GATE 2026 Physics Question Paper with Solutions

ZOLLEGE

General Instructions

(i) This booklet contains 65 questions, each provided with a complete, step-by-step solution.

(ii) It comprises 38 single-correct multiple-choice questions, 10 multiple-correct questions and 17 numerical / integer-type questions.

(iii) Attempt each question on your own before reviewing the given solution.

(iv) For multiple-correct questions, all correct options must be selected to earn full marks.

(v) For numerical questions, report the answer rounded exactly as asked.

. "He often ______ the numbers. False claims are not going to help. Honesty ______ trust", said the manager.
Choose the option with the correct order of words to fill the blanks.

(A) exaggerates; engenders

(B) excels; encourages

(C) aggravates; alleviates

(D) diminishes; eliminates

Correct Answer: (A) exaggerates; engenders

SOLUTION:

Step 1: Test which words can even pair with "the numbers."
A completion question like this often gets solved fastest by checking which verbs naturally take the noun right after the blank.
"Exaggerates the numbers" and "diminishes the numbers" both work grammatically as direct verb plus object. "Excels the numbers" is

wrong because excel needs "at" or "in." "Aggravates the numbers" is odd because aggravate is used for problems, not figures. This already narrows the first blank to (A) or (D).

Step 2: Use the clue sentence to choose between the two.

"False claims are not going to help" tells us the manager is criticizing a habit of overstating things, not understating them. "Exaggerates" fits an overstated, false claim. "Diminishes" would mean understating the numbers, which does not match "false claims" at all. So the first blank must be "exaggerates."

Step 3: Test the second blank the same way, by meaning.

"Honesty ______ trust" needs a word meaning "produces" or "leads to." "Engenders" fits exactly, since it means to bring something into being. "Encourages" also sounds positive, but it belongs to option (B), which already failed the first blank. "Alleviates" (ease a problem) and "eliminates" (remove) both work against the idea that honesty builds trust, so they are wrong.

Step 4: Combine both results.

The first blank must be "exaggerates" and the second must be "engenders." Checking the four choices, this exact pair appears only in option (A).

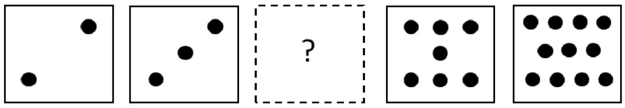
Final Answer:

The sentence reads correctly as "He often exaggerates the numbers... Honesty engenders trust," confirming option (A).

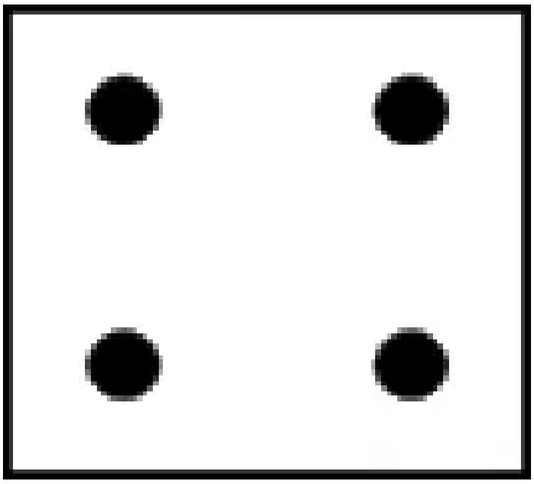
$\boxed{\text{Option (A)}}$

Quick Tip: **Hint:**
Look at what "false claims are not going to help" tells you about the first blank, then find a word for the second blank that shows honesty creating a good outcome.

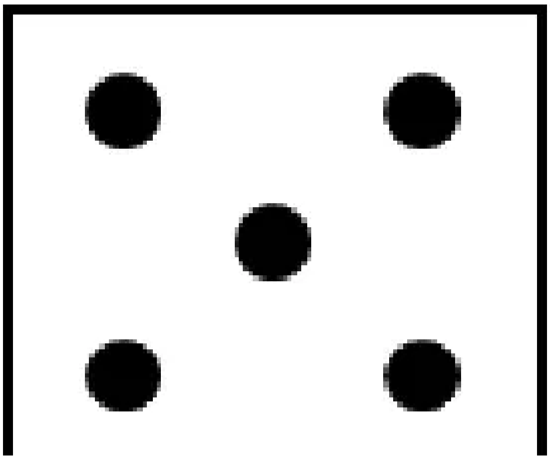
. In the sequence of tiles shown below, the missing tile indicated by the question mark should be:



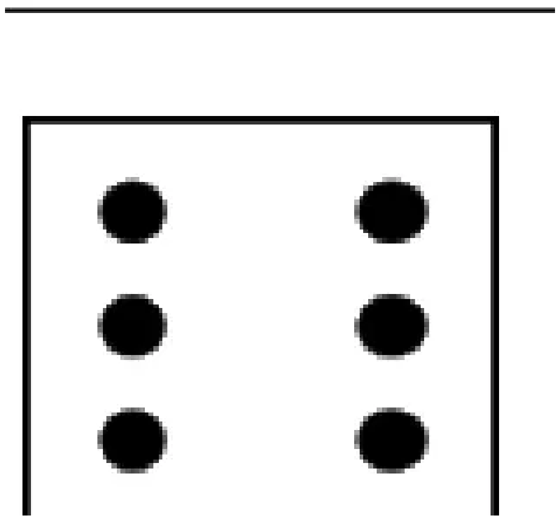
(A)



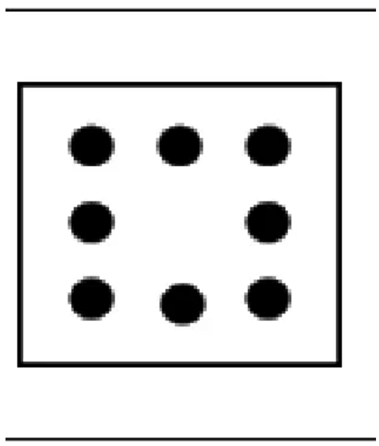
(B)



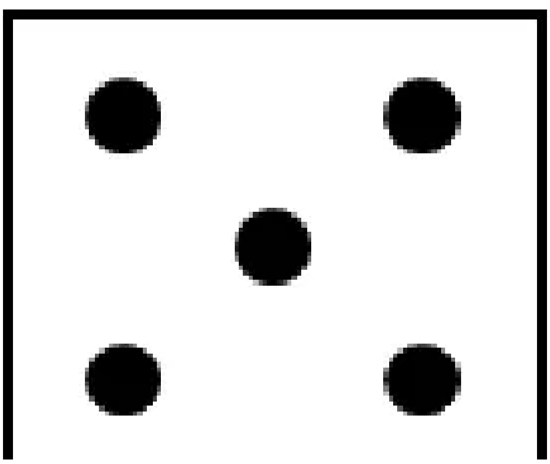
(C)



(D)



Correct Answer: (B)



SOLUTION:

Step 1: List the totals you can already see.

Counting the dots tile by tile gives 2, 3, a missing value, 7, 11.

Step 2: Test whether these follow simple addition or multiplication.

The gaps between the visible numbers are not constant: from 2 to 3 is a jump of 1, but from 7 to 11 is a jump of 4, so there is no single fixed common difference. It is not a doubling pattern either, since 3 times 2 equals 6, not 7. Both a constant-difference rule and a constant-ratio rule fail here.

Step 3: Check each visible number for prime factors instead.

2 has no factors besides 1 and itself. 3 has no factors besides 1 and itself. 7 has no factors besides 1 and itself. 11 has no factors besides 1 and itself. Every visible number in the row is prime, and none of them is composite.

Step 4: Find the prime that belongs between 3 and 7.

Listing primes in order gives $2, 3, 5, 7, 11, 13, \ldots$. The only prime sitting between 3 and 7 in this list is 5, so the missing tile must carry 5 dots.

Step 5: Confirm against the answer choices.

Checking each option for primality: 4 is composite (2 times 2), 5 is prime, 6 is composite (2 times 3), 8 is composite (2 times 2 times 2). Only option (B) is prime and matches the required value.

Final Answer:

The sequence of dot counts is exactly the list of prime numbers 2, 3, 5, 7, 11, so the missing tile has 5 dots.

$\boxed{\text{Option (B)}}$

Quick Tip: Hint:
Count the dots in the tiles you can already see (2, 3, then a gap, then 7, 11) and check whether these numbers share a special property, such as all being prime, rather than following a simple addition pattern.

• • •

. A school has 100 students distributed among 1st to 10th standards. Based on this, which one of the following statements is always correct?

(A) There are at least 10 students who belong to the same standard.

(B) There is at least one student in each standard.

(C) There are at most 10 students in 10th standard.

(D) The total number of students from 1st to 5th standards is at least 50.

Correct Answer: (A) There are at least 10 students who belong to the same standard.

SOLUTION:

Step 1: Try to disprove each option by building a counterexample.
A fast way to solve "which statement is always true" questions is to hunt for one valid arrangement of the 100 students that breaks each option. If you find even a single arrangement where a statement fails, that option is eliminated, since "always correct" cannot allow any exception.

Step 2: Break option (B).
Put all 100 students in the 1st standard, with 0 students in standards 2 to 10. The total is still 100, so this is allowed, but standards 2 through

10 now have no students at all. This contradicts "at least one student in each standard," so (B) fails.

Step 3: Break option (C).

Put all 100 students in the 10th standard, with 0 elsewhere. The total is still 100, but now the 10th standard alone has 100 students, far more than "at most 10." So (C) fails.

Step 4: Break option (D).

Put all 100 students in standards 6 to 10 only, with 0 in standards 1 to 5. This still totals 100, but the count from 1st to 5th standards is 0, which is nowhere close to "at least 50." So (D) fails.

Step 5: Try to break option (A), and see why you cannot.

Suppose, for the sake of argument, that every one of the 10 standards has 9 or fewer students. The largest possible total in that case would be $10 \times 9 = 90$ students. But we are told there are 100 students, and $100 > 90$, so this assumption is impossible. There is no way to keep every standard at 9 or below and still reach a total of 100.

Step 6: Conclude that option (A) cannot be broken.

Since assuming "every standard has fewer than 10 students" leads to a contradiction (a maximum total of only 90, not 100), it must be true that at least one standard has 10 or more students, in every possible arrangement.

Final Answer:

Every attempt to break option (A) fails, while (B), (C), and (D) can each be broken by a simple example, so option (A) is the statement that is always correct.

$\boxed{\text{Option (A)}}$

Quick Tip: **Hint:**
Try to build a distribution of 100 students across 10 standards that breaks each statement. The one statement you cannot break, no matter how you arrange the students, is the answer.

• • •

. How many 3-digit numbers can be formed using three distinct single digit prime numbers?

(A) 64

(B) 24

(C) 12

(D) 4

Correct Answer: (B) 24

SOLUTION:

Step 1: List the digits available.
The single digit primes are 2, 3, 5, and 7, so there are 4 digits to choose from in total.

Step 2: Fill the 3-digit number one position at a time, using the multiplication principle.
Instead of splitting the problem into "choose the set" and "arrange the set" separately, we can directly count how many digits are available for each of the three positions (hundreds, tens, units) of the number, one after another.

Step 3: Fill the hundreds place.
Any of the 4 primes (2, 3, 5, 7) can go in the hundreds place, since none of them is zero and all are allowed to lead the number. This gives 4 choices for the first digit.

Step 4: Fill the tens place.
The digits must all be distinct, so the tens digit cannot repeat whichever digit was used in the hundreds place. That leaves $4 - 1 = 3$ digits still available for the tens place.

Step 5: Fill the units place.
By the same logic, two digits are already used up in the hundreds and tens places, so only $4 - 2 = 2$ digits remain available for the units place.

Step 6: Multiply the choices for each position.

$$4 \times 3 \times 2 = 24$$

Each step's choice is independent of how we count it, whether by first fixing the set of digits or by filling positions left to right, so this matches the combinations-then-arrangements method exactly.

Step 7: Confirm the wrong options.
Getting 64 (option A) would mean allowing a digit to be reused in more than one position, which is ruled out by the word "distinct" in the question. Getting 12 or 4 (options C and D) means one of the three position-fill steps was skipped or miscounted.

Final Answer:

Filling the hundreds, tens, and units positions one at a time with 4, then 3, then 2 remaining digits gives $4 \times 3 \times 2 = 24$.

$$\boxed{24}$$

Quick Tip: **Hint:**
First find how many single digit prime numbers exist, then work out how many ways you can pick 3 of them and arrange those 3 in order.

• • •

. In a group of students, 10 students like Mathematics, 12 students like English, 4 students like both Mathematics and English, and 6 students like neither Mathematics nor English. The number of students in the group is

_____

(A) 18

(B) 20

(C) 24

(D) 32

Correct Answer: (C) 24

SOLUTION:

Step 1: Split the students into four non-overlapping groups.
Instead of using the union formula directly, we can split every student in the class into exactly one of four categories: only Mathematics, only English, both subjects, and neither subject. Since these four categories do not overlap with each other, adding them up gives the total group size directly.

Step 2: Work out how many students like only Mathematics.
Out of the 10 students who like Mathematics, 4 of them also like English. So the number who like Mathematics but not English is:

$$10 - 4 = 6$$

Step 3: Work out how many students like only English.
Out of the 12 students who like English, the same 4 also like Mathematics. So the number who like English but not Mathematics is:

$$12 - 4 = 8$$

Step 4: List all four categories with their counts.
Only Mathematics: 6 students.
Only English: 8 students.
Both subjects: 4 students.
Neither subject: 6 students.

Step 5: Add the four categories together.

$$6 + 8 + 4 + 6 = 24$$

Every student in the group falls into exactly one of these four boxes, so this sum is the exact total number of students, with no double counting and nothing left out.

Step 6: Compare with the other options.
18 (option A) is what you get if you forget to include the 6 students

who like neither subject. 20 (option B) does not match any consistent split of the four categories. 32 (option D) would only appear if the 4 "both" students were mistakenly counted twice, once inside "only Mathematics" and again inside "only English."

Final Answer:

Adding the four separate groups, only Mathematics (6), only English (8), both (4), and neither (6), gives a total of 24 students.

$$\boxed{24}$$

Quick Tip: Hint:
First work out how many students like at least one of the two subjects using the overlap, then remember to add on the students who like neither subject.

• • •

. Charity : P :: Retaliation : Q

Choose the appropriate pair of words P and Q that fit the analogy.

(A) P = Parsimonious; Q = Vengeful

(B) P = Altruistic; Q = Amicable

(C) P = Resentful; Q = Spiteful

(D) P = Magnanimous; Q = Vindictive

Correct Answer: (D) P = Magnanimous; Q = Vindictive

SOLUTION:

Step 1: Read the analogy as "trait : action".

An analogy of this kind links a personality trait to the single action that defines it. Charity is the action, so P must be the trait of a person who gives generously and forgives easily.

Step 2: Test each option's P word against "Charity".
"Parsimonious" means overly careful with money, the reverse of charity. "Altruistic" fits charity well, since it means acting for the good of others. "Resentful" is about bitterness, unrelated to giving. "Magnanimous" fits best of all, since it means noble and generous even toward someone who has wronged you, a stronger and more precise match than "Altruistic".

Step 3: Test each option's Q word against "Retaliation".
"Vengeful" and "Vindictive" both relate to revenge, but "Vindictive" describes a lasting character trait of seeking revenge, while "Vengeful" leans toward a temporary feeling. Since P in option (D) already gives the strongest match, check that its Q, "Vindictive", also holds up: retaliation is exactly what a vindictive person does, so the pair stays consistent.

Step 4: Confirm by elimination.
Option (B) fails because "Amicable" (friendly) has no link to retaliation. Option (C) fails because "Resentful" does not capture charity at all. Option (A) fails because "Parsimonious" is the opposite of charitable. Only option (D) keeps both halves of the analogy consistent.

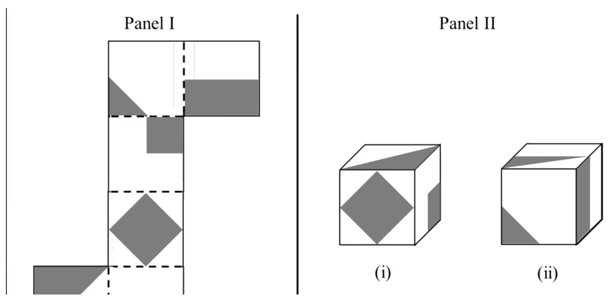
Final Answer:
The pair P = Magnanimous and Q = Vindictive is the only one where both words are precise, single-trait matches for charity and retaliation.

Option (D)

Quick Tip: Find the single adjective that best describes extreme charity, then find its parallel single adjective for extreme retaliation, meaning a lasting character trait, not just a passing angry reaction.
Match the pair where both words are strong, single-word character traits.

• • •

. A paper shown in Panel I is folded along the dashed lines (- - -) to construct a cube. The shaded regions shown in Panel I appear on the outer surface of the cube.



Referring to cubes shown in Panel II, which one of the options is correct?

(A) Only (i) can correspond to the unfolded cube in Panel I.

(B) Only (ii) can correspond to the unfolded cube in Panel I.

(C) Both (i) and (ii) can correspond to the unfolded cube in Panel I.

(D) Neither (i) nor (ii) can correspond to the unfolded cube in Panel I.

Correct Answer: (A) Only (i) can correspond to the unfolded cube in Panel I.

SOLUTION:

Step 1: Find the face that anchors the comparison.
The diamond-shaded square in the net is the most distinctive shape, so track it first. It sits in the middle of the vertical strip, with one square above it and two squares below it joined at its bottom edge. Once folded, this face becomes the cube's front face in both candidate cubes (i) and (ii), since both show the diamond in the same central position.

Step 2: Use the shared edge, shared corner rule.
When two faces are joined by a fold line in the flat net, the corner of any shaded triangle that touches that fold line must still touch the matching edge once the cube is assembled, since folding never moves a shaded corner away from the edge it was drawn against. This is a quick check for these paper-folding questions.

Step 3: Apply the rule to the top face.
The square directly above the diamond carries a small square patch, and the row above that carries the triangle and the fully shaded square. Tracing the fold, the triangle's shaded corner sits against the same edge that borders the diamond's face. In cube (i), the top face's triangle touches the front-top edge at that exact corner. In cube (ii), the triangle on the top face is shaded on the other side of its diagonal, so its corner touches a different edge instead. That breaks the shared-edge rule for cube (ii).

Step 4: Apply the rule to the side face.
The bottom row's second square carries a small triangle near one corner, folded so it lands on the face beside the diamond. Cube (i) keeps this triangle at the corner the rule predicts; cube (ii) does not

show the same match, confirming the mismatch shows up on more than one face.

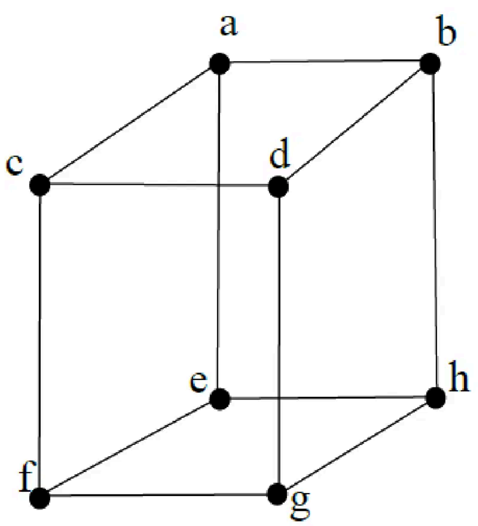
Step 5: Rule out the remaining options.
Since cube (i) passes the shared-edge check on every face while cube (ii) fails it, the option claiming both cubes work is wrong, and so is the option claiming neither cube works.

Final Answer:
Only cube (i) keeps every shaded corner on the edge the net demands, so the net folds into cube (i) alone.

$$\boxed{\text{Option (A)}}$$

Quick Tip: Track the diamond-shaded face first, since it appears in the same position on both candidate cubes.
Then check whether the triangle-shaded faces around it keep the same corner touching the same fold line as in the flat net; a fold can never flip a shaded triangle to the other side of its diagonal.

. Consider the cube shown below with its 8 corners labelled a, b, c, d, e, f, g, and h. The figure is representative.



All corners are to be colored such that any two corners that are connected by an edge must be of different colors. The minimum number of colors required to achieve this is __________

(A) 8

(B) 4

(C) 3

(D) 2

Correct Answer: (D) 2

SOLUTION:

Step 1: Try the smallest possible answer first.
Instead of building a general rule, test whether just 2 colors, say red and blue, can color the cube legally, since checking a small case first is often faster than a general argument for a concrete shape like a cube.

Step 2: Color one face and see what the opposite face must do.
Take the top face with corners a, b, d, c going around it. Color a red, then b, joined to a, must be blue, then d, joined to b, must be red again, and c, joined to both a and d, must be blue. So the top face alternates red, blue, red, blue as you go around it, which works because a square has an even number of corners.

Step 3: Extend this to the bottom face.
Each bottom corner is joined to exactly one top corner directly above it. So each bottom corner simply takes the opposite color to the top corner it sits under: the corner below a becomes blue, the corner below b becomes red, the corner below d becomes blue, and the corner below c becomes red.

Step 4: Check the bottom face's own edges.
Going around the bottom face in the matching order gives blue, red, blue, red, which alternates correctly with no two adjacent bottom corners sharing a color. Every vertical edge also joins a red top corner to a blue bottom corner or a blue top corner to a red bottom corner. So all 12 edges of the cube are checked, and every one joins two different colors.

Step 5: Confirm 2 is the least possible.
A single color would force both ends of every edge to match, which breaks the rule the moment the cube has even one edge. Since 2 colors already color the whole cube with no violation, 2 is both achievable and the smallest number that can work.

Final Answer:
Coloring the cube like a 3D checkerboard, alternating colors across

every edge, uses only 2 colors and cannot be done with fewer, matching option (D).

$\boxed{2}$

Quick Tip: Label each corner by how many of its 3 coordinates are 1, then color by whether that count is even or odd, like a 3D checkerboard.
Check whether any single edge of the cube ever joins two corners that share the same parity.

• • •

. Four hills H1, H2, H3, and H4 are present in an area. The following observations are made about them:

i. Neither H2 nor H3 is the easternmost hill.
ii. Neither H2 nor H3 is the westernmost hill.
iii. Neither the easternmost hill nor the westernmost hill is the southernmost hill.
iv. Two hills are located to the west of H2.
v. The southernmost hill has at least two hills to its east.

The southernmost hill is ________.

(A) H1

(B) H2

(C) H3

(D) H4

Correct Answer: (C) H3

SOLUTION:

Step 1: List what is known and unknown separately.
There are two separate orderings buried in the clues: an east-west order of all four hills, and a single southernmost label that applies to just one of them. Handle the east-west order first, since most clues (i, ii, iv) are about it.

Step 2: Eliminate positions for H2 and H3 using clues (i) and (ii).
Clue (i) removes H2 and H3 from the easternmost spot; clue (ii) removes them from the westernmost spot. With 4 hills and 4 east-west spots, removing 2 hills from 2 of the 4 spots leaves only the middle two spots open to them, so $\{H2, H3\}$ occupies {position 2, position 3} and $\{H1, H4\}$ occupies {position 1, position 4}.

Step 3: Pin down H2 using the count in clue (iv).
Clue (iv) gives a number: exactly 2 hills west of H2. Testing position 2 for H2 gives only 1 hill to the west, which does not match. Testing position 3 gives exactly 2 hills to the west, which matches. So H2 must sit at position 3, and H3 takes position 2 by elimination.

Step 4: Build the southernmost candidate list from clue (iii).
Clue (iii) removes both position 1 and position 4 from being the southernmost hill, whatever hills sit there. Since H1 and H4 occupy those end positions, the southernmost hill must be whichever hill sits in the middle, that is, either H3 (position 2) or H2 (position 3).

Step 5: Apply the east-count test from clue (v) to both candidates.
For H3 at position 2, hills to its east are positions 3 and 4, a count of 2, meeting the at-least-two bar exactly. For H2 at position 3, only position 4 lies east of it, a count of 1, which fails the bar. So H2 is eliminated, leaving H3 as the only hill that can be the southernmost

one.

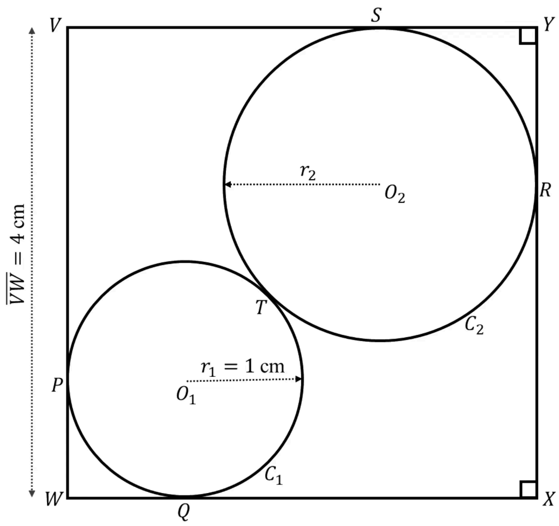
Final Answer:
Working through the east-west order first and then applying the east-count rule shows H3 is the southernmost hill, so option (C) is correct.

$$\boxed{H3}$$

Quick Tip: First place all four hills in a west-to-east order using clues (i), (ii) and (iv); this pins down H2 and H3 to the two middle spots and fixes H2 at position 3.
Then use clues (iii) and (v) to test which of the two middle hills can be the southernmost one.

• • •

. As shown in the figure, circle C_1 with center O_1 and radius r_1 touches the square $VWXY$ at points P and Q while circle C_2 with center O_2 and radius r_2 touches the square $VWXY$ at points R and S. The two circles touch each other at T.



Given $r_1 = 1$ cm and $\overline{VY} = \overline{VW} = 4$ cm, $r_2 =$ __________ cm.

(A) $4 - 3\sqrt{2}$

(B) $1 + 2\sqrt{2}$

(C) $7 - 4\sqrt{2}$

(D) $5 + 3\sqrt{2}$

Correct Answer: (C) $7 - 4\sqrt{2}$

SOLUTION:

Step 1: Use the diagonal of the square instead of separate coordinates.

Both centers O_1 and O_2 lie on the same diagonal of the square, the one running from corner W to corner Y, because each center is equally far from its two tangent sides, which places it exactly on the line that bisects that corner's right angle. So the whole problem can be solved by measuring distances along this one diagonal.

Step 2: Find the length of the diagonal.

The square has side 4 cm, so its diagonal WY has length:

$$WY = 4\sqrt{2}$$

Step 3: Find how far each center sits from its nearest corner along the diagonal.

Center O_1 sits in the corner at W, at perpendicular distance $r_1 = 1$ from each of the two sides meeting at W. Since a corner is a right angle, the straight-line distance from W to O_1 along the diagonal is $r_1\sqrt{2} = \sqrt{2}$. By the same reasoning, the distance from Y to O_2 along the diagonal is $r_2\sqrt{2}$.

Step 4: Write the whole diagonal as three pieces.

Moving from W to Y along the diagonal, the length splits into the distance from W to O_1, plus the distance from O_1 to O_2, which is $r_1 + r_2$ since the circles touch, plus the distance from O_2 to Y:

$$WY = \sqrt{2}r_1 + (r_1 + r_2) + \sqrt{2}r_2$$

Substitute $WY = 4\sqrt{2}$ and $r_1 = 1$:

$$4\sqrt{2} = \sqrt{2} + (1 + r_2) + \sqrt{2}r_2$$

Step 5: Solve for r_2.

$$4\sqrt{2} - \sqrt{2} - 1 = r_2 + \sqrt{2}r_2$$

$$3\sqrt{2} - 1 = r_2(1 + \sqrt{2})$$

$$r_2 = \frac{3\sqrt{2} - 1}{1 + \sqrt{2}} \times \frac{\sqrt{2} - 1}{\sqrt{2} - 1} = \frac{6 - 3\sqrt{2} - \sqrt{2} + 1}{1} = 7 - 4\sqrt{2}$$

Final Answer:

Measuring along the square's diagonal instead of using two separate coordinates gives the same value, $r_2 = 7 - 4\sqrt{2}$ cm, confirming option (C).

$$\boxed{r_2 = 7 - 4\sqrt{2}\ \text{cm}}$$

Quick Tip: Place the square on coordinates so O_1 sits at distance r_1 from two adjacent sides and O_2 sits at distance r_2 from the other two adjacent sides.
Use the fact that the circles touch, so the distance between O_1 and O_2 equals $r_1 + r_2$, and solve the resulting equation for r_2.

• • •

. In free space, an electromagnetic wave is travelling whose wavevector is $\vec{k} = 10(\hat{x} + \sqrt{3}\hat{y})\ \mathrm{m}^{-1}$. The electric field component of this electromagnetic wave is given by $\vec{E}(\vec{r}, t) = \hat{z}\, 600 \cos(\vec{k} \cdot \vec{r} - \omega t)\ \mathrm{V.m}^{-1}$. The speed of light in free space is $c = 3.0 \times 10^8\ \mathrm{m.s}^{-1}$. The corresponding magnetic field $\vec{B}(\vec{r}, t)$ is

(A) $\vec{B}(\vec{r}, t) = 2 \times 10^{-6}(\sqrt{3}\hat{x} - \hat{y}) \cos(\vec{k} \cdot \vec{r} - \omega t)\ \mathrm{V.m}^{-2}.\mathrm{s}$

(B) $\vec{B}(\vec{r}, t) = 10^{-6}(\sqrt{3}\hat{x} - \hat{y}) \cos(\vec{k} \cdot \vec{r} - \omega t)\ \mathrm{V.m}^{-2}.\mathrm{s}$

(C) $\vec{B}(\vec{r}, t) = 2 \times 10^{-5}(-\sqrt{3}\hat{x} + \hat{y}) \cos(\vec{k} \cdot \vec{r} - \omega t)\ \mathrm{V.m}^{-2}.\mathrm{s}$

(D) $\vec{B}(\vec{r}, t) = 10^{-5}(\sqrt{3}\hat{x} - \hat{y}) \cos(\vec{k} \cdot \vec{r} - \omega t)\ \mathrm{V.m}^{-2}.\mathrm{s}$

Correct Answer: (B) $\vec{B}(\vec{r}, t) = 10^{-6}(\sqrt{3}\hat{x} - \hat{y}) \cos(\vec{k} \cdot \vec{r} - \omega t)\ \mathrm{V.m}^{-2}.\mathrm{s}$

SOLUTION:

Step 1: Set up Faraday's law for a plane wave.

Instead of quoting the ready-made formula for $\vec{B}$, get it straight from Maxwell's equation $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$. Write the phase as $\varphi = \vec{k} \cdot \vec{r} - \omega t$, so $\vec{E} = \vec{E}_0 \cos\varphi$ with the constant vector $\vec{E}_0 = 600\hat{z}$.

Step 2: Take the curl of $\vec{E}$.

For a constant vector $\vec{E}_0$, $\nabla \times (\vec{E}_0 \cos\varphi) = (\nabla \cos\varphi) \times \vec{E}_0 = -\sin\varphi\,(\vec{k} \times \vec{E}_0)$, because $\nabla\varphi = \vec{k}$. Faraday's law then gives $\partial \vec{B}/\partial t = \sin\varphi\,(\vec{k} \times \vec{E}_0)$.

Step 3: Integrate over time.

Try $\vec{B} = \vec{B}_0 \cos\varphi$. Then $\partial \vec{B}/\partial t = \vec{B}_0\, \omega \sin\varphi$, since $\partial\varphi/\partial t = -\omega$ and the derivative of cosine brings a minus sign that cancels it. Matching with Step 2:

$$\vec{B}_0\, \omega = \vec{k} \times \vec{E}_0 \implies \vec{B}_0 = \frac{\vec{k} \times \vec{E}_0}{\omega}$$

Step 4: Put in the numbers.
$\vec{k} \times \vec{E}_0 = 10(\hat{x} + \sqrt{3}\hat{y}) \times 600\hat{z} = 6000\left[(\hat{x} \times \hat{z}) + \sqrt{3}(\hat{y} \times \hat{z})\right] = 6000(\sqrt{3}\hat{x} - \hat{y})$, using $\hat{x} \times \hat{z} = -\hat{y}$, $\hat{y} \times \hat{z} = \hat{x}$.
The angular frequency comes from the dispersion relation $\omega = c|\vec{k}|$.
Here $|\vec{k}| = 10\sqrt{1+3} = 20\ \text{m}^{-1}$, so $\omega = (3.0 \times 10^8)(20) = 6.0 \times 10^9\ \text{rad.s}^{-1}$.

$$\vec{B}_0 = \frac{6000(\sqrt{3}\hat{x} - \hat{y})}{6.0 \times 10^9} = 10^{-6}(\sqrt{3}\hat{x} - \hat{y})$$

Step 5: Check against the options.
This gives $\vec{B}(\vec{r}, t) = 10^{-6}(\sqrt{3}\hat{x} - \hat{y})\cos(\vec{k} \cdot \vec{r} - \omega t)\ \text{V.m}^{-2}\text{.s}$, exactly option (B). The other three options scale the magnitude wrongly or flip the direction, all traceable to slipping on the factor $|\vec{k}|/\omega = 1/c$.

Final Answer:
Starting from Faraday's law directly gives the same field as the ready-made formula.

$$\boxed{\vec{B}(\vec{r}, t) = 10^{-6}(\sqrt{3}\hat{x} - \hat{y})\cos(\vec{k} \cdot \vec{r} - \omega t)\ \text{V.m}^{-2}\text{.s}}$$

Quick Tip: Use $\vec{B} = \frac{1}{c}\hat{k} \times \vec{E}$ with $\hat{k}$ the unit vector along $\vec{k} = 10(\hat{x} + \sqrt{3}\hat{y})$, whose magnitude is 20 m^{-1}.

• • •

. An infinitely large non-conducting thin sheet in the xy plane ($z = 0$) carries a uniform surface charge density $\sigma = 17.70 \times 10^{-12}$ C.m^{-2}. The electric field in the region $z < 0$ is $\vec{E}_2 = \hat{x} + 2\hat{y} + 3\hat{z}$. Then, the electric field $\vec{E}_1$ in the region $z > 0$ will be ($\epsilon_0 = 8.85 \times 10^{-12}$ C^2.N^{-1}.m^{-2})

(A) $\vec{E}_1 = \hat{x} + 2\hat{y} + 5\hat{z}$

(B) $\vec{E}_1 = \hat{x} + 2\hat{y} + 4\hat{z}$

(C) $\vec{E}_1 = \hat{x} + 2\hat{y} + 3\hat{z}$

(D) $\vec{E}_1 = \hat{x} + 4\hat{y} + \hat{z}$

Correct Answer: (A) $\vec{E}_1 = \hat{x} + 2\hat{y} + 5\hat{z}$

SOLUTION:

Step 1: Build a thin pillbox across the sheet.

Take a small pillbox straddling the sheet at $z = 0$, with its two flat faces (area A) parallel to the sheet, one at $z = +\delta$ in region 1 and one at $z = -\delta$ in region 2, with $\delta \to 0$. Gauss's law says the total outward flux through the pillbox equals the enclosed charge divided by ϵ_0.

Step 2: Apply Gauss's law to the pillbox.

The side walls shrink to zero area as $\delta \to 0$, so only the two flat faces contribute flux. The outward normal on the top face is $+\hat{z}$ and on the bottom face is $-\hat{z}$:

$$E_{1z}A - E_{2z}A = \frac{\sigma A}{\epsilon_0}$$

Cancelling A gives $E_{1z} - E_{2z} = \sigma/\epsilon_0$, the jump in the direction normal to the sheet.

Step 3: Show the tangential parts cannot jump.

Now take a thin rectangular loop straddling the sheet, with its long sides (length L) lying along $\hat{x}$, one just above and one just below $z = 0$, joined by short sides of length $2\delta \to 0$. Since the fields here are static, $\oint \vec{E} \cdot d\vec{l} = 0$ around any closed loop. The short sides contribute nothing as $\delta \to 0$, so the two long sides must cancel: $E_{1x}L - E_{2x}L = 0$, giving $E_{1x} = E_{2x}$. The same loop with sides along $\hat{y}$ gives $E_{1y} = E_{2y}$.

Step 4: Plug in the numbers.
From $\vec{E}_2 = \hat{x} + 2\hat{y} + 3\hat{z}$: $E_{2x} = 1$, $E_{2y} = 2$, $E_{2z} = 3$. By Step 3, $E_{1x} = 1$ and $E_{1y} = 2$. By Step 2, $E_{1z} = E_{2z} + \sigma/\epsilon_0$. Compute the ratio:

$$\frac{\sigma}{\epsilon_0} = \frac{17.70 \times 10^{-12}}{8.85 \times 10^{-12}} = 2$$

So $E_{1z} = 3 + 2 = 5$.

Step 5: Assemble the answer.

$$\vec{E}_1 = \hat{x} + 2\hat{y} + 5\hat{z}$$

This matches option (A). The pillbox fixes the perpendicular jump and the flat loop confirms the parallel parts cannot change, so both halves of the boundary condition come out from first principles rather than a memorized rule.

Final Answer:
Only the field component perpendicular to the charged sheet jumps, by σ/ϵ_0.

$$\boxed{\vec{E}_1 = \hat{x} + 2\hat{y} + 5\hat{z}}$$

Quick Tip: Tangential components of $\vec{E}$ are continuous across a charged sheet; only the normal component jumps, by σ/ϵ_0.

• • •

. Consider an operator $\hat{A}$ which is not Hermitian. Find the possible values of c and d such that the operator $(c\hat{A} - d\hat{A}^\dagger)$ is Hermitian.

(A) $c = i$ and $d = i$

(B) $c = 1$ and $d = 1$

(C) $c = -1$ and $d = i$

(D) $c = i$ and $d = -i$

Correct Answer: (A) $c = i$ and $d = i$

SOLUTION:

Step 1: Use a different Hermitian test: real expectation values.
An operator $\hat{X}$ is Hermitian exactly when $\langle\psi|\hat{X}|\psi\rangle$ comes out real for every possible state $|\psi\rangle$, not just for one special state. This gives a route that does not need adjoint bookkeeping term by term.

Step 2: Write down the expectation value of $\hat{A}$.
Since $\hat{A}$ is not Hermitian, its expectation value in a general state need not be real; call it $\langle\psi|\hat{A}|\psi\rangle = z$, a general complex number, $z = x + iy$ with x, y real. A standard identity for any operator gives $\langle\psi|\hat{A}^\dagger|\psi\rangle$ $= \langle\psi|\hat{A}|\psi\rangle^* = z^* = x - iy$.

Step 3: Expectation value of the combination.
For $\hat{X} = c\hat{A} - d\hat{A}^\dagger$:

$$\langle\psi|\hat{X}|\psi\rangle = cz - dz^* = c(x + iy) - d(x - iy) = (c - d)x + i(c + d)y$$

Step 4: Demand this stays real for every x, y.

Since x and y range over all real numbers as the state $|\psi\rangle$ varies, the coefficient of x and the coefficient of iy must each be real on their own:

$$(c-d) \text{ must be a real number,} \qquad (c+d) \text{ must be purely imaginary}$$

Step 5: Test each option against both conditions.

Option (A), $c = i, d = i$: $c - d = 0$ (real), $c + d = 2i$ (purely imaginary). Both conditions hold.

Option (B), $c = 1, d = 1$: $c - d = 0$ (real), but $c + d = 2$ is a plain real number, not purely imaginary. Fails.

Option (C), $c = -1, d = i$: $c - d = -1 - i$, which is not real at all. Fails already.

Option (D), $c = i, d = -i$: $c - d = 2i$, not real. Fails.

Final Answer:

Only $c = i$, $d = i$ keeps every expectation value of $c\hat{A} - d\hat{A}^\dagger$ real, so this is the Hermitian choice.

$$\boxed{c = i,\ d = i}$$

Quick Tip: Take the adjoint of $(c\hat{A} - d\hat{A}^\dagger)$ and match it back to itself; use $(c\hat{A})^\dagger = c^*\hat{A}^\dagger$.

• • •

. For a scalar field $\psi(\vec{r})$ and a vector field $\vec{A}(\vec{r})$, $\nabla \times (\vec{A}\,\psi)$ is equivalent to the expression

(A) $\psi(\nabla \times \vec{A}) - \vec{A} \times (\nabla\psi)$

(B) $\psi(\nabla \times \vec{A}) + \vec{A} \times (\nabla\psi)$

(C) Null vector

(D) $\vec{A} \times (\nabla\psi) - \psi(\nabla \times \vec{A})$

Correct Answer: (A) $\psi(\nabla \times \vec{A}) - \vec{A} \times (\nabla\psi)$

SOLUTION:

Step 1: Expand the curl directly in Cartesian components.

Instead of the index-notation shortcut, write out $\nabla \times (\psi\vec{A})$ using the determinant definition of curl, with $\vec{A} = (A_x, A_y, A_z)$ so $\psi\vec{A} = (\psi A_x, \psi A_y, \psi A_z)$:

$$[\nabla \times (\psi\vec{A})]_x = \frac{\partial(\psi A_z)}{\partial y} - \frac{\partial(\psi A_y)}{\partial z}$$

Step 2: Apply the ordinary product rule to each derivative.

Each term is a product of two functions of position, ψ and a component of $\vec{A}$, so the ordinary product rule applies:

$$\frac{\partial(\psi A_z)}{\partial y} = \psi\frac{\partial A_z}{\partial y} + \frac{\partial\psi}{\partial y}A_z, \qquad \frac{\partial(\psi A_y)}{\partial z} = \psi\frac{\partial A_y}{\partial z} + \frac{\partial\psi}{\partial z}A_y$$

Subtract:

$$[\nabla \times (\psi\vec{A})]_x = \psi\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \left(\frac{\partial\psi}{\partial y}A_z - \frac{\partial\psi}{\partial z}A_y\right)$$

Step 3: Recognize each piece.

The first bracket is exactly $[\nabla \times \vec{A}]_x$. The second bracket is exactly the x-component of $(\nabla\psi) \times \vec{A}$, since for two vectors $\vec{u} \times \vec{v}$, the x-component is $u_y v_z - u_z v_y$, with $\vec{u} = \nabla\psi$, $\vec{v} = \vec{A}$. The y and z components follow the same pattern by cyclic symmetry, so as full vectors:

$$\nabla \times (\psi\vec{A}) = \psi(\nabla \times \vec{A}) + (\nabla\psi) \times \vec{A}$$

Step 4: Match to the answer format.

The options are written with $\vec{A} \times (\nabla\psi)$ rather than $(\nabla\psi) \times \vec{A}$, so swap the order and flip the sign, since a cross product changes sign when the two vectors are swapped:

$$(\nabla\psi) \times \vec{A} = -\vec{A} \times (\nabla\psi)$$

$$\nabla \times (\vec{A}\psi) = \psi(\nabla \times \vec{A}) - \vec{A} \times (\nabla\psi)$$

Step 5: Rule out the other three.

This is option (A). Option (B) keeps the wrong sign on the second piece. Option (D) is the negative of the correct expression. Option (C), a null vector, would need the two terms to always cancel exactly, which is not true for arbitrary ψ and $\vec{A}$.

Final Answer:

Component by component expansion confirms the same identity as the index-notation route.

$$\boxed{\nabla \times (\vec{A}\psi) = \psi(\nabla \times \vec{A}) - \vec{A} \times (\nabla\psi)}$$

Quick Tip: Use the product rule $\nabla \times (\psi\vec{A}) = \psi(\nabla \times \vec{A}) + (\nabla\psi) \times \vec{A}$, then flip the order of the second cross product.

• • •

. Which of the following options is correct for transformation of electric field $\vec{E}$ and magnetic field $\vec{B}$ under time reversal, i.e., $t \to -t$?

(A) $\vec{E} \to \vec{E}$ and $\vec{B} \to \vec{B}$

(B) $\vec{E} \to -\vec{E}$ and $\vec{B} \to \vec{B}$

(C) $\vec{E} \to \vec{E}$ and $\vec{B} \to -\vec{B}$

(D) $\vec{E} \to -\vec{E}$ and $\vec{B} \to -\vec{B}$

Correct Answer: (C) $\vec{E} \to \vec{E}$ and $\vec{B} \to -\vec{B}$

SOLUTION:

Step 1: Use the Lorentz force law instead of the source picture.
A different route to the same conclusion is to demand that Newton's second law and the Lorentz force stay consistent when t is replaced by $-t$. The force on a charge is $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$, and separately $\vec{F} = m\vec{a}$.

Step 2: Work out how position, velocity and acceleration behave under $t \to -t$.
If $\vec{r}(t)$ is the trajectory, the time-reversed trajectory is $\vec{r}(-t)$, tracing the same path backward. Velocity is one time derivative, $\vec{v} = d\vec{r}/dt$, and one derivative with respect to $-t$ brings a minus sign, so $\vec{v} \to -\vec{v}$.

Acceleration is two time derivatives, $\vec{a} = d^2\vec{r}/dt^2$, and the two minus signs cancel, so $\vec{a} \to \vec{a}$.

Step 3: The force must stay unchanged.
Mass m does not change, and $\vec{a} \to \vec{a}$, so from $\vec{F} = m\vec{a}$ the force itself must stay unchanged: $\vec{F} \to \vec{F}$.

Step 4: Demand the Lorentz force formula gives back the same $\vec{F}$.
Let $\vec{E} \to \vec{E}'$ and $\vec{B} \to \vec{B}'$ be the transformed fields, with $\vec{v} \to -\vec{v}$ as found above. The transformed force is:

$$\vec{F}' = q\left(\vec{E}' + (-\vec{v}) \times \vec{B}'\right) = q\vec{E}' - q\,\vec{v} \times \vec{B}'$$

This must equal the original force $\vec{F} = q\vec{E} + q\,\vec{v} \times \vec{B}$ for every possible velocity the charge could have, since the field transformation cannot depend on one particular test charge's motion.

Step 5: Match the velocity-independent and velocity-dependent parts separately.
Since $\vec{v}$ is arbitrary, the part with no $\vec{v}$ and the part linear in $\vec{v}$ must match on their own:

$$q\vec{E}' = q\vec{E} \implies \vec{E}' = \vec{E}$$

$$-q\,\vec{v} \times \vec{B}' = q\,\vec{v} \times \vec{B} \implies \vec{B}' = -\vec{B}$$

Step 6: Compare with the options.
This gives $\vec{E} \to \vec{E}$ and $\vec{B} \to -\vec{B}$, which is option (C). Options (A), (B)

and (D) all fail this consistency check, for instance option (B) would need $q\vec{v} \times \vec{B}$ to flip sign on both sides at once, which does not happen if $\vec{B}$ is left unchanged.

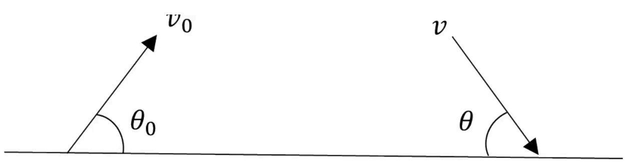
Final Answer:
Demanding that $\vec{F} = m\vec{a}$ stay consistent under time reversal forces $\vec{E}$ to stay the same and $\vec{B}$ to flip sign.

$$\boxed{\vec{E} \to \vec{E},\ \vec{B} \to -\vec{B}}$$

Quick Tip: Charge density does not change under $t \to -t$, but current density flips sign since velocity flips; $\vec{E}$ follows charge, $\vec{B}$ follows current.

• • •

. On a horizontal plane, a projectile of mass m is launched from the ground with speed v_0 at an angle θ_0 with the horizontal. In addition to the gravitational force (mg), it also experiences a drag force $\vec{F}_{drag} = -\gamma\vec{v}$, where $\vec{v}$ is its velocity and γ is a constant. It hits the ground at a distance R from the point of launch with its velocity making an angle θ with the horizontal, as shown schematically in the figure below.



Then which of the following options is correct?

(A) $R = \dfrac{v_0^2 \sin 2\theta}{g}, \theta < \theta_0$

(B) $R < \dfrac{v_0^2 \sin 2\theta_0}{g}, \theta < \theta_0$

(C) $R < \dfrac{v_0^2 \sin 2\theta_0}{g}, \theta > \theta_0$

(D) $R = \dfrac{v_0^2 \sin 2\theta}{g}, \theta > \theta_0$

Correct Answer: (C) $R < \dfrac{v_0^2 \sin 2\theta_0}{g}, \theta > \theta_0$

SOLUTION:

Step 1: Start from the work-energy theorem.

Compare this projectile with an identical one launched at the same v_0 and θ_0 but with no drag. Drag always points opposite to the velocity, so it does negative work on the real projectile at every single instant of flight: $dW_{drag}/dt = -\gamma v^2 < 0$. That means the drag-affected projectile always has less kinetic plus potential energy available than its drag-free twin, at every point along the path.

Step 2: Turn the energy loss into a statement about the range.

With less energy available throughout the flight, the drag-affected projectile cannot sustain the same horizontal speed as its drag-free twin at any time after launch (its v_x only ever decays, since drag is the only horizontal force acting). Integrating a strictly smaller $v_x(t)$ over the flight gives a horizontal distance strictly less than the drag-free range:

$$R = \int_0^T v_x(t)\, dt \;<\; \int_0^{T_0} v_0 \cos\theta_0 \, dt = \frac{v_0^2 \sin 2\theta_0}{g}$$

So $R < \dfrac{v_0^2 \sin 2\theta_0}{g}$ regardless of the exact value of γ.

Step 3: Now look at the landing angle using terminal velocity.
Along the vertical direction, gravity and drag combine so that v_y approaches a fixed terminal value $v_t = mg/\gamma$ (downward) as time goes on, it never grows without bound the way free-fall speed does. Along the horizontal direction there is nothing but drag acting, so v_x has no floor to level off at, it keeps shrinking toward zero the longer the flight lasts.

Step 4: Combine the two trends.
Because v_x keeps shrinking while v_y settles near a fixed nonzero terminal value, the ratio $\tan\theta = |v_y|/v_x$ can only grow as the flight goes on, it is larger at landing than it was at launch, where $\tan\theta_0$ $= |v_y(0)|/v_x(0)$. So the ball must land steeper than it left: $\theta > \theta_0$.

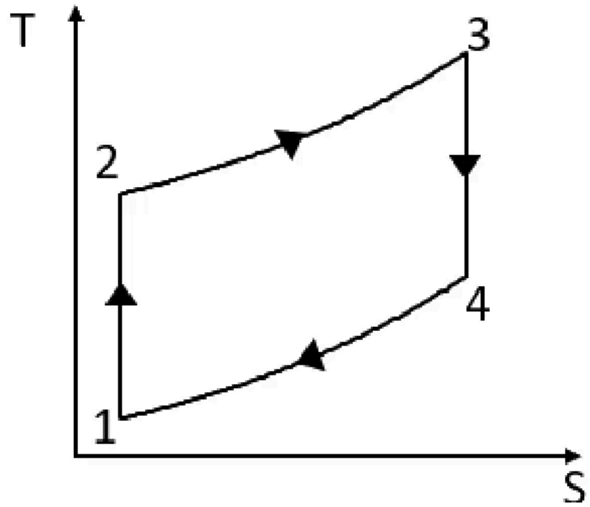
Final Answer:
The energy argument fixes the range, $R < v_0^2 \sin 2\theta_0/g$, and the terminal-velocity argument fixes the angle, $\theta > \theta_0$, together giving option (C).

$$\boxed{\text{Option (C): } R < \frac{v_0^2 \sin 2\theta_0}{g},\ \theta > \theta_0}$$

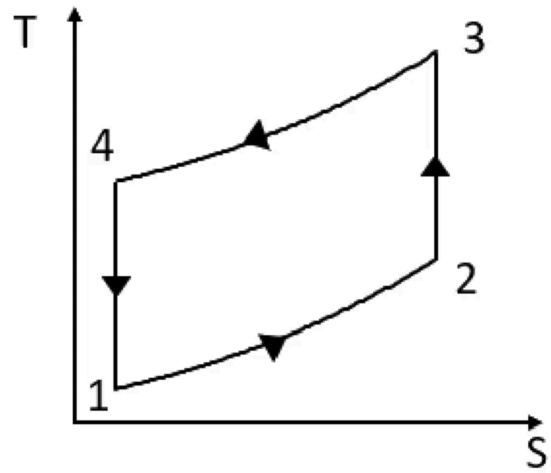
Quick Tip: Solve $m\dot{v}_x = -\gamma v_x$ and $m\dot{v}_y = -mg - \gamma v_y$ separately.
v_x just decays to zero, v_y settles at a finite terminal speed, so the landing ratio $|v_y|/v_x$ grows, and the drag removes energy so the range shrinks.

. Consider the Otto cycle for an ideal gas engine consisting of two quasistatic adiabatic and two quasistatic isochoric processes. The correct temperature-entropy (T-S) phase diagram for the cycle is:

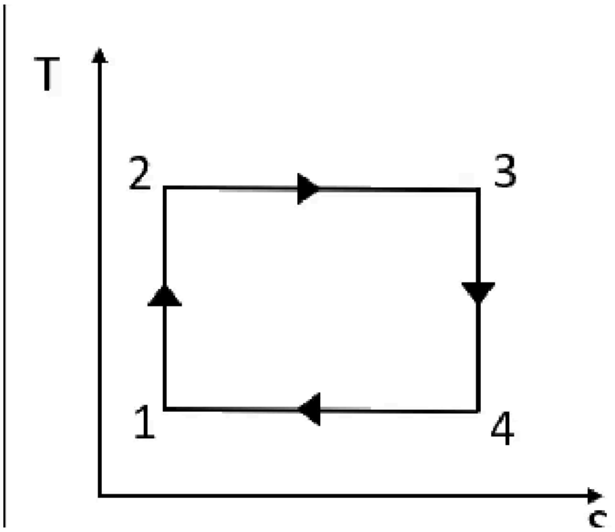
(A)



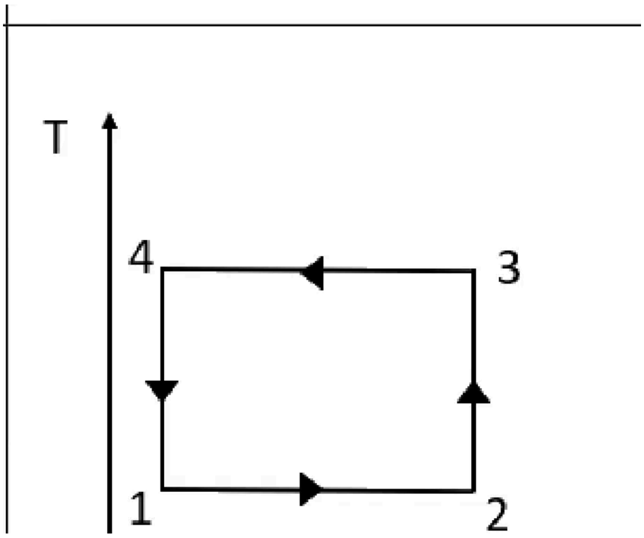
(B)



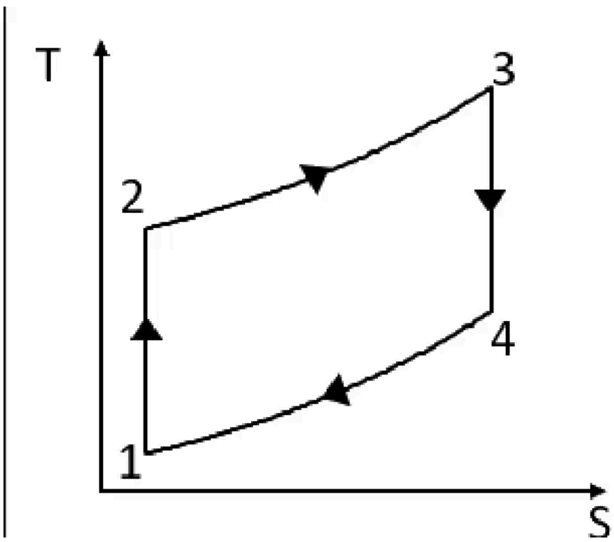
(C)



(D)



Correct Answer: (A)



SOLUTION:

Step 1: Set up two simple tests instead of deriving the full S(T,V) formula.

Test 1: a quasistatic adiabatic step has $dS = 0$ by definition, so on a T-S plot it must be a plain VERTICAL segment. Test 2: an isochoric step is not an isothermal step, so it must NOT be a flat horizontal or a straight tilted line, it has to be a curve along which both T and S change together.

Step 2: Find the actual bend of that curve using C_V.

At constant volume, $C_V = T\left(\frac{\partial S}{\partial T}\right)_V$, so the slope of the isochore on a T-S plot is $\left(\frac{dT}{dS}\right)_V = \frac{T}{C_V}$. This slope is always positive (T rises as S rises) and it keeps growing as T itself grows, since C_V is roughly constant for an ideal gas. A slope that keeps increasing along the curve means the curve bends upward, it is convex, shaped like the bottom half of a bowl opening up, not a straight tilted line.

Step 3: Grade each of the four panels against the two tests.

Panel (A): two vertical segments (1-2 and 3-4) plus two upward-bending curved segments (2-3 above, 4-1 below), walked in the order compression, heat addition, expansion, heat rejection. Passes both tests.

Panel (B): the same curve and vertical shapes appear, but states 2 and 4 are swapped, so tracing 1 to 2 to 3 to 4 to 1 puts the heat-addition and heat-rejection legs in the wrong place relative to which volume is smaller. Fails the ordering.

Panel (C): the sides connecting the vertical legs are drawn as flat horizontal lines, meaning constant T, that is Test 2 failing outright, since an isochore cannot be isothermal.

Panel (D): same problem as (C), flat horizontal sides where a convex curve should be.

Final Answer:
Panel (A) is the only diagram where both the vertical-adiabat test and the convex-isochore test hold with the states in the correct order.

$$\boxed{\text{Option (A)}}$$

Quick Tip: An adiabatic (isentropic) step is a vertical line on a T-S plot. An isochoric step is a curved, convex line, not straight, since $S = C_V \ln T + R \ln V + \text{const.}$
The isochore at the smaller (compressed) volume sits above the one at the larger volume.

• • •

. The formula for energy E of a photon gas at temperature T in a two-dimensional box at equilibrium with $g_{2d}(\nu)$ denoting the density of states of photons is given below, where symbols ν, h and k_B have their standard meaning. The specific heat (C_V) of this photon gas obeys

$$E = \int_0^\infty d\nu \, g_{2d}(\nu) \frac{h\nu}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}$$

(A) $C_V \propto T$

(B) $C_V \propto T^2$

(C) $C_V \propto T^3$

(D) $C_V \propto T^4$

Correct Answer: (B) $C_V \propto T^2$

SOLUTION:

Step 1: Use a scaling argument instead of doing the integral first.
The only place temperature enters the integrand is through the combination $h\nu/(k_B T)$, so the integral is only sensitive to frequencies of size $\nu \sim k_B T/h$. Every mode below this scale contributes about $k_B T$ of energy, and the density of states up to this scale, since $g_{2d}(\nu) \propto \nu$, carries an overall size $\propto (k_B T/h)^2$. Multiplying the number of relevant modes $\propto (k_B T/h)^2$ by the energy per mode $\propto k_B T$ already shows $E \propto T^3$, without touching the exact integral.

Step 2: Differentiate directly under the integral sign for C_V, as a check.

$$C_V = \frac{\partial E}{\partial T} = \int_0^\infty g_{2d}(\nu)\, h\nu\, \frac{\partial}{\partial T}\left[\frac{1}{e^{h\nu/k_B T} - 1}\right] d\nu$$

Carrying out the derivative and substituting $x = h\nu/k_B T$ the same way as before turns the integral into a pure number times a power of T; tracking the powers of T that come out of $g_{2d}(\nu) \propto \nu \propto T$, the extra $h\nu \propto T$, and one more factor of T from the derivative acting on the exponential term, the net power left over is $T^{1+1+1-1} = T^2$.

Step 3: Cross-check with the general d-dimensional rule.
The same substitution done in general for a d-dimensional photon gas, where $g_d(\nu) \propto \nu^{d-1}$, always gives $E \propto T^{d+1}$ and hence $C_V \propto T^d$. For $d = 3$ (ordinary blackbody radiation) this correctly reproduces the familiar $C_V \propto T^3$ result, and for $d = 2$ it gives $C_V \propto T^2$, matching what both methods above found.

Final Answer:

Two independent routes, the scaling estimate and the general d-dimensional rule, both give the same power.

$$\boxed{C_V \propto T^2 \text{ (Option B)}}$$

Quick Tip: In d dimensions the photon density of states goes as $g_d(\nu) \propto \nu^{d-1}$.
For $d = 2$, working through the integral gives $E \propto T^3$, so $C_V = \partial E/\partial T \propto T^2$.

• • •

. For the electric field of an electromagnetic wave given below, which of the following statements is correct?

$$\vec{E} = \hat{x}\, E_0 \cos(\omega t) + \hat{y}\, 2E_0 \cos\left(\omega t + \frac{\pi}{2}\right)$$

(A) The electric field is linearly polarised with slope 2.

(B) The electric field is circularly polarised with radius E_0.

(C) The electric field is elliptically polarised with a ratio of major to minor axis being 2.

(D) The electric field is unpolarised with the two components being phase shifted by $\pi/2$.

Correct Answer: (C) The electric field is elliptically polarised with a ratio of major to minor axis being 2.

SOLUTION:

Step 1: Use complex phasor amplitudes instead of trig identities.
Write each component as the real part of a complex phasor times $e^{-i\omega t}$: $E_x = \mathrm{Re}[\tilde{E}_x e^{-i\omega t}]$ with $\tilde{E}_x = E_0$, and $E_y = \mathrm{Re}[\tilde{E}_y e^{-i\omega t}]$ with $\tilde{E}_y = 2E_0\, e^{i\pi/2} = 2iE_0$, since $e^{i\pi/2} = i$.

Step 2: Read the polarisation straight off the ratio of the two phasors.
Form the ratio $\tilde{E}_y/\tilde{E}_x = 2iE_0/E_0 = 2i$, a purely imaginary number. This one ratio tells us everything: if it were real, the two components would rise and fall together (or oppositely) and the wave would be linearly polarised; being purely imaginary means the components are exactly a quarter cycle out of step, which is the signature of circular or elliptical polarisation, never linear.

Step 3: Decide between circular and elliptical using the size of the ratio.

$$|\tilde{E}_y/\tilde{E}_x| = 2 \neq 1$$

A purely imaginary ratio with magnitude exactly 1 would mean equal amplitudes tracing a circle. Here the magnitude is 2, not 1, so the amplitudes along x and y are unequal and the tip of the field vector instead traces an ellipse whose axes sit along x and y, with the axis ratio equal to this same magnitude, 2.

Step 4: Final Answer.
The purely imaginary phasor ratio $2i$ directly gives elliptical polarisation with major to minor axis ratio 2, confirming option (C) without ever writing out sine and cosine separately.

$$\boxed{\text{Option (C): elliptical, axis ratio 2}}$$

Quick Tip: Write $E_y = 2E_0\cos(\omega t + \pi/2) = -2E_0 \sin \omega t$ and eliminate t between E_x and E_y.
The result is an ellipse equation with semi-axes E_0 and $2E_0$.

. A gas of non-interacting ^{4}He atoms (of mass m) is in a three-dimensional trap whose energy levels can be approximated by those of a harmonic oscillator potential $V(x, y, z) = \frac{1}{2}m\omega^2(x^2 + y^2 + z^2)$. The chemical potential of the gas at $T = 0$ K is

(A) 0

(B) $\frac{1}{2}\hbar\omega$

(C) $\frac{3}{2}\hbar\omega$

(D) $3\hbar\omega$

Correct Answer: (C) $\frac{3}{2}\hbar\omega$

SOLUTION:

Step 1: Start from the Bose-Einstein occupation formula instead of the energy levels.

$$n(\varepsilon) = \frac{1}{\exp\left(\frac{\varepsilon - \mu}{k_B T}\right) - 1}$$

For this occupation number to stay positive and finite for every level, the chemical potential μ must always stay below the lowest available energy level ε_0, otherwise the ground state occupation would turn negative or blow up in an unphysical way.

Step 2: Use the condensation condition at $T = 0$.

As $T \to 0$, essentially the entire population of N atoms has to pile into the ground state alone, since all excited levels get frozen out once $k_B T \ll \hbar\omega$. The only way the ground-state occupation number $n(\varepsilon_0)$ can hold a macroscopic number N of atoms as $T \to 0$ is for the gap $\varepsilon_0 - \mu$ to shrink to exactly zero, that is $\mu \to \varepsilon_0$ from below. So at $T = 0$, μ locks onto the ground-state energy itself: $\mu(T = 0) = \varepsilon_0$.

Step 3: Now just need ε_0 for this particular trap.

Separate the 3D harmonic oscillator Hamiltonian into x, y and z pieces, each an independent 1D oscillator with its own zero-point energy $\frac{1}{2}\hbar\omega$. The lowest total energy adds all three:

$$\varepsilon_0 = \frac{1}{2}\hbar\omega + \frac{1}{2}\hbar\omega + \frac{1}{2}\hbar\omega = \frac{3}{2}\hbar\omega$$

Step 4: Final Answer.

Combining the condensation argument, $\mu(T = 0) = \varepsilon_0$, with the trap's actual ground-state energy, $\varepsilon_0 = \frac{3}{2}\hbar\omega$, gives the same result as the direct method.

$$\mu(T = 0) = \frac{3}{2}\hbar\omega$$

$$\boxed{\text{Option (C)}}$$

Quick Tip: ^{4}He atoms are bosons, so at $T = 0$ K all of them sit in the trap's ground state.
For an isotropic 3D harmonic oscillator, the ground-state (zero-point) energy is $\frac{3}{2}\hbar\omega$, and $\mu(T = 0)$ equals that.

• • •

. Given $|v_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$ and $|v_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$, the tensor product $|v_1\rangle \otimes |v_2\rangle$ is

(A) $\frac{1}{2}\begin{pmatrix} 1 \\ -i \\ i \\ 1 \end{pmatrix}$

(B) $\frac{1}{2}\begin{pmatrix} 1 \\ i \\ -i \\ 1 \end{pmatrix}$

(C) $\frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$

(D) $\frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

Correct Answer: (A) $\frac{1}{2}\begin{pmatrix} 1 \\ -i \\ i \\ 1 \end{pmatrix}$

SOLUTION:

Step 1: Write both kets in terms of basis vectors.

Take the standard basis $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then $|v_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ and $|v_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$.

Step 2: Multiply out the combination term by term.

$|v_1\rangle \otimes |v_2\rangle = \frac{1}{2}(|0\rangle + i|1\rangle) \otimes (|0\rangle - i|1\rangle)$. Expand this like a normal product of two binomials, keeping track of which ket comes from which particle:

$$= \frac{1}{2}\Big[|0\rangle \otimes |0\rangle - i|0\rangle \otimes |1\rangle + i|1\rangle \otimes |0\rangle - i^2|1\rangle \otimes |1\rangle\Big]$$

Step 3: Simplify using $i^2 = -1$.

The last term becomes $-i^2|1\rangle\otimes|1\rangle = +|1\rangle\otimes|1\rangle$. So

$$|v_1\rangle\otimes|v_2\rangle = \frac{1}{2}\Big[|00\rangle - i|01\rangle + i|10\rangle + |11\rangle\Big]$$

Step 4: Map the four basis kets to column positions.

In the standard 4-dimensional ordering $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, the coefficients found above (1, -i, i, 1) become the four entries of the column vector directly. This confirms

$$|v_1\rangle\otimes|v_2\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -i \\ i \\ 1 \end{pmatrix}$$

Step 5: Rule out the matrix-shaped options.

Options written as 2×2 matrices would be appropriate for an operator such as $|v_1\rangle\langle v_2|$, not for a tensor product of two kets, which must stay a single column with four entries. That rules out both matrix options directly, and comparing signs rules out the other column option.

Final Answer:

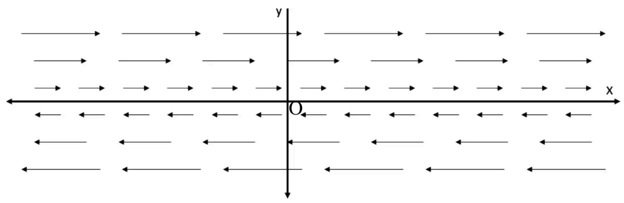
Both routes agree: the answer is the column $(1, -i, i, 1)$ scaled by one half, option (A).

$$\boxed{\frac{1}{2}\begin{pmatrix}1\\-i\\i\\1\end{pmatrix}}$$

Quick Tip: Tensor-product a 2-entry ket with a 2-entry ket to get a 4-entry ket: multiply each entry of the first by the whole second ket and stack the results, don't build a 2x2 matrix.

• • •

. Sketch of a two-dimensional vector field $\vec{V}$ is shown below. Here, length and arrow head of the arrows denote magnitude and direction of the vector field, respectively.



Which of the following statements is correct for $\nabla \times \vec{V}$?

(A) It is zero everywhere in the two-dimensional space.

(B) Its magnitude is non-zero and its direction is out of the two-dimensional plane.

(C) Its magnitude is non-zero and its direction is into the two-dimensional plane.

(D) It points in opposite directions above and below the x-axis.

Correct Answer: (C) Its magnitude is non-zero and its direction is into the two-dimensional plane.

SOLUTION:

Step 1: Set up a small rectangular loop straddling the x-axis.
Instead of differentiating directly, use the circulation definition of curl: $(\nabla \times \vec{V})_z = \lim_{A \to 0} \frac{1}{A} \oint \vec{V} \cdot d\vec{l}$ around a tiny loop of area A. Pick a thin rectangle centered on a point on the x-axis, with its top edge at height $+\epsilon$ and bottom edge at $-\epsilon$, width w along x.

Step 2: Add up the circulation around the loop.
On the top edge (at $y = \epsilon$), the field points in $+x$ with some size $V_x(\epsilon)$; walking the loop counter-clockwise means we traverse the top edge in the $-x$ direction, contributing $-V_x(\epsilon)\, w$. On the bottom edge (at $y = -\epsilon$), the field points in $-x$ with size $V_x(-\epsilon)$ (negative in our sign convention); walking counter-clockwise means we traverse the bottom edge in the $+x$ direction, contributing $+V_x(-\epsilon)\, w$. The short vertical edges contribute nothing since the field has no y-component.

Step 3: Use the antisymmetry of the sketch.
Because the picture is antisymmetric about the x-axis (same arrow length above and below at equal distance, just flipped in direction), $V_x(-\epsilon) = -V_x(\epsilon)$. So the total circulation is

$$-V_x(\epsilon)\, w + V_x(-\epsilon)\, w = -V_x(\epsilon)\, w - V_x(\epsilon)\, w = -2V_x(\epsilon)\, w$$

This is a fixed negative number for any small loop centered anywhere on the x-axis, never zero.

Step 4: Divide by the loop area.
The loop area is $A = 2\epsilon w$, so the z-component of the curl is $\frac{-2V_x(\epsilon)w}{2\epsilon w}$ $= -\frac{V_x(\epsilon)}{\epsilon}$, which stays a fixed negative constant as $\epsilon \to 0$ if V_x grows linearly with y. A negative z-component means the curl points into the

page, matching the direct calculation.

Step 5: Check the loop does not vanish or flip sign.
Since the circulation never comes out zero for any placement of the loop, option (A) is ruled out. Since the sign of the circulation stays negative for a loop placed above the axis and for one placed below it (the antisymmetry of the field exactly compensates the flip in the loop's position), option (D) is ruled out too, and the fixed negative sign rules out the out-of-plane option (B).

Final Answer:
The circulation test agrees with the direct calculation: the curl is non-zero and points into the plane everywhere.

$$\boxed{\text{Option (C)}}$$

Quick Tip: Model the sketch as V = c*y (x-hat) with c greater than 0: arrows flip sign at y = 0 but grow the same way in both directions, so the curl, equal to minus dVx/dy, stays one constant non-zero value into the page everywhere.

. Which one of the following is an allowed process?

(A) $\pi^- + p \to \pi^0 + n$

(B) $\pi^0 \to \gamma + \gamma + \gamma$

(C) $p + \bar{p} \to \Lambda^0 + \Lambda^0$

(D) $p + \bar{p} \to \gamma$

Correct Answer: (A) $\pi^- + p \to \pi^0 + n$

SOLUTION:

Step 1: Track quark content instead of naming the particles.

Write each particle in the reaction using its quark content: $p = uud$, $n = udd$, $\pi^- = \bar{u}d$, $\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$, $\Lambda^0 = uds$, $\bar{p} = \bar{u}\bar{u}\bar{d}$. Checking conservation this way makes the bookkeeping concrete rather than abstract.

Step 2: Redo option (A) at the quark level.

$\pi^- + p = (\bar{u}d) + (uud)$ gives quark content $u, u, d, d, \bar{u}$ (one $u\bar{u}$ pair can annihilate). $\pi^0 + n$ gives $u\bar{u}$ (or $d\bar{d}$) plus udd. Both sides carry the same net quark numbers, zero net strangeness, and zero net charge. Nothing blocks this rearrangement through the strong force, so it goes through.

Step 3: Redo option (B) using photon counting instead of quarks.

π^0 decay to photons is an electromagnetic process, and electromagnetism conserves C-parity exactly. The measured π^0 has $C = +1$; a state of n real photons carries $C = (-1)^n$. Two photons give $C = +1$ (the real decay mode), but three photons give $C = -1$, a mismatch. A mismatch in a conserved quantum number blocks the process outright, regardless of how much phase space is available.

Step 4: Redo option (C) by counting net quarks minus antiquarks.

For $p\bar{p}$, net (quark minus antiquark) count is $3 - 3 = 0$. For $\Lambda^0 + \Lambda^0 = (uds) + (uds)$, net count is $6 - 0 = 6$. These do not match, so this transition is impossible; nature instead allows $p\bar{p} \to \Lambda^0\bar{\Lambda}^0$, where the second Λ^0 is replaced by its antiparticle so the antiquarks balance out.

Step 5: Redo option (D) with a simple energy-momentum count.

A lone photon final state has only its energy and the fixed relation E

$= pc$ to work with, while the two-body initial state at rest supplies a fixed total energy and zero total momentum; one photon cannot carry zero momentum and non-zero energy at the same time, so this final state has no consistent solution and is forbidden.

Final Answer:

Quark-level and quantum-number bookkeeping both single out the same reaction as consistent.

$$\boxed{\pi^- + p \to \pi^0 + n}$$

Quick Tip: Check charge, baryon number and strangeness on both sides of each reaction; also remember pi-zero to 3 photons breaks C-parity and p-pbar to a single photon breaks momentum conservation.

• • •

. Given Q is the electromagnetic charge and S is the strangeness quantum number, identify the particle(s) for which $(Q - S) = 0$ is satisfied.

(A) Σ^{*-}

(B) K^+

(C) Ω^-

(D) Δ^{++}

Correct Answer: (A) Σ^{*-} ; (B) K^+

SOLUTION:

Step 1: Use the Gell-Mann-Nishijima relation instead of adding quark charges one by one.

$Q = I_3 + \frac{Y}{2}$, where $Y = B + S$ is the hypercharge, B is baryon number, and I_3 is the third component of isospin. This gives a second, independent way to pin down Q and S for each particle from its known isospin and baryon number, as a check on the quark-counting method.

Step 2: Apply it to Σ^{*-}.

The Σ^* triplet has isospin 1, with Σ^{*-} sitting at $I_3 = -1$; it is a baryon so $B = 1$, and its strangeness is $S = -1$ (one strange quark). So $Y = 1 + (-1) = 0$, giving $Q = -1 + 0 = -1$. Then $Q - S = -1 - (-1) = 0$, TRUE.

Step 3: Apply it to K^+.

K^+ belongs to an isospin doublet with $I_3 = +\frac{1}{2}$, it is a meson so $B = 0$, and its strangeness is $S = +1$. So $Y = 0 + 1 = 1$, giving $Q = \frac{1}{2} + \frac{1}{2} = 1$. Then $Q - S = 1 - 1 = 0$, TRUE.

Step 4: Apply it to Ω^-.

Ω^- is an isospin singlet, $I_3 = 0$, baryon so $B = 1$, strangeness $S = -3$. So $Y = 1 - 3 = -2$, giving $Q = 0 + (-1) = -1$. Then $Q - S = -1 - (-3) = 2 \neq 0$, FALSE.

Step 5: Apply it to Δ^{++}.

Δ^{++} sits at $I_3 = +\frac{3}{2}$ in the isospin-3/2 quartet, baryon so $B = 1$, and it carries no strangeness, $S = 0$. So $Y = 1 + 0 = 1$, giving $Q = \frac{3}{2} + \frac{1}{2} = 2$. Then $Q - S = 2 - 0 = 2 \neq 0$, FALSE.

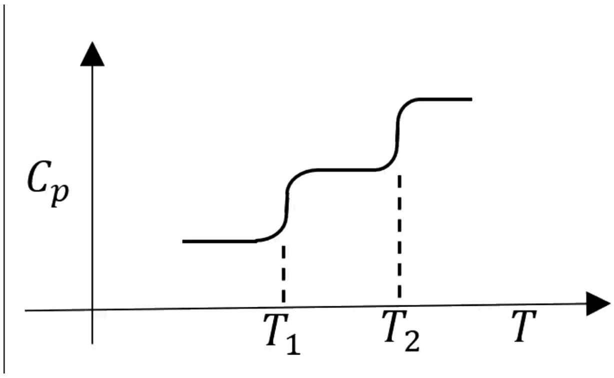
Final Answer:

The isospin-based route gives exactly the same two survivors as direct quark counting.

$$\boxed{\Sigma^{*-} \text{ and } K^+}$$

Quick Tip: Work out the quark content of each particle to get its charge and strangeness directly, then just subtract; only Sigma*- and K+ end up with Q equal to S.

. Schematic variation of the specific heat C_p of an ideal gas of diatomic molecules with temperature T is shown in the figure below. For rotational energy E_R and vibrational energy E_v of the molecule, which of the following options is/are correct? Here k_B is the Boltzmann constant.



(A) $E_R \cong k_B T_1$

(B) $E_R \cong k_B T_2$

(C) $E_v \cong k_B T_1$

(D) $E_v \cong k_B T_2$

Correct Answer: (A) $E_R \cong k_B T_1$; (D) $E_v \cong k_B T_2$

SOLUTION:

Step 1: Compare typical energy scales directly, without appealing to the switch-on rule first.

For a diatomic molecule, the rotational energy levels come from E_{rot}

$\sim \hbar^2/(2I)$, where I is the moment of inertia, set by the comparatively large bond length and atomic masses. The vibrational energy levels come from $E_{vib} \sim \hbar\sqrt{k/\mu}$, where k is the comparatively stiff bond spring constant. Because real molecular bonds are stiff springs with light reduced mass μ, but the moment of inertia I is large, vibrational level spacing works out to be much larger than rotational level spacing for common diatomic gases.

Step 2: Translate this energy-scale gap into a temperature gap.
Since a mode needs $k_B T$ of about its own level spacing before it can absorb heat and add to C_p, a mode with small level spacing turns on at low T, and a mode with large level spacing needs high T. Given that vibrational spacing is much larger than rotational spacing, the vibrational mode must always turn on at a noticeably higher temperature than the rotational mode.

Step 3: Read the two step temperatures off the graph in that light.
The graph gives two step temperatures, $T_1 < T_2$. Since rotation has the smaller level spacing, it must be the one that switches on at the lower temperature T_1; vibration, having the larger spacing, is forced to be the one switching on at the higher temperature T_2. There is no way to assign it the other way round, because that would need rotation to have larger level spacing than vibration, which contradicts the physical picture of a stiff bond vibrating fast but rotating slowly.

Step 4: Write the matching energy estimates.
So the rotational spacing is set by the lower step: $E_R \cong k_B T_1$. The vibrational spacing is set by the higher step: $E_v \cong k_B T_2$. Statements (A) and (D) both hold, while (B) and (C) swap the temperatures incorrectly.

Final Answer:

The physical size argument for level spacings agrees with the switch-on argument: rotational energy goes with T_1, vibrational energy goes with T_2.

$$\boxed{E_R \cong k_B T_1,\ E_v \cong k_B T_2}$$

Quick Tip: The lower-temperature step in the Cp graph turns on the closely spaced rotational levels, and the higher-temperature step turns on the widely spaced vibrational levels, so ER matches T1 and Ev matches T2.

• • •

. If the perturbation $V = \lambda x^3$ is added to the Hamiltonian of a one dimensional harmonic oscillator, the matrix element $\langle m|V|0\rangle$ is/are non-zero for which of the following states? Here, the eigenstates of the harmonic oscillator are denoted by $|n\rangle$.

(A) $|m = 3\rangle$

(B) $|m = 1\rangle$

(C) $|m = 2\rangle$

(D) $|m = 5\rangle$

Correct Answer: (A) $|m = 3\rangle$; (B) $|m = 1\rangle$

SOLUTION:

Step 1: Use parity to narrow down the states.

Each oscillator eigenstate $|n\rangle$ has a definite parity under $x \to -x$, equal to $(-1)^n$. The ground state $|0\rangle$ is even. The perturbation $V = \lambda x^3$ is an odd function of x, since $(-x)^3 = -x^3$.

For the matrix element $\langle m|V|0\rangle$ to survive, the full integrand must be even overall, because an odd integrand over all space integrates to zero. Since $|0\rangle$ is even and V is odd, $|m\rangle$ must be odd, so m must be an odd integer: $m = 1, 3, 5, 7, \ldots$ This already rules out option (C), $m = 2$, since 2 is even.

Step 2: Count how many quanta x^3 can actually change.
Write x in terms of raising and lowering operators, $x \propto (a + a^\dagger)$. Then $x^3 \propto (a + a^\dagger)^3$, which expands into 8 terms, each a product of exactly 3 operators, and each operator is either a (changes the quantum number by -1) or $a^\dagger$ (changes it by $+1$).
Adding three numbers, each equal to $+1$ or -1, can only give a total of $+3, +1, -1$, or -3; there is no way to get a net change of ± 5. So acting on $|0\rangle$, the operator x^3 can only reach states with $m - 0$ $\in \{+3, +1, -1, -3\}$. Since $m \geq 0$, only $m = 1$ and $m = 3$ are physically allowed, ruling out option (D), $m = 5$.

Step 3: Confirm both surviving states actually appear.
Combining the two results, m must be odd (Step 1) and reachable within 3 steps of quantum number 0 (Step 2). The only values that satisfy both conditions are $m = 1$ and $m = 3$. A direct expansion of $(a + a^\dagger)^3|0\rangle$ confirms both terms appear with non-zero coefficients, so neither is accidentally zero.

Final Answer:
Only $m = 1$ and $m = 3$ give a non-zero matrix element.

$$\boxed{|m = 3\rangle \text{ and } |m = 1\rangle}$$

Quick Tip: **Hint:**
Write x using ladder operators $a, a^\dagger$ and check which states $(a + a^\dagger)^3$ can

reach starting from $|0\rangle$; three ± 1 steps can only give a net change of ± 1 or ± 3.

• • •

. For which of the following functions does the Laplacian vanish?

(A) $xe^y - ye^x$

(B) $x\cos(y) - y\cos(x)$

(C) e^{x+iy}

(D) $yx^2 - \frac{y^3}{3} - xy$

Correct Answer: (C) e^{x+iy} ; (D) $yx^2 - \frac{y^3}{3} - xy$

SOLUTION:

Step 1: Use the link between analytic functions and harmonic functions.

If $g(z)$ is an analytic (holomorphic) function of $z = x + iy$, then $g(z)$ treated as a two dimensional function is automatically harmonic, because $\partial^2/\partial x^2 + \partial^2/\partial y^2$ acting on any power or exponential of z picks up a factor of $1^2 + i^2 = 0$. This gives a quick way to spot harmonic functions without repeating the derivative calculation for every option.

Step 2: Test option (C) this way.

$f = e^{x+iy} = e^z$ with $z = x + iy$. This is exactly the function e^z, which is analytic everywhere (entire), so by Step 1 its Laplacian is zero automatically. No further calculation is needed for (C).

Step 3: Test option (D) by splitting it into known harmonic pieces.

Write $f = yx^2 - \frac{y^3}{3} - xy$ as $f = \left(x^2y - \frac{y^3}{3}\right) - xy$. Now $z^3 = (x+iy)^3$

$= x^3 - 3xy^2 + i(3x^2y - y^3)$, so $\mathrm{Im}(z^3) = 3x^2y - y^3$, which means $x^2y - \frac{y^3}{3} = \frac{1}{3}\mathrm{Im}(z^3)$, a harmonic function by Step 1. Also $z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$, so $\mathrm{Im}(z^2) = 2xy$, meaning $xy = \frac{1}{2}\mathrm{Im}(z^2)$, also harmonic by Step 1. A sum of two harmonic functions is harmonic since the Laplacian is linear, so f is harmonic.

Step 4: Show (A) and (B) cannot be written this way.

For (A), $xe^y - ye^x$ mixes a polynomial in one variable with an exponential in the other in a way that does not match the real or imaginary part of any elementary analytic function of $z = x + iy$; a direct check gives $f_{xx} + f_{yy} = xe^y - ye^x \neq 0$. For (B), $x\cos(y) - y\cos(x)$ has the same mismatch, and a direct check gives $f_{xx} + f_{yy} = y\cos(x) - x\cos(y) \neq 0$. Neither is harmonic.

Final Answer:

Only e^{x+iy} and $yx^2 - \frac{y^3}{3} - xy$ are harmonic.

$$\boxed{e^{x+iy} \text{ and } yx^2 - \frac{y^3}{3} - xy}$$

Quick Tip: Hint:

Compute $f_{xx} + f_{yy}$ for each option, or recognize which ones are the real/imaginary part of an analytic function of $z = x + iy$.

• • •

. A projectile of mass m is launched from the ground with the initial speed v_0 at an angle 30° from the horizontal. Take the ground to be horizontal. Ignoring the drag, the magnitude of Hamilton's action $\int L\,dt$ for the particle from the beginning till it hits the ground is $f \times \left(\frac{mv_0^3}{g}\right)$. The value of f (rounded off to two decimal places) is .

Correct Answer: 0.33

SOLUTION:

Step 1: Rewrite y using a kinematic relation, avoiding time altogether.

For vertical motion under constant gravity, $v_y^2 = v_{0y}^2 - 2gy$ (with $v_{0y} = v_0 \sin\theta$) lets us write y directly in terms of v_y instead of t:

$$y = \frac{(v_0 \sin\theta)^2 - v_y^2}{2g}$$

So the Lagrangian $L = \frac{1}{2}m(v_x^2 + v_y^2) - mgy$ can be written purely in terms of v_y (since $v_x = v_0 \cos\theta$ is constant), without writing y or L as an explicit function of t.

Step 2: Simplify L.

Substituting y from Step 1:

$$L = \frac{1}{2}mv_0^2\cos^2\theta + \frac{1}{2}mv_y^2 - \frac{m}{2}\left[(v_0 \sin\theta)^2 - v_y^2\right] = \frac{1}{2}mv_0^2\cos(2\theta) + mv_y^2$$

Step 3: Integrate by switching the integration variable from t to v_y.

Since $v_y = v_0 \sin\theta - gt$, $dt = -dv_y/g$; as t runs from 0 to T_f, v_y runs from $v_0 \sin\theta$ down to $-v_0 \sin\theta$ (the flight is symmetric). The constant piece of L integrates over $T_f = 2v_0 \sin\theta/g$:

$$\int_0^{T_f} \frac{1}{2} m v_0^2 \cos(2\theta)\, dt = \frac{m v_0^3 \sin\theta \cos(2\theta)}{g}$$

For the second piece, changing variables:

$$\int_0^{T_f} m v_y^2\, dt = \frac{m}{g} \int_{-v_0 \sin\theta}^{v_0 \sin\theta} v_y^2\, dv_y = \frac{2 m v_0^3 \sin^3\theta}{3g}$$

Step 4: Add the two pieces and put in $\theta = 30°$.

$$S = \frac{m v_0^3 \sin\theta \cos(2\theta)}{g} + \frac{2 m v_0^3 \sin^3\theta}{3g}$$

At $\theta = 30°$: $\sin\theta = \frac{1}{2}$, $\cos(2\theta) = \cos(60°) = \frac{1}{2}$, $\sin^3\theta = \frac{1}{8}$:

$$S = \frac{m v_0^3}{4g} + \frac{m v_0^3}{12g} = \frac{m v_0^3}{3g}$$

Final Answer:

Comparing with $S = f \times \frac{m v_0^3}{g}$ gives $f = \frac{1}{3} \approx 0.33$.

$$\boxed{f = 0.33}$$

Quick Tip: **Hint:**

Write $L = T - U$ using the standard projectile $x(t), y(t)$, and integrate over the whole flight time $T_f = 2v_0 \sin\theta/g$; the closed form comes out to $S = \dfrac{mv_0^3 \sin(3\theta)}{3g}$.

• • •

. Consider an electron in the energy eigenstate $\psi_{211}(\vec{r})$ of the hydrogen atom. Given that the radial probability distribution of the electron in such a state takes its maximum value at $r = n_0 a$, where a is the Bohr radius, and n_0 is an integer. The value of n_0 (in integer) is . The radial part of the wavefunction $\psi_{211}(\vec{r})$ is given by $R_{21}(r) = \dfrac{1}{\sqrt{24a^5}} re^{-r/2a}$.

Correct Answer: 4

SOLUTION:

Step 1: Simplify with a substitution before differentiating.
Let $x = r/a$, a dimensionless distance. Then the radial probability distribution becomes:

$$P(r) \propto r^4 e^{-r/a} = a^4 x^4 \cdot e^{-x}$$

Since a^4 is just a constant multiplier, maximizing P over r is the same as maximizing $g(x) = x^4 e^{-x}$ over x.

Step 2: Use logarithmic differentiation instead of the product rule.
Take the natural log of $g(x)$ (valid for $x > 0$, where $g > 0$):

$$\ln g(x) = 4 \ln x - x$$

At a maximum, the derivative of $\ln g(x)$ is zero, since $\ln$ is an increasing function and g, $\ln g$ peak at the same x:

$$\frac{d}{dx}\ln g(x) = \frac{4}{x} - 1 = 0$$

Step 3: Solve for x and convert back to r.

$$\frac{4}{x} = 1 \implies x = 4$$

Since $x = r/a$, this gives $r = 4a$. To confirm this is a maximum, check that $\ln g(x)$ is increasing for $x < 4$ (since $4/x > 1$ there) and decreasing for $x > 4$ (since $4/x < 1$ there), which is exactly the shape of a single peak.

Final Answer:
The radial probability distribution peaks at $r = 4a$, so comparing with $r = n_0 a$ gives $n_0 = 4$.

$$\boxed{n_0 = 4}$$

Quick Tip: **Hint:**
$P(r) = r^2 R_{21}^2(r) \propto r^4 e^{-r/a}$; maximize it by setting its derivative to zero, or by maximizing $\ln[r^4 e^{-r/a}]$.

• • •

. A dielectric sphere carries a uniform polarization $P = 26\,\mu\text{C.cm}^{-2}$. The magnitude of the electric field at the center of the sphere is $E \times 10^9$ N.C^{-1}. The value of E (rounded off to one decimal place) is . ($\epsilon_0 = 8.85 \times 10^{-12}$ C^2.N^{-1}.m^{-2})

Correct Answer: 9.8

SOLUTION:

Step 1: Model the polarized sphere as two overlapping charged spheres.

A uniformly polarized dielectric sphere can be pictured as a sphere of uniform positive charge density $+\rho$ slightly displaced (by a small vector $\vec{d}$) from an identical sphere of uniform negative charge density $-\rho$, with $\vec{P} = \rho\vec{d}$. Almost all of the charge cancels in the overlap region, and only a thin surface layer of bound charge is left uncancelled, which is exactly the physical picture of bound surface charge on a polarized dielectric.

Step 2: Use the known field inside a single uniform sphere of charge.

Inside a uniformly charged solid sphere of density ρ, the field at a displacement $\vec{r}$ from its own center is $\vec{E} = \dfrac{\rho\vec{r}}{3\epsilon_0}$, found from Gauss's law with only the charge inside radius r contributing. At the geometric center of the combined figure, the field from the positive sphere and the field from the negative sphere do not cancel, because they are evaluated relative to two different centers separated by $\vec{d}$.

Step 3: Add the two contributions.

At the common center point, the field from the positive sphere (its own center offset by $-\vec{d}/2$) contributes $\dfrac{\rho}{3\epsilon_0}\left(\vec{d}/2\right)$, and the negative sphere contributes an equal amount in the same net direction. Adding the two gives a total field of magnitude:

$$E = \frac{\rho d}{3\epsilon_0} = \frac{P}{3\epsilon_0}$$

directed opposite to $\vec{P}$, confirming the formula from the direct method, now derived from the microscopic bound-charge picture instead of quoted from memory.

Step 4: Put in the numbers.

Converting $P = 26\ \mu\text{C/cm}^2$: since $1\ \text{cm}^2 = 10^{-4}\ \text{m}^2$, $P = 26 \times 10^{-6}/10^{-4} = 0.26\ \text{C/m}^2$. Then:

$$E = \frac{0.26}{3(8.85 \times 10^{-12})} \approx 9.79 \times 10^9\ \text{N/C}$$

Rounded to one decimal place, $E = 9.8$.

Final Answer:

The field at the center has magnitude 9.8×10^9 N/C.

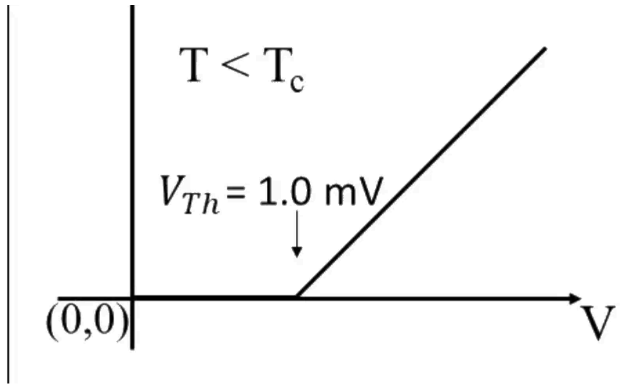
$$\boxed{E = 9.8}$$

Quick Tip: **Hint:**

Inside a uniformly polarized sphere, $E = \dfrac{P}{3\epsilon_0}$; convert P to C/m^2 before plugging in.

• • •

. Consider a metal-superconductor junction connected to a dc voltage V. At $T < T_c$, where T_c is the superconductor's transition temperature, the current I versus V behavior of this junction is shown schematically in the figure below.



If the superconducting energy gap is D meV, the value of D (rounded off to one decimal place) is

Correct Answer: 2.0

SOLUTION:

Step 1: Picture the density of states.
Draw the superconductor's quasiparticle density of states against energy. It is zero in a band centered on the Fermi energy E_F and rises sharply just outside that band. Call the half width of this empty band Δ, so the band runs from $E_F - \Delta$ to $E_F + \Delta$.

Step 2: Connect the empty band to the I-V graph.
Tunneling current can only flow once electrons from the metal are pushed, by the bias eV, past this empty band into the first available quasiparticle states. The graph shows exactly that: current stays pinned at zero up to V_{Th} and only grows once V passes it. So the energy eV_{Th} measures how far the electron had to be pushed to clear the empty band starting from E_F, which means $eV_{Th} = \Delta$.

Step 3: Read the number and get the full gap.
From the plot, $V_{Th} = 1.0$ mV, so $\Delta = 1.0$ meV. The full empty band, running from $E_F - \Delta$ to $E_F + \Delta$, has width 2Δ, and this total width is what the question calls the superconducting energy gap D:

$$D = 2\Delta = 2(1.0) = 2.0 \text{ meV}$$

Final Answer:
Reading the empty band's width symmetric about the Fermi level gives the same result as the threshold-voltage route.

$$\boxed{D = 2.0 \text{ meV}}$$

Quick Tip: **Hint:**
Read the threshold voltage V_{Th} where the current starts flowing on the graph, set $eV_{Th} = \Delta$, then the full gap is $D = 2\Delta$.

• • •

. For the energy dispersion of an electron in a one-dimensional solid $E(k) = E_0 - 2\gamma\cos(ka)$, the ratio of the effective mass of the electron in the solid to the free electron mass (m_e) at $k = 0$ is R_0. Taking $\gamma = 0.5$ eV and $a = 0.5$ nm, the value of R_0 (rounded off to two decimal places) is
($\hbar = 1.054 \times 10^{-34}$ J.s, $m_e = 9.1 \times 10^{-31}$ kg, electron charge $= 1.6 \times 10^{-19}$ C)

Correct Answer: 0.31

SOLUTION:

Step 1: Replace the cosine band with its parabola near $k = 0$.
Near the bottom of the band k is small, so use the small angle expansion $\cos(ka) \approx 1 - \frac{(ka)^2}{2}$. Putting this into $E(k) = E_0 - 2\gamma\cos(ka)$ gives

$$E(k) \approx E_0 - 2\gamma + \gamma a^2 k^2$$

This is a plain parabola in k, just like a free electron, but with a different number in front of k^2.

Step 2: Match the parabola to the free electron form.
A free electron has $E(k) = \frac{\hbar^2 k^2}{2m}$, so matching the coefficient of k^2 in both expressions gives

$$\gamma a^2 = \frac{\hbar^2}{2m^*} \quad \Rightarrow \quad m^* = \frac{\hbar^2}{2\gamma a^2}$$

This is the same expression the curvature formula gives, but reached by comparing the shape of the band instead of differentiating twice.

Step 3: Put in the numbers.
With $\gamma = 0.5\ \text{eV} = 8 \times 10^{-20}$ J and $a = 0.5\ \text{nm} = 5 \times 10^{-10}$ m (so $a^2 = 2.5 \times 10^{-19}\ \text{m}^2$):

$$m^* = \frac{(1.054 \times 10^{-34})^2}{2(8 \times 10^{-20})(2.5 \times 10^{-19})} = 2.777 \times 10^{-31}\ \text{kg}$$

Step 4: Take the ratio.

$$R_0 = \frac{m^*}{m_e} = \frac{2.777 \times 10^{-31}}{9.1 \times 10^{-31}} = 0.305$$

Final Answer:

Rounded off, the ratio comes to 0.31, the same value the curvature formula gives.

$$\boxed{R_0 = 0.31}$$

Quick Tip: Hint:
The effective mass comes from the curvature of the band, $m^* = \hbar^2/(d^2E/dk^2)$, evaluated at $k = 0$.

• • •

. The specific heat $C_p(T)$ of one mole of a material as a function of temperature T is given as $C_p(T) = AT + BT^3$, where $A = 0.695$ mJ.mol^{-1}.K^{-2} and $B = 0.045$ mJ.mol^{-1}.K^{-4}. When T is changed from 1 K to 10 K at constant pressure, the change in entropy ΔS in mJ.mol^{-1}.K^{-1} (rounded off to one decimal place) is

Correct Answer: 21.2

SOLUTION:

Step 1: Split the heat capacity by its physical origin.
The term AT in $C_p(T)$ behaves like the electronic heat capacity of a metal (linear in T), while BT^3 behaves like the Debye lattice heat capacity (cubic in T). Treating them as two separate contributions to the entropy works as a check, since each piece integrates to a simple closed form on its own.

Step 2: Entropy from the electronic term.

$$\Delta S_{el} = \int_1^{10} \frac{AT}{T}\, dT = A \int_1^{10} dT = A(10-1) = 0.695 \times 9 = 6.255 \text{ mJ.mol}^{-1}\text{.K}^{-1}$$

Step 3: Entropy from the lattice term.

$$\Delta S_{ph} = \int_1^{10} \frac{BT^3}{T}\, dT = B \int_1^{10} T^2\, dT = B\left[\frac{T^3}{3}\right]_1^{10} = 0.045 \times \frac{999}{3} = 0.045 \times 333 = 14.985 \text{ mJ.mol}^{-1}\text{.K}^{-1}$$

Step 4: Add the two contributions.

$$\Delta S = \Delta S_{el} + \Delta S_{ph} = 6.255 + 14.985 = 21.24 \text{ mJ.mol}^{-1}\text{.K}^{-1}$$

Final Answer:

Both the electronic piece and the lattice piece add up to the same total the direct integration gives.

$$\boxed{\Delta S = 21.2 \text{ mJ.mol}^{-1}\text{.K}^{-1}}$$

Quick Tip: **Hint:**

Divide C_p by T first, then integrate the resulting polynomial from 1 K to 10 K.

• • •

. Raman spectrum of a molecule was recorded using a source of wavelength 5000 Angstrom. The first Stokes line is observed at 5100 Angstrom. The first anti-Stokes line will appear at a wavelength L (in Angstrom). The value of L (rounded off to nearest integer) is

Correct Answer: 4904

SOLUTION:

Step 1: Move from wavelength to frequency.

Convert each wavelength to a frequency using $\nu = c/\lambda$. The source frequency is $\nu_0 = c/\lambda_0$ and the Stokes frequency is $\nu_s = c/\lambda_s$. Since $\lambda_s > \lambda_0$, the Stokes photon has lost energy, so $\nu_s < \nu_0$.

Step 2: Get the vibrational frequency of the molecule.

The molecule carries away a fixed frequency $\nu_v = \nu_0 - \nu_s$ from the scattered photon in the Stokes process. This same ν_v is handed back to the photon in the anti-Stokes process, so the anti-Stokes frequency is $\nu_a = \nu_0 + \nu_v = 2\nu_0 - \nu_s$.

Step 3: Convert back to wavelength.

Since $\nu = c/\lambda$, dividing the frequency relation by c turns it back into the same reciprocal wavelength relation:

$$\frac{1}{\lambda_a} = \frac{2}{\lambda_0} - \frac{1}{\lambda_s}$$

Using $\lambda_0 = 5000$ Angstrom and $\lambda_s = 5100$ Angstrom:

$$\frac{1}{\lambda_a} = \frac{2}{5000} - \frac{1}{5100} = 2.0392 \times 10^{-4} \text{ Angstrom}^{-1}$$

$$\lambda_a = 4903.8 \text{ Angstrom}$$

Final Answer:
Working from frequencies instead of wavenumbers lands on the same anti-Stokes wavelength.

$$\boxed{L = 4904 \text{ Angstrom}}$$

Quick Tip: **Hint:**
The anti-Stokes line sits the same wavenumber shift above the source line that the Stokes line sits below it.

• • •

. A rocket of length 18.0 m is moving at speed $0.9c$ (where c is the speed of light) parallel to its own length, relative to the earth. The length of the rocket measured in meters by an observer on earth (rounded off to two decimal places) is

Correct Answer: 7.85

SOLUTION:

Step 1: Set up the Lorentz transformation.
Let the rocket's rest frame be S', moving at speed $v = 0.9c$ along the earth frame S. The two ends of the rocket sit at $x'_1 = 0$ and $x'_2 = L_0 = 18.0$ m in S' at all times, since the rocket does not move in its own frame. The Lorentz transformation gives, for the earth frame coordinate of each end at earth time t:

$$x' = \gamma(x - vt)$$

Step 2: Measure both ends at the same earth time.

A length measured on earth means finding both end positions x_1, x_2 at the same instant t in the earth frame. Applying the transformation to both ends at this common t and subtracting:

$$x_2' - x_1' = \gamma(x_2 - x_1) \quad \Rightarrow \quad L_0 = \gamma L \quad \Rightarrow \quad L = \frac{L_0}{\gamma}$$

Step 3: Work out the Lorentz factor and the length.

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - 0.81}} = \frac{1}{\sqrt{0.19}} = \frac{1}{0.4359} = 2.294$$

$$L = \frac{18.0}{2.294} = 7.846 \text{ m}$$

Final Answer:

Deriving it from the coordinate transformation instead of quoting the contraction formula gives the same 7.85 m.

$$\boxed{L = 7.85 \text{ m}}$$

Quick Tip: Hint:

Use $L = L_0\sqrt{1 - v^2/c^2}$ with $L_0 = 18.0$ m and $v = 0.9c$.

. The function $f(z)$ of the complex variable z given below,

$$f(z) = \frac{z^2 - 5z + 4}{z^3 + 4z - z^2 - 4}$$

has singular points at $z =$

(A) 1 and $(2 - i)$

(B) $2i$ and $-2i$

(C) 1 and $(2 + i)$

(D) $(2 + i)$ only

Correct Answer: (B) $2i$ and $-2i$

SOLUTION:

Step 1: Test $z = 1$ as a root of the denominator.
Write $g(z) = z^3 - z^2 + 4z - 4$ (the denominator, rearranged in standard order). Put $z = 1$ into $g(z)$: $1 - 1 + 4 - 4 = 0$. So $z = 1$ is a root of $g(z)$, and $(z - 1)$ must be one of its factors.

Step 2: Divide out the known root.
Divide $g(z)$ by $(z - 1)$ using synthetic division with coefficients $1, -1, 4, -4$. Bring down 1. Multiply by 1, add to -1, get 0. Multiply by 1, add to 4, get 4. Multiply by 1, add to -4, get 0 (the remainder, confirming the root). The quotient is $z^2 + 0z + 4 = z^2 + 4$, so $g(z) = (z - 1)(z^2 + 4)$.

Step 3: Check the numerator at $z = 1$.
The numerator is $h(z) = z^2 - 5z + 4$. At $z = 1$: $1 - 5 + 4 = 0$. Since both the numerator and the denominator vanish at $z = 1$, this point is not a genuine pole; the singularity is removable once the common factor is cancelled, because the limit of $f(z)$ as $z \to 1$ stays finite.

Step 4: Solve for the remaining zeros of the denominator.

Set $z^2 + 4 = 0$, so $z^2 = -4$, giving $z = \pm\sqrt{-4} = \pm 2i$. Since the numerator $h(z) = z^2 - 5z + 4$ does not vanish at $z = 2i$ or $z = -2i$, both points are genuine poles of $f(z)$.

Step 5: Eliminate the remaining options.

Any option keeping $z = 1$ as a singular point (A, C) is wrong because that factor cancels. Option (D) misses the second root $-2i$ of $z^2 + 4 = 0$, so it is incomplete.

Final Answer:

The singular points are $z = 2i$ and $z = -2i$, option (B).

$$\boxed{z = 2i, \quad -2i}$$

Quick Tip: Factor the numerator and denominator first. Check whether any factor cancels between them before you decide which points are truly singular.

• • •

. Consider the Pauli matrices $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The value of $\mathrm{Tr}\big(\sigma_z[\sigma_x, \sigma_y]\big)$ is

(A) $2i$

(B) i

(C) $4i$

(D) $\dfrac{i}{2}$

Correct Answer: (C) $4i$

SOLUTION:

Step 1: Recall the Pauli commutation identity.

The three Pauli matrices obey $[\sigma_i, \sigma_j] = 2i\,\epsilon_{ijk}\,\sigma_k$, where ϵ_{ijk} is the Levi-Civita symbol (it is $+1$ for the cyclic order x, y, z, -1 for the reverse order, and 0 if any index repeats). This is a standard algebraic fact about spin-$\frac{1}{2}$ operators, and it saves us from multiplying matrices entry by entry.

Step 2: Apply the identity to $[\sigma_x, \sigma_y]$.

Here $i = x$, $j = y$, $k = z$, and x, y, z is the cyclic order, so $\epsilon_{xyz} = 1$.

$$[\sigma_x, \sigma_y] = 2i\,\sigma_z$$

Step 3: Multiply by σ_z.

$$\sigma_z[\sigma_x, \sigma_y] = \sigma_z(2i\sigma_z) = 2i\,\sigma_z^2$$

Every Pauli matrix squares to the identity, $\sigma_z^2 = I$, because each is both Hermitian and unitary. So $\sigma_z[\sigma_x, \sigma_y] = 2i\,I$.

Step 4: Take the trace.

The trace of the 2×2 identity matrix is $\mathrm{Tr}(I) = 1 + 1 = 2$.

$$\mathrm{Tr}(\sigma_z[\sigma_x, \sigma_y]) = \mathrm{Tr}(2i\,I) = 2i \times 2 = 4i$$

Step 5: Rule out the distractors.

$2i$ (option A) is the coefficient sitting in front of σ_z in the commutator itself, before it gets multiplied by σ_z again and traced, so picking it directly skips a step. i and $i/2$ (options B and D) do not follow from this identity under any natural slip, so they are just there to catch factor-of-2 errors.

Final Answer:
Using the Pauli algebra identity, $\mathrm{Tr}(\sigma_z[\sigma_x, \sigma_y]) = 4i$, option (C).

$$\boxed{4i}$$

Quick Tip: Multiply out $\sigma_x\sigma_y$ and $\sigma_y\sigma_x$ as 2×2 matrices first, then subtract to get the commutator before multiplying by σ_z.

• • •

. Which one of the following statements is true?

(A) In the decay $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$, CPT is violated.

(B) The decay $\Lambda \to p^+ + \pi^-$ is allowed and strangeness is violated.

(C) The decay $p^+ \to e^+ + \gamma$ is allowed.

(D) The decay $\Omega^- \to \Xi^0 + K^-$ is allowed.

Correct Answer: (B) The decay $\Lambda \to p^+ + \pi^-$ is allowed and strangeness is violated.

SOLUTION:

Step 1: Set up a quantum number table.
For each decay, list the baryon number B, lepton number L, strangeness S and charge Q on both sides. A decay through the strong or electromagnetic interaction must keep B, L, S and Q all exactly equal on both sides; a weak decay must still conserve B, L and Q, but is allowed to change S by exactly one unit. Any decay also needs the

parent's rest mass to be at least as large as the sum of the daughters' rest masses.

Step 2: Statement (A), $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$.
Lepton numbers: the muon side carries muon lepton number $L_\mu = -1$ (since it's an antimuon) and electron lepton number $L_e = 0$; the products carry $L_e = -1 + 1 = 0$ (positron plus electron neutrino) and $L_\mu = -1$ (from the muon antineutrino). Both totals match, so lepton number is conserved separately in each flavor. Charge, energy and momentum all check out too. CPT is a general theorem that holds for any Lorentz-invariant local field theory, and nothing in this bookkeeping breaks it, so this decay does not violate CPT. Statement (A) fails.

Step 3: Statement (B), $\Lambda \to p^+ + \pi^-$.
B: $1 \to 1 + 0 = 1$ (conserved). Q: $0 \to 1 + (-1) = 0$ (conserved). S: $-1 \to 0 + 0 = 0$, a change of one unit. Since this is a weak process (matching the Λ's known lifetime scale) and weak decays are allowed exactly this one-unit change in S, the decay proceeds and strangeness is indeed violated by it. Statement (B) holds up.

Step 4: Statement (C), $p^+ \to e^+ + \gamma$.
B: $1 \to 0 + 0 = 0$. This mismatch alone rules the decay out completely, regardless of any other quantum number, since baryon number must be conserved in every known interaction. Statement (C) fails.

Step 5: Statement (D), $\Omega^- \to \Xi^0 + K^-$.
Every quantum number matches: $Q : -1 \to 0 + (-1) = -1$, $B : 1 \to 1 + 0 = 1$, $S : -3 \to -2 + (-1) = -3$. So this table alone would suggest the decay is fine. But quantum numbers matching is only a necessary condition, not a sufficient one; energy bookkeeping also has to work.

Adding the masses of Ξ^0 (about 1315 MeV) and K^- (about 494 MeV) gives about 1809 MeV, which is more than the Ω^- mass of about 1672 MeV. The parent is lighter than what the decay would need to produce, so it cannot happen. Statement (D) fails.

Final Answer:

Statement (B) is the only one that survives every check.

$$\boxed{\text{(B)}}$$

Quick Tip: Check charge, baryon number and strangeness for each decay, then also check whether the parent particle actually has enough mass to produce the listed final particles.

• • •

. The Hamiltonian for a quantum particle of mass m is given below, where $\omega < \Omega$. The Schrodinger equation for this system can be solved exactly using the orthogonal transformation $x = \dfrac{x_1 - x_2}{\sqrt{2}}$ and $y = \dfrac{x_1 + x_2}{\sqrt{2}}$.

$$H = -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] + \frac{1}{2}m\Omega^2(x^2 + y^2) + m\omega^2 xy$$

The ground state energy of this system is

(A) $\dfrac{\hbar}{2}\left[\sqrt{\Omega^2 - \omega^2} + \sqrt{\Omega^2 + \omega^2}\right]$

(B) $\hbar\left[\sqrt{\Omega^2 - \omega^2} + \sqrt{\Omega^2 + \omega^2}\right]$

(C) $\dfrac{\hbar}{2}\left[\sqrt{\Omega^2 + \Omega\omega} + \sqrt{\Omega^2 - \Omega\omega}\right]$

(D) $\hbar\left[\sqrt{\Omega^2 + \Omega\omega} - \sqrt{\Omega^2 - \Omega\omega}\right]$

Correct Answer: (A) $\dfrac{\hbar}{2}\left[\sqrt{\Omega^2 - \omega^2} + \sqrt{\Omega^2 + \omega^2}\right]$

SOLUTION:

Step 1: Write the potential as a matrix.

The potential part of H is $V(x, y) = \frac{1}{2}m\Omega^2x^2 + \frac{1}{2}m\Omega^2y^2 + m\omega^2xy$. This can be written as $V = \frac{1}{2}m\begin{pmatrix} x & y \end{pmatrix} M \begin{pmatrix} x \\ y \end{pmatrix}$ with the symmetric matrix $M = \begin{pmatrix} \Omega^2 & \omega^2 \\ \omega^2 & \Omega^2 \end{pmatrix}$, since expanding this product gives back $\Omega^2x^2 + \Omega^2y^2 + 2\omega^2xy$, matching twice the cross term coefficient we need.

Step 2: Find the eigenvalues of M.

The normal mode frequencies squared are the eigenvalues of M. Solve $\det(M - \lambda I) = 0$:

$$(\Omega^2 - \lambda)^2 - \omega^4 = 0$$

$$\Omega^2 - \lambda = \pm\omega^2$$

So $\lambda_1 = \Omega^2 + \omega^2$ and $\lambda_2 = \Omega^2 - \omega^2$. These are the squares of the two normal mode frequencies, $\omega_1 = \sqrt{\Omega^2 + \omega^2}$ and $\omega_2 = \sqrt{\Omega^2 - \omega^2}$. The condition $\omega < \Omega$ keeps λ_2 positive, so both frequencies stay real.

Step 3: Note that the kinetic term stays diagonal under this rotation.

The eigenvectors of a real symmetric 2×2 matrix with equal diagonal entries are always the 45° rotated directions, exactly the x_1, x_2 combination given in the question. Since this eigenvector rotation is orthogonal, it leaves the kinetic energy $-\frac{\hbar^2}{2m}(\partial_x^2 + \partial_y^2)$ unchanged in form, so in the rotated frame the full Hamiltonian splits into two separate one-dimensional oscillators with frequencies ω_1 and ω_2.

Step 4: Add the two zero-point energies.

Each one-dimensional harmonic oscillator contributes a ground state energy of $\frac{1}{2}\hbar\omega_i$. Adding both normal modes:

$$E_0 = \frac{1}{2}\hbar\left(\sqrt{\Omega^2 + \omega^2} + \sqrt{\Omega^2 - \omega^2}\right)$$

Step 5: Rule out the other options.

Option (B) drops the $\frac{1}{2}$ that comes from summing two zero-point energies of $\frac{1}{2}\hbar\omega_i$ each. Options (C) and (D) would only appear if the eigenvalues of M had come out as $\Omega^2 \pm \Omega\omega$, which is not what solving $\det(M - \lambda I) = 0$ gives here.

Final Answer:

The eigenvalue approach gives the same result, $E_0 = \frac{\hbar}{2}\left[\sqrt{\Omega^2 - \omega^2} + \sqrt{\Omega^2 + \omega^2}\right]$, option (A).

$$\boxed{E_0 = \frac{\hbar}{2}\left[\sqrt{\Omega^2 - \omega^2} + \sqrt{\Omega^2 + \omega^2}\right]}$$

Quick Tip: Rewrite $x^2 + y^2$ and xy in terms of x_1, x_2 using the given rotation, then group terms to see two separate uncoupled oscillators.

• • •

. Consider two particles with angular momenta $j_1 = 2\hbar$ and $j_2 = \hbar/2$. If the expression

$$|j = \tfrac{5}{2}, m = \tfrac{3}{2}\rangle = c_1|j_1 = 2, m_1 = 1\rangle|j_2 = \tfrac{1}{2}, m_2 = \tfrac{1}{2}\rangle + c_2|j_1 = 2, m_1 = 2\rangle|j_2 = \tfrac{1}{2}, m_2 = -\tfrac{1}{2}\rangle$$

gives an eigenstate of the total angular momentum of the two particles, using standard notation, which of the following is true?
(Hint: $\hat{J}_{\pm}|j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)}\,|j, m \pm 1\rangle$)

(A) $c_1 = \dfrac{2}{\sqrt{5}}, \quad c_2 = \dfrac{1}{\sqrt{5}}$

(B) $c_1 = \dfrac{1}{\sqrt{5}}, \quad c_2 = \dfrac{2}{\sqrt{5}}$

(C) $c_1 = \dfrac{1}{\sqrt{2}}, \quad c_2 = \dfrac{1}{\sqrt{2}}$

(D) $c_1 = 0, \quad c_2 = 1$

Correct Answer: (A) $c_1 = \dfrac{2}{\sqrt{5}}, \quad c_2 = \dfrac{1}{\sqrt{5}}$

SOLUTION:

Step 1: Recognise this as a stretched coupling case.
Combining $j_1 = 2$ with $j_2 = \frac{1}{2}$ can only give total angular momentum $j = \frac{5}{2}$ (the stretched, maximum case) or $j = \frac{3}{2}$. Since we want $j = \frac{5}{2}$, which is $j_1 + j_2$, there is a standard closed formula for the Clebsch-Gordan coefficients that mix a spin-$\frac{1}{2}$ particle into a state of total $j = j_1 + \frac{1}{2}$, without needing to build up from the top state by hand.

Step 2: Quote the standard formula.
For coupling j_1 with a spin-$\frac{1}{2}$ particle to reach $j = j_1 + \frac{1}{2}$ with projection m, the general result is:

$$|j_1 + \tfrac{1}{2}, m\rangle = \sqrt{\frac{j_1 + m + \frac{1}{2}}{2j_1 + 1}}\,\left|j_1, m - \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle + \sqrt{\frac{j_1 - m + \frac{1}{2}}{2j_1 + 1}}\,\left|j_1, m + \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle$$

This formula comes from the general Clebsch-Gordan tables for adding spin-$\frac{1}{2}$ to any j_1, and it directly gives both mixing coefficients without any extra derivation.

Step 3: Plug in $j_1 = 2$ and $m = \frac{3}{2}$.
Here $2j_1 + 1 = 5$. The first coefficient uses $m_1 = m - \frac{1}{2} = 1$, matching the term $|2,1\rangle|\frac{1}{2},\frac{1}{2}\rangle$:

$$\sqrt{\frac{j_1 + m + \frac{1}{2}}{2j_1 + 1}} = \sqrt{\frac{2 + \frac{3}{2} + \frac{1}{2}}{5}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$$

The second coefficient uses $m_1 = m + \frac{1}{2} = 2$, matching the term $|2,2\rangle|\frac{1}{2},-\frac{1}{2}\rangle$:

$$\sqrt{\frac{j_1 - m + \frac{1}{2}}{2j_1 + 1}} = \sqrt{\frac{2 - \frac{3}{2} + \frac{1}{2}}{5}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}}$$

Step 4: Match to the question's labeling.
The question calls the coefficient of $|2,1\rangle|\frac{1}{2},\frac{1}{2}\rangle$ as c_1 and the coefficient of $|2,2\rangle|\frac{1}{2},-\frac{1}{2}\rangle$ as c_2, so directly $c_1 = \dfrac{2}{\sqrt{5}}$ and $c_2 = \dfrac{1}{\sqrt{5}}$.

Step 5: Check this against a quick consistency test.
A genuine quantum state must be normalised, meaning $c_1^2 + c_2^2 = 1$.
Here $\left(\dfrac{2}{\sqrt{5}}\right)^2 + \left(\dfrac{1}{\sqrt{5}}\right)^2 = \dfrac{4}{5} + \dfrac{1}{5} = 1$, which checks out, confirming these are valid coefficients. Option (C), by contrast, also happens to satisfy $c_1^2 + c_2^2 = 1$ but does not match the actual weighting of the two terms from the formula, so passing the normalisation check alone is

not enough to pick it.

Final Answer:

The standard stretched-coupling formula gives $c_1 = \frac{2}{\sqrt{5}}$ and $c_2 = \frac{1}{\sqrt{5}}$, option (A).

$$\boxed{c_1 = \frac{2}{\sqrt{5}},\ c_2 = \frac{1}{\sqrt{5}}}$$

Quick Tip: Start from the top state $|j = 5/2, m = 5/2\rangle$, which is just the simple product state, then apply the lowering operator to both sides and match coefficients.

• • •

. The energy E and degeneracy d of the second excited state of a three-dimensional, isotropic quantum harmonic oscillator with angular frequency ω are

(A) $E = \frac{7}{2}\hbar\omega,\ d = 6$

(B) $E = \frac{7}{2}\hbar\omega,\ d = 3$

(C) $E = \frac{5}{2}\hbar\omega,\ d = 3$

(D) $E = \frac{5}{2}\hbar\omega,\ d = 6$

Correct Answer: (A) $E = \frac{7}{2}\hbar\omega,\ d = 6$

SOLUTION:

Step 1: Set up the counting problem differently.

Instead of listing triples by hand, treat each of the three directions as a bin that can hold any number of energy quanta, and ask how many quanta n the second excited level carries.

Step 2: Find n for the second excited state.

Level 0 (ground state) carries $n = 0$ quanta, level 1 carries $n = 1$, so the second excited level carries $n = 2$ quanta above the ground state.

The energy of this level is $E = (n + \frac{3}{2})\hbar\omega$, so for $n = 2$:

$$E = \left(2 + \frac{3}{2}\right)\hbar\omega = \frac{7}{2}\hbar\omega$$

Step 3: Count the degeneracy with stars and bars.

The degeneracy is the number of ways to split $n = 2$ identical quanta among 3 bins (n_x, n_y, n_z). This is a standard stars-and-bars count: choose 2 dividers among $n + 2$ slots, giving $\binom{n+2}{2}$ arrangements.

$$d = \binom{n+2}{2} = \binom{4}{2} = \frac{4!}{2!\,2!} = 6$$

This counts the same six states as before ($(2, 0, 0)$ and permutations, $(1, 1, 0)$ and permutations) without listing them one by one.

Step 4: Rule out the distractors.

A degeneracy of 3 (option B) is what stars-and-bars gives for $n = 1$, not $n = 2$, since $\binom{3}{2} = 3$. An energy of $\frac{5}{2}\hbar\omega$ (options C, D) belongs to $n = 1$, one level too low.

Final Answer:

Matching energy $\frac{7}{2}\hbar\omega$ with degeneracy 6 fixes the answer as option (A).

$$\boxed{E = \frac{7}{2}\hbar\omega,\ d = 6}$$

Quick Tip: Second excited state of a 3D isotropic oscillator is $n = 2$: use $E_n = (n + 3/2)\hbar\omega$ and degeneracy $d(n) = (n + 1)(n + 2)/2$.

• • •

. Two identical particles with a fixed total energy $E = 2\hbar\omega$ are in thermal equilibrium in a one-dimensional harmonic oscillator potential $\frac{1}{2}m\omega^2x^2$. Let the entropy of the particles be denoted by S_F if they are fermions with spin $\frac{1}{2}$ ($S_z = \pm\frac{\hbar}{2}$) and by S_B if they are bosons with spin 0. Then, which of the following options is correct? (k_B is the Boltzmann constant)

(A) $S_F = k_B \ln 2,\ S_B = k_B \ln 2$

(B) $S_F = 2k_B \ln 2,\ S_B = 0$

(C) $S_F = 4k_B \ln 2,\ S_B = 0$

(D) $S_F = 2k_B \ln 2,\ S_B = k_B \ln 2$

Correct Answer: (B) $S_F = 2k_B \ln 2,\ S_B = 0$

SOLUTION:

Step 1: Find which single-particle levels can be occupied.

The single particle energies are $\varepsilon_n = (n + \frac{1}{2})\hbar\omega$, so two particles with total energy $E = 2\hbar\omega$ need $n_1 + n_2 = 1$. The only non-negative integer pair is $\{0, 1\}$: one particle in the ground level, one in the first excited level. No pair with $n_1 = n_2$ works, since that would need $n_1 + n_2$ to be even, and 1 is odd.

Step 2: List the fermion microstates directly.

Label each single-particle state by (level, spin). The particle in level 0 can carry spin $+\frac{1}{2}$ or $-\frac{1}{2}$, and independently, the particle in level 1 can also carry spin $+\frac{1}{2}$ or $-\frac{1}{2}$. Listing all four combinations: (0,up)(1,up), (0,up)(1,down), (0,down)(1,up), (0,down)(1,down). Since the two particles already sit in different spatial levels, exchange antisymmetry never removes any of these four combinations, they are all valid two-fermion states. So $\Omega_F = 4$, giving

$$S_F = k_B \ln 4 = 2k_B \ln 2$$

Step 3: List the boson microstates directly.

Spin-0 bosons carry no spin label, so the only information is which levels are occupied: one particle in level 0, one in level 1. For identical particles, swapping "which particle" is in which level is not a new state, it is the same physical configuration counted once. So there is exactly one microstate, $\Omega_B = 1$, giving

$$S_B = k_B \ln 1 = 0$$

Step 4: Match against the options.

Only the pair $S_F = 2k_B \ln 2$ and $S_B = 0$ appears together, which is option (B). Any option with $S_B \neq 0$ is wrong because the identical bosons have just one way to arrange themselves here, and $4k_B \ln 2$ overshoots the fermion count, which stops at 4 combinations, not 16.

Final Answer:

Direct enumeration of the allowed (level, spin) combinations gives $\Omega_F = 4$ and $\Omega_B = 1$.

$$\boxed{S_F = 2k_B \ln 2,\ S_B = 0}$$

Quick Tip: Total energy fixes $n_1 + n_2 = 1$, so the two particles must sit in levels 0 and 1. Count the allowed spin/label combinations for fermions and for bosons separately, then use $S = k_B \ln \Omega$.

• • •

. In the circuit shown, $V(t) = 2\sin(2000\pi t)$ Volts, where t is in seconds. The source $V(t)$ drives the non-inverting (+) input of an opamp directly, the inverting (-) input is tied to ground (0 V), and the opamp is powered from +15 V and -15 V rails with no feedback network between its output and either input. Take the opamp to be ideal. Which of the following options is correct?

(A) $V_{out}(t)$ is square wave with peak-to-peak voltage = 30 V and time period is 1 ms.

(B) $V_{out}(t)$ is a sine wave with peak-to-peak voltage = 4 V and time period is 1 ms.

(C) $V_{out}(t)$ is sine wave with peak-to-peak voltage = 30 V and time period of 1 ms.

(D) $V_{out}(t)$ is square wave with peak-to-peak voltage = 4 V and time period is 1 ms.

Correct Answer: (A) $V_{out}(t)$ is square wave with peak-to-peak voltage = 30 V and time period is 1 ms.

SOLUTION:

Step 1: Identify the circuit function.
There is no resistor feeding the output back into either input, so this is not a negative-feedback amplifier stage. With an ideal (infinite gain) opamp and no feedback, output can only rail to $+15$ V or -15 V,

whichever input is higher wins completely.

Step 2: Track the sign of $V(t)$ over one cycle.
The inverting input sits at 0 V (grounded), so the comparison reduces to the sign of $V(t) = 2\sin(2000\pi t)$ at the non-inverting input. Checking a few instants: at $t = 0$, $V = 0$ and about to go positive; for small $t > 0$, $\sin(2000\pi t) > 0$ so $V_{out} = +15$ V; once $2000\pi t$ passes π, $\sin(\cdot) < 0$ so V_{out} flips to -15 V. This toggling repeats every time $\sin(2000\pi t)$ changes sign.

Step 3: Get the frequency, then the period.
Comparing $2000\pi t$ with the standard form $2\pi f t$ gives $f = \dfrac{2000\pi}{2\pi}$ $= 1000$ Hz. So the period is

$$T = \frac{1}{f} = \frac{1}{1000\ \text{Hz}} = 1\ \text{ms}$$

Since the output only ever sits at one of the two rails, its peak-to-peak swing is just the rail separation:

$$V_{pp} = 15\ \text{V} - (-15\ \text{V}) = 30\ \text{V}$$

Step 4: Eliminate the wrong shapes and amplitudes.
A sine-shaped output (options B, C) would need the opamp working in its linear region with feedback, which is not present here, so both are ruled out on shape alone. A 4 V swing (option D) is far too small for a rail-to-rail comparator running off ±15 V supplies.

Final Answer:

The comparator toggles between +15 V and −15 V at the input's zero crossings, giving a square wave of 30 V peak-to-peak and 1 ms period.

$$\boxed{V_{pp} = 30\text{ V},\ T = 1\text{ ms}}$$

Quick Tip: With no feedback, an ideal opamp comparator output only ever sits at one of its two supply rails, comparing $V(t)$ against the grounded inverting input tells you when it flips.

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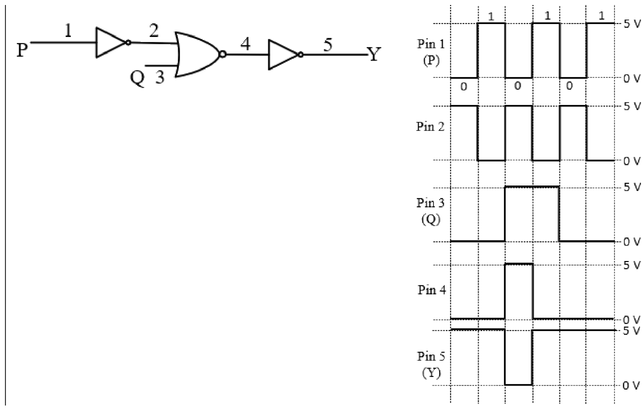
Considering the circuit and the associated signals measured at different pins (numbered as 1, 2, 3, 4, 5) shown in the figure, the correct option is



(A) NOT gate between pins 1 and 2 is faulty.

(B) NOR gate is faulty.

(C) NOT gate between pins 4 and 5 is faulty.

(D) The NOR and output NOT gates, are both faulty.

Correct Answer: (B) NOR gate is faulty.

SOLUTION:

Step 1: Write the ideal output as a single function of P and Q.

If every gate worked, $Y = \overline{\overline{\overline{P} + Q}} = \overline{P} + Q$ (inverting a NOR twice just gives back the OR of its inputs). So a fully healthy circuit should give $Y = \overline{P} + Q$ directly from P and Q, regardless of what happens at the intermediate pins.

Step 2: Compute this ideal Y from the given P, Q waveform.
Reading P: 0,1,0,1,0,1 and Q: 0,0,1,1,0,0 across the six slots, $\overline{P}$ is 1,0,1,0,1,0, so $\overline{P} + Q$ is:
slot1: $1 + 0 = 1$, slot2: $0 + 0 = 0$, slot3: $1 + 1 = 1$, slot4: $0 + 1 = 1$, slot5: $1 + 0 = 1$, slot6: $0 + 0 = 0$.
Ideal Y = 1, 0, 1, 1, 1, 0.

Step 3: Compare with the measured Y (pin 5).
The measured pin 5 trace reads 1, 1, 0, 0, 1, 1, which disagrees with the ideal 1, 0, 1, 1, 1, 0 in slots 2, 3, 4 and 6. Since the fully-correct circuit and the actual circuit disagree, at least one gate must be broken; now isolate which.

Step 4: Isolate the faulty gate using the intermediate pins.
Pin 2 measured is 1,0,1,0,1,0, which equals $\overline{P}$ exactly, so gate 1 (the input NOT) is fine.
Pin 4 measured is 0,0,1,1,0,0, but $\overline{(\text{pin 2}) + Q}$ works out to 0,1,0,0,0,1, so the NOR gate's output does not match its own inputs, this is the broken stage.
Pin 5 measured, 1,1,0,0,1,1, is exactly $\overline{\text{pin 4 (measured)}}$, so the last NOT gate is correctly inverting the bad signal it was handed; it is not itself broken.

Step 5: Confirm against the options.

Since the mismatch traces back to the NOR stage alone, with both NOT gates verified healthy, the answer is that the NOR gate is faulty.

Final Answer:
Comparing the ideal $Y = \overline{P} + Q$ against the measured waveform, and then checking each pin in turn, pins the fault on the NOR gate.

$$\boxed{\text{NOR gate is faulty}}$$

Quick Tip: Compute what pin 2, pin 4 and pin 5 should be from P and Q using the stated NOT-NOR-NOT chain, then check each measured pin against that expectation to isolate the bad gate.

• • •

. A gas of N classical particles that can occupy energy levels ϵ_1 and $\epsilon_2 = \epsilon_1 + \Delta$ is in equilibrium with a reservoir at temperature T. From the schematics shown below, choose the correct dependence of the internal energy U on T.

(A) A curve that starts flat at a low value for small T, rises smoothly through an S-shaped (sigmoid) transition, and saturates to a higher flat value at large T.

(B) A curve that starts at zero and keeps rising with an ever-increasing slope as T grows, with no upper flat limit.

(C) A curve that starts high at small T and falls off smoothly to a lower flat value as T increases.

(D) A curve that starts at a small nonzero value and keeps rising with an ever-increasing slope as T grows, with no upper flat limit.

Correct Answer: (A) A curve that starts flat at a low value for small T, rises smoothly through an S-shaped (sigmoid) transition, and saturates to a higher flat value at large T.

SOLUTION:

Step 1: Work with occupation fractions instead of the partition function.

Let p_1 and p_2 be the fraction of the N particles sitting in levels ϵ_1 and ϵ_2 at temperature T. Detailed balance for a classical gas in contact with a reservoir gives the Boltzmann ratio

$$\frac{p_2}{p_1} = e^{-\Delta/k_B T}$$

Step 2: Turn the ratio into normalized fractions.

Since $p_1 + p_2 = 1$, substituting $p_2 = p_1 e^{-\Delta/k_B T}$ gives

$$p_1 = \frac{1}{1 + e^{-\Delta/k_B T}}, \qquad p_2 = \frac{e^{-\Delta/k_B T}}{1 + e^{-\Delta/k_B T}}$$

Step 3: Build $U(T)$ from the populations.

The internal energy is just the population-weighted average energy times N:

$$U(T) = N(p_1\epsilon_1 + p_2\epsilon_2) = N\epsilon_1 + Np_2\Delta = N\epsilon_1 + \frac{N\Delta\, e^{-\Delta/k_B T}}{1 + e^{-\Delta/k_B T}}$$

which is the same expression as before, written from the populations rather than from $-\partial \ln z/\partial\beta$.

Step 4: Read off the low- and high-temperature behavior.

At low T, $e^{-\Delta/k_BT} \to 0$, so $p_2 \to 0$: almost every particle stays in the ground level and $U \to N\epsilon_1$, a flat plateau.

At high T, $e^{-\Delta/k_BT} \to 1$, so $p_1 \to p_2 \to \frac{1}{2}$: the levels become equally populated and $U \to N(\epsilon_1 + \Delta/2)$, a second flat plateau.

Between these limits p_2 climbs smoothly from 0 to $\frac{1}{2}$, so $U(T)$ rises smoothly between the two plateaus, an S-shaped curve, never overshooting the upper plateau and never decreasing.

Step 5: Rule out the wrong shapes.

Curves that keep growing without bound (options B, D) contradict p_2 saturating at $\frac{1}{2}$. A curve that decreases with T (option C) contradicts p_2 only ever increasing as T rises.

Final Answer:

The population fractions saturate, so $U(T)$ must saturate too, rising from $N\epsilon_1$ to $N(\epsilon_1 + \Delta/2)$ in a sigmoid shape, which is graph (A).

$$\boxed{U(T):\ N\epsilon_1 \to N\left(\epsilon_1 + \frac{\Delta}{2}\right),\ \text{option (A)}}$$

Quick Tip: A two-level system's average energy per particle is bounded between ϵ_1 (all particles in the ground level) and $\epsilon_1 + \Delta/2$ (levels equally populated), and it rises smoothly between the two.

• • •

. The Lagrangian $L_0 = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2$ with the generalized coordinate q is transformed to $L = L_0 + \alpha\frac{df(q)}{dt}$. Consider the following statements:

(i) Expression for the canonical momentum does not change.

(ii) The equation of motion does not change.

Which of the following options is correct for the above statements?

(A) Both (i) and (ii) are correct.

(B) Both (i) and (ii) are not correct.

(C) (i) is correct and (ii) is not correct.

(D) (i) is not correct and (ii) is correct.

Correct Answer: (D) (i) is not correct and (ii) is correct.

SOLUTION:

Step 1: Recall the general rule about adding a total time derivative.
There is a standard result in Lagrangian mechanics: if you add $\frac{d}{dt}F(q,t)$ to any Lagrangian, the equations of motion never change, because the extra piece contributes only a boundary term to the action $\int L\,dt$, and the variation of a boundary term vanishes when the endpoints of $q(t)$ are held fixed. Here $F(q) = \alpha f(q)$, so this rule applies directly and predicts that the dynamics cannot change. What the rule does not protect is the canonical momentum, which is defined pointwise as $p = \partial L/\partial\dot{q}$ and is free to shift by whatever extra $\dot{q}$ dependence the added term carries.

Step 2: Check statement (i), the momentum.
Write out the added piece explicitly: $\alpha\frac{df(q)}{dt} = \alpha f'(q)\dot{q}$, a term proportional to $\dot{q}$. Differentiating the full L with respect to $\dot{q}$ picks up this whole term as a plain additive shift:

$$p_{\text{new}} = m\dot{q} + \alpha f'(q), \qquad p_{\text{old}} = m\dot{q}$$

The two momenta differ by $\alpha f'(q)$, a nonzero quantity for a generic f. Since the question does not restrict f to have $f' = 0$, the momentum genuinely changes, so (i) is FALSE.

Step 3: Check statement (ii), the equation of motion, using the boundary-term argument directly on the action.
The action changes as $SoS + \alpha \int \frac{df(q)}{dt}\, dt = S + \alpha\big[f(q(t_2)) - f(q(t_1))\big]$. Varying $q(t)$ with the endpoints $q(t_1)$ and $q(t_2)$ held fixed (as is always assumed in deriving the Euler-Lagrange equation) makes this extra piece a constant under variation, so $\delta\,(\alpha[f(q(t_2)) - f(q(t_1))]) = 0$. The stationary-action condition on the new L therefore reduces to exactly the stationary-action condition on L_0, giving the same Euler-Lagrange equation $m\ddot{q} + m\omega^2 q = 0$. So (ii) is TRUE.

Step 4: Cross-check with the direct algebra.
As a consistency check, note that the extra momentum term $\alpha f'(q)$ contributes $\frac{d}{dt}[\alpha f'(q)] = \alpha f''(q)\dot{q}$ to $\frac{d}{dt}(\partial L/\partial \dot{q})$, and the same $\alpha f''(q)\dot{q}$ appears with the opposite sign in $\partial L/\partial q$ (from differentiating $\alpha f'(q)\dot{q}$ with respect to q). These identical terms cancel in the Euler-Lagrange equation, matching the boundary-term argument above.

Final Answer:
The momentum picks up an extra term and is not preserved, while the dynamics stay exactly the same as for L_0: (i) is not correct, (ii) is correct.

$$\boxed{\text{(D)}}$$

Quick Tip: Adding $\alpha\frac{df(q)}{dt} = \alpha f'(q)\dot{q}$ to a Lagrangian only adds a boundary term to the action, so the Euler-Lagrange equation is unchanged; but since $p = \partial L/\partial\dot{q}$ is evaluated pointwise, the momentum does shift by $\alpha f'(q)$.

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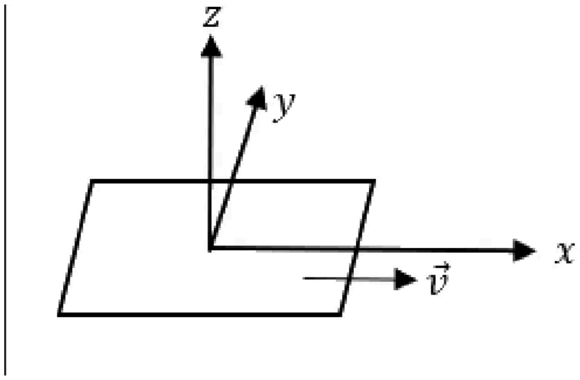
. An infinitely large thin sheet in the xy-plane carries uniform positive charge density and is moving with constant velocity $\vec{v}$ in the $+x$ direction (see figure below). The direction of the corresponding Poynting vector is



(A) $+x$ for both $z < 0$ and $z > 0$

(B) $+x$ for $z < 0$ and $-x$ for $z > 0$

(C) $-x$ for $z < 0$ and $+x$ for $z > 0$

(D) $-x$ for both $z < 0$ and $z > 0$

Correct Answer: (A) $+x$ for both $z < 0$ and $z > 0$

SOLUTION:

Step 1: Set up the fields with a general height coordinate.

Let h stand for the field point's z-coordinate, positive or negative. Both fields on either side of the sheet can be written compactly using the sign function sgn(h):

$$\vec{E} = \frac{\sigma}{2\epsilon_0}\,\text{sgn}(h)\,\hat{z}, \qquad \vec{B} = -\frac{\mu_0 \sigma v}{2}\,\text{sgn}(h)\,\hat{y}$$

This single-formula form already shows that $\vec{E}$ and $\vec{B}$ each pick up exactly one factor of $\operatorname{sgn}(h)$ when you cross the sheet.

Step 2: Take the cross product symbolically before plugging in signs.

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} = \frac{1}{\mu_0}\left(\frac{\sigma}{2\epsilon_0}\operatorname{sgn}(h)\hat{z}\right) \times \left(-\frac{\mu_0 \sigma v}{2}\operatorname{sgn}(h)\hat{y}\right)$$

$$\vec{S} = -\frac{\sigma^2 v}{4\epsilon_0}\left[\operatorname{sgn}(h)\right]^2(\hat{z} \times \hat{y})$$

Step 3: Use $[\operatorname{sgn}(h)]^2 = 1$.
Whatever the sign of h, squaring it always gives 1, since $\operatorname{sgn}(h) = \pm 1$. That single fact already tells you the final direction cannot depend on whether z is positive or negative, before even working out the cross product of the unit vectors. This immediately rules out options (B), (C) and (D), which all claim the direction is different on the two sides.

Step 4: Work out the fixed direction.
With $\hat{z} \times \hat{y} = -\hat{x}$:

$$\vec{S} = -\frac{\sigma^2 v}{4\epsilon_0}(1)(-\hat{x}) = \frac{\sigma^2 v}{4\epsilon_0}\hat{x}$$

which is along $+x$, matching the direction the sheet itself is moving in.

Step 5: Physical sanity check.
This makes sense energetically: a moving charged sheet is effectively carrying electromagnetic energy along with it as it travels, and the

electromagnetic momentum density $\vec{S}/c^2$ should point the same way the charge is physically moving, on both faces of the sheet. A field pattern that reversed on the two sides would not match this simple picture of energy being dragged along by the moving charge.

Final Answer:

The Poynting vector is $+x$ for both $z < 0$ and $z > 0$.

$$\boxed{\text{(A)}}$$

Quick Tip: Use $E = \sigma/2\epsilon_0$ away from the sheet and $B = \mu_0 K/2$ (with $K = \sigma v$) circling the current sheet like a solenoid winding; both E and B flip sign together across the sheet, so $\vec{S} = \vec{E} \times \vec{B}/\mu_0$ keeps the same sign on both sides.

• • •

. A positive point charge is fixed at the origin. At some distance from it on the x axis, a point dipole is kept pointing in the $+y$ direction. The force on the dipole is

(A) 0

(B) in the $+y$ direction

(C) in the $-y$ direction

(D) in the $+x$ direction

Correct Answer: (B) in the $+y$ direction

SOLUTION:

Step 1: Model the dipole as two nearby point charges.

Instead of using the dipole-force formula directly, replace the point dipole by two real charges: $+q_d$ at $(d, \delta/2, 0)$ and $-q_d$ at $(d, -\delta/2, 0)$, with δ very small, so that the dipole moment is $p = q_d\delta$ pointing along $+y$ (from the negative charge to the positive one), matching the direction given in the problem.

Step 2: Write the force from the fixed charge Q on each of the two charges.

Let Q (positive) sit at the origin. The distance from Q to each of the two charges is nearly the same for small δ:

$$r_+ = \sqrt{d^2 + \delta^2/4} \approx d, \qquad r_- = \sqrt{d^2 + \delta^2/4} \approx d$$

The Coulomb force on $+q_d$ is $\vec{F}_+ = \dfrac{kQq_d}{r_+^3}(d,\ \delta/2,\ 0)$, and on $-q_d$ it is $\vec{F}_- = -\dfrac{kQq_d}{r_-^3}(d,\ -\delta/2,\ 0)$.

Step 3: Add the two forces and keep only the leading order in δ.

Since r_+^3 and r_-^3 agree with d^3 up to corrections of order δ^2, we can replace both by d^3 when adding the forces to leading order in δ:

$$\vec{F}_+ + \vec{F}_- \approx \frac{kQq_d}{d^3}\Big[(d,\ \delta/2,\ 0) - (d,\ -\delta/2,\ 0)\Big] = \frac{kQq_d}{d^3}(0,\ \delta,\ 0)$$

Notice the x-components of the two forces are equal and cancel out when we form $\vec{F}_+ + \vec{F}_-$ (both contribute $kQq_d d/d^3$, but with opposite overall sign from the charges), while the y-components add constructively.

Step 4: Write the total force in terms of p.

$$\vec{F} = \frac{kQq_d\delta}{d^3}\hat{y} = \frac{kQp}{d^3}\hat{y}$$

since $p = q_d\delta$. This is the same result as the gradient formula, but reached by directly adding up two real Coulomb forces instead of differentiating the field.

Step 5: Interpret the sign.
The $+q_d$ end sits slightly further along $+y$ where the field of Q has a slightly larger y-component than at the $-q_d$ end (because moving along $+y$ at fixed $x = d$ increases the radial distance and rotates the field vector to pick up more y-component up to this order), so the pull on $+q_d$ in $+y$ outweighs the pull on $-q_d$ in $-y$, giving a net push of the whole dipole toward $+y$.

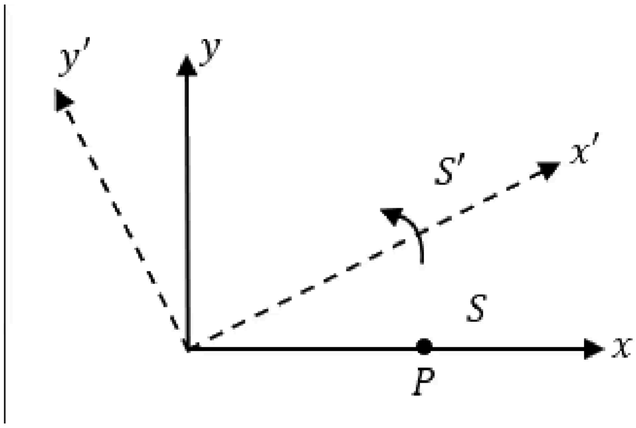
Final Answer:
Adding the two Coulomb forces gives a net force along $+y$, the same direction the dipole already points in.

$$\boxed{\text{in the } +y \text{ direction}}$$

Quick Tip: Use $\vec{F} = (\vec{p} \cdot \nabla)\vec{E}$ for a dipole in an external field; since $\vec{p} = p\hat{y}$, only $\partial E_y/\partial y$ at the dipole's location matters, and for a point charge this comes out positive on the x-axis.

• • •

. Two frames S (solid lines) and S' (dashed lines) with common origin are shown in the figure below. Frame S is inertial while S' is rotating about the common z-axis. There is a point mass fixed at P on the x-axis of the S frame. The magnitude of the centrifugal force and the Coriolis force experienced by the mass in the S' frame is F_{cen} and F_{cor}, respectively. Which of the following options is correct for these forces?



(A) $F_{cen} = 0$ and $F_{cor} = 0$

(B) $F_{cen} \neq 0$ and $F_{cor} \neq 0$ and $F_{cen} = \frac{F_{cor}}{2}$

(C) $F_{cen} \neq 0$ and $F_{cor} \neq 0$ and $F_{cen} = 2F_{cor}$

(D) $F_{cen} \neq 0$ and $F_{cor} \neq 0$ and $F_{cen} = F_{cor}$

Correct Answer: (B) $F_{cen} \neq 0$ and $F_{cor} \neq 0$ and $F_{cen} = \frac{F_{cor}}{2}$

SOLUTION:

Step 1: Picture the motion directly as seen from the rotating frame. Instead of starting from the velocity-addition formula, think about what an observer sitting on S' actually sees. The mass at P never moves in the inertial frame S, but since S' itself spins relative to S at rate ω, the S'-observer sees P trace out a circle of radius r (the fixed distance of P from the common origin) at angular speed ω, going around in the sense opposite to how S' spins relative to S. This is a purely kinematic fact, true even before any forces are considered.

Step 2: Get the speed and true (kinematic) acceleration of this apparent circular path.
For circular motion of radius r at angular speed ω, the speed is $u = \omega r$ and the acceleration needed to keep tracing that circle points toward the center with magnitude $a' = u^2/r = \omega^2 r$. This is simply the standard centripetal acceleration formula for any object moving on a circle, applied to the apparent path of P as tracked using S''s own axes.

Step 3: Match this kinematic acceleration to the pseudo-forces.
Since P feels zero real force (it just sits still in the inertial frame S), Newton's second law written in the non-inertial frame S' must produce this same net inward acceleration purely from pseudo-forces: $ma'_{\text{net}} = F_{cor} - F_{cen}$, where F_{cor} points inward (since it opposes the apparent circular drift) and F_{cen} points outward. Using the standard expressions $F_{cen} = m\omega^2 r$ and $F_{cor} = 2m\omega^2 r$ (from $F_{cor} = 2m\omega u = 2m\omega(\omega r)$, since the Coriolis force magnitude is $2m\Omega v'$ for velocity perpendicular to the rotation axis):

$$F_{cor} - F_{cen} = 2m\omega^2 r - m\omega^2 r = m\omega^2 r = ma'_{\text{net}}$$

which exactly matches $ma' = m\omega^2 r$ found by pure kinematics in Step 2. This cross-check confirms both magnitudes are consistent with each other.

Step 4: Read off the ratio.
From $F_{cen} = m\omega^2 r$ and $F_{cor} = 2m\omega^2 r$, both are clearly nonzero, and dividing gives $F_{cen}/F_{cor} = 1/2$, i.e. $F_{cen} = F_{cor}/2$.

Final Answer:

Both pseudo-forces are present and the centrifugal force is exactly half the Coriolis force.

$$\boxed{\text{(B)}}$$

Quick Tip: A mass fixed in the inertial frame S appears in the rotating frame S' to move on a circle of radius r at speed ωr; use $F_{cen} = m\omega^2 r$ and $F_{cor} = 2m\Omega v' = 2m\omega^2 r$ to get the ratio 1 : 2.

• • •

. Which of the following operators is/are self-adjoint?

(A) $x^2\dfrac{d^2}{dx^2} + 3x\dfrac{d}{dx} + x^2$

(B) $(1-x^2)\dfrac{d^2}{dx^2} - 2x\dfrac{d}{dx} + 3x$

(C) $(3x-4x^3)\dfrac{d^2}{dx^2} + (3-12x^2)\dfrac{d}{dx} + 12$

(D) $x\dfrac{d^2}{dx^2} + x^2\dfrac{d}{dx} + \dfrac{5x}{3}$

Correct Answer: (B) $(1-x^2)\dfrac{d^2}{dx^2} - 2x\dfrac{d}{dx} + 3x$; (C) $(3x-4x^3)\dfrac{d^2}{dx^2} + (3 - 12x^2)\dfrac{d}{dx} + 12$

SOLUTION:

Step 1: Derive the self-adjointness condition from scratch using integration by parts.

For $L = A(x)D^2 + B(x)D + C(x)$ (with $D = d/dx$) acting on functions vanishing at the boundary points, the formal adjoint is found by moving derivatives off one function u onto another w in $\int w\,(Lu)\,dx$. Integrating $\int wAu''\,dx$ by parts twice gives $\int (Aw)''u\,dx$ (boundary terms drop), and integrating $\int wBu'\,dx$ by parts once gives $-\int (Bw)'u$

dx. Collecting terms, the adjoint operator turns out to be $L^\dagger w = (Aw)'' - (Bw)' + Cw = Aw'' + (2A' - B)w' + (A'' - B' + C)w$.

Step 2: Compare $L^\dagger$ with L to get the condition.
L itself is $Aw'' + Bw' + Cw$. For $L^\dagger = L$, the coefficients of w' must match: $2A' - B = B$, which gives $B = A'$. (The coefficients of w'' already match automatically since both are A.) So self-adjointness is equivalent to $B(x) = A'(x)$, the same condition used in the other method, but here it comes directly out of the integration-by-parts bookkeeping instead of being quoted.

Step 3: Apply the condition option by option, starting from (D).
For (D): $A = x \Rightarrow A' = 1$, but $B = x^2$. Since $x^2 \neq 1$ except at isolated points, this fails everywhere except a single value of x, so (D) is NOT self-adjoint.

Step 4: Check (A).
$A = x^2 \Rightarrow A' = 2x$, while $B = 3x$. The ratio $B/A' = 3/2 \neq 1$ for all $x \neq 0$, so (A) is NOT self-adjoint.

Step 5: Check (C).
$A = 3x - 4x^3 \Rightarrow A' = 3 - 12x^2$, and directly reading off $B = 3 - 12x^2$ from the given operator shows B and A' are the identical polynomial, so (C) IS self-adjoint.

Step 6: Check (B).
$A = 1 - x^2 \Rightarrow A' = -2x$, and the operator gives $B = -2x$ directly, so $B = A'$ again holds exactly, and (B) IS self-adjoint. This is the well-known Legendre differential operator.

Final Answer:

Working the adjoint out from first principles confirms the same pair: (B) and (C) are self-adjoint, while (A) and (D) are not.

$$\boxed{\text{(B), (C)}}$$

Quick Tip: For $L = A(x)D^2 + B(x)D + C(x)$, the operator is formally self-adjoint exactly when $B(x) = A'(x)$, which lets it be written as $\frac{d}{dx}\left(A(x)\frac{d}{dx}\right) + C(x)$. Check this coefficient match for each option.

• • •

. Consider two operators $\hat{A}$ and $\hat{B}$ which are related as $\hat{A} = \exp(i\theta\hat{B})$. If θ is a non-zero real number, which of the following statements is/are true?

(A) If $\hat{B}$ is Hermitian, then $\hat{A}$ is unitary.

(B) If $\hat{B}$ is anti-Hermitian, then $\hat{A}$ is unitary.

(C) If $\hat{B}$ is Hermitian, then $|\mathrm{Det}(\hat{A})| = 1$.

(D) If $\hat{B}$ is anti-Hermitian, then $\hat{A}$ is Hermitian.

Correct Answer: (A) If $\hat{B}$ is Hermitian, then $\hat{A}$ is unitary. ; (C) If $\hat{B}$ is Hermitian, then $|\mathrm{Det}(\hat{A})| = 1$. ; (D) If $\hat{B}$ is anti-Hermitian, then $\hat{A}$ is Hermitian.

SOLUTION:

Step 1: Work in the eigenbasis of B.

Both Hermitian and anti-Hermitian operators are normal operators, so B always has a complete set of orthonormal eigenvectors. Call its eigenvalues λ_n, so in this basis B is diagonal with entries λ_n, and $A = e^{i\theta B}$ is diagonal in the same basis with entries $e^{i\theta\lambda_n}$.

Step 2: Case where B is Hermitian.

A Hermitian operator has only real eigenvalues, so each λ_n is real. Each diagonal entry of A is $e^{i\theta\lambda_n}$, a complex number sitting exactly on the unit circle, since $\theta\lambda_n$ is real. A diagonal matrix whose every entry has magnitude 1 is unitary, which confirms statement (A) is TRUE.

Step 3: Determinant in the Hermitian case.

The determinant of a diagonal matrix is the product of its entries, so $\mathrm{Det}(A) = \prod_n e^{i\theta\lambda_n}$. This is a product of unit circle numbers, and such a product always has magnitude 1. So $|\mathrm{Det}(A)| = 1$, confirming statement (C) is TRUE.

Step 4: Case where B is anti-Hermitian.

An anti-Hermitian operator has purely imaginary eigenvalues, so write $\lambda_n = i\mu_n$ with μ_n real. Each diagonal entry of A becomes $e^{i\theta(i\mu_n)}$ $= e^{-\theta\mu_n}$, a plain real number, not a phase. Since $\theta\mu_n$ need not be zero, this entry does not sit on the unit circle in general, so A is not unitary here, and statement (B) is FALSE.

Step 5: Hermiticity in the anti-Hermitian case.

Every diagonal entry of A works out to the real number $e^{-\theta\mu_n}$, and the basis used is orthonormal, so A is a diagonal matrix with real entries in an orthonormal basis. Such a matrix always equals its own adjoint, so A is Hermitian. This confirms statement (D) is TRUE.

Final Answer:

Diagonalizing B shows directly that (A), (C) and (D) hold while (B) fails.

$$\boxed{\text{A, C, D}}$$

Quick Tip: Test each case using $(e^{\hat{X}})^\dagger = e^{\hat{X}^\dagger}$ and $\text{Det}(e^{\hat{X}}) = e^{\text{Tr}(\hat{X})}$.
A Hermitian generator times i becomes anti-Hermitian, so its exponential is unitary; an anti-Hermitian generator times i becomes Hermitian, so its exponential is Hermitian.

• • •

. Consider operators $\hat{A}$, $\hat{B}$, and $\hat{C}$ for three observables of a quantum system satisfying $[\hat{A}, \hat{B}] = 0$, $[\hat{B}, \hat{C}] = 0$, and $[\hat{A}, \hat{C}] \neq 0$, with uncertainties $\Delta A, \Delta B, \Delta C$, respectively. From the options given below, which is/are implied by the commutation relations among $\hat{A}$, $\hat{B}$, and $\hat{C}$?

(A) $\Delta A\, \Delta B > 0$

(B) $\Delta A\, \Delta C > 0$

(C) $\hat{A}, \hat{B}$ can be simultaneously diagonalized.

(D) $\hat{A}, \hat{B}, \hat{C}$ can be simultaneously diagonalized.

Correct Answer: (B) $\Delta A\, \Delta C > 0$; (C) $\hat{A}, \hat{B}$ can be simultaneously diagonalized.

SOLUTION:

Step 1: Translate each commutator into a physical statement.
A zero commutator means the two observables are compatible, they can be measured together with no fundamental limit, and their operators share eigenvectors. A non-zero commutator means the observables are incompatible, no state can be an exact eigenstate of both at once.

Step 2: Read off compatibility for each pair.
$\hat{A}$ and $\hat{B}$ are compatible ($[\hat{A}, \hat{B}] = 0$). $\hat{B}$ and $\hat{C}$ are compatible ($[\hat{B}, \hat{C}] = 0$). $\hat{A}$ and $\hat{C}$ are incompatible ($[\hat{A}, \hat{C}] \neq 0$).

Step 3: Check statement (C) using compatibility.
Compatible observables always admit a shared eigenbasis, that is the entire meaning of simultaneous diagonalization. Since $\hat{A}$ and $\hat{B}$ are compatible, this shared basis exists, so (C) is TRUE.

Step 4: Check statement (D) using compatibility.
A single basis that diagonalizes all of $\hat{A}, \hat{B}, \hat{C}$ together would in particular be a shared eigenbasis for $\hat{A}$ and $\hat{C}$ too, forcing $[\hat{A}, \hat{C}] = 0$. Since we are told the opposite, no such triple-diagonalizing basis can exist, so (D) is FALSE.

Step 5: Check statement (A) using compatibility.
$\hat{A}$ and $\hat{B}$ are compatible, meaning the system can sit in a state that is a simultaneous eigenstate of both, where each uncertainty is individually zero. That makes $\Delta A \, \Delta B$ equal to zero in such a state, so it is not always greater than zero, and (A) is FALSE.

Step 6: Check statement (B) using compatibility.
$\hat{A}$ and $\hat{C}$ are incompatible. Because no state exists in which both are simultaneously sharp, the system cannot reach $\Delta A = 0$ and $\Delta C = 0$ together, so their product $\Delta A \, \Delta C$ stays above zero. Statement (B) is TRUE.

Final Answer:
Reading each commutator as a compatible or incompatible pair of observables again shows (B) and (C) are the two statements that must hold.

$$\boxed{\text{B, C}}$$

Quick Tip: A zero commutator means the pair can share eigenstates, so their uncertainty product can hit zero; a non-zero commutator blocks a full common eigenbasis, so the pair's uncertainty product must stay positive. Apply this separately to each pair among $\hat{A}, \hat{B}, \hat{C}$.

• • •

. Consider the distribution of outcomes generated by N ($N \gg 1$) independent throws of (i) a coin or (ii) a six-sided dice. A coin (dice) is unbiased if both (all) its sides have equal probability to show up in a throw; it is biased otherwise. For the cases (i) and (ii) above, which of the following statements is/are true?

(A) The entropy of an unbiased coin is smaller than that of an unbiased dice.

(B) The entropy of an unbiased coin is greater than that of an unbiased dice.

(C) The entropy of a biased dice is smaller than that of an unbiased dice.

(D) The entropy of a biased coin is greater than that of an unbiased coin.

Correct Answer: (A) The entropy of an unbiased coin is smaller than that of an unbiased dice. ; (C) The entropy of a biased dice is smaller than that of an unbiased dice.

SOLUTION:

Step 1: Think in terms of counting typical sequences instead of a single-throw formula.

For N independent throws with k possible symbols, the number of distinct sequences that actually appear (the typical outcomes) grows like k^N when every symbol is equally likely, since each of the N throws freely picks any of the k symbols. The entropy per throw works out to about $\frac{1}{N}$log(number of typical sequences), so more freely mixed symbols means a bigger count of sequences and hence higher entropy.

Step 2: Apply the counting idea to the coin and the dice.
For the coin, each throw has 2 choices, so N throws give about 2^N equally probable sequences, and the entropy rate works out to $\log 2$. For the dice, each throw has 6 choices, so N throws give about 6^N equally probable sequences, giving entropy rate $\log 6$. Since 6^N is a far larger count than 2^N for the same N, the dice output is more spread out, hence more uncertain.

Step 3: Check statement (A).
Because $\log 2 < \log 6$, the coin's entropy rate is smaller than the dice's entropy rate. So (A) is TRUE.

Step 4: Check statement (B).
This claims the reverse ordering, which contradicts Step 3, so (B) is FALSE.

Step 5: Check statement (C).
Biasing the dice means some faces come up more often than others. The sequences of N throws then cluster around the favored faces instead of spreading over all 6^N possibilities equally, so the effective count of typical sequences shrinks. A smaller pool of typical sequences means lower entropy, so the biased dice has smaller entropy than the unbiased dice. Statement (C) is TRUE.

Step 6: Check statement (D).
By the same counting logic, biasing the coin favors one face over the other, shrinking the pool of typical sequences below the 2^N count of the unbiased coin. That means the biased coin's entropy is lower, not

higher, than the unbiased coin's. Statement (D) is FALSE.

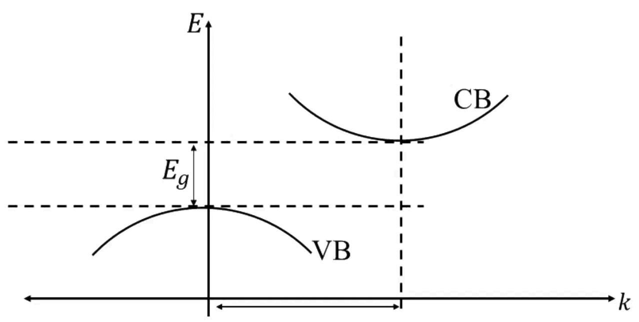
Final Answer:

Counting how many typical outcome sequences each source can produce again picks out (A) and (C) as correct.

$$\boxed{A, C}$$

Quick Tip: Entropy of k equally likely outcomes is $\log k$, and this is the maximum possible entropy for that many outcomes.
Compare $\log 2$ (coin) with $\log 6$ (dice), and remember any bias away from equal probabilities can only lower the entropy.

. The dispersion ($E(k)$) of the conduction band (CB) and valence band (VB) for a semiconductor are shown schematically in the figure. Considering the possibility of an electron making a transition from the bottom of the CB to the top of the VB, which of the following options is/are correct?



(A) The transition is forbidden.

(B) A photon can be emitted with an energy exactly equal to E_g.

(C) A photon can be emitted with an energy less than E_g.

(D) A phonon can be created with a crystal momentum $\hbar q$.

Correct Answer: (C) A photon can be emitted with an energy less than E_g. ; (D) A phonon can be created with a crystal momentum $\hbar q$.

SOLUTION:

Step 1: Write down conservation laws for the transition.
Energy conservation requires

$$E_{CB}(k = q) - E_{VB}(k = 0) = E_g$$

to be released, and crystal momentum must change from q to 0, a change of $\hbar q$. Any emitted particles in the process must together balance both of these.

Step 2: Test a photon-only process.
If only a photon of energy E_γ is emitted, energy conservation would need $E_\gamma = E_g$, while momentum conservation would need the photon momentum to equal $\hbar q$. A photon's momentum is $p = E_\gamma / c$, and for typical band gaps and lattice constants this is many orders of magnitude smaller than $\hbar q$. A photon-only process cannot satisfy both equations at once, so this pathway is ruled out, which is why statement (B), a photon energy of exactly E_g, fails as the actual process.

Step 3: Add a phonon to the balance.
Let a phonon of energy E_{ph} and crystal momentum $\hbar q$ also take part. Momentum conservation is now satisfied exactly, since the phonon supplies the full $\hbar q$ the photon could not reach. Energy conservation becomes $E_\gamma + E_{ph} = E_g$, so $E_\gamma = E_g - E_{ph}$.

Step 4: Read off the answer for statement (C).
Since $E_{ph} > 0$ for a real phonon, $E_\gamma = E_g - E_{ph}$ works out strictly

smaller than E_g. A photon can indeed be emitted with energy less than E_g, confirming statement (C) is TRUE, while statement (B) stays FALSE since the photon alone never carries the full E_g in this process.

Step 5: Read off the answer for statement (D).
The phonon brought in during Step 3 to fix the momentum balance carries crystal momentum $\hbar q$ by construction. Since this is exactly what closes the momentum equation from Step 1, such a phonon can indeed be created, confirming statement (D) is TRUE.

Step 6: Read off the answer for statement (A).
Steps 3 and 4 show a consistent process, a photon plus a phonon together, that satisfies both conservation laws. So the transition is not forbidden outright, it only needs this two-particle route. Statement (A) is FALSE.

Final Answer:
Balancing energy and crystal momentum separately shows the transition needs a phonon to go with the photon, matching (C) and (D).

$$\boxed{\text{C, D}}$$

Quick Tip: The CB minimum and VB maximum sit at different values of k in the figure, so the transition needs a change in crystal momentum of $\hbar q$. A photon alone cannot supply that momentum change, so a phonon must be created alongside it, and the photon then carries less than E_g.

. A symmetric rigid body has moment of inertia I_1, I_2, I_3 about its principal axes 1, 2, and 3, respectively, with $I_1 = I_3 = I_\perp$ and $I_2 \neq I_\perp$. It is rotating in space with no torque on it so that its angular momentum $\vec{L}$ is constant. Let $\omega_1, \omega_2, \omega_3$ be the components of its angular velocity along the principal axes 1, 2, and 3, respectively. Which of the following quantities is/are constant during the motion of this rigid body?

(A) $\omega_1 + \omega_3$

(B) $\omega_1^2 + \omega_3^2$

(C) Angle between axis 2 and $\vec{L}$

(D) ω_2

Correct Answer: (B) $\omega_1^2 + \omega_3^2$; (C) Angle between axis 2 and $\vec{L}$; (D) ω_2

SOLUTION:

Step 1: List the two conserved quantities available without touching Euler's equations directly.

For a torque-free rigid body, the space-frame angular momentum vector $\vec{L}$ is completely fixed in magnitude and direction, and the rotational kinetic energy T is conserved since no torque does any work. In body-axis components, using $I_1 = I_3 = I_\perp$, these read

$$L^2 = I_\perp^2(\omega_1^2 + \omega_3^2) + I_2^2\,\omega_2^2$$

$$2T = I_\perp(\omega_1^2 + \omega_3^2) + I_2\,\omega_2^2$$

with L^2 and T both constant in time.

Step 2: Solve this pair of equations for the two combinations that appear.

Treat $X = \omega_1^2 + \omega_3^2$ and $Y = \omega_2^2$ as unknowns in the linear system $I_\perp^2 X + I_2^2 Y = L^2$ and $I_\perp X + I_2 Y = 2T$. Since $I_\perp \neq I_2$, this system has a unique solution for X and Y in terms of the constants $L^2, T, I_\perp, I_2$. So both $X = \omega_1^2 + \omega_3^2$ and $Y = \omega_2^2$ come out as fixed numbers, not functions of time.

Step 3: Read off statements (B) and (D).
$X = \omega_1^2 + \omega_3^2$ being constant directly confirms statement (B) is TRUE. $Y = \omega_2^2$ being constant means ω_2 keeps a fixed magnitude throughout the motion; since ω_2 varies continuously in time from one starting value, it cannot jump in sign, so ω_2 itself stays constant, confirming statement (D) is TRUE.

Step 4: Check statement (A) using the same constants.
Knowing only that $\omega_1^2 + \omega_3^2$ is fixed does not fix the individual values of ω_1 and ω_3, they can still trade off with each other over time, tracing a circle of fixed radius in the ω_1-ω_3 plane. Their plain sum $\omega_1 + \omega_3$ is not one of the conserved combinations X or Y, and it changes as the body turns, so statement (A) is FALSE.

Step 5: Check statement (C) using L^2 and the body-frame components.
The angle θ between the symmetry axis (axis 2) and $\vec{L}$ satisfies $L_2 = L\cos\theta$, where $L_2 = I_2\omega_2$ is the body-frame component of $\vec{L}$ along axis 2. Since I_2 is fixed, ω_2 is constant (Step 3), and $L = |\vec{L}|$ is constant, the ratio $\cos\theta = I_2\omega_2/L$ is built from constants alone. So θ does not change with time, confirming statement (C) is TRUE.

Final Answer:
Using only the conservation of $\vec{L}$ and kinetic energy, without solving

Euler's equations directly, again shows that $\omega_1^2 + \omega_3^2$, ω_2, and the axis-2 to $\vec{L}$ angle stay fixed, while $\omega_1 + \omega_3$ does not.

$$\boxed{\text{B, C, D}}$$

Quick Tip: Since $I_1 = I_3$, Euler's equation for ω_2 has a zero coefficient, so ω_2 stays fixed.
That in turn fixes $\omega_1^2 + \omega_3^2$ (via the conserved L^2 and kinetic energy) and fixes the angle between axis 2 and $\vec{L}$, while $\omega_1 + \omega_3$ alone keeps oscillating.

• • •

. An alpha particle moves towards a fixed nucleus carrying charge Ze, with initial speed v_0 and impact parameter b. Starting from a large distance from the nucleus, its distance of closest approach is r_m and its speed there is v_m. Then which of the following options is correct?

$$\left(k = \frac{1}{4\pi\epsilon_0} \text{ and } r_0 = k\frac{Ze^2}{mv_0^2}\right)$$

(A) $v_0 b = v_m r_m$

(B) $v_0 b = 2v_m r_0$

(C) For $\frac{b}{r_0} \ll 1$, $r_m = 4r_0 + \frac{b^2}{2r_0}$, ignoring higher order corrections in $\frac{b}{r_0}$

(D) For $\frac{b}{r_0} \ll 1$, $r_m = 4r_0 + \frac{b^2}{8r_0}$, ignoring higher order corrections in $\frac{b}{r_0}$

Correct Answer: (A) $v_0 b = v_m r_m$

SOLUTION:

Step 1: Set up the orbit as a hyperbola.

For a repulsive inverse square force, the alpha particle path around the nucleus is one branch of a hyperbola with the nucleus at the outer focus. Two constants fix this hyperbola: the semi latus rectum l, set by the angular momentum, and the eccentricity e, set by the energy.

Step 2: Find the semi latus rectum l.

For a force $F = k'/r^2$ with $k' = 2kZe^2$ (the alpha particle carries charge $2e$), the orbit relation gives $l = L^2/(mk')$, where $L = mv_0 b$ is the conserved angular momentum. Since $r_0 = kZe^2/(mv_0^2)$ gives $k' = 2r_0 m v_0^2$,

$$l = \frac{(mv_0 b)^2}{m(2r_0 m v_0^2)} = \frac{b^2}{2r_0}$$

Step 3: Find the eccentricity e.

The eccentricity of this scattering hyperbola is $e = \sqrt{1 + \frac{2EL^2}{mk'^2}}$, with total energy $E = \frac{1}{2}mv_0^2$. Substituting $L = mv_0 b$ and $k' = 2r_0 m v_0^2$,

$$e = \sqrt{1 + \frac{2\left(\frac{1}{2}mv_0^2\right)(mv_0 b)^2}{m(2r_0 m v_0^2)^2}} = \sqrt{1 + \frac{b^2}{4r_0^2}}$$

Step 4: Write the closest approach distance and check $b = 0$.

For the repulsive branch, $r_m = l/(e-1)$. At $b = 0$ this is $0/0$, so take the small b limit: $e - 1 \approx b^2/(8r_0^2)$, giving

$$r_m \to \frac{b^2/(2r_0)}{b^2/(8r_0^2)} = 4r_0 \quad \text{as } b \to 0$$

This already fixes the head on distance at $4r_0$, matching the constant term in options (C) and (D).

Step 5: Expand r_m to order b^2.

Carrying the expansion of $e-1$ one order further and dividing,

$$r_m = \frac{l}{e-1} = 4r_0 + \frac{b^2}{4r_0} + O(b^4)$$

The coefficient of b^2 is $1/(4r_0)$. Option (C) claims $1/(2r_0)$ and option (D) claims $1/(8r_0)$, so both are FALSE.

Step 6: Check the momentum relation directly.

The Coulomb force always points along the radius vector to the nucleus, so it exerts zero torque and angular momentum never changes: $mv_0 b = mv_m r_m$ at every point of the path, in particular at closest approach where the velocity is purely tangential. So $v_0 b = v_m r_m$ holds exactly for any b, confirming option (A). Option (B), $v_0 b = 2v_m r_0$, would need r_m to always equal $2r_0$, which Step 5 already rules out in general, so (B) is FALSE.

Final Answer:

Only option (A), the angular momentum relation, is an identity that holds for every impact parameter.

$$\boxed{\text{(A)}}$$

Quick Tip: Hint:

The Coulomb force on the alpha particle always points along the line to the nucleus, so it produces zero torque. Use that to relate v_0, b, v_m, r_m, then bring in energy conservation and expand for small b/r_0.

• • •

. Consider a particle of mass $m = 9.0 \times 10^{-5}$ kg and charge $q = 3.0 \times 10^{-4}$ C in a uniform electromagnetic field $\vec{E} = 2\,\hat{x}\,\text{V.m}^{-1}$, $\vec{B} = 3\,\hat{z}\,\text{V.m}^{-2}\text{.s}$. The particle is released from the coordinates $(0, 5\text{ m}, 0)$ at time $t = 0$. Starting from initial speed zero, it comes back to the y-axis for the first time at time t. The value of t in seconds (rounded off to two decimal places) is ________

Correct Answer: 0.63

SOLUTION:

Step 1: Picture the motion as a drift plus circular motion.

In crossed uniform fields $\vec{E} = E\hat{x}$ and $\vec{B} = B\hat{z}$, a charged particle's velocity always splits into a steady drift velocity $\vec{v}_d$ plus a circular rotation at the cyclotron frequency $\omega = qB/m$ around that drift. Here $\vec{v}_d = \vec{E} \times \vec{B}/B^2$ points along $-\hat{y}$ with size E/B.

Step 2: Locate the start on the velocity circle.

The particle starts at rest, $\vec{v}(0) = 0$. In velocity space this is a point on a circle of radius E/B centred at $\vec{v}_d = (0, -E/B)$; the origin sits at distance E/B directly above this centre, so the starting velocity is at the top of the circle. As time passes, the velocity vector sweeps around this circle at the steady rate ω.

Step 3: Relate v_x to the swept angle.

Measuring the angle $\phi = \omega t$ swept from the start, $v_x(t) = \frac{E}{B}\sin(\omega t)$: positive for $0 < \omega t < \pi$, negative for $\pi < \omega t < 2\pi$, and it returns to zero once every half circle ($\omega t = \pi, 2\pi, 3\pi, \ldots$).

Step 4: Connect this to the return on the y-axis.

The x-displacement is the running area under $v_x(t)$: $x(t)$ $= \frac{1}{\omega}\int_0^{\omega t} \frac{E}{B}\sin\phi\, d\phi = \frac{E}{B\omega}(1 - \cos\omega t)$. This is zero only when $\cos\omega t = 1$, a full turn, $\omega t = 2\pi, 4\pi, \ldots$ (the half turn at $\omega t = \pi$ gives $x = 2E/(B\omega)$)

$\neq 0$, the farthest point of the loop, not a return). So the particle is back on the y-axis only after one complete revolution of the velocity vector, at $t = 2\pi/\omega$.

Step 5: Put in the numbers.

$$\omega = \frac{qB}{m} = \frac{(3.0 \times 10^{-4}\ \text{C})(3\ \text{T})}{9.0 \times 10^{-5}\ \text{kg}} = 10\ \text{rad/s}$$

$$t = \frac{2\pi}{10} = 0.6283\ \text{s}$$

Final Answer:

This rounds to

$$\boxed{t = 0.63\ \text{s}}$$

Quick Tip: Hint:

A particle starting from rest in crossed $\vec{E}$ and $\vec{B}$ traces a cycloid: it returns to its starting x-line once every full cyclotron period $2\pi/\omega$ with $\omega = qB/m$. Solve the coupled v_x, v_y equations, or just find ω and use this period directly.

• • •

. An atom has two energy levels with energy difference 2.2 eV between them. A gas of these atoms has 8×10^{20} atoms in the upper state and 5×10^{20} atoms in the lower state. Ignoring spontaneous emission, the maximum possible energy released by this gas of atoms by stimulated emission is E Joules. The value of E (rounded off to one decimal place) is ________ ($e = 1.6 \times 10^{-19}$ C)

Correct Answer: 52.8

SOLUTION:

Step 1: Think in terms of the final equalized population.
Stimulated absorption and stimulated emission use the same rate constant here, so the process can only keep pushing atoms from the upper to the lower level while $N_2 > N_1$; it must stop exactly when the two populations become equal, since beyond that point absorption and emission balance and cancel. So instead of tracking the difference, look directly at the common final population both levels settle to.

Step 2: Find the final common population.
The total number of atoms, $N_1 + N_2 = 5 \times 10^{20} + 8 \times 10^{20} = 13 \times 10^{20}$, stays fixed since only redistribution between the two levels happens (no atoms enter or leave). When the populations equalize, each level ends up with half this total:

$$N_f = \frac{13 \times 10^{20}}{2} = 6.5 \times 10^{20}$$

Step 3: Find how many atoms moved down.
The upper level goes from 8×10^{20} down to 6.5×10^{20}, so the number of atoms that made the downward transition is

$$\Delta N = 8 \times 10^{20} - 6.5 \times 10^{20} = 1.5 \times 10^{20}$$

This matches checking the lower level too: it rises from 5×10^{20} to 6.5×10^{20}, a gain of 1.5×10^{20}, consistent with ΔN.

Step 4: Convert to energy in joules.

Each transferred atom releases one photon carrying the level gap of 2.2 eV, so

$$E = \Delta N \times 2.2\ \text{eV} \times e = (1.5 \times 10^{20})(2.2)(1.6 \times 10^{-19})$$

$$E = 3.3 \times 10^{20} \times 1.6 \times 10^{-19} = 52.8\ \text{J}$$

Final Answer:

Rounded to one decimal place,

$$\boxed{E = 52.8\ \text{J}}$$

Quick Tip: **Hint:**
Stimulated absorption and stimulated emission share the same rate constant, so the net process stops once the populations equalize. Find how many atoms must move from the upper to the lower state for that to happen, then multiply by the photon energy.

• • •

. The Hamiltonian of two interacting spin-1/2 particles is $H = \frac{A}{\hbar^2}\vec{S}_1 \cdot \vec{S}_2$, where $\vec{S}_1$ and $\vec{S}_2$ are the spin angular momenta of particles 1 and 2, respectively. Here, $A = 10.56$ eV. The energy in eV required to induce an excitation from the ground state to the excited state (rounded off to two decimal places) is ________

Correct Answer: 10.56

SOLUTION:

Step 1: Rewrite the Hamiltonian using Pauli matrices.

For a spin-1/2 particle, $\vec{S}_i = \frac{\hbar}{2}\vec{\sigma}_i$, where $\vec{\sigma}_i$ are the Pauli matrices. So

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{\hbar^2}{4}\,\vec{\sigma}_1 \cdot \vec{\sigma}_2, \qquad H = \frac{A}{4}\,\vec{\sigma}_1 \cdot \vec{\sigma}_2$$

Step 2: Get the eigenvalues of $\vec{\sigma}_1 \cdot \vec{\sigma}_2$ from $(\vec{\sigma}_1 + \vec{\sigma}_2)^2$.

Since each $\vec{\sigma}_i^2 = 3$ (as $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$),

$$(\vec{\sigma}_1 + \vec{\sigma}_2)^2 = \vec{\sigma}_1^2 + \vec{\sigma}_2^2 + 2\vec{\sigma}_1 \cdot \vec{\sigma}_2 = 6 + 2\vec{\sigma}_1 \cdot \vec{\sigma}_2$$

The left side is the total spin operator built from Pauli matrices, with eigenvalue $4S(S+1)$ for total spin S: this is 0 for the singlet ($S = 0$) and 8 for the triplet ($S = 1$).

Step 3: Solve for $\vec{\sigma}_1 \cdot \vec{\sigma}_2$ in each case.

Singlet: $0 = 6 + 2\vec{\sigma}_1 \cdot \vec{\sigma}_2 \Rightarrow \vec{\sigma}_1 \cdot \vec{\sigma}_2 = -3$.

Triplet: $8 = 6 + 2\vec{\sigma}_1 \cdot \vec{\sigma}_2 \Rightarrow \vec{\sigma}_1 \cdot \vec{\sigma}_2 = 1$.

Step 4: Turn these into energies.

$$E_{singlet} = \frac{A}{4}(-3) = -\frac{3A}{4}, \qquad E_{triplet} = \frac{A}{4}(1) = \frac{A}{4}$$

With $A = 10.56$ eV > 0, the singlet sits lower, so it is the ground state and the triplet is the excited state.

Step 5: Find the gap.

$$\Delta E = \frac{A}{4} - \left(-\frac{3A}{4}\right) = A = 10.56 \text{ eV}$$

Final Answer:

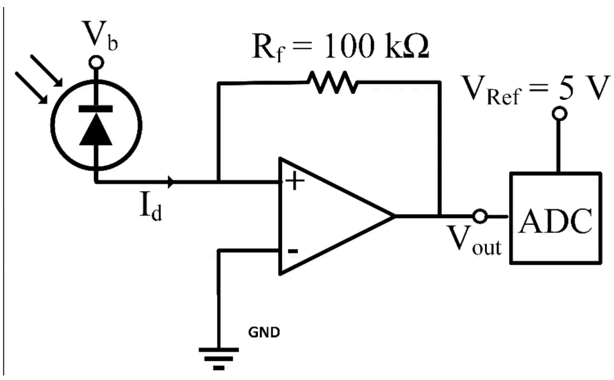
Rounded to two decimal places,

$$\boxed{\Delta E = 10.56 \text{ eV}}$$

Quick Tip: **Hint:**
Use $\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}(S^2 - S_1^2 - S_2^2)$ with total spin $S = 0$ (singlet, ground state) or $S = 1$ (triplet, excited state), then subtract the two energies.

• • •

. The output signal (current I_d) of a reversed biased (with a voltage V_b) photodiode, on which light is incident, is fed to an amplifier (see figure). The output voltage is digitized by a 10 bit Analogue to Digital convertor (ADC) which has a reference voltage of 5 V. The smallest current which can be measured by the circuit in nano-Amperes (rounded off to one decimal place) is ________



Correct Answer: 48.8

SOLUTION:

Step 1: Find the full scale current the ADC can register.

Since $V_{out} = I_d R_f$ from the transimpedance stage, the largest current the ADC can fully register corresponds to its full reference voltage:

$$I_{d,max} = \frac{V_{Ref}}{R_f} = \frac{5\ \text{V}}{100 \times 10^3\ \Omega} = 5 \times 10^{-5}\ \text{A} = 50000\ \text{nA}$$

Step 2: Split this range into the ADC steps.

A 10 bit converter divides its full input range into $2^{10} = 1024$ equal current steps, since V_{out} and I_d are directly proportional through the fixed resistor R_f. So each digital step, the smallest change the ADC can register, corresponds to a current of

$$I_{d,min} = \frac{I_{d,max}}{1024} = \frac{50000\ \text{nA}}{1024}$$

Step 3: Compute the value.

$$I_{d,min} = 48.828\ \text{nA}$$

This is exactly the same number as scaling the reference voltage directly by R_f and the bit count, just reached by normalizing the full scale current instead of the full scale voltage.

Final Answer:

Rounded to one decimal place,

$$\boxed{I_{d,min} = 48.8\ \text{nA}}$$

Quick Tip: Hint:

The transimpedance stage gives $V_{out} = I_d R_f$. Find the ADC least significant bit, $V_{Ref}/2^{10}$, then divide by R_f to get the smallest resolvable current.

• • •

. A capacitor is made of two circular metal plates of radius 1 m, separated by a distance $d = 1$ mm. The space between the plates is filled with a dielectric of permittivity $\epsilon_r = 5$. The capacitor is connected to a voltage $V = 10\sin(2\pi \times 10^6 \times t)$ volts, where t is in seconds. The maximum value of the magnetic field between the plates, at a radial distance $r = 0.5$ m from the centre, is $B \times 10^{-6}$ T. The value of B (rounded off to two decimal places) is ________.
(Speed of light in vacuum $c = 3 \times 10^8$ m.s^{-1})

Correct Answer: 0.87

SOLUTION:

Step 1: Set up the electric field between the plates.
Treat the capacitor as an ideal parallel plate capacitor with a uniform field $E(t) = V(t)/d$ between the plates. This is a fair approximation near the axis, well inside the plate edges.

Step 2: Find the enclosed displacement current.
The displacement current density is $J_d = \epsilon\, dE/dt$, where $\epsilon = \epsilon_r\epsilon_0$. For an Amperian loop of radius r centred on the axis, the enclosed displacement current is:

$$I_{d,enc} = \epsilon \frac{dE}{dt}(\pi r^2) = \epsilon_r\epsilon_0 \frac{r^2}{d}\frac{dV}{dt}\pi$$

Step 3: Apply Ampere's law to get B.
Ampere's law gives $B(2\pi r) = \mu_0 I_{d,enc}$, so:

$$B = \frac{\mu_0 I_{d,enc}}{2\pi r} = \frac{\mu_0 \epsilon_r \epsilon_0\, r}{2d}\frac{dV}{dt}$$

Using $\mu_0\epsilon_0 = 1/c^2$ (with the value of c given in the question) avoids needing ϵ_0 separately:

$$B = \frac{\epsilon_r r}{2dc^2}\frac{dV}{dt}$$

Step 4: Substitute the peak rate of change of voltage.
$V(t) = 10\sin(\omega t)$ with $\omega = 2\pi \times 10^6$ rad/s, so the largest value dV/dt can take is $10\omega = 2\pi \times 10^7$ V/s. Plugging in $\epsilon_r = 5$, $r = 0.5$ m, $d = 10^{-3}$ m and $c = 3 \times 10^8$ m/s:

$$B_{max} = \frac{5 \times 0.5}{2 \times 10^{-3} \times (3 \times 10^8)^2} \times 2\pi \times 10^7 = 8.73 \times 10^{-7}\ \text{T}$$

Final Answer:
Written in the form asked in the question, $B_{max} = 0.87 \times 10^{-6}$ T.

$$\boxed{B = 0.87}$$

Quick Tip: **Hint:**
Use the Ampere-Maxwell law with the displacement current between the plates: $B = \frac{\epsilon_r r}{2dc^2}\frac{dV}{dt}$, and use the maximum rate of change of the given sinusoidal voltage.

• • •

. Rotational spectrum of a diatomic molecule consists of lines of equal spacing with an interval of 20.0 cm^{-1}. Its moment of inertia is found to be $I_0 \times 10^{-47}$ kg.m^2, the value of I_0 (rounded off to one decimal place) is

______.

($h = 6.6 \times 10^{-34}$ J.s, speed of light in vacuum $c = 3 \times 10^8$ m.s^{-1})

Correct Answer: 2.8

SOLUTION:

Step 1: Recall how rotational lines are spaced.

In the rigid rotor model, the pure rotational absorption lines of a diatomic molecule sit at $\tilde{\nu}_J = 2B_e(J+1)$ for $J = 0, 1, 2, \ldots$, so any two neighbouring lines are separated by exactly $2B_e$, a constant independent of J. This is why the spectrum shows evenly spaced lines.

Step 2: Write the spacing directly in terms of I.

Since $B_e = h/(8\pi^2 cI)$, the spacing itself can be written directly in terms of the moment of inertia, without solving for B_e as a separate number:

$$\Delta\tilde{\nu} = 2B_e = \frac{h}{4\pi^2 cI}$$

Step 3: Solve for I.

Rearranging:

$$I = \frac{h}{4\pi^2 c\,\Delta\tilde{\nu}}$$

This single expression uses the measured line spacing directly, so there is no need to compute B_e as an intermediate step.

Step 4: Substitute the numbers.

With $h = 6.6 \times 10^{-34}$ J.s, $c = 3 \times 10^{10}$ cm/s (converted to cm/s since $\Delta\tilde{\nu}$ is in cm^{-1}), and $\Delta\tilde{\nu} = 20.0$ cm^{-1}:

$$I = \frac{6.6 \times 10^{-34}}{4\pi^2 \times 3 \times 10^{10} \times 20.0} = \frac{6.6 \times 10^{-34}}{2.369 \times 10^{13}} = 2.79 \times 10^{-47} \text{ kg.m}^2$$

Final Answer:

Rounded to one decimal place, $I_0 = 2.8$.

$$\boxed{I_0 = 2.8}$$

Quick Tip: **Hint:**
The spacing between adjacent rotational lines equals $2B_e$, and $B_e = h/(8\pi^2 cI)$.
Use c in cm/s since the spacing is given in cm^{-1}.

• • •

. Copper has an electron number density of 8.3×10^{28} m^{-3}. Its Fermi energy in eV (rounded off to one decimal place) is ________.
($\hbar = 1.06 \times 10^{-34}$ J.s, mass of electron $m_e = 9.10 \times 10^{-31}$ kg, charge of electron $= 1.60 \times 10^{-19}$ C)

Correct Answer: 7.0

SOLUTION:

Step 1: Find the Fermi wave vector first.

Instead of jumping straight to the energy formula, first find the radius of the Fermi sphere in k-space, k_F, which is fixed by how many electron states must fit inside it:

$$k_F = (3\pi^2 n)^{1/3}$$

Step 2: Evaluate k_F numerically.

With $n = 8.3 \times 10^{28}\ \mathrm{m}^{-3}$:

$$3\pi^2 n = 2.458 \times 10^{30}\ \mathrm{m}^{-3}$$

$$k_F = (2.458 \times 10^{30})^{1/3} = 1.350 \times 10^{10}\ \mathrm{m}^{-1}$$

Step 3: Get the Fermi energy from k_F.

A free electron with wave vector k_F has energy $E_F = \hbar^2 k_F^2/(2m_e)$, treating the electron like a free particle carrying momentum $\hbar k_F$.

$$E_F = \frac{(1.06 \times 10^{-34})^2 \times (1.350 \times 10^{10})^2}{2 \times 9.10 \times 10^{-31}}$$

$$E_F = \frac{1.1236 \times 10^{-68} \times 1.822 \times 10^{20}}{1.82 \times 10^{-30}} = 1.125 \times 10^{-18}\ \mathrm{J}$$

Step 4: Convert to electron volts.

$1\ \mathrm{eV} = 1.60 \times 10^{-19}\ \mathrm{J}$, so:

$$E_F = \frac{1.125 \times 10^{-18}}{1.60 \times 10^{-19}} = 7.03\ \mathrm{eV}$$

Final Answer:

Rounded to one decimal place, the Fermi energy of copper is 7.0 eV.

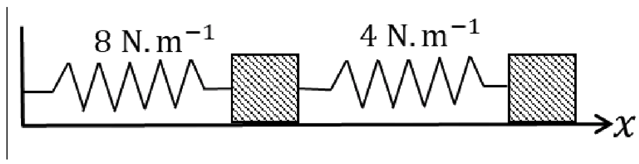
$$\boxed{E_F = 7.0 \text{ eV}}$$

Quick Tip: Hint:

Use $E_F = \frac{\hbar^2}{2m_e}(3\pi^2 n)^{2/3}$, then convert the answer from joules to electron volts by dividing by the electron charge.

• • •

. Two 1 kg blocks are connected to two massless springs of spring constants 8 N.m^{-1} and 4 N.m^{-1}. The system is kept on a frictionless horizontal floor with one end of a spring attached to a wall (see figure below). They are performing oscillatory motion along the x-axis with the normal mode frequencies ω_H and ω_L ($\omega_H > \omega_L$). The ratio $\frac{\omega_H}{\omega_L}$ (rounded off to two decimal places) is ______



Correct Answer: 2.41

SOLUTION:

Step 1: Write the kinetic and potential energy.

Take x_1, x_2 as the displacements of the two blocks from equilibrium. The kinetic energy of the system is:

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$$

The potential energy stored in the two springs (wall-to-block-1, and block-1-to-block-2) is:

$$U = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2$$

Step 2: Get the equations of motion from the Lagrangian.

With $L = T - U$, the Euler-Lagrange equations $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0$ give:

$$m\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0, \qquad m\ddot{x}_2 - k_2x_1 + k_2x_2 = 0$$

This is the same as writing $M\ddot{\mathbf{x}} = -K\mathbf{x}$ with mass matrix $M = mI$ and stiffness matrix $K = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}$.

Step 3: Solve the generalized eigenvalue problem.

Normal modes need $\det(K - \omega^2 M) = 0$. With $m = 1$, $k_1 = 8$, $k_2 = 4$:

$$\det\begin{pmatrix} 12 - \omega^2 & -4 \\ -4 & 4 - \omega^2 \end{pmatrix} = 0$$

$$(12 - \omega^2)(4 - \omega^2) - 16 = 0 \implies \omega^4 - 16\omega^2 + 32 = 0$$

Step 4: Solve for the two roots.

$$\omega^2 = 8 \pm 4\sqrt{2}$$

so $\omega_H = \sqrt{8+4\sqrt{2}} \approx 3.70$ rad/s and $\omega_L = \sqrt{8-4\sqrt{2}} \approx 1.53$ rad/s.

Final Answer:

The ratio is $\omega_H/\omega_L = \sqrt{(8+4\sqrt{2})/(8-4\sqrt{2})} = 1+\sqrt{2} \approx 2.41.$

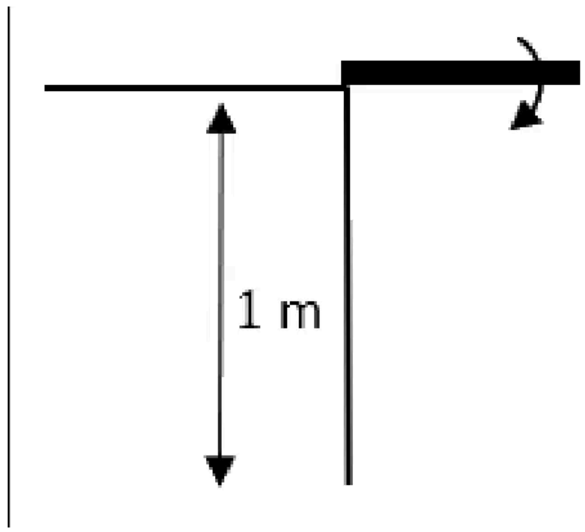
$$\boxed{\frac{\omega_H}{\omega_L} = 2.41}$$

Quick Tip: **Hint:**
Write Newton's second law for both blocks, assume $x_i \propto e^{i\omega t}$, and set the determinant of the resulting 2×2 system to zero to get $\omega^4 - 16\omega^2 + 32 = 0$.

• • •

. A 15 cm long scale is held horizontally with one of its ends on the edge of a 1 m high table and the other end resting on one's index finger. As the finger is removed (see figure below), the scale starts rotating about its end on the table. After 0.1 s, during which it has rotated by a negligibly small angle but has gained a rotational speed, it leaves the table and falls vertically towards the ground. When its centre of mass has fallen by 0.5 m, it has rotated by an angle θ. The value of θ in degrees (rounded off to one decimal place) is ________

$(g = 9.8\ m.s^{-2})$



Correct Answer: 179.4

SOLUTION:

Step 1: Get the spin rate at the moment of separation.

Before it leaves the table, the scale swings about the table edge like a rod pivoted at one end. Newton's law for rotation gives $\alpha = \tau/I$, and for a rod of length L pivoted at an end, $I = mL^2/3$ while gravity supplies torque $\tau \approx mgL/2$ (valid since the turn angle stays small in this stage). This gives a constant angular acceleration:

$$\alpha = \frac{3g}{2L} = \frac{3(9.8)}{2(0.15)} = 98\ \text{rad/s}^2$$

Step 2: Use $\omega = \alpha t$ over the given 0.1 s window.
Starting from rest, after the stated $t_1 = 0.1$ s the rod has picked up a spin rate:

$$\omega_0 = (98)(0.1) = 9.8 \text{ rad/s}$$

This is the value carried over into the next stage, since the pivot lets go right at this instant.

Step 3: Treat the second stage as independent free fall plus steady spin.
Once separated, the only force on the rod is gravity acting through its centre of mass, so there is zero net torque about the centre of mass and the spin rate ω_0 stays fixed for the rest of the motion. The vertical drop of the centre of mass, since the rotation up to separation is negligible, can be treated on its own using simple free fall from rest:

$$0.5 = \frac{1}{2}(9.8)t_2^2 \implies t_2 = \sqrt{\frac{1}{9.8}} = 0.3194 \text{ s}$$

Step 4: Combine the two to get the angle turned.
Because the spin rate is constant through this stage, the angle swept equals the product of the spin rate and this time, independent of the vertical motion itself:

$$\theta = \omega_0 t_2 = (9.8)(0.3194) = 3.130 \text{ rad}$$

Converting radians to degrees using $180/\pi$:

$$\theta = 3.130 \times 57.3^\circ = 179.4^\circ$$

Final Answer:

By the time the centre of mass has dropped 0.5 m, the scale has rotated by about 179.4°, which is nearly half a full turn.

$$\boxed{\theta = 179.4^\circ}$$

Quick Tip: **Hint:**

Find the angular speed gained while pivoted using $\alpha = 3g/(2L)$ and $\omega_0 = \alpha t_1$, then find the time to fall 0.5 m from rest, and multiply that time by ω_0 to get the angle turned.