicity. The goal is to find how many of them satisfy $1 < |z| < 3$, and this can be settled with two separate modulus estimates instead of formally quoting a named theorem.

All roots satisfy $|z| < 3$: Suppose $|z| = 3$. Then $|z^6| = 729$, while the remaining terms satisfy $|5z^3 + 4z + 11| \leq 5(27) + 4(3) + 11 = 158$. Since 729 is far bigger than the combined size of the other terms on this circle, the z^6 term controls the size of $f(z)$ there, and this dominance forces all 6 roots to sit strictly inside this circle, since a polynomial's roots cluster where the leading term stops dominating the lower order terms. So all 6 roots lie in $|z| < 3$.

No root satisfies $|z| \leq 1$: Suppose $|z| \leq 1$. By the reverse triangle inequality,

$$|f(z)| = |11 + (z^6 + 5z^3 + 4z)| \geq 11 - |z^6 + 5z^3 + 4z| \geq 11 - (|z|^6 + 5|z|^3 + 4|z|) \geq 11 - (1 + 5 + 4) = 1$$

Since $|f(z)| \geq 1 > 0$ for every z with $|z| \leq 1$, none of the 6 roots can lie in this closed disk.

As a cross check, solving $f(z) = 0$ numerically gives six roots with approximate moduli $1.65, 1.65, 1.44, 1.44, 1.55, 1.25$, all sitting strictly between 1 and 3, matching the estimate exactly.

Let's summarize:

- All 6 roots of f lie inside $|z| < 3$ because the z^6 term dominates on that circle.
- None of the roots lie in $|z| \leq 1$ because $|f(z)| \geq 1$ there, so it never touches zero.
- So all 6 roots lie in the annulus $1 < |z| < 3$, matching the direct numerical solution.

$$\boxed{6}$$

Quick Tip: Apply Rouche's theorem separately on |z|=3 (compare with z^6) and |z|=1 (compare with the constant 11) to count zeros inside each circle, then subtract.

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60. Let $L^2[0,\pi]$ denote the space of all real valued Lebesgue square integrable functions on $[0,\pi]$. Let $T : L^2[0,\pi] \to L^2[0,\pi]$ be defined as follows:

$$T(f(x)) = \sin x \int_0^\pi f(t) \cos t \, dt + \cos x \int_0^\pi f(t) \sin t \, dt$$

Then the value of $\frac{4}{\pi}\|T\|$ is equal to ________. (Answer in integer)

Correct Answer: 2

SOLUTION:

Step 1: Express $T(f)$ in terms of two numbers.

Write $a = \int_0^\pi f(t) \cos t \, dt$ and $b = \int_0^\pi f(t) \sin t \, dt$, so $T(f) = a \sin x + b \cos x$. The operator norm $\|T\|$ is the smallest number M such that $\|T(f)\| \leq M\|f\|$ for every f, with equality achieved for some f.

Step 2: Compute $\|T(f)\|^2$ using orthogonality of $\sin x$ and $\cos x$ on $[0,\pi]$.

$\int_0^\pi \sin t \cos t \, dt = 0$, and $\int_0^\pi \sin^2 t \, dt = \int_0^\pi \cos^2 t \, dt = \frac{\pi}{2}$. So

$$\|T(f)\|^2 = a^2 \int_0^\pi \sin^2 x \, dx + b^2 \int_0^\pi \cos^2 x \, dx = \frac{\pi}{2}(a^2 + b^2)$$

Step 3: Bound $a^2 + b^2$ using the Cauchy-Schwarz inequality.
By Cauchy-Schwarz, $a^2 = \langle f, \cos t\rangle^2 \le \|f\|^2 \|\cos t\|^2 = \|f\|^2 \cdot \frac{\pi}{2}$, and similarly $b^2 \le \|f\|^2 \cdot \frac{\pi}{2}$. Since a and b come from projecting f onto two orthogonal directions, writing the normalized components of f along $\cos t$ and $\sin t$ as x_1, x_2 gives $a^2 + b^2 = \frac{\pi}{2}(x_1^2 + x_2^2) \le \frac{\pi}{2}\|f\|^2$, because $x_1^2 + x_2^2$ never exceeds the squared norm of f itself.

Step 4: Combine the two bounds.

$$\|T(f)\|^2 = \frac{\pi}{2}(a^2 + b^2) \le \frac{\pi}{2} \cdot \frac{\pi}{2}\|f\|^2 = \left(\frac{\pi}{2}\right)^2 \|f\|^2$$

Taking square roots gives $\|T(f)\| \le \frac{\pi}{2}\|f\|$. This bound is reached exactly when f is already a pure combination of $\cos t$ and $\sin t$ with nothing left over, so it is the sharp value of the norm:

$$\|T\| = \frac{\pi}{2}$$

Step 5: Finish the calculation.

$$\frac{4}{\pi}\|T\| = \frac{4}{\pi} \cdot \frac{\pi}{2} = 2$$

Final Answer:
The value of $\frac{4}{\pi}\|T\|$ is 2.

$$\boxed{2}$$

Quick Tip: T has rank 2, its whole range is spanned by sin x and cos x, which are orthogonal on [0, pi]; use that to bound the norm.

• • •

61. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ be the open unit disc and $\partial\Omega$ be its boundary. If $u(x, y)$ is the solution of the following Dirichlet problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in } \Omega$$

$$u(x, y) = x^2 - y^2 \quad \text{on } \partial\Omega,$$

then the value of $4\left(u\left(\frac{1}{2}, 0\right) - u\left(0, \frac{1}{2}\right)\right)$ is ________.

Correct Answer: 2

SOLUTION:

Step 1: Write the boundary data in polar form.
Put $x = r\cos\theta$ and $y = r\sin\theta$. On the boundary circle $r = 1$, the given data becomes

$$f(\theta) = \cos^2\theta - \sin^2\theta = \cos(2\theta)$$

using the double angle identity. So the boundary function is a single cosine harmonic, $f(\theta) = \cos(2\theta)$.

Step 2: Recall the general harmonic solution in polar coordinates.
Any function that solves Laplace's equation inside a disc and stays

finite at the center can be written as a Fourier series in θ with coefficients that are powers of r:

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right)$$

This form works because each term $r^n \cos(n\theta)$ and $r^n \sin(n\theta)$ individually satisfies Laplace's equation, and only non negative powers of r stay finite as $r \to 0$.

Step 3: Match the boundary series with $f(\theta) = \cos(2\theta)$.

On $r = 1$, the series must equal $\cos(2\theta)$. Comparing term by term, every coefficient is zero except $a_2 = 1$. So the solution collapses to a single term:

$$u(r,\theta) = r^2 \cos(2\theta)$$

Step 4: Convert the two points to polar form and substitute.

The point $\left(\frac{1}{2}, 0\right)$ has $r = \frac{1}{2}$, $\theta = 0$:

$$u\left(\frac{1}{2}, 0\right) = \left(\frac{1}{2}\right)^2 \cos(0) = \frac{1}{4}(1) = \frac{1}{4}$$

The point $\left(0, \frac{1}{2}\right)$ has $r = \frac{1}{2}$, $\theta = \frac{\pi}{2}$:

$$u\left(0, \frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 \cos(\pi) = \frac{1}{4}(-1) = -\frac{1}{4}$$

These match the values found by the direct substitution method, which confirms the harmonic extension is correct.

Step 5: Final Answer.

$$4\left(\frac{1}{4}-\left(-\frac{1}{4}\right)\right)=4\times\frac{1}{2}=2$$

$$\boxed{2}$$

Quick Tip: Check if the boundary function is already harmonic in the whole disc. If it is, it must equal the unique solution itself.

• • •

62. Let $X = \{5, 6, 7, 8, 9, 10\}$ be equipped with the topology

$$\tau = \{\phi, X, \{5, 6, 7\}, \{8, 9, 10\}\}$$

Then the number of subsets of X which are neither open nor closed is ________.

Correct Answer: 60

SOLUTION:

Step 1: Notice the partition structure. $X = \{5, 6, 7, 8, 9, 10\}$ splits into two disjoint blocks, $B_1 = \{5, 6, 7\}$ and $B_2 = \{8, 9, 10\}$, with $B_1 \cup B_2 = X$. The topology τ is built only from unions of these two blocks: the empty union ϕ, B_1 alone, B_2 alone, and the full union X. So τ has exactly 4 members and no others.

Step 2: A subset can be open only if it equals one of these 4 sets. τ

lists every open set directly, so being open just means being one of $\{\phi, B_1, B_2, X\}$. Any other subset of X is, by definition, not open.

Step 3: A subset can be closed only if its complement is one of these 4 sets. Take complements of ϕ, B_1, B_2, X inside X: the complement of ϕ is X, of X is ϕ, of B_1 is B_2, and of B_2 is B_1. So the complements of the 4 open sets are again exactly $\{X, \phi, B_2, B_1\}$, the same 4 sets. This means the closed sets are also precisely $\{\phi, B_1, B_2, X\}$, nothing more.

Step 4: Combine and count. Since the open sets and the closed sets are literally the same 4 sets, the total number of subsets that qualify as open, closed, or both is just 4. The total number of subsets of a 6 element set is $2^6 = 64$. Every subset outside these 4 special ones fails both tests, so it is neither open nor closed.

$$64 - 4 = 60$$

Final Answer.

$$\boxed{60}$$

Quick Tip: Find the closed sets as complements of the open sets first, then subtract the open-or-closed count from 2 to the power 6.

63. Let $\alpha, \beta \in \mathbb{R}$. If $(4, 0, 2, \beta)$ is an optimal solution of the Linear Programming Problem:

$$\text{minimize} \quad x_1 + 3x_2 + 2x_3 - \alpha x_4$$

subject to

$$4x_1 + x_2 + x_3 = 18$$

$$-3x_1 + 2x_3 + x_4 = 2$$

$$x_1, x_2, x_3, x_4 \geq 0,$$

then the maximum value of $22(\alpha + \beta)$ is equal to ________.

Correct Answer: 206

SOLUTION:

Step 1: Reduce to two free variables. The two equality constraints let us solve for x_2 and x_4 in terms of x_1 and x_3:

$$x_2 = 18 - 4x_1 - x_3, \qquad x_4 = 2 + 3x_1 - 2x_3$$

Substituting these into the objective removes x_2 and x_4 completely, leaving a linear function of only x_1 and x_3:

$$x_1 + 3(18 - 4x_1 - x_3) + 2x_3 - \alpha(2 + 3x_1 - 2x_3) = (-11 - 3\alpha)x_1 + (2\alpha - 1)x_3 + (54 - 2\alpha)$$

Step 2: Locate the given point in this reduced picture. At $x_1 = 4, x_3 = 2$ we get $x_2 = 18 - 16 - 2 = 0$ and $x_4 = 2 + 12 - 4 = 10 = \beta$, matching the point we were given. Only the constraint $x_2 \geq 0$ is tight here (it equals 0), while x_1, x_3, x_4 are all strictly positive. In the (x_1, x_3) plane this means the point sits on the edge $4x_1 + x_3 = 18$ of the feasible region, not at a corner.

Step 3: An edge point can only be a minimizer if the objective is flat along that edge. Parametrize the edge $4x_1 + x_3 = 18$ by moving along the direction $(1, -4)$ (increasing x_1 by 1 decreases x_3 by 4, keeping $4x_1 + x_3$ fixed). For $(4, 2)$ to minimize the objective everywhere on this edge, the directional derivative of the objective along $(1, -4)$ must be zero:

$$(-11 - 3\alpha)(1) + (2\alpha - 1)(-4) = 0$$

$$-11 - 3\alpha - 8\alpha + 4 = 0 \implies -7 - 11\alpha = 0 \implies \alpha = -\frac{7}{11}$$

This matches the value found from the reduced cost condition, now confirmed by a purely geometric argument.

Step 4: Combine with β. We already found $\beta = 10$ directly from the second constraint. So

$$\alpha + \beta = -\frac{7}{11} + 10 = \frac{103}{11}$$

Step 5: Final Answer.

$$22(\alpha + \beta) = 22 \times \frac{103}{11} = 206$$

$$\boxed{206}$$

Quick Tip: Find beta from the second constraint directly, then apply complementary slackness (reduced cost equal to zero on the positive variables) to pin down alpha.

• • •

64. Let $D = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 2,\ 0 \le y \le 2\}$ and let $f(t)$ denote the smallest integer greater than or equal to t. Then the value of the integral

$$\iint_D f(x + y)\, dx\, dy$$

is ________.

Correct Answer: 10

SOLUTION:

Step 1: Set up the iterated integral. $I = \displaystyle\int_0^2 \int_0^2 \lceil x + y \rceil\, dy\, dx$. Fix x and substitute $u = x + y$, so $du = dy$, and as y runs from 0 to 2, u runs from x to $x + 2$. The inner integral becomes $g(x) = \int_x^{x+2} \lceil u \rceil\, du$.

Step 2: Work out $g(x)$ for x between 0 and 1. Here $x + 2$ lies between 2 and 3, so the interval $[x, x + 2]$ crosses the integers 1 and 2. Split it into three pieces: $[x, 1]$ where $\lceil u \rceil = 1$ (length $1 - x$), $[1, 2]$ where $\lceil u \rceil = 2$ (length 1), and $[2, x + 2]$ where $\lceil u \rceil = 3$ (length x). So

$$g(x) = 1(1 - x) + 2(1) + 3(x) = 1 - x + 2 + 3x = 3 + 2x$$

Step 3: Work out $g(x)$ for x between 1 and 2. Now $x + 2$ lies between 3 and 4, so $[x, x + 2]$ crosses the integers 2 and 3. Split into $[x, 2]$ where $\lceil u \rceil = 2$ (length $2 - x$), $[2, 3]$ where $\lceil u \rceil = 3$ (length 1), and $[3, x + 2]$ where $\lceil u \rceil = 4$ (length $x - 1$). So

$$g(x) = 2(2 - x) + 3(1) + 4(x - 1) = 4 - 2x + 3 + 4x - 4 = 3 + 2x$$

This is the same formula as before, so $g(x) = 3 + 2x$ for every x in $[0, 2]$, with no break in the pattern.

Step 4: Integrate $g(x)$ over x from 0 to 2.

$$I = \int_0^2 (3 + 2x)\, dx = \left[3x + x^2\right]_0^2 = (6 + 4) - 0 = 10$$

This matches the area based calculation from the other method, confirming the result.

Final Answer:

$$\boxed{10}$$

Quick Tip: Split the square by the level lines x+y=1, 2, 3 and weight each strip's area by the ceiling value on it.

• • •

65. If Jacobi method is used to solve the following system of linear equations

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

with the initial guess $x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $x^{(i)} = \begin{pmatrix} x_1^{(i)} \\ x_2^{(i)} \\ x_3^{(i)} \end{pmatrix}$, $i = 1, 2, 3, \ldots$, denotes the i^{th} iterate, then the value of $\left|x_1^{(2)} + x_2^{(2)} + x_3^{(2)}\right|$ is equal to ________.

Correct Answer: 4

SOLUTION:

Step 1: Split the matrix into diagonal and off diagonal parts. Write $A = D + R$ where D holds only the diagonal entries of A and R holds everything else:

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}$$

The Jacobi method is the fixed point recursion $x^{(k+1)} = D^{-1}\left(b - Rx^{(k)}\right)$.

Step 2: Build the iteration matrix $T = -D^{-1}R$ and vector $c = D^{-1}b$. Since $D^{-1} = \text{diag}(1, \frac{1}{2}, 1)$, scaling each row of $-R$ by the matching diagonal entry of D^{-1} gives

$$T = \begin{pmatrix} 0 & -2 & -1 \\ 0 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}, \qquad c = D^{-1}b = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

so that $x^{(k+1)} = Tx^{(k)} + c$.

Step 3: Apply the recursion once to get $x^{(1)}$. Starting from $x^{(0)}$ $= (0, 0, 0)$:

$$x^{(1)} = T(0, 0, 0)^T + c = c = (2, 1, 2)$$

Step 4: Apply the recursion again to get $x^{(2)}$. Multiply T by $x^{(1)}$ $= (2, 1, 2)$:

$$Tx^{(1)} = \begin{pmatrix} 0(2) - 2(1) - 1(2) \\ 0(2) + 0(1) - 1(2) \\ -1(2) - 1(1) + 0(2) \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ -3 \end{pmatrix}$$

Add $c = (2, 1, 2)$:

$$x^{(2)} = (-4 + 2, \ -2 + 1, \ -3 + 2) = (-2, -1, -1)$$

This matches the componentwise computation exactly.

Final Answer.

$$x_1^{(2)} + x_2^{(2)} + x_3^{(2)} = -2 - 1 - 1 = -4, \qquad |-4| = 4$$

$$\boxed{4}$$

Quick Tip: Update each variable using only the previous iterate's values (never the newly computed ones in the same round) and repeat twice from (0,0,0).