# GATE 2026 Statistics Question Paper with Solutions

ZOLLEGE

## General Instructions

**(i)** This booklet contains 65 questions, each provided with a complete, step-by-step solution.

**(ii)** It comprises 35 single-correct multiple-choice questions, 13 multiple-correct questions and 17 numerical / integer-type questions.

**(iii)** These questions follow the GATE Statistics examination pattern.

**(iv)** Questions carry 1 or 2 marks. MCQs have negative marking (1/3 or 2/3 of the marks); MSQ and numerical-answer (NAT) questions carry no negative marking.

**(v)** GATE is a 3-hour paper of 65 questions (100 marks).

**(vi)** A virtual scientific calculator is provided; physical calculators are not allowed.

**(vii)** Attempt each question on your own before reviewing the given solution.

**(viii)** For multiple-correct questions, all correct options must be selected to earn full marks.

**(ix)** For numerical questions, report the answer rounded exactly as asked.

**1.** The antonym of the word protagonist is __________.

(A) agnostic

(B) antagonist

(C) arsonist

(D) anarchist

**Correct Answer:** (B) antagonist

**SOLUTION:**

**Step 1:** Break the word into parts.

The word "protagonist" comes from the Greek "protos" meaning first and "agonistes" meaning actor or competitor. So a protagonist is the lead character, the one the story is built around.

**Step 2:** Think about what an antonym needs to do.

An antonym must flip the core idea. Here the core idea is "the person the story favors and follows". The opposite idea is "the person who fights against that lead character".

**Step 3:** Match the prefix.

The prefix "ant-" or "anti-" means against. Adding it to "agonist" (a competitor) gives "antagonist", a competitor who stands against someone. This fits exactly what we need.

**Step 4:** Rule out the sound-alike options.

Agnostic, arsonist and anarchist all sound close to antagonist because they share letters and a similar ending, but their meanings point elsewhere. Agnostic is about doubt over religious belief. Arsonist is about setting fires. Anarchist is about rejecting government rule. None of them describe someone opposing a story's hero.

**Step 5:** Confirm the pick.

Only antagonist carries the built in sense of "against the main character", which is the true opposite of protagonist.

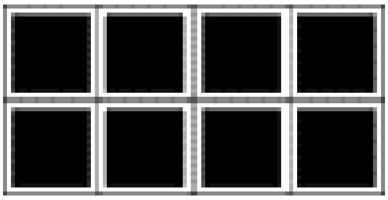
$$\boxed{\text{antagonist}}$$

**Quick Tip:** Think about the character in a story who opposes the hero; the prefix "ant-" means against.

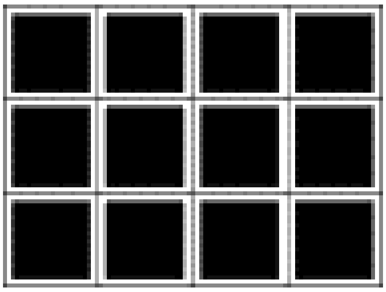
**2.** The figure shows two 4-tile patterns.

Either one or both of the patterns can be used any number of times and in any orientation to construct a new pattern. Which one of the options below **cannot** be constructed by using only these two 4-tile patterns, assuming there are no overlaps among them?

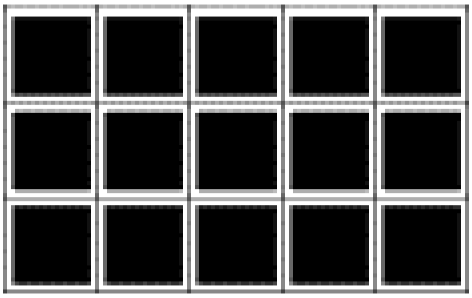
(A)



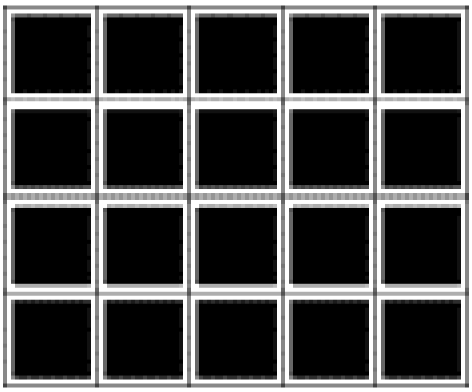
(B)



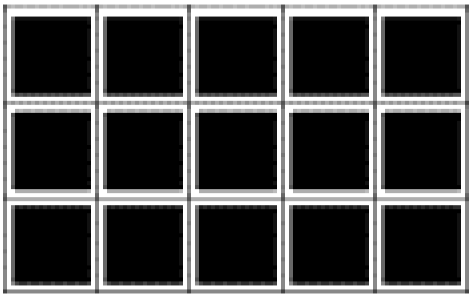
(C)



(D)



**Correct Answer:** (C)



**SOLUTION:**

**Step 1:** Work out the area covered by one piece.
Both given shapes, the small square and the straight bar, are made of exactly 4 unit tiles. This means every time we place one piece down, we add exactly 4 tiles to our picture, never more and never less.

**Step 2:** Turn this into a rule about totals.
If we use $k$ pieces in total, a mix of squares and bars in any orientation, the finished shape must contain exactly $4k$ tiles. So a valid shape always has a tile count that leaves remainder 0 when divided by 4.

**Step 3:** Work out the tile count of each option first, before worrying about shape.
Option (A) has $4 \times 2 = 8$ tiles. Option (B) has $4 \times 3 = 12$ tiles. Option (C) has $5 \times 3 = 15$ tiles. Option (D) has $5 \times 4 = 20$ tiles.

**Step 4:** Apply the divide by 4 test.
$8 \div 4 = 2$, exact. $12 \div 4 = 3$, exact. $20 \div 4 = 5$, exact. But $15 \div 4 = 3$ remainder 3, not exact.

**Step 5:** Interpret the failing case.
Since option (C) needs 15 tiles and no combination of 4-tile pieces can

ever sum to a number that is not a multiple of 4, option (C) fails before we even try to fit the pieces together. The other three options pass this test, and each can indeed be filled: squares stacked for (B), a row of squares for (A), and a mix for (D).

**Step 6:** State the conclusion.
The shape that cannot be built is the one whose total tile count is not a multiple of 4.

$$\boxed{\text{Option (C)}}$$

**Quick Tip:** Each piece always covers exactly 4 tiles, so check whether the total number of tiles in each option is a multiple of 4.

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**3.** Consider a knock-out women's badminton singles tournament where there are no ties. The loser in each game is eliminated from the tournament. Every player plays until she is defeated or remains the last undefeated player. The last undefeated player is declared the winner of the tournament. If there are 64 players in the beginning of the tournament, how many games should be played in total to declare the winner of the tournament?

(A) 127

(B) 64

(C) 63

(D) 32

**Correct Answer:** (C) 63

**SOLUTION:**

**Step 1:** Set up the rounds.

With 64 players, the first round pairs them into $64/2 = 32$ matches, so 32 players survive to round 2.

**Step 2:** Continue halving round by round.

Round 2 has $32/2 = 16$ matches. Round 3 has $16/2 = 8$ matches. Round 4 has $8/2 = 4$ matches. Round 5 has $4/2 = 2$ matches. Round 6, the final, has $2/2 = 1$ match.

**Step 3:** Add up every round's matches.

$$32 + 16 + 8 + 4 + 2 + 1$$

Adding these step by step: $32 + 16 = 48$, then $48 + 8 = 56$, then $56 + 4 = 60$, then $60 + 2 = 62$, then $62 + 1 = 63$.

**Step 4:** Cross check with a shortcut.

A single elimination bracket always needs one fewer game than the number of starting players, because every game removes exactly one player and only one player, the champion, is never removed. So the total is $64 - 1 = 63$, matching the round by round sum.

**Step 5:** Rule out the distractors.

64 would mean an extra pointless game is played after a winner is already found. 32 is just the first round alone. 127 does not match a simple knock-out format at all.

**Step 6:** Conclude.

$$\boxed{63}$$

**Quick Tip:** Each game eliminates exactly one player, and the tournament ends with exactly one player left undefeated.

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**4.** A student needs to enroll for a minimum of 60 credits. A student cannot enroll for more than 70 credits. The credits are divided amongst project and three distinct sets of courses namely, core courses, specialization courses, and elective courses. It is compulsory for a student to enroll for exactly 15 credits of core courses and exactly 20 credits of project. In addition, a student has to enroll for a minimum of 10 credits of specialization courses. The maximum credits of elective courses that a student can enroll for is _______

(A) 10

(B) 15

(C) 20

(D) 25

**Correct Answer:** (D) 25

**SOLUTION:**

**Step 1:** Fix the two exact numbers first.

Core courses always use exactly 15 credits, and project always uses exactly 20 credits. Together these two fixed parts use $15 + 20 = 35$ credits no matter what.

**Step 2:** See what is left for the flexible parts.

The overall cap on total credits is 70. So whatever is left for specialization plus elective together cannot cross $70 - 35 = 35$.

**Step 3:** Remember specialization has its own floor.
The question sets a rule that specialization credits cannot go below 10. This 10 is credits that elective can never grab, because specialization must keep at least that much for itself.

**Step 4:** Give the rest fully to elective.
Out of the 35 credits shared between specialization and elective, at least 10 must sit with specialization. That leaves at most $35 - 10 = 25$ for elective.

**Step 5:** Verify with real numbers.
Take core 15, project 20, specialization 10, elective 25. Adding all four gives $15 + 20 + 10 + 25 = 70$, right at the allowed ceiling and respecting every stated minimum. So 25 is genuinely achievable, not just a theoretical bound.

**Step 6:** Why smaller answers are wrong.
Picking 10, 15, or 20 for elective would mean specialization is holding more than its required minimum of 10 credits without any rule forcing it to. That does not break any constraint, but it fails to reach the true maximum, since specialization can safely be trimmed down to exactly 10 to free up more room for elective.

**Step 7:** Conclude.

$$\boxed{25}$$

**Quick Tip:** Fix core and project at their exact values, push specialization to its minimum of 10, then use the leftover credits for elective.

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**5.** 'When the teacher is in the room, all students stand silently.'
If the above statement is true, which one of the following statements is **not necessarily** true?

(A) If any student is not standing silently, then the teacher is not in the room.

(B) When the teacher is in the room, all students are silent.

(C) If all students are standing, then the teacher is in the room.

(D) When the teacher is in the room, all students are standing.

**Correct Answer:** (C) If all students are standing, then the teacher is in the room.

**SOLUTION:**

**Step 1:** Translate the sentence into an if-then form.
Let $T$ stand for "teacher is in the room" and $S$ stand for "all students stand silently". The given fact is the conditional $T \Rightarrow S$. A conditional only makes a promise in one direction, from $T$ towards $S$.

**Step 2:** Recall the safe moves with a conditional.
From $T \Rightarrow S$, we can safely form the contrapositive not $S \Rightarrow$ not $T$, which is always equally true. We can also split $S$ into its two parts, standing and being silent, and each part still follows whenever $T$ holds. What we cannot safely do is flip the arrow around to get $S \Rightarrow T$, since that is a different claim called the converse, and a conditional never guarantees its converse.

**Step 3:** Sort the four options by which move they use.

Option (A) is exactly the contrapositive of $T \Rightarrow S$, so it is safe and necessarily true. Option (B) only pulls out the silent half of $S$, still following from $T \Rightarrow S$, so it is safe. Option (D) only pulls out the standing half of $S$, again safe for the same reason.

**Step 4:** Examine option (C) closely.

Option (C) states that students standing forces the teacher to be present, which is the pattern $S \Rightarrow T$. This is the converse of the given rule. Nothing in the original sentence stops students from standing for a completely different reason when the teacher is away, so this converse claim can fail even though the original statement is true.

**Step 5:** Conclude which option is not guaranteed.

Since a conditional statement never guarantees its converse, option (C) is the one that is not necessarily true.

$$\boxed{\text{Option (C)}}$$

**Quick Tip:** A statement "if P then Q" does not guarantee "if Q then P"; watch for the option that reverses the original logic.

**6.** Combinatorics deals with problems involving counting. For example, "How many distinct arrangements of N distinct objects in M spaces on a circle are possible?" is a typical problem in combinatorics. This kind of counting is sometimes used in the modeling of several physical phenomena. Often, in such models, the different combinatorial possibilities are assigned probability values. Assigning probabilities enables the computation of the average values of physical quantities.

Consider the following statements:

P: Combinatorics is always invoked in the modeling of physical phenomena.

Q: Modeling some physical phenomena involves assigning probabilities to combinatorial possibilities in order to compute average values of physical quantities.

Based on the passage above, what can be inferred about statements P and Q?

(A) P is False and Q is False

(B) P is False and Q is True

(C) P is True and Q is False

(D) P is True and Q is True

**Correct Answer:** (B) P is False and Q is True

**SOLUTION:**

**Step 1:** Separate what the passage states as fact from what the statements claim.

The passage gives two pieces of information: combinatorics is "sometimes" used to model physical phenomena, and in such models probabilities are often assigned to combinatorial outcomes so that average values of physical quantities can be worked out.

**Step 2:** Test statement Q against the passage.
Q says modeling some physical phenomena uses probabilities on combinatorial possibilities to get average values. This is exactly what the passage describes in its last line. Nothing in Q goes beyond what the passage supports. So Q is TRUE.

**Step 3:** Test statement P against the passage.
P says combinatorics is always invoked in modeling physical phenomena. The word used in the passage is "sometimes", which is a weaker claim than "always". An "always" claim needs every case to hold, and the passage never says that. So P is FALSE.

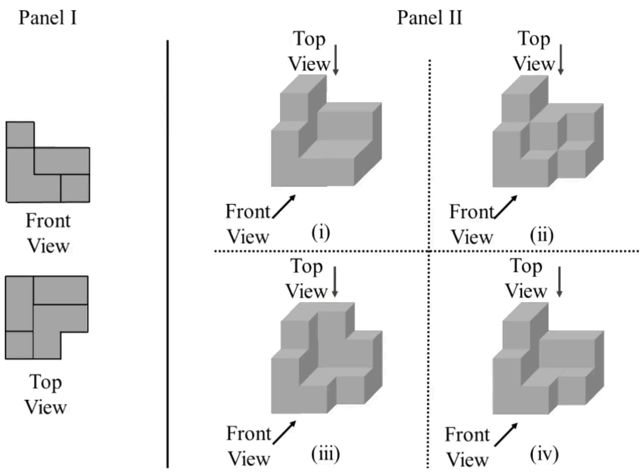
**Step 4:** Match with the answer choices.
We need P false and Q true. Checking the four choices, only one option pairs a false P with a true Q, and the rest either make P true or make Q false, both of which go against what we found.

**Step 5:** State the result.
P is false because the passage only claims sometimes, not always. Q is true because it restates the passage directly.

$$\boxed{\text{P is False and Q is True}}$$

**Quick Tip:** Look for words like "always" versus "sometimes" in the passage before accepting a strong claim like P.

**7.** In Panel I of the figure below, the front view and top view of a structure are shown. Which one of the 3D structures shown in Panel II possesses the views shown in Panel I?




(A) (i)

(B) (ii)

(C) (iii)

(D) (iv)

**Correct Answer:** (D) (iv)

**SOLUTION:**

**Step 1:** Fix the rule for matching a 3D block structure to its views.
A block structure is correct only if looking at it from the front gives the exact front view outline and looking at it from directly above gives the exact top view outline, both at the same time. A structure that matches only one of the two views is not the answer.

**Step 2:** Start with the top view this time.
Panel I's top view has a specific footprint with a notch cut into it. Any candidate whose footprint from above is a plain rectangle or is missing that notch can be dropped right away, since the footprint of a structure never changes no matter which way you look at it from the

side.

**Step 3:** Drop the options whose footprint is wrong.
Options (i) and (ii) both change the footprint, one by shifting a cube out of the notch area and one by adding a cube where Panel I has none. Neither reproduces the top view of Panel I, so both are out.

**Step 4:** Compare the remaining candidates using the front view.
Between the structures left standing, their side placement differs from Panel I once we look from the front. Structure (iii) places a cube one step back, which changes the staircase pattern of the front view, so its front outline no longer matches Panel I.

**Step 5:** Confirm the last candidate.
Structure (iv) is the one candidate left once the wrong top views and the wrong front view are removed. Checking it directly against Panel I, both its top view notch and its front view staircase line up exactly.

**Step 6:** State the result.
Working from the top view down to the front view leads to the same single answer.

$$\boxed{\text{(iv)}}$$

**Quick Tip:** Match the front view outline first to eliminate options, then confirm with the top view.

**8.** For positive real numbers $S$ and $K$, the function $H_K(S)$ is defined as:

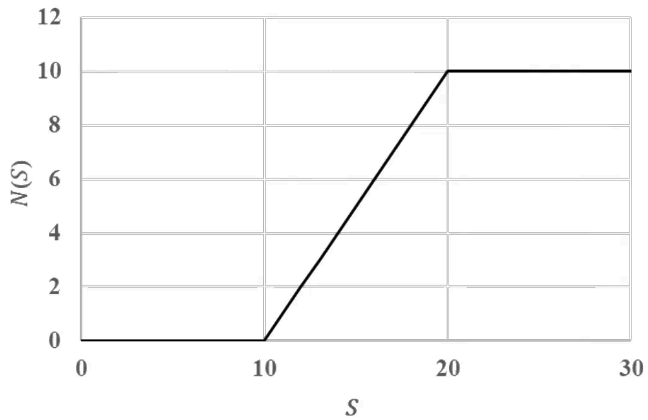
$$H_K(S) = \max(S - K, 0).$$

The max function is defined as:

$$\max(a, b) = \begin{cases} a, & \text{when } a > b \\ b, & \text{when } a \leq b \end{cases}$$

The graph below shows the plot of a function $N(S)$ versus $S$.

$N(S)$ can be expressed as ________.




(A) $H_{10}(S) - H_{20}(S)$

(B) $H_{10}(S) - 2H_{20}(S)$

(C) $-H_{10}(S) + H_{20}(S)$

(D) $H_{15}(S) - H_{20}(S)$

**Correct Answer:** (A) $H_{10}(S) - H_{20}(S)$

**SOLUTION:**

**Step 1:** Look at where the slope of $N(S)$ changes.

From the graph, the slope of $N(S)$ is 0 for $S$ between 0 and 10, then jumps to 1 for $S$ between 10 and 20, then drops back to 0 for $S$ beyond 20. So the slope changes by $+1$ at $S = 10$ and by $-1$ at $S = 20$.

**Step 2:** Recall the slope behavior of a single hinge function.

The function $H_K(S) = \max(S - K, 0)$ has slope 0 before $S = K$ and slope 1 after $S = K$. So adding a term $H_K(S)$ to a sum bumps the slope up by 1 right at $S = K$, and does nothing before that point.

**Step 3:** Build $N(S)$ as a sum of slope changes.

We need a +1 change in slope at $S = 10$ and a −1 change in slope at $S = 20$. A +1 change comes from adding $H_{10}(S)$, and a −1 change comes from subtracting $H_{20}(S)$. So

$$N(S) = H_{10}(S) - H_{20}(S)$$

**Step 4:** Check this against the value at $S = 20$.

At $S = 20$: $H_{10}(20) = 10$ and $H_{20}(20) = 0$, giving $N(20) = 10$, which matches the graph's flat height of 10 after the ramp. For $S > 20$ both terms grow at the same rate of 1 per unit, so their difference stays fixed at 10, matching the flat part of the graph.

**Step 5:** Eliminate the wrong options by their slope pattern.

$H_{10}(S) - 2H_{20}(S)$ has a slope change of −2 at $S = 20$, which sends the graph sloping downward for large $S$ instead of flattening out, so this is wrong. Reversing the signs to $-H_{10}(S) + H_{20}(S)$ gives a slope change of −1 at $S = 10$ and +1 at $S = 20$, the mirror image of what is needed, so this is wrong too. Using $H_{15}(S) - H_{20}(S)$ shifts the first hinge to $S = 15$ instead of $S = 10$, which does not match where the graph starts to rise.

**Step 6:** Conclude.

Only $H_{10}(S) - H_{20}(S)$ has the right hinge points and the right flat height.

$$\boxed{H_{10}(S) - H_{20}(S)}$$

**Quick Tip:** A ramp that starts rising at one point and flattens out at another can always be built as the difference of two shifted $\max(S - K, 0)$ functions.

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**9.** In the 2020 summer Olympics' Javelin throw finals, Neeraj Chopra exhibited a spectacular performance to win the gold medal. The silver medal was won by Jakub Vadlejch and the bronze medal was won by Vitezlav Vesely. There were six rounds of throws with each athlete having one throw per round. The best of all the throws of each athlete is considered for the medal. Following were the observations about the throws:

(i) The first and second rounds were dominated by Neeraj Chopra with a gold medal performance in his second throw, while the other two athletes did not have any medal winning throws in these rounds.

(ii) The throws in the last round by both Jakub Vadlejch and Vitezlav Vesely were fouls and were not considered for scoring.

(iii) After four rounds, Vitezlav Vesely was in the second position and could not improve upon his best throw in the succeeding rounds.

(iv) In the fourth round, the throw by Jakub Vadlejch was the best in that round.

In which round did Vitezlav Vesely have his best throw?

(A) Third

(B) Fourth

(C) Fifth

(D) Sixth

**Correct Answer:** (A) Third

**SOLUTION:**

**Step 1:** Set up the picture round by round for VV.
There are six rounds and we need to find which single round holds Vitezlav Vesely's (VV) best throw, the one that decided his bronze medal. Go through each clue and mark rounds as possible or impossible for VV.

**Step 2:** Mark rounds 1 and 2 as impossible.
The passage tells us rounds 1 and 2 belonged entirely to Neeraj Chopra, with the other two athletes, including VV, having no medal winning throw there. So rounds 1 and 2 are out for VV.

**Step 3:** Mark round 6 as impossible.
VV's round 6 throw was a foul. A foul throw carries no distance and cannot be the throw that won him bronze. So round 6 is out.

**Step 4:** Mark round 5 as impossible using the no-improvement clue.
We are told that after round 4, VV could not beat his own best throw in the rounds that came after, which covers rounds 5 and 6. So whatever his best throw was, it was already achieved by the end of round 4. Round 5 is out.

**Step 5:** Now only rounds 3 and 4 remain, so test round 4 directly.
Suppose, just to check, that VV's best throw came in round 4. We are told Jakub Vadlejch (JV) had the single best throw of round 4 among all three athletes, so JV beat VV in round 4. If VV's best throw is exactly his round 4 throw, and JV's round 4 throw beats it, then JV's best throw so far, which is at least his round 4 throw, already beats VV's best throw so far. That would place JV above VV in the standings after four rounds.

**Step 6:** Compare with the given standing after four rounds.
We are told VV stood second after four rounds, ahead of JV. This directly clashes with what Step 5 just produced, so the assumption in Step 5 must be false. VV's best throw is therefore not in round 4.

**Step 7:** Settle on the remaining round.
With rounds 1, 2, 4, 5 and 6 all ruled out, round 3 is the only one left for VV's best throw.

$$\boxed{\text{Third}}$$

**Quick Tip:** First narrow VV's best throw down to round 3 or round 4 using the medal and foul clues, then use the round 4 comparison with Jakub Vadlejch to see which one is actually possible.

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**10.** An unbiased six-faced dice whose faces are marked with numbers 1, 2, 3, 4, 5, and 6 is rolled twice in succession and the number on the top face is recorded each time. The probability that the number appearing in the second roll is an integer multiple of the number appearing in the first roll is ________.

(A) $\frac{1}{6}$

(B) $\frac{5}{18}$

(C) $\frac{7}{18}$

(D) $\frac{5}{6}$

**Correct Answer:** (C) $\frac{7}{18}$

**SOLUTION:**

**Step 1:** Picture all outcomes as a $6 \times 6$ grid.

Write the first roll $x$ along the rows and the second roll $y$ along the columns, both running from 1 to 6. Each of the 36 cells is equally likely, since the two rolls are independent and the dice is fair.

**Step 2:** Go row by row and mark the cells where $y$ is a multiple of $x$.

Row $x = 1$: every column from 1 to 6 works, since any number is a multiple of 1. That is 6 cells. Row $x = 2$: columns $2, 4, 6$ work. That is 3 cells. Row $x = 3$: columns $3, 6$ work. That is 2 cells. Row $x = 4$: only column 4 works. That is 1 cell. Row $x = 5$: only column 5 works. That is 1 cell. Row $x = 6$: only column 6 works. That is 1 cell.

**Step 3:** Total the marked cells.

Adding the row counts,

$$6 + 3 + 2 + 1 + 1 + 1 = 14$$

So 14 of the 36 equally likely cells satisfy the condition.

**Step 4:** Turn the count into a probability.

$$P = \frac{14}{36}$$

Divide the top and bottom by 2:

$$P = \frac{7}{18}$$

**Step 5:** Sanity check against the answer choices.

A quick check: $\frac{1}{6}$ equals $\frac{6}{36}$, which only accounts for the row $x = 1$

and misses everything else, so it is too small. $\frac{5}{6}$ equals $\frac{30}{36}$, which is far more than the 14 cells actually marked, so it is too big. Only $\frac{7}{18}$ matches the count of 14 out of 36.

**Step 6:** Conclude.

$$\boxed{\frac{7}{18}}$$

**Quick Tip:** Fix the first roll and count how many second-roll values are multiples of it, then add up over all six first-roll values.

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**11.** Consider the polynomial

$$p(t) = t(t-1)(t-2)$$

and define

$$F(x) = \int_{1/2}^{x} \frac{1}{p(t)}\,dt, \qquad x \in \left[\frac{1}{2}, 1\right).$$

Which of the following statements is correct?

(A) $F$ is strictly decreasing

(B) $F$ is strictly increasing

(C) $\lim_{x\to 1} F(x)$ exists

(D) $F'\left(\frac{3}{4}\right) = 1$

**Correct Answer:** (B) $F$ is strictly increasing

**SOLUTION:**

**Step 1:** What FTC gives us.

Since $F(x) = \int_{1/2}^{x} \frac{1}{p(t)}\, dt$ with $p(t) = t(t-1)(t-2)$, the Fundamental Theorem of Calculus tells us $F'(x) = \frac{1}{p(x)}$ at every point where $p(x) \neq 0$. So we just need to track the sign and size of $p$ on the interval $[1/2, 1)$.

**Step 2:** Sign check using a test point.

Take a test point $t = 0.75$ inside $(1/2, 1)$: $t = 0.75 > 0$, $t - 1 = -0.25 < 0$, $t - 2 = -1.25 < 0$. Multiplying a positive number by two negative numbers gives a positive result, so $p(0.75) > 0$. Since $p$ has no root inside the open interval $(1/2, 1)$ (its roots are exactly $0, 1, 2$), the sign of $p$ cannot flip inside this interval, so $p(t) > 0$ for every $t$ in $(1/2, 1)$.

**Step 3:** Read off monotonicity.

Because $p(t) > 0$ throughout $(1/2, 1)$, we get $F'(x) = 1/p(x) > 0$ there too. A function with a positive derivative on an interval is strictly increasing on it, so option (B) holds and option (A) is ruled out.

**Step 4:** Test the numeric claim in option (D).

At $x = 3/4$:

$$p(3/4) = \frac{3}{4} \cdot \left(-\frac{1}{4}\right) \cdot \left(-\frac{5}{4}\right) = \frac{15}{64}$$

so $F'(3/4) = 64/15$, about 4.27, not 1. Option (D) fails.

**Step 5:** Test whether the limit at $x = 1$ exists.

Write $p(t) = t(t-1)(t-2)$ and let $t \to 1^-$. Both $t$ and $(t-2)$ tend to the finite nonzero values 1 and $-1$, while $(t-1) \to 0^-$. So near $t = 1$, $p(t)$ behaves like a constant times $(t-1)$, a simple zero. This makes $1/p(t)$ blow up like $1/(1-t)$, and integrating $1/(1-t)$ up to $x \to 1$ gives a log term that diverges to infinity. So $F(x) \to \infty$, meaning the limit in option (C) does not exist as a finite number.

**Step 6:** Conclude.

Only the monotonicity claim survives all checks.

$$\boxed{\text{F is strictly increasing}}$$

**Quick Tip:** Check the sign of p(t) on the interval first: this fixes the sign of F'(x)=1/p(t), which tells you if F increases or decreases.

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**12.** Let $f : (-1, 1) \to \mathbb{R}$ be a differentiable function. Consider the following statements:

(I) Suppose $f(0) \geq 0$, and $f'(x) > 0$ whenever $f(x) = 0$, for any $x \geq 0$. Then $f(x) > 0$, for any $x > 0$.

(II) Suppose $f(0) \leq 0$, and $f'(x) > 0$ whenever $f(x) = 0$, for any $x \leq 0$. Then $f(x) < 0$, for any $x < 0$.

Which of the following statements is correct?

(A) Only statement (I) is correct

(B) Only statement (II) is correct

(C) Both statements (I) and (II) are correct

(D) Neither statement (I) nor statement (II) is correct

**Correct Answer:** (C) Both statements (I) and (II) are correct

**SOLUTION:**

**Step 1:** Restate what each claim is really saying.

Both statements describe a function that cannot come back down through zero with a non-positive slope. Statement (I) looks at $x \geq 0$ and rules out $f$ dropping to zero or below once it starts non-negative at 0. Statement (II) is the same idea but read backwards from 0 toward negative $x$.

**Step 2:** Set up statement (I) by contradiction.
Assume $f(0) \geq 0$ and $f'(x) > 0$ whenever $f(x) = 0$ for $x \geq 0$, but suppose $f(x_1) \leq 0$ for some $x_1 > 0$. Continuity of $f$ (it is differentiable) plus the Intermediate Value Theorem gives a smallest crossing point $x_0 \in (0, x_1]$ with $f(x_0) = 0$ and $f > 0$ just before $x_0$.

**Step 3:** Compare slopes at the crossing point.
Approaching $x_0$ from values where $f > 0$ and hitting $f(x_0) = 0$ means the function is falling into zero from above, so its derivative there cannot be positive: $f'(x_0) \leq 0$. This clashes with the rule $f'(x_0) > 0$ (since $f(x_0) = 0$ and $x_0 \geq 0$). The contradiction kills the assumption, so $f$ stays positive for every $x > 0$. If instead $f(0) = 0$ itself, the same rule at $x = 0$ forces $f'(0) > 0$, pushing $f$ positive just after 0, and the same crossing-point argument then keeps it positive forever after. So (I) holds.

**Step 4:** Prove (II) directly with the same style of argument, run leftward.
Assume $f(0) \leq 0$ and $f'(x) > 0$ whenever $f(x) = 0$ for $x \leq 0$. Suppose, for contradiction, $f(x_1) \geq 0$ for some $x_1 < 0$. By the Intermediate Value Theorem there is a value $x_0 \in [x_1, 0)$ closest to 0 with $f(x_0) = 0$, and $f < 0$ for all $x$ strictly between $x_0$ and 0.

**Step 5:** Check the slope at that crossing.
Since $f < 0$ immediately to the right of $x_0$ (between $x_0$ and 0) and $f(x_0) = 0$, the function rises from 0 down into negative values as $x$

moves away from $x_0$. This means the right-hand difference quotient satisfies $\frac{f(x)-f(x_0)}{x-x_0} \le 0$ as $x \to x_0^+$, so $f'(x_0) \le 0$. But the hypothesis of (II) demands $f'(x_0) > 0$ since $f(x_0) = 0$ and $x_0 \le 0$. This is a contradiction, so no such $x_1 < 0$ exists, meaning $f(x) < 0$ for all $x < 0$.

**Step 6:** Put both results together.

Both (I) and (II) survive their own contradiction test, so both statements are correct.

$$\boxed{\text{Both statements (I) and (II) are correct}}$$

**Quick Tip:** Use a contradiction argument: if f touches zero with a positive derivative there, it cannot fall back to zero or below right after, since that would force a non-positive slope at the crossing point. Reflect x to -x to turn statement (II) into statement (I).

• • •

**13.** Consider the following two subspaces of $\mathbb{R}^4$

$$W_1 = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 + x_4 = 0\}$$

$$W_2 = \{(x_1, x_2, x_3, x_4) : x_1 + 2x_2 + 3x_3 + 4x_4 = 0\}.$$

Which of the following statements is correct?

(A) $\dim(W_1 \cap W_2) = 0$

(B) $\dim(W_1 \cap W_2) = 1$

(C) $\dim(W_1 \cap W_2) = 2$

(D) $\dim(W_1 \cap W_2) = 3$

**Correct Answer:** (C) $\dim(W_1 \cap W_2) = 2$

**SOLUTION:**

**Step 1:** Think of $W_1$ and $W_2$ as hyperplanes with normal vectors.
$W_1$ is the set of vectors orthogonal to $n_1 = (1, 1, 1, 1)$, and $W_2$ is the set of vectors orthogonal to $n_2 = (1, 2, 3, 4)$. Each hyperplane in $\mathbb{R}^4$ has dimension 3, since one equation removes one degree of freedom.

**Step 2:** Use the dimension formula for an intersection of two hyperplanes.
For two hyperplanes through the origin in $\mathbb{R}^n$ given by independent normal vectors,

$$\dim(W_1 \cap W_2) = n - 2$$

as long as $n_1$ and $n_2$ point in genuinely different directions (are linearly independent). Here $n = 4$.

**Step 3:** Check that $n_1$ and $n_2$ are independent.
$n_1 = (1, 1, 1, 1)$ and $n_2 = (1, 2, 3, 4)$ would be dependent only if $n_2 = k\, n_1$ for some scalar $k$. Matching the first entries gives $k = 1$, but then $n_1$ scaled by 1 would need every entry equal to 1, while $n_2$ has entries $1, 2, 3, 4$ that are not all equal. So no such $k$ works, and $n_1, n_2$ are linearly independent.

**Step 4:** Apply the formula.

$$\dim(W_1 \cap W_2) = 4 - 2 = 2$$

**Step 5:** Confirm with an explicit parametrization.
Solving $x_1 + x_2 + x_3 + x_4 = 0$ and $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$ together, subtract to get $x_2 + 2x_3 + 3x_4 = 0$, so $x_2 = -2x_3 - 3x_4$, and then $x_1$

$= x_3 + 2x_4$. Treating $x_3, x_4$ as free parameters gives a 2-dimensional family of solutions, confirming the count.

**Step 6:** Compare with the answer choices.
A dimension of 0 or 1 would need the normals to force more independent constraints than we actually have, and a dimension of 3 would need $n_1$ and $n_2$ to be scalar multiples of each other, which Step 3 ruled out. Only dimension 2 fits.

$$\boxed{\dim(W_1 \cap W_2) = 2}$$

**Quick Tip:** Two independent linear equations in 4 unknowns cut down the solution space by 2 dimensions, so the intersection of the two hyperplanes has dimension 4-2=2.

• • •

**14.** Let $X$ and $Y$ be two independent discrete random variables such that the moment generating functions of $X$ and $X + Y$ are given by

$$M_X(t) = \frac{1 + 2e^{-t} + 3e^{2t}}{6}, \quad t \in \mathbb{R},$$

and

$$M_{X+Y}(t) = \frac{2 + e^{t} + 3e^{3t}}{6}, \quad t \in \mathbb{R},$$

respectively. Then which of the following statements is correct?

(A) $P(XY = 0) = \dfrac{1}{6}$

(B) $P(XY = 2) = \dfrac{1}{6}$

(C) $E(X) = 0$

(D) $\text{Var}(Y) = 1$

**Correct Answer:** (A) $P(XY = 0) = \dfrac{1}{6}$

**SOLUTION:**

**Step 1:** Split $M_X(t)$ into probability weighted exponentials.

Write $M_X(t) = \frac{1}{6} + \frac{2}{6}e^{-t} + \frac{3}{6}e^{2t}$. Matching each term $p\,e^{xt}$ to a point mass, $X$ is $-1$ with chance $1/3$, 0 with chance $1/6$, and 2 with chance $1/2$.

**Step 2:** Do the same for $M_{X+Y}(t)$.

$$M_{X+Y}(t) = \frac{2}{6} + \frac{1}{6}e^{t} + \frac{3}{6}e^{3t}$$

So $X + Y$ takes the value 0 with chance $1/3$, 1 with chance $1/6$, and 3 with chance $1/2$.

**Step 3:** Compare the two lists side by side.

$X$ values: $-1, 0, 2$ with weights $1/3, 1/6, 1/2$. $X + Y$ values: $0, 1, 3$ with the same weights $1/3, 1/6, 1/2$ in the same order. Each value of $X + Y$ is exactly 1 more than the matching value of $X$, and the weights line up perfectly. This pattern only happens if $Y$ adds a fixed shift of 1 to every outcome of $X$, that is, $Y$ is the constant 1.

**Step 4:** Confirm algebraically.

If $Y \equiv 1$, then $M_Y(t) = e^t$, and multiplying,

$$M_X(t)M_Y(t) = e^t \cdot \frac{1 + 2e^{-t} + 3e^{2t}}{6} = \frac{e^t + 2 + 3e^{3t}}{6}$$

which is exactly $M_{X+Y}(t)$ as given. So the guess checks out and $Y = 1$ with probability 1, meaning $\text{Var}(Y) = 0$.

**Step 5:** Since $Y$ is the constant 1, $XY = X$.

So the question reduces to reading probabilities of $X$ directly:

$$P(XY = 0) = P(X = 0) = \frac{1}{6}, \qquad P(XY = 2) = P(X = 2) = \frac{1}{2}$$

This confirms (A) is right and (B) is wrong (it claims 1/6 but the true value is 1/2).

**Step 6:** Check the remaining options quickly.

$$E(X) = -1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{6} + 2 \cdot \frac{1}{2} = \frac{2}{3} \neq 0$$

so (C) fails, and since $Y$ never varies, $\text{Var}(Y) = 0 \neq 1$, so (D) fails too.

**Step 7:** Conclude.

$$\boxed{P(XY = 0) = \tfrac{1}{6}}$$

**Quick Tip:** Read the distribution of X directly off its MGF, then divide the MGF of X+Y by the MGF of X to find the MGF of Y. A constant random variable equal to c has MGF e^{ct}.

• • •

**15.** Let $X$ and $Y$ be identically distributed random variables with variance $\sigma^2 \in (0, \infty)$. Then the correlation coefficient between $X$ and $Y$ is

(A) $1 - \dfrac{E(X-Y)^2}{2\sigma^2}$

(B) $1 - \dfrac{2E(X-Y)^2}{\sigma^2}$

(C) $1 - \dfrac{E(X-Y)^2}{\sigma^2}$

(D) $1 - \dfrac{E(X+Y)^2}{\sigma^2}$

**Correct Answer:** (A) $1 - \dfrac{E(X-Y)^2}{2\sigma^2}$

**SOLUTION:**

**Step 1:** Start from the definition of correlation.

Since $X$ and $Y$ are identically distributed, both have variance $\sigma^2$, so

$$\rho = \frac{\text{Cov}(X,Y)}{\sigma^2}$$

The whole problem is just about writing $\text{Cov}(X,Y)$ in terms of $E(X-Y)^2$.

**Step 2:** Expand $E(X-Y)^2$ directly using expectations.

$$E(X-Y)^2 = E(X^2) - 2E(XY) + E(Y^2)$$

**Step 3:** Rewrite each term using variance and covariance.

Since $X, Y$ have the same mean $\mu$, $E(X^2) = \sigma^2 + \mu^2$ and $E(Y^2) = \sigma^2 + \mu^2$, and $E(XY) = \text{Cov}(X,Y) + \mu^2$. Substituting,

$$E(X-Y)^2 = (\sigma^2 + \mu^2) - 2(\text{Cov}(X,Y) + \mu^2) + (\sigma^2 + \mu^2) = 2\sigma^2 - 2\,\text{Cov}(X,Y)$$

(the $\mu^2$ terms cancel completely, which makes sense since correlation should not depend on the common mean at all).

**Step 4:** Isolate the covariance.

$$\mathrm{Cov}(X, Y) = \sigma^2 - \frac{1}{2}E(X - Y)^2$$

**Step 5:** Divide by $\sigma^2$ to get $\rho$.

$$\rho = \frac{\mathrm{Cov}(X, Y)}{\sigma^2} = 1 - \frac{E(X - Y)^2}{2\sigma^2}$$

**Step 6:** Sanity check with a simple case.
If $X = Y$ exactly, then $\rho$ should be 1 (perfect correlation), and indeed $E(X - Y)^2 = E(0) = 0$, giving $\rho = 1 - 0 = 1$. This matches, which supports the formula.

**Step 7:** Rule out the other listed formulas.
Formula (B) has an extra factor of 4 compared to ours (it uses $2E(X - Y)^2/\sigma^2$ instead of $E(X - Y)^2/(2\sigma^2)$), which would break the sanity check above whenever $E(X - Y)^2 \neq 0$. Formula (C) is missing the factor of 2 in the denominator. Formula (D) uses $E(X + Y)^2$, which mixes in the mean $\mu$ and does not cancel the way our derivation needs. Only our derived formula survives.

**Step 8:** Conclude.

$$\boxed{\rho(X, Y) = 1 - \frac{E(X - Y)^2}{2\sigma^2}}$$

**Quick Tip:** Expand E(X-Y)^2 using Var(X-Y) and the fact that identically distributed variables have equal means and variances, then solve for the covariance and divide by sigma squared.

• • •

**16.** Let $X$ and $Y$ be independent and identically distributed normal random variables. If

$$P(X + 2Y \leq 3) = P(2X - Y \geq 4),$$

then $E(X)$ is

(A) $\frac{7}{3}$

(B) $\frac{7}{4}$

(C) $\frac{3}{7}$

(D) $\frac{4}{7}$

**Correct Answer:** (B) $\frac{7}{4}$

**SOLUTION:**

**Step 1:** Note what stays the same across the two combinations.

Call the common mean $\mu = E(X) = E(Y)$ and the common variance $\sigma^2$. Build $U = X + 2Y$ and $V = 2X - Y$. Because $X$ and $Y$ are independent normal variables, any linear combination of them is again normal, so both $U$ and $V$ are normal random variables.

**Step 2:** Compute variance first, since it turns out to be the shortcut.

$$Var(U) = 1^2 \cdot \sigma^2 + 2^2 \cdot \sigma^2 = 5\sigma^2$$

$$Var(V) = 2^2 \cdot \sigma^2 + (-1)^2 \cdot \sigma^2 = 5\sigma^2$$

Both variances come out equal to $5\sigma^2$. This match happens because $1^2 + 2^2 = 2^2 + 1^2$, and it is the reason the standardizing step below becomes so clean.

**Step 3:** Standardize both probability statements.

Since $U \sim N(3\mu, 5\sigma^2)$ and $V \sim N(\mu, 5\sigma^2)$, write each probability as a statement about a standard normal variable $Z$:

$$P(U \leq 3) = P\left(Z \leq \frac{3 - 3\mu}{\sigma\sqrt{5}}\right)$$

$$P(V \geq 4) = P\left(Z \geq \frac{4 - \mu}{\sigma\sqrt{5}}\right) = P\left(Z \leq \frac{\mu - 4}{\sigma\sqrt{5}}\right)$$

The second line uses the mirror symmetry of the standard normal curve around 0: the area to the right of a point equals the area to the left of its negative.

**Step 4:** Match the two z-values.

Because the given condition says these two probabilities are equal, and because the standard normal cdf never repeats a value, the two z-values behind them must themselves be equal:

$$\frac{3 - 3\mu}{\sigma\sqrt{5}} = \frac{\mu - 4}{\sigma\sqrt{5}}$$

**Step 5:** Clear the denominator and solve.

The $\sigma\sqrt{5}$ on each side cancels directly, so

$$3 - 3\mu = \mu - 4$$

Collect the $\mu$ terms on one side and the plain numbers on the other:

$$3 + 4 = 4\mu$$

$$\mu = \frac{7}{4}$$

**Step 6:** Rule out the distractors.

7/3 appears if you swap the roles of the coefficients 3 and 4 while collecting terms. 3/7 and 4/7 appear if the equation is inverted by mistake. None of these track the algebra correctly.

**Step 7:** State the result.

The common mean of $X$ and $Y$ works out to 7/4, so

$$\boxed{E(X) = \frac{7}{4}}$$

**Quick Tip:** $X + 2Y$ and $2X - Y$ are both normal with the same variance $5\sigma^2$. Equal probabilities mean equal standardized cutoffs, so set $\frac{3 - 3\mu}{\sqrt{5}} = \frac{\mu - 4}{\sqrt{5}}$ and solve for $\mu$.

• • •

**17.** Let $X$ be a random variable with the following probability density function

$$f(x) = \begin{cases} 4x^2 e^{-2x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

If $Y = \ln X$, then which of the following statements is correct?

(A) $E(Y)$ is not finite

(B) $E(Y) = \ln \frac{3}{2}$

(C) $E(Y) > \ln \frac{3}{2}$

(D) $E(Y) < \ln \frac{3}{2}$

**Correct Answer:** (D) $E(Y) < \ln \frac{3}{2}$

**SOLUTION:**

**Step 1:** Recognize the density as a Gamma distribution.

The density $f(x) = 4x^2 e^{-2x}$ for $x > 0$ is a Gamma density with shape $\alpha = 3$ and rate $\beta = 2$, since the normalizing constant $\frac{\beta^\alpha}{\Gamma(\alpha)} = \frac{2^3}{2!} = 4$ matches.

**Step 2:** Recall the formula for the mean log of a Gamma variable.

For $X \sim Gamma(\alpha, \beta)$, there is a standard result using the digamma function $\psi$:

$$E[\ln X] = \psi(\alpha) - \ln \beta$$

For an integer shape $\alpha = 3$, the digamma function has the closed form

$$\psi(3) = -\gamma + 1 + \frac{1}{2} = \frac{3}{2} - \gamma$$

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant.

**Step 3:** Plug in the numbers.

$$\psi(3) = 1.5 - 0.5772 = 0.9228$$

$$\ln \beta = \ln 2 = 0.6931$$

$$E[\ln X] = 0.9228 - 0.6931 = 0.2297$$

So $E(Y) \approx 0.2297$, a finite number. This already tells us option (A) is wrong.

**Step 4:** Compute the benchmark value $\ln(3/2)$.

$$\ln \frac{3}{2} = \ln 1.5 \approx 0.4055$$

**Step 5:** Compare.
Since $0.2297 < 0.4055$, we get $E(Y) < \ln(3/2)$ strictly. This rules out (B), which needs exact equality, and (C), which needs the opposite direction.

**Step 6:** Cross check with the concavity idea.
This makes sense because ln bends downward, so averaging $x$ first and then taking the log always gives a bigger number than taking the log first and then averaging, as long as $X$ is not a fixed constant. Here $E(X) = \alpha/\beta = 3/2$, so $\ln E(X) = \ln(3/2)$ sits above $E(\ln X)$, matching our direct calculation. This concavity gap is not a small rounding effect either: the difference $\ln(3/2) - E[\ln X] \approx 0.4055 - 0.2297 \approx 0.1758$ is a genuine, strictly positive gap. Since $X$ has a proper spread coming from its Gamma(3,2) density, that gap can never close.

**Step 7:** Conclude.

$$\boxed{E(Y) < \ln \frac{3}{2}}$$

**Quick Tip:** $X$ follows a Gamma(3,2) distribution with $E(X) = 3/2$. Since ln is strictly concave, Jensen's inequality gives $E(\ln X) < \ln(E(X))$.

• • •

**18.** Let $\{X_n : n \geq 0\}$ be a homogeneous Markov chain with state space $S = \{1, 2, \dots, 7\}$ and transition probability matrix

$$P = \begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

Then which of the following statements is correct?

(A) 2 is a transient state

(B) 3 is a transient state

(C) 4 is a transient state

(D) 5 is a transient state

**Correct Answer:** (D) 5 is a transient state

**SOLUTION:**

**Step 1:** Draw the picture in words.

Look at where probability mass can flow out of each state, just the 'can go to' arrows: state 1 only talks to state 3 and itself; state 3 only talks to state 1 and itself. So $\{1, 3\}$ is a self contained loop, nothing escapes it.

**Step 2:** Find the second loop.

State 2 only talks to $2, 4, 6$; state 4 only talks to $2, 4, 6$; state 6 only talks to $2, 4, 6$. So $\{2, 4, 6\}$ is a second self contained loop. Together these two loops use up six of the seven states, and neither loop sends anything to the other.

**Step 3:** Recall what closed implies for a finite chain.

On a finite state space, a closed set of states you can never leave once entered is always recurrent. So every state in $\{1, 3\}$ and every state in $\{2, 4, 6\}$ is recurrent. That already rules out state 2, state 3 and state 4 from being transient.

**Step 4:** Look at what is left over.

States 5 and 7 are the two states not inside either closed loop. Row 5 sends 1/2 to state 1, 1/4 to itself, and 1/4 to state 6. Both 1 and 6 belong to closed loops. So over time, with probability 1, the chain started at 5 gets pulled into one of the two closed loops and stays trapped there.

**Step 5:** Confirm state 5 never comes back.

Since neither closed loop ever sends anything back out, once the chain leaves 5 for state 1 or state 6, it can never revisit 5 again. A state the chain eventually leaves forever is transient.

**Step 6:** Match against the answer choices.

(A) says 2 is transient, but 2 sits in the closed loop $\{2, 4, 6\}$, so this is wrong. (B) says 3 is transient, but 3 sits in the closed loop $\{1, 3\}$, so this is wrong too. (C) says 4 is transient, same problem. (D) says 5 is transient, and that is exactly what Step 5 showed.

**Step 7:** Conclude.

$$\boxed{5 \text{ is a transient state}}$$

**Quick Tip:** States $\{1,3\}$ and $\{2,4,6\}$ each form a closed set the chain can never leave. State 5 can jump into either closed set and never return, so it is transient.

• • •

**19.** Let $X_1, X_2$ be a random sample from the following probability density function

$$f_\alpha(x) = \begin{cases} \alpha x^{\alpha-1} e^{-x^\alpha} & \text{if } x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha \in (0, \infty)$ is an unknown parameter. Let

$$X_{(1)} = \min\{X_1, X_2\} \quad \text{and} \quad X_{(2)} = \max\{X_1, X_2\}.$$

Then which of the following statements is correct?

(A) $(X_1 + X_2,\ X_{(1)})$ is a minimal sufficient statistic

(B) $(X_1,\ X_{(2)})$ is a minimal sufficient statistic

(C) $(X_1 - X_2,\ X_{(1)})$ is a sufficient statistic but not a minimal sufficient statistic

(D) $(X_1 - X_2,\ X_{(2)})$ is a complete sufficient statistic

**Correct Answer:** (A) $(X_1 + X_2,\ X_{(1)})$ is a minimal sufficient statistic

**SOLUTION:**

**Step 1:** Write down the joint density.

For $X_1, X_2$ iid with density $f_\alpha(x) = \alpha x^{\alpha-1} e^{-x^\alpha}$ for $x > 0$, the joint density is

$$f_\alpha(x_1, x_2) = \alpha^2(x_1x_2)^{\alpha-1}e^{-(x_1^\alpha+x_2^\alpha)}$$

**Step 2:** Try to factor this using the Fisher-Neyman theorem.
The factorization theorem says $T(X_1, X_2)$ is sufficient if we can write the joint density as $g(T; \alpha) \cdot h(x_1, x_2)$, where $h$ does not involve $\alpha$. Here the whole expression depends on $(x_1, x_2)$ only through $x_1x_2$ and $x_1^\alpha + x_2^\alpha$, and both of these only depend on which two numbers $x_1, x_2$ are, not which one is labeled first. So the natural candidate for a sufficient statistic is the unordered pair, the order statistics $(X_{(1)}, X_{(2)})$.

**Step 3:** Show no further reduction is possible.
Could we get away with just $X_1 + X_2$ alone? No, because $\alpha$ shows up as an exponent, and this Weibull-type family is not an exponential family with a fixed sufficient statistic of dimension one. Two different pairs with the same sum but different individual values, like $(1, 3)$ and $(2, 2)$, give different values of $x_1^\alpha + x_2^\alpha$ for general $\alpha$.

**Step 4:** Translate $(X_{(1)}, X_{(2)})$ into the form given in option (A).
The sum of the two observations does not depend on order: $X_1 + X_2 = X_{(1)} + X_{(2)}$. So if we know $X_1 + X_2$ and $X_{(1)}$ together, we can solve for the other order statistic:

$$X_{(2)} = (X_1 + X_2) - X_{(1)}$$

This shows $(X_1 + X_2, X_{(1)})$ carries exactly the same information as $(X_{(1)}, X_{(2)})$.

**Step 5:** Test option (B) with a concrete pair.

Take $X_1 = 5, X_2 = 2$. Then $X_{(2)} = \max = 5 = X_1$. The pair $(X_1, X_{(2)})$ $= (5, 5)$ gives no way to recover $X_2 = 2$. So this statistic does not preserve enough information, and option (B) fails.

**Step 6:** Test option (C) with the same idea.
$(X_1 - X_2, X_{(1)})$ can be inverted back to the exact ordered pair $(X_1, X_2)$, more detailed than the order statistics actually require.

**Step 7:** Test option (D) the same way.
$(X_1 - X_2, X_{(2)})$ similarly recovers the full labeled sample $(X_1, X_2)$, so it also fails to be minimal.

**Step 8:** Conclude.

$$\boxed{\left(X_1 + X_2,\ X_{(1)}\right) \text{ is a minimal sufficient statistic}}$$

**Quick Tip:** Use the Lehmann-Scheffe likelihood ratio method: the ratio $f(x_1, x_2)/f(y_1, y_2)$ stays free of $\alpha$ only when $\{x_1, x_2\} = \{y_1, y_2\}$, which is exactly the order statistics $(X_{(1)}, X_{(2)})$, the same information as $(X_1 + X_2, X_{(1)})$.

• • •

**20.** Let $X_1$ and $X_2$ be independent and identically distributed random variables following normal distribution with mean $\theta \in (-\infty, \infty)$ and variance 1. Then which of the following estimators of their expected values attains the Cramer-Rao lower bound?

(A) $X_1 + X_2$

(B) $\dfrac{X_1^2 + X_2^2 + 5}{2}$

(C) $(X_1 + X_2)^2$

(D) $2X_1 - X_2$

**Correct Answer:** (A) $X_1 + X_2$

**SOLUTION:**

**Step 1:** Recall what attaining the Cramer-Rao bound really means.
An estimator hits the Cramer-Rao bound exactly when it is a constant times the score function of the likelihood, plus a shift. For a normal model with known variance, this happens only for statistics that are linear combinations of the data with equal weights on every observation.

**Step 2:** Write the score function for this specific model.
With $X_1, X_2 \sim N(\theta, 1)$ independently,

$$\frac{\partial}{\partial\theta}\log L(\theta) = (x_1 - \theta) + (x_2 - \theta) = (x_1 + x_2) - 2\theta$$

This is a straight line in $x_1 + x_2$, with slope 1 and intercept $-2\theta$.

**Step 3:** Match option (A) against this score.
Option (A) is $T = X_1 + X_2$. Subtracting its mean $\psi(\theta) = 2\theta$ gives $(x_1 + x_2) - 2\theta$, which is identical to the score function from Step 2. Whenever an estimator minus its own mean equals a constant times the score, the Cramer-Rao inequality becomes an equality. So $X_1 + X_2$ attains the bound.

**Step 4:** Confirm with a direct variance computation.
Fisher information per observation for $N(\theta, 1)$ is 1, so for two independent observations, $I_2(\theta) = 2$. Since $\psi(\theta) = 2\theta$ has derivative $\psi'(\theta) = 2$, the Cramer-Rao bound for $T$ is

$$\frac{(\psi'(\theta))^2}{I_2(\theta)} = \frac{4}{2} = 2$$

The actual variance is $Var(X_1 + X_2) = 1 + 1 = 2$, an exact match.

**Step 5:** Check option (B), the quadratic mix $\dfrac{X_1^2 + X_2^2 + 5}{2}$.
Subtracting its own mean does not give anything proportional to $(x_1 + x_2) - 2\theta$, since squared terms cannot be written as a constant multiple of a linear expression. So this cannot achieve equality.

**Step 6:** Check option (C), the squared sum $(X_1 + X_2)^2$.
Even though this is built only from $X_1 + X_2$, squaring it makes it a quadratic function, not a straight line, of the score. Only the straight line case gives equality.

**Step 7:** Check option (D), the mismatched combination $2X_1 - X_2$.
Here the weights on $X_1$ and $X_2$ are 2 and $-1$, not equal, so this is not a constant multiple of $(x_1 + x_2) - 2\theta$. A quick variance check confirms it: $Var(2X_1 - X_2) = 4 + 1 = 5$, while estimating $\theta$ needs a bound of $1/2$. Since 5 is nowhere near $1/2$, this estimator is far from efficient.

**Step 8:** Conclude.
Only the equal weighted sum $X_1 + X_2$ lines up exactly with the score function.

$$\boxed{X_1 + X_2 \text{ attains the Cramer-Rao lower bound}}$$

**Quick Tip:** An estimator attains the Cramer-Rao bound only if it is an affine function of the score. Since the score is $(x_1 + x_2) - 2\theta$, only $X_1 + X_2$ matches it exactly.

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**21.** For testing a null hypothesis $H_0$ against an alternative hypothesis $H_1$ at level of significance $\alpha \in (0, 1)$, which of the following statements is correct?

(A) A uniformly most powerful test always exists

(B) If a uniformly most powerful test exists then its size is $\alpha$

(C) If a uniformly most powerful unbiased test exists then it is necessarily a uniformly most powerful test

(D) If a uniformly most powerful test exists then it is necessarily an unbiased test

**Correct Answer:** (D) If a uniformly most powerful test exists then it is necessarily an unbiased test

**SOLUTION:**

**Step 1:** Set the definitions straight.
A test of $H_0$ against $H_1$ has size $\alpha$ if it wrongly rejects a true $H_0$ with probability at most $\alpha$. Among all such tests, a uniformly most powerful, or UMP, test is one whose power curve sits above every other level $\alpha$ test's power curve, at every point of $H_1$.

**Step 2:** Test the claim that a UMP test always exists.
This is too strong. UMP tests show up mainly in one sided problems with a monotone likelihood ratio. In many two sided problems, different level $\alpha$ tests win at different alternatives, so no single test dominates everywhere. Statement (A) fails.

**Step 3:** Test the claim about size equal to $\alpha$.
Being UMP only means the power is maximised among level $\alpha$ tests, where level $\alpha$ allows the actual rejection probability under $H_0$ to be anywhere up to $\alpha$. Nothing in the definition forces this probability to

land exactly on $\alpha$ in all cases, so the blanket statement in (B) fails.

**Step 4:** Test the claim linking UMP unbiased and UMP.
The UMP unbiased test is chosen as the best performer only inside the smaller family of unbiased tests. When the full class of level $\alpha$ tests has no single winner, for instance a common two sided testing setup, restricting to unbiased tests can still produce a best one. That best unbiased test is not automatically best over the whole class, so (C) fails.

**Step 5:** Test the claim that a UMP test is unbiased.
Take a UMP test $\phi$ of level $\alpha$. Set it against the plain test that ignores the data completely and rejects with fixed probability $\alpha$. This plain test still has level $\alpha$, and its power is flat at $\alpha$ everywhere on $H_1$. Because $\phi$ beats every level $\alpha$ test in power, $\phi$ must beat this plain one too, giving

$$\text{Power}_{\phi}(\theta) \geq \alpha \text{ for all } \theta \in H_1.$$

Power never dropping below $\alpha$ is exactly what unbiasedness means. So (D) holds in general.

**Step 6:** Conclude.
Only the statement tying UMP tests to unbiasedness survives the check.

$$\boxed{\text{If a UMP test exists then it is necessarily an unbiased test}}$$

**Quick Tip:** Compare a UMP test's power against the trivial test that rejects with fixed probability $\alpha$; its power can never fall below $\alpha$, which is the definition of unbiasedness.

• • •

**22.** Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$, $n \geq 2$, be a random sample from a continuous bivariate distribution with joint distribution function $F_{X,Y}$. Further, $F_X$ and $F_Y$ are the marginal distribution functions of $X$ and $Y$, respectively. If

$$F_{X,Y}(x, y) = F_X(x)F_Y(y), \quad \forall (x, y),$$

then, for any two independent pairs $(X_i, Y_i)$ and $(X_j, Y_j)$,

$$P\left[(X_i - X_j)(Y_i - Y_j) > 0\right]$$

equals

(A) $\frac{1}{4}$

(B) $\frac{1}{2}$

(C) $\frac{3}{4}$

(D) $\frac{3}{8}$

**Correct Answer:** (B) $\frac{1}{2}$

**SOLUTION:**

**Step 1:** Translate the quantity being asked into a familiar idea.

The event $(X_i - X_j)(Y_i - Y_j) > 0$ is what statisticians call a concordant pair: as $X$ moves from the smaller value to the larger, $Y$ moves the same way. This probability of concordance is tied to Kendall's measure of association, $\tau = 2P(\text{concordant}) - 1$, so finding this probability is the same as finding how strongly $X$ and $Y$ move together.

**Step 2:** Use the given factoring condition.

We are told $F_{X,Y}(x, y) = F_X(x)F_Y(y)$ for every $(x, y)$. This is precisely the statement that $X$ and $Y$ are independent. So knowing $X$ tells us nothing about $Y$ inside a pair.

**Step 3:** Build four independent variables.
The two pairs $(X_i, Y_i)$ and $(X_j, Y_j)$ come from independent draws of the sample. Along with the within pair independence from Step 2, all of $X_i, Y_i, X_j, Y_j$ act as four separate, mutually independent draws.

**Step 4:** Find the chance each difference is positive.
$X_i$ and $X_j$ share the same continuous distribution, so by symmetry $X_i - X_j$ is just as likely to be positive as negative, and the chance of an exact tie is 0. This gives $P(X_i - X_j > 0) = \frac{1}{2}$. The same reasoning for $Y_i - Y_j$ gives $P(Y_i - Y_j > 0) = \frac{1}{2}$.

**Step 5:** Combine using independence.
Since the sign of $X_i - X_j$ does not depend on the sign of $Y_i - Y_j$, the product $(X_i - X_j)(Y_i - Y_j)$ is positive only when both differences share a sign. Adding the two matching cases,

$$P[(X_i - X_j)(Y_i - Y_j) > 0] = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{2}.$$

**Step 6:** Read the answer through Kendall's tau.
With independent $X$ and $Y$, Kendall's tau is 0, and $\tau = 2P(\text{concordant}) - 1 = 0$ gives $P(\text{concordant}) = \frac{1}{2}$, matching the direct computation.

$$\boxed{\frac{1}{2}}$$

**Quick Tip:** With $X$ and $Y$ independent, the four values $X_i, Y_i, X_j, Y_j$ are mutually independent, so the signs of $X_i - X_j$ and $Y_i - Y_j$ are independent fair coin flips.

• • •

**23.** For a random sample $X_1, \ldots, X_n$, $n \geq 2$, from a population with distribution function $F_X$, let $Z_n(x)$ be the proportion of sample values less than or equal to $x$, $x \in \mathbb{R}$. Which of the following statements is/are true?

(A) $E\big(Z_n(x)\big) = F_X(x)$

(B) $\mathrm{Var}\big(Z_n(x)\big) = \dfrac{F_X(x)(1 - F_X(x))}{n^2}$

(C) $\mathrm{Cov}\big(Z_n(x), Z_n(y)\big) = 0$, for all $x \neq y$

(D) $Z_n(x)$ is a consistent estimator of $F_X(x)$

**Correct Answer:** (A) $E\big(Z_n(x)\big) = F_X(x)$ ; (D) $Z_n(x)$ is a consistent estimator of $F_X(x)$

**SOLUTION:**

**Step 1:** Recognise $Z_n(x)$ as the empirical distribution function.

$Z_n(x)$ counts what fraction of the sample falls at or below $x$, so it can be written as an average of indicator variables:

$$Z_n(x) = \frac{1}{n} \sum_{i=1}^{n} I(X_i \leq x)$$

Each $I(X_i \leq x)$ takes the value 1 with probability $F_X(x)$ and 0 otherwise, so it is a Bernoulli variable with success probability $F_X(x)$, and the $n$ of them are independent because the sample is random.

**Step 2:** Get the mean, statement (A).

Averaging $n$ Bernoulli($F_X(x)$) variables gives back the same probability, so

$$E(Z_n(x)) = F_X(x)$$

This is a direct match with statement (A), so it holds.

**Step 3:** Get the variance, statement (B).

A single Bernoulli($p$) variable has variance $p(1-p)$. Summing $n$ independent copies scales the variance by $n$, and averaging (dividing by $n$) scales the variance by $1/n^2$. Multiplying these together,

$$\mathrm{Var}(Z_n(x)) = \frac{n\,F_X(x)(1-F_X(x))}{n^2} = \frac{F_X(x)(1-F_X(x))}{n}$$

The $n$'s do not fully cancel to leave $n^2$ in the denominator, they leave a single $n$. So statement (B), which claims a denominator of $n^2$, does not hold.

**Step 4:** Test the covariance claim, statement (C).

Pick $x < y$. The same observation $X_i$ feeds both $I(X_i \le x)$ and $I(X_i \le y)$, and being below $x$ forces being below $y$ too, so these two indicators move together rather than being unrelated. Working out the covariance of one pair and scaling by $n$ for the sum gives

$$\mathrm{Cov}(Z_n(x), Z_n(y)) = \frac{F_X(x)\big(1 - F_X(y)\big)}{n}$$

which is not zero except in boundary cases. So the claim of zero covariance at every pair of points fails.

**Step 5:** Test consistency, statement (D).

$Z_n(x)$ has no bias, since its mean is exactly $F_X(x)$, and its variance shrinks toward 0 as $n$ grows, since it carries a factor $1/n$. An estimator with vanishing bias and vanishing variance converges in probability to the true value. So $Z_n(x)$ is a consistent estimator of $F_X(x)$, which is just the familiar fact that the empirical distribution function estimates the true distribution function well for large samples. Statement (D) holds.

**Step 6:** Collect the results.

Only the mean statement and the consistency statement survive scrutiny.

$$\boxed{\text{(A) and (D)}}$$

**Quick Tip:** Write $Z_n(x)$ as an average of Bernoulli($F_X(x)$) indicators; its mean is $F_X(x)$ and its variance is $F_X(x)(1 - F_X(x))/n$, not $/n^2$, and values at different points are correlated.

• • •

**24.** The value of

$$\lim_{n\to\infty} n \int_{1-\frac{1}{2n}}^{1+\frac{1}{2n}} e^{t^2-1}\, dt$$

equals __________ (answer in integer).

**Correct Answer:** 1

**SOLUTION:**

**Step 1:** Spot that this is an averaging limit.

The expression is

$$L = \lim_{n\to\infty} n \int_{1-\frac{1}{2n}}^{1+\frac{1}{2n}} e^{t^2-1}\, dt$$

Note that $n = \dfrac{1}{\text{width of interval}}$, since the interval $\left[1 - \dfrac{1}{2n}, 1 + \dfrac{1}{2n}\right]$ has length $\dfrac{1}{n}$. So $n$ times the integral is just the average value of $e^{t^2-1}$ over that shrinking interval.

**Step 2:** Use a first order expansion near $t = 1$.
Write $t = 1 + s$ where $s$ ranges over $\left[-\dfrac{1}{2n}, \dfrac{1}{2n}\right]$, a small interval around 0. Then

$$t^2 - 1 = (1 + s)^2 - 1 = 2s + s^2$$

For small $s$, $2s + s^2 \approx 2s$, so

$$e^{t^2-1} \approx e^{2s} \approx 1 + 2s$$

using the fact that $s \to 0$ as $n \to \infty$.

**Step 3:** Average this approximation over the small interval.

$$n \int_{-\frac{1}{2n}}^{\frac{1}{2n}} (1 + 2s)\, ds = n\left[s + s^2\right]_{-\frac{1}{2n}}^{\frac{1}{2n}}$$

Evaluating, the $s$ term gives $n\left(\dfrac{1}{2n} - \left(-\dfrac{1}{2n}\right)\right) = n \cdot \dfrac{1}{n} = 1$, and the $s^2$ term gives $n\left(\dfrac{1}{4n^2} - \dfrac{1}{4n^2}\right) = 0$ since the square is symmetric and cancels out.

**Step 4:** Take the limit.
So the leading behaviour gives exactly 1, and all the higher order correction terms from the approximation vanish faster than $\dfrac{1}{n}$ as $n \to \infty$, so they contribute nothing in the limit.

**Step 5:** Confirm using the direct evaluation at the centre.

More simply, as the interval shrinks to the single point $t = 1$, the average value of the continuous function $e^{t^2-1}$ over the interval converges to its value at the centre point, which is

$$e^{1^2-1} = e^0 = 1$$

This matches the expansion method.

$$\boxed{1}$$

**Quick Tip:** As $n \to \infty$, the interval shrinks to the single point $t = 1$, so the expression tends to the value of $e^{t^2-1}$ at $t = 1$.

• • •

**25.** Consider the system of linear equations

$$x + y + z = 1,$$

$$2x + y + 3z = 6,$$

$$3x + 2y + kz = k + 1.$$

If the above system has no solution, then the value of $k$ equals __________ (answer in integer).

**Correct Answer:** 4

**SOLUTION:**

**Step 1:** Set up the coefficient picture.

The system is

$$x + y + z = 1, \quad 2x + y + 3z = 6, \quad 3x + 2y + kz = k + 1$$

Notice that the coefficients of $x$ and $y$ in the third equation, namely 3 and 2, equal the sum of the coefficients in the first two equations: $1 + 2 = 3$ for $x$ and $1 + 1 = 2$ for $y$. This hints that adding equation (i) and equation (ii) will connect directly to equation (iii).

**Step 2:** Add the first two equations.

$$(x + y + z) + (2x + y + 3z) = 1 + 6$$

$$3x + 2y + 4z = 7$$

**Step 3:** Compare with the third equation.

The third equation is $3x + 2y + kz = k + 1$. Since the $x$ and $y$ coefficients in this sum, 3 and 2, exactly match the third equation's coefficients, subtract the third equation from the sum found in Step 2:

$$(3x + 2y + 4z) - (3x + 2y + kz) = 7 - (k + 1)$$

$$(4 - k)z = 6 - k$$

**Step 4:** Read off when this breaks down.

This is the same relation as before, just written with signs flipped: $(4 - k)z = 6 - k$ is the same equation as $z(k - 4) = k - 6$. If $k \neq 4$, this fixes a unique $z$, and the first two original equations then fix $x$ and $y$ uniquely, so a solution exists and is unique.

**Step 5:** Check the special value $k = 4$.

Putting $k = 4$ into $(4 - k)z = 6 - k$ gives

$$(4-4)z = 6-4$$

$$0 = 2$$

which is impossible for any $z$. So at $k = 4$ the three planes represented by the equations do not share a common point, meaning the system has no solution. This also matches the geometric fact that the first two planes intersect in a line, and at $k = 4$ the third plane becomes parallel to that line without containing it.

**Step 6:** State the answer.

No solution occurs exactly at

$$\boxed{4}$$

**Quick Tip:** Eliminate $x$ and $y$ to get an equation of the form $(k-4)z = k - 6$; the system has no solution when this reduces to $0 =$ nonzero, that is at $k = 4$.

• • •

**26.** Consider the subspace

$$U = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_4 = 0,\ x_3 = 0\}.$$

Let $U^\perp$ be its orthogonal complement. The value of $\dim(U^\perp)$ equals __________ (answer in integer).

**Correct Answer:** 2

**SOLUTION:**

**Step 1:** Recall the row space and null space relation.
For a matrix $A$, the null space of $A$ and the row space of $A$ are orthogonal complements of each other inside $\mathbb{R}^n$. Here $U$ is exactly the null space of the matrix built from the two defining equations.

**Step 2:** Write down that matrix.
The equations $x_1 + x_2 + x_4 = 0$ and $x_3 = 0$ come from the rows

$$a_1 = (1, 1, 0, 1), \qquad a_2 = (0, 0, 1, 0).$$

So $U = \{v : a_1 \cdot v = 0, a_2 \cdot v = 0\}$, which is precisely the set of vectors perpendicular to both $a_1$ and $a_2$.

**Step 3:** Identify U perp directly.
Since $U$ consists of all vectors orthogonal to $a_1$ and $a_2$, the orthogonal complement $U^\perp$ is the span of $a_1$ and $a_2$ themselves:

$$U^\perp = \operatorname{span}\{a_1, a_2\}.$$

**Step 4:** Find the dimension of that span.
$a_1 = (1, 1, 0, 1)$ and $a_2 = (0, 0, 1, 0)$ are not scalar multiples of each other, so they are linearly independent. A span of 2 independent vectors has dimension 2.

**Step 5:** Conclude.

$$\dim(U^\perp) = 2.$$

This matches the count from the rank-nullity route, confirming the same answer without first finding a basis for $U$ itself.

$$\boxed{2}$$

**Quick Tip:** **Hint:**
For any subspace $U$ of $\mathbb{R}^n$, $\dim(U) + \dim(U^\perp) = n$. Find $\dim(U)$ using rank-nullity on the two defining equations.

• • •

**27.** Let $X$ be a random variable such that

$$P(X = i) = 2P(X = i - 1), \quad i = 2, 3, \dots, n, \; n \geq 7,$$

and $\sum_{i=1}^{n} P(X = i) = 1$. Then $(2^n - 1)P(X = 7)$ equals __________ (answer in integer).

**Correct Answer:** 64

**SOLUTION:**

**Step 1:** Guess a form that fits the doubling rule.
Since each term is twice the previous one, any sequence of the form $P(X = i) = C \cdot 2^i$ for a constant $C$ automatically satisfies $P(X = i)$ $= 2P(X = i - 1)$, because $C \cdot 2^i = 2 \cdot (C \cdot 2^{i-1})$.

**Step 2:** Use the total probability condition to find C.

$$\sum_{i=1}^{n} P(X = i) = C \sum_{i=1}^{n} 2^i = 1.$$

**Step 3:** Sum the geometric series starting from $2^1$.

$$\sum_{i=1}^{n} 2^i = 2 + 4 + \cdots + 2^n = 2(2^n - 1) = 2^{n+1} - 2.$$

So

$$C(2^{n+1} - 2) = 1 \quad \Rightarrow \quad C = \frac{1}{2(2^n - 1)}.$$

**Step 4:** Write P(X=7) using this C.

$$P(X = 7) = C \cdot 2^7 = \frac{128}{2(2^n - 1)} = \frac{64}{2^n - 1}.$$

**Step 5:** Multiply by $(2^n - 1)$.

$$(2^n - 1)P(X = 7) = (2^n - 1) \cdot \frac{64}{2^n - 1} = 64.$$

**Final Answer:**

The extra constant factor cancels neatly, giving the same clean number both ways.

$$\boxed{64}$$

**Quick Tip:** **Hint:**

The recursion $P(X = i) = 2P(X = i - 1)$ makes $\{P(X = i)\}$ a geometric sequence. Write $P(X = i) = 2^{i-1}P(X = 1)$ and use the normalization condition to find $P(X = 1)$.

• • •

**28.** Let $X$ and $Y$ be two continuous random variables having the following joint probability density function

$$f(x, y) = \begin{cases} x + y & \text{if } 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $72\big(\mathrm{Var}(X) + \mathrm{Var}(Y)\big)$ equals __________ (answer in integer).

**Correct Answer:** 11

**SOLUTION:**

**Step 1:** Set the target formula.

We want $\mathrm{Var}(X) + \mathrm{Var}(Y)$. Using $\mathrm{Var}(X) = E(X^2) - E(X)^2$ and the same for $Y$, we need four numbers: $E(X), E(Y), E(X^2), E(Y^2)$.

**Step 2:** Compute E(X) as a direct double integral.

$$E(X) = \int_0^1 \int_0^1 x(x + y)\, dy\, dx = \int_0^1 \left(x^2 + \frac{x}{2}\right) dx = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

**Step 3:** Compute E(Y) the same way.

By swapping the roles of $x$ and $y$ in the integrand $x + y$ (which is symmetric), the same computation gives

$$E(Y) = \int_0^1 \int_0^1 y(x + y)\, dx\, dy = \frac{7}{12}.$$

So $E(X) = E(Y)$ without needing a separate calculation.

**Step 4:** Compute E(X^2) as a direct double integral.

$$E(X^2) = \int_0^1 \int_0^1 x^2(x+y)\,dy\,dx = \int_0^1 \left(x^3 + \frac{x^2}{2}\right)dx = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}.$$

**Step 5:** Get E(Y^2) by the same symmetry shortcut.
Swapping $x$ and $y$ again in the symmetric integrand gives $E(Y^2) = \frac{5}{12}$ too, so both squared moments match.

**Step 6:** Combine into the variance sum.

$$\mathrm{Var}(X) + \mathrm{Var}(Y) = \left[E(X^2) - E(X)^2\right] + \left[E(Y^2) - E(Y)^2\right] = 2\left(\frac{5}{12} - \frac{49}{144}\right) = 2 \cdot \frac{11}{144} = \frac{11}{72}.$$

**Step 7:** Scale by 72.

$$72 \cdot \frac{11}{72} = 11.$$

**Final Answer:**
Working straight from the double integral, without stopping to write out the marginal density formula, gives the same clean value.

$$\boxed{11}$$

**Quick Tip: Hint:**
Find the marginal density of X by integrating out y, then compute $E(X)$ and $E(X^2)$ to get $\text{Var}(X)$. Use the symmetry of $f(x, y) = x + y$ to get $\text{Var}(Y)$ the same way.

• • •

**29.** Let $X_1, X_2$ and $X_3$ be three independent random variables such that $X_k$ $(k = 1, 2, 3)$ has the following probability density function

$$f_k(x) = \begin{cases} k\,e^{-kx} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = \min\{X_1, X_2, X_3\}$. Then the value of $E(3Y^2 - Y)$ equals __________ (answer in integer).

**Correct Answer:** 0

**SOLUTION:**

**Step 1:** Write the survival function of Y from scratch.
$Y = \min\{X_1, X_2, X_3\}$ exceeds $y$ only if all three variables exceed $y$. Since $X_k$ has rate $k$, $P(X_k > y) = e^{-ky}$ for $y > 0$. By independence,

$$P(Y > y) = P(X_1 > y)\,P(X_2 > y)\,P(X_3 > y) = e^{-y} \cdot e^{-2y} \cdot e^{-3y} = e^{-6y}.$$

**Step 2:** Get the density of Y.
The CDF is $F_Y(y) = 1 - e^{-6y}$, so differentiating gives the density

$$g(y) = \frac{d}{dy} F_Y(y) = 6e^{-6y}, \qquad y > 0,$$

which is an exponential density with rate 6.

**Step 3:** Compute E(Y) by direct integration.

$$E(Y) = \int_0^{\infty} y \cdot 6e^{-6y}\, dy.$$

Using the standard gamma-integral result $\int_0^\infty y\, e^{-\lambda y}\, dy = 1/\lambda^2$ with $\lambda = 6$, this becomes

$$E(Y) = 6 \cdot \frac{1}{6^2} = \frac{1}{6}.$$

**Step 4:** Compute E(Y^2) by direct integration.
Using $\int_0^\infty y^2 e^{-\lambda y}\, dy = 2/\lambda^3$ with $\lambda = 6$,

$$E(Y^2) = 6 \cdot \frac{2}{6^3} = \frac{12}{216} = \frac{1}{18}.$$

**Step 5:** Put the pieces together.

$$E(3Y^2 - Y) = 3\left(\frac{1}{18}\right) - \frac{1}{6} = \frac{1}{6} - \frac{1}{6} = 0.$$

**Final Answer:**
Building the density from the survival function and integrating directly gives the same cancellation.

$$\boxed{0}$$

**Quick Tip:** **Hint:**
The minimum of independent exponential random variables with rates $\lambda_1, \lambda_2, \lambda_3$ is exponential with rate $\lambda_1 + \lambda_2 + \lambda_3$. Use this to find the moments of $Y$.

• • •

**30.** Let $X$ be a continuous random variable having distribution function $F$. If

$$Y = -3 \ln F(X),$$

then $E(Y)$ equals ________ (answer in integer).

**Correct Answer:** 3

**SOLUTION:**

**Step 1:** Apply the probability integral transform.
Whenever $X$ is continuous with CDF $F$, the quantity $U = F(X)$ is always Uniform(0,1), no matter what $F$ looks like. This is a standard fact used to simulate any continuous distribution from uniform random numbers.

**Step 2:** Recognize the transform of a uniform variable.
A well known result is that if $U \sim \text{Uniform}(0, 1)$, then $Z = -\ln U$ has the standard exponential distribution with rate 1, that is $Z \sim \text{Exp}(1)$. This is because the survival function of $Z$ is

$$P(Z > z) = P(-\ln U > z) = P(U < e^{-z}) = e^{-z},$$

which is exactly the Exp(1) survival function.

**Step 3:** Express Y in terms of Z.
Since $Y = -3 \ln U = 3(-\ln U) = 3Z$, and we just showed $Z \sim \text{Exp}(1)$, the variable $Y$ is simply 3 times a standard exponential variable.

**Step 4:** Use the known mean of an exponential variable.
For $Z \sim \text{Exp}(1)$, $E(Z) = 1$. Scaling a random variable by a constant scales its mean by the same constant, so

$$E(Y) = 3E(Z) = 3 \times 1 = 3.$$

**Final Answer:**
Recognizing $-\ln U$ as a standard exponential variable gives the mean directly, without doing the integral by hand.

$$\boxed{3}$$

**Quick Tip:** **Hint:**
For continuous X with CDF F, $U = F(X)$ is Uniform(0,1). Rewrite Y in terms of U and use $\int_0^1 \ln u \, du = -1$.

• • •

**31.** Let $\{W(t) : t \geq 0\}$ be a standard Brownian motion, with $W(0) = 0$. Define

$$Z_1 = W(1) + W(2) \quad \text{and} \quad Z_2 = W(2) + W(3).$$

Let $\rho$ be the correlation coefficient between $Z_1$ and $Z_2$. Then the value of $10\rho$ is ________ (round off to two decimal places).

**Correct Answer:** 8.94

**SOLUTION:**

**Step 1:** Break the Brownian motion into independent increments.

Write $A = W(1)$, $B = W(2) - W(1)$, $C = W(3) - W(2)$. Since Brownian motion has independent increments, $A$, $B$, $C$ are independent, each with variance equal to its time length, so $\text{Var}(A) = \text{Var}(B) = \text{Var}(C) = 1$.

**Step 2:** Rewrite Z1 and Z2 in terms of A, B, C.

Since $W(1) = A$, $W(2) = A + B$, $W(3) = A + B + C$,

$$Z_1 = W(1) + W(2) = 2A + B$$

$$Z_2 = W(2) + W(3) = 2A + 2B + C$$

**Step 3:** Get Var(Z1) from the independent pieces.

Since $A, B, C$ are independent,

$$\text{Var}(Z_1) = 4\text{Var}(A) + \text{Var}(B) = 4(1) + 1 = 5$$

**Step 4:** Get Var(Z2) the same way.

$$\mathrm{Var}(Z_2) = 4\mathrm{Var}(A) + 4\mathrm{Var}(B) + \mathrm{Var}(C) = 4 + 4 + 1 = 9$$

**Step 5:** Get Cov(Z1,Z2).
Only matching independent terms survive:

$$\mathrm{Cov}(Z_1, Z_2) = \mathrm{Cov}(2A + B, 2A + 2B + C) = 4\mathrm{Var}(A) + 2\mathrm{Var}(B) = 4(1) + 2(1) = 6$$

**Step 6:** Form the correlation and scale by 10.

$$\rho = \frac{6}{\sqrt{5 \times 9}} = \frac{6}{3\sqrt{5}} = \frac{2}{\sqrt{5}} \approx 0.8944$$

$$10\rho \approx 8.94$$

$$\boxed{10\rho \approx 8.94}$$

**Quick Tip: Hint:**
Use $\mathrm{Cov}(W(s), W(t)) = \min(s, t)$ to expand $\mathrm{Var}(Z_1)$, $\mathrm{Var}(Z_2)$ and $\mathrm{Cov}(Z_1, Z_2)$, then form $\rho = \mathrm{Cov}(Z_1, Z_2)/\sqrt{\mathrm{Var}(Z_1)\mathrm{Var}(Z_2)}$.

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**32.** Let $x_1, x_2, \dots, x_n$ ($n \geq 2$) be the observed values of a random sample from the following probability density function

$$f(x) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha \in (0, \infty)$ and $\lambda \in (0, \infty)$ are unknown parameters. If

$$\frac{x_1 + x_2 + \cdots + x_n}{n} = 2 \quad \text{and} \quad \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} = 5,$$

then the method of moments estimate of $\alpha$ equals ________ (answer in integer).

**Correct Answer:** 4

**SOLUTION:**

**Step 1:** Use the sample mean and sample variance instead of raw moments.

The sample mean is $\bar{x} = 2$. The sample variance about the mean is

$$s^2 = m_2' - \bar{x}^2 = 5 - 4 = 1$$

**Step 2:** Match these to the Gamma parameters.

For the Gamma density, $\bar{x}$ estimates $\alpha/\lambda$ and $s^2$ estimates $\alpha/\lambda^2$. So

$$\frac{\alpha}{\lambda} = 2, \qquad \frac{\alpha}{\lambda^2} = 1$$

**Step 3:** Take the ratio of the two equations.

Dividing the variance equation by the mean equation cancels $\alpha$:

$$\frac{\alpha/\lambda^2}{\alpha/\lambda} = \frac{1}{2} \quad \Rightarrow \quad \frac{1}{\lambda} = \frac{1}{2} \quad \Rightarrow \quad \lambda = 2$$

**Step 4:** Recover alpha from the mean equation.

$$\alpha = 2\lambda = 2 \times 2 = 4$$

**Step 5:** Confirm with the variance.

With $\alpha = 4, \lambda = 2$: $\alpha/\lambda^2 = 4/4 = 1$, matching $s^2 = 1$ exactly.

$$\boxed{\alpha = 4}$$

**Quick Tip: Hint:**
Equate the population mean $\alpha/\lambda$ and second moment $\alpha/\lambda^2 + (\alpha/\lambda)^2$ to the given sample moments 2 and 5, then solve the two equations together.

• • •

**33.** Let $X_1, X_2, \ldots, X_{10}$ be a random sample from the following probability density function

$$f(x) = \begin{cases} 2(x-\mu)e^{-(x-\mu)^2} & \text{if } x > \mu \\ 0 & \text{otherwise,} \end{cases}$$

where $\mu \in (-\infty, \infty)$ is an unknown parameter. It is given that the observed value of $\min\{X_1, X_2, \ldots, X_{10}\}$ is 1. Using the pivot $\min\{X_1, X_2, \ldots, X_{10}\} - \mu$, suppose a 95% confidence interval of $\mu$ is of the form $(c, 1)$, then $c$ equals _______ (rounded off to two decimal places).

**Correct Answer:** 0.45

**SOLUTION:**

**Step 1:** Reduce the minimum to a simple exponential variable.
Let $Y_i = X_i - \mu$, which has density $g(y) = 2ye^{-y^2}$ for $y > 0$, free of $\mu$. Let $T = \min\{X_i\} - \mu = \min\{Y_i\}$ for 10 such independent variables. Its survival function works out to $P(T > t) = e^{-10t^2}$.

**Step 2:** Square and rescale T.
Define $S = 10T^2$. Then for $s > 0$,

$$P(S > s) = P\left(T > \sqrt{s/10}\right) = e^{-10(s/10)} = e^{-s}$$

So $S = 10T^2$ is a standard Exponential random variable with mean 1, a much simpler pivot to work with.

**Step 3:** Find the 95th percentile of this exponential pivot.
We want $s_0$ with $P(S \leq s_0) = 0.95$, that is $1 - e^{-s_0} = 0.95$, so

$$e^{-s_0} = 0.05 \quad \Rightarrow \quad s_0 = \ln(20) \approx 2.9957$$

**Step 4:** Undo the transformation to get back to T.

Since $S = 10T^2 \leq s_0$ with probability 0.95, and $T > 0$,

$$T \leq \sqrt{\frac{s_0}{10}} = \sqrt{0.29957} \approx 0.5473$$

So $P(0 < T \leq 0.5473) = 0.95$.

**Step 5:** Turn this into a confidence interval for mu.

Since $T = \min\{X_i\} - \mu$, the event $0 < T \leq 0.5473$ rearranges to

$$\min\{X_i\} - 0.5473 \leq \mu < \min\{X_i\}$$

With $\min\{X_i\} = 1$, the interval is $(1 - 0.5473,\ 1) = (0.4527,\ 1)$.

**Step 6:** Read off c.

$$c \approx 0.45$$

$$\boxed{c \approx 0.45}$$

**Quick Tip: Hint:**

Show that $Y = X - \mu$ has distribution function $1 - e^{-y^2}$, find the survival function of $\min\{X_i\} - \mu$, and use it as a pivot to build a one-sided confidence interval.

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**34.** Let $X$ be a single observation from a distribution having the probability density function

$$f_\theta(x) = \begin{cases} 1 & \text{if } \theta < x < \theta + 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta \in (-\infty, \infty)$. For testing $H_0 : \theta \leq 0$ against $H_1 : \theta > 1$, let $\beta$ be the power of the uniformly most powerful test of level 0.05. Then $225\beta$ equals _______ (answer in integer).

**Correct Answer:** 225

**SOLUTION:**

**Step 1:** Picture the two families of intervals.

The random variable $X$ always lands inside an interval of length 1 starting at $\theta$, that is $(\theta, \theta + 1)$. Under $H_0$, $\theta \leq 0$, so this interval sits entirely at or below $(0, 1)$. Under $H_1$, $\theta > 1$, so the interval sits entirely above 1.

**Step 2:** Note the hypotheses do not touch.

Because $\theta \leq 0$ forces $X \leq \theta + 1 \leq 1$, and $\theta > 1$ forces $X > \theta > 1$, any observed value bigger than 1 rules out $H_0$ completely. That already gives a test region $X > 1$ with zero chance of a wrong rejection, using none of our allowed 5 percent error budget.

**Step 3:** Spend the leftover error budget.

A good test should use all the allowed level, not less, since rejecting for more values of $x$, all of which favor a large $\theta$, only helps catch true alternatives. The natural region to add is values of $x$ just under 1, so we look for a cutoff $k$ and reject when $X > k$.

**Step 4:** Pin down k from the worst case inside H0.

Among all $\theta \le 0$, the chance of wrongly landing above $k$ is largest when the interval $(\theta, \theta + 1)$ sits as high as possible, which happens at $\theta = 0$, giving $X \sim \text{Uniform}(0, 1)$. The chance of exceeding $k$ is then simply the leftover length, $1 - k$. Setting the worst case chance to the allowed level,

$$1 - k = 0.05$$

$$k = 0.95$$

So we reject $H_0$ whenever $X > 0.95$.

**Step 5:** Work out the power.

Take any $\theta > 1$ under $H_1$. Its interval $(\theta, \theta + 1)$ lies completely above 1, and $1 > 0.95$, so every single value $X$ can take already satisfies $X > 0.95$. The test rejects with certainty, so

$$\beta = P(X > 0.95 \mid \theta) = 1$$

for every $\theta$ bigger than 1, so this constant value is the power asked for.

**Step 6:** Final number.

$$225\beta = 225 \times 1 = 225$$

$$\boxed{225\beta = 225}$$

**Quick Tip:** **Hint:**
Since $\theta \leq 0$ forces $X < 1$ and $\theta > 1$ forces $X > 1$, find the cutoff $k$ using $X \sim$ Uniform$(0, 1)$ at $\theta = 0$, then check that the power is the same for every $\theta > 1$.

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**35.** Let $(X_1, X_2, X_3)^T$ follow a trivariate normal distribution with mean vector $\mu$ and covariance matrix $\Sigma$ given by

$$\mu = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Then $\mathrm{Var}(X_1 \mid X_2 = 1, X_3 = -1)$ equals ________ (rounded off to two decimal places).

**Correct Answer:** 1.33

**SOLUTION:**

**Step 1:** Set up the best linear predictor of X1.
For jointly normal variables, the conditional mean of $X_1$ given $X_2, X_3$ is a linear function $a + b_2X_2 + b_3X_3$, and the conditional variance is the leftover variance of $X_1$ after removing what this line explains. The coefficients $b_2, b_3$ solve the normal equations

$$\mathrm{Cov}(X_1, X_2) = b_2\mathrm{Var}(X_2) + b_3\mathrm{Cov}(X_2, X_3)$$

$$\mathrm{Cov}(X_1, X_3) = b_2\mathrm{Cov}(X_2, X_3) + b_3\mathrm{Var}(X_3)$$

**Step 2:** Plug in the numbers from Sigma.

From the matrix, $\mathrm{Var}(X_2) = \mathrm{Var}(X_3) = 2$, $\mathrm{Cov}(X_2, X_3) = 1$, and $\mathrm{Cov}(X_1, X_2) = \mathrm{Cov}(X_1, X_3) = 1$. So

$$2b_2 + b_3 = 1$$

$$b_2 + 2b_3 = 1$$

**Step 3:** Solve the pair by elimination.

From the first equation, $b_3 = 1 - 2b_2$. Put this in the second equation:

$$b_2 + 2(1 - 2b_2) = 1$$

$$b_2 + 2 - 4b_2 = 1$$

$$-3b_2 = -1 \quad \Rightarrow \quad b_2 = \frac{1}{3}$$

Then $b_3 = 1 - \frac{2}{3} = \frac{1}{3}$. By the symmetry of the covariance matrix this equal split makes sense.

**Step 4:** Find the leftover, unexplained variance.

The conditional variance equals the total variance of $X_1$ minus the part explained by the best linear fit:

$$\mathrm{Var}(X_1 \mid X_2, X_3) = \mathrm{Var}(X_1) - [b_2 \mathrm{Cov}(X_1, X_2) + b_3 \mathrm{Cov}(X_1, X_3)]$$

$$= 2 - \left[\frac{1}{3}(1) + \frac{1}{3}(1)\right] = 2 - \frac{2}{3} = \frac{4}{3}$$

**Step 5:** Note that the given values do not matter.

The numbers $X_2 = 1$ and $X_3 = -1$ only shift the conditional mean, not this leftover variance, so they play no role in the final answer.

$$\frac{4}{3} \approx 1.33$$

$$\boxed{\mathrm{Var}(X_1 \mid X_2 = 1, X_3 = -1) \approx 1.33}$$

**Quick Tip:** **Hint:**

Use $\mathrm{Var}(X_1 \mid X_2, X_3) = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$; the conditioning values themselves do not affect the conditional variance.

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**36.** Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x_1, x_2) = 2x_1^4 + x_2^2 + x_2 x_1^2.$$

Which of the following statements is correct?

(A) There are two distinct stationary points of $f$

(B) $(0,0)$ is a saddle point of $f$

(C) $(0,0)$ is a local maximum of $f$

(D) $(0,0)$ is a local minimum of $f$

**Correct Answer:** (D) $(0, 0)$ is a local minimum of $f$

**SOLUTION:**

**Step 1:** Recall what stationary point and Hessian test mean.

A stationary point of a function of two variables is a point where both first partial derivatives vanish. Once we find such a point, the usual next step is the second derivative (Hessian) test: if the Hessian matrix of second partial derivatives is positive definite there, the point is a local minimum; if negative definite, a local maximum; if it has both a positive and a negative eigenvalue, it is a saddle point. When the Hessian is only semi-definite, meaning its determinant is zero, this test gives no answer and we must argue directly from the function.

**Step 2:** Find the stationary point a different way.

Set the first partial derivative to zero first: $\dfrac{\partial f}{\partial x_1} = 2x_1(4x_1^2 + x_2) = 0$ gives $x_1 = 0$ or $x_2 = -4x_1^2$.

**Step 3:** Feed each branch into the second equation.

The second equation is $\dfrac{\partial f}{\partial x_2} = 2x_2 + x_1^2 = 0$. If $x_1 = 0$, this gives $x_2 = 0$, so we get the point $(0, 0)$. If instead $x_2 = -4x_1^2$, substitute to get $2(-4x_1^2) + x_1^2 = -7x_1^2 = 0$, so again $x_1 = 0$ and then $x_2 = 0$. Both branches collapse to the same point, so $(0, 0)$ is the only stationary point and there cannot be a second distinct one. This already rules out option (A), which claims two distinct stationary points.

**Step 4:** Test positivity with a discriminant argument.

View $f$ as a quadratic in $x_2$ for a fixed $x_1$:

$$f = x_2^2 + x_1^2 x_2 + 2x_1^4.$$

The coefficient of $x_2^2$ is $1 > 0$, so this quadratic opens upward in $x_2$. Its discriminant is

$$D = (x_1^2)^2 - 4(1)(2x_1^4) = x_1^4 - 8x_1^4 = -7x_1^4.$$

**Step 5:** Read the discriminant.
Since $x_1^4 \geq 0$, we have $D = -7x_1^4 \leq 0$ for every real $x_1$, with $D = 0$ only at $x_1 = 0$. A upward quadratic with $D \leq 0$ never goes negative, so $f \geq 0$ for every $(x_1, x_2)$. When $x_1 \neq 0$ the discriminant is strictly negative, so $f > 0$ strictly, and when $x_1 = 0$ the quadratic reduces to $x_2^2 \geq 0$, zero only at $x_2 = 0$.

**Step 6:** Cross check with a couple of test points.
Take $x_1 = 1, x_2 = -2$: $f = 2(1) + 4 + (-2)(1) = 2 + 4 - 2 = 4 > 0$. Take $x_1 = 1, x_2 = 0$: $f = 2(1) + 0 + 0 = 2 > 0$. Take $x_1 = 0, x_2 = 1$: $f = 0 + 1 + 0 = 1 > 0$. None of these ever beat $f(0,0) = 0$, consistent with $(0,0)$ being the minimum.

**Step 7:** Conclude.
So $f(x_1, x_2) \geq 0 = f(0,0)$ everywhere, and this value is achieved nowhere else. That makes $(0,0)$ a strict global minimum, so it is a local minimum, not a saddle point or a local maximum.

$$\boxed{(0,0) \text{ is a local minimum of } f}$$

**Quick Tip:** Write $f$ as a quadratic in $x_2$ and complete the square in $x_2$; the result is a sum of two squared terms, so $f \geq 0$ everywhere with equality only at the origin.

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**37.** Let $x_1 \in (0,4)$ and consider the sequence $\{x_n\}_{n\geq 1}$ defined iteratively by

$$x_{n+1} = 2 - (4 - x_n)^{\frac{1}{2}}, \quad n \geq 1.$$

Consider the following statements:

(I) $\{x_n\}$ converges to 0.

(II) $\left\{\dfrac{x_{n+1}}{x_n}\right\}$ converges to $\dfrac{1}{4}$.

Which of the following statements is correct?

(A) Only statement (I) is correct

(B) Only statement (II) is correct

(C) Both statements (I) and (II) are correct

(D) Neither statement (I) nor statement (II) is correct

**Correct Answer:** (C) Both statements (I) and (II) are correct

**SOLUTION:**

**Step 1:** Bound and monotonicity directly from an inequality.

For $0 < x < 4$ we compare $x$ with $x_{n+1} = 2 - \sqrt{4-x}$. We claim $\sqrt{4-x} > 2 - x$ for every $x$ in $(0,4)$. If $x \leq 2$, then $2 - x \geq 0$, and squaring both sides gives $4 - x > 4 - 4x + x^2$, that is $x(3-x) > 0$, true because $0 < x \leq 2 < 3$. If $2 < x < 4$, then $2 - x < 0 \leq \sqrt{4-x}$, so the inequality holds automatically. So $x > 2 - \sqrt{4-x} = x_{n+1}$.

**Step 2:** The sequence is trapped and shrinking.

Step 1 shows $x_{n+1} < x_n$ for every term, so the sequence is strictly decreasing. Also $\sqrt{4-x_n} \in (0,2)$ whenever $x_n \in (0,4)$, so $x_{n+1} = 2 - \sqrt{4-x_n} \in (0,2)$, keeping every later term positive. A decreasing sequence bounded below by 0 must converge to some limit $L \geq 0$.

**Step 3:** Pin down $L$ using the fixed point equation.
Passing to the limit gives $L = 2 - \sqrt{4-L}$, so $\sqrt{4-L} = 2 - L$. Squaring gives $L^2 - 3L = 0$, giving $L = 0$ or $L = 3$. Because the sequence stays in $(0, 2)$, the value $L = 3$ cannot be the limit, so $L = 0$. Statement (I) holds.

**Step 4:** Get the ratio from a first order expansion near 0.
Near $x = 0$, write $\sqrt{4-x} = 2\sqrt{1 - x/4}$. Using $\sqrt{1-u} \approx 1 - \frac{u}{2}$ for small $u = x/4$, gives $\sqrt{4-x} \approx 2 - \frac{x}{4}$. So for small $x_n$,

$$x_{n+1} = 2 - \sqrt{4 - x_n} \approx 2 - \left(2 - \frac{x_n}{4}\right) = \frac{x_n}{4}.$$

**Step 5:** Take the limit of the ratio.
So $\frac{x_{n+1}}{x_n} \approx \frac{1}{4}$ once $x_n$ is small, and since $x_n \to 0$ this becomes exact in the limit, giving $\frac{x_{n+1}}{x_n} \to \frac{1}{4}$. Statement (II) also holds.

Both statements are correct.

$$\boxed{\text{Both statements (I) and (II) are correct}}$$

**Quick Tip:** Rationalise the recursion to get $x_{n+1} = \frac{x_n}{2 + \sqrt{4 - x_n}}$; this shows the sequence decreases to 0, and the ratio $x_{n+1}/x_n$ tends to $1/(2+2) = 1/4$.

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**38.** Let $A \in M_n(\mathbb{R})$ be an $n \times n$ real matrix, $n \geq 2$. Consider the following two statements:
(I) If $\lambda \in \mathbb{C}$ is an eigenvalue of $A$, then its complex conjugate $\bar{\lambda}$ is also an eigenvalue.
(II) If $v = (v_1, v_2, \dots, v_n) \in \mathbb{C}^n$ is an eigenvector corresponding to eigenvalue $\lambda = x + iy$, $y \neq 0$, then

$$\mathrm{Re}(v) = (\mathrm{Re}(v_1), \dots, \mathrm{Re}(v_n)) \quad \text{and} \quad \mathrm{Im}(v) = (\mathrm{Im}(v_1), \dots, \mathrm{Im}(v_n))$$

are linearly independent vectors over $\mathbb{R}$.

Which of the following statements is correct?

(A) Only statement (I) is correct

(B) Only statement (II) is correct

(C) Both statements (I) and (II) are correct

(D) Neither statement (I) nor statement (II) is correct

**Correct Answer:** (C) Both statements (I) and (II) are correct

**SOLUTION:**

**Step 1:** Statement (I).
For a real matrix, the characteristic equation $\det(A - \lambda I) = 0$ expands into a polynomial whose coefficients come only from sums and products of the real entries of $A$, so every coefficient is real. A standard algebra fact says that if a polynomial with real coefficients has a complex root $\lambda$, its conjugate $\bar{\lambda}$ is automatically a root too. Applying this to the characteristic polynomial proves statement (I) is TRUE for every real matrix.

**Step 2:** Statement (II), attack it through a $2 \times 2$ real block.

Let $v = u + iw$ with $u, w \in \mathbb{R}^n$, and $\lambda = x + iy$, $y \neq 0$. Comparing real and imaginary parts of $Av = \lambda v$ gives $Au = xu - yw$ and $Aw = yu + xw$. This says the pair $(u, w)$ spans a 2-dimensional real subspace on which $A$ acts exactly like the $2 \times 2$ real matrix $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$.

**Step 3:** Use that this $2 \times 2$ matrix has no real eigenvector.
The matrix $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ has eigenvalues $x \pm iy$, which are not real because $y \neq 0$. If $u$ and $w$ were linearly dependent, say $w = cu$ with $u \neq 0$, then $A$ restricted to that line would have to send $u$ to a real multiple of itself, meaning $u$ is a real eigenvector for a real eigenvalue, contradicting that the eigenvalues $x \pm iy$ are non-real.

**Step 4:** Handle the case where one of $u, w$ is exactly zero.
If $w = 0$, the relation $Aw = yu + xw$ reduces to $0 = yu$, and since $y \neq 0$ this forces $u = 0$ too, which cannot happen because $v = u + iw \neq 0$. The symmetric argument rules out $u = 0$. So $u, w$ are both nonzero and, by Step 3, cannot be proportional either.

**Step 5:** Conclude.
So $u = \text{Re}(v)$ and $w = \text{Im}(v)$ are linearly independent over $\mathbb{R}$, proving statement (II) TRUE as well. Both statements hold.

$$\boxed{\text{Both statements (I) and (II) are correct}}$$

**Quick Tip:** Real-coefficient polynomials have roots in conjugate pairs, which settles statement (I); for statement (II), split $Av = \lambda v$ into real and imaginary parts and check that $\text{Re}(v)$ and $\text{Im}(v)$ cannot be proportional when $y \neq 0$.

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**39.** Consider the system of equations

$$x + 2y - z = a$$

$$x + y + 3z = b$$

$$2x + 3y + 2z = c$$

Consider the following statements:
(I) For every $(a, b, c) \in \mathbb{R}^3$, the above system has a solution.
(II) For $(a, b, c) = (0, 0, 0)$, the solution set is given by

$$\{(-7t,\ 4t,\ t) : t \in \mathbb{R}\}.$$

Which of the following statements is correct?

(A) Only statement (I) is correct

(B) Only statement (II) is correct

(C) Both statements (I) and (II) are correct

(D) Neither statement (I) nor statement (II) is correct

**Correct Answer:** (B) Only statement (II) is correct

**SOLUTION:**

**Step 1:** Recall when a square system is solvable for every right-hand side.

A system $Mx = v$ has exactly one solution for every choice of $v$

precisely when the rows of $M$ are linearly independent. If one row can be written as a combination of the others, then most choices of $(a, b, c)$ will fail.

**Step 2:** Test statement (I) by trying to eliminate directly.
Subtract the first equation from the second: $(x + y + 3z) - (x + 2y - z) = b - a$, giving $-y + 4z = b - a$.

**Step 3:** Build the third equation from the first two.
$2(x + 2y - z) + (-y + 4z) = 2x + 4y - 2z - y + 4z = 2x + 3y + 2z$, exactly the left side of equation 3. So equation 3 is not independent. Tracking the right-hand side: $2a + (b - a) = a + b$. So consistency needs $c = a + b$; this cannot hold for an arbitrary $(a, b, c)$, e.g. $(0, 0, 1)$ has no solution. So statement (I) is FALSE.

**Step 4:** Confirm $(0, 0, 0)$ passes the consistency test.
With $a = b = c = 0$, $c = a + b$ reads $0 = 0$, true.

**Step 5:** Solve using the relation found in Step 2.
With $a = b = 0$, $-y + 4z = 0$ gives $y = 4z$. Put into $x + 2y - z = 0$: $x + 8z - z = 0$, so $x = -7z$.

**Step 6:** Write and verify the solution set.
Letting $z = t$ gives $(x, y, z) = (-7t, 4t, t)$. Check the third equation: $2(-7t) + 3(4t) + 2(t) = 0$ for every $t$. This matches statement (II), so it is TRUE.

**Step 7:** Sanity check with $t = 1$.
$(x, y, z) = (-7, 4, 1)$: eq1: $-7 + 8 - 1 = 0$; eq2: $-7 + 4 + 3 = 0$; eq3: $-14 + 12 + 2 = 0$. All hold.

Only statement (II) is correct.

$$\boxed{\text{Only statement (II) is correct}}$$

**Quick Tip:** The coefficient matrix has determinant 0, so the system is not solvable for every $(a, b, c)$; solving the homogeneous system $a = b = c = 0$ by elimination gives $(x, y, z) = (-7t, 4t, t)$.

• • •

**40.** Let $(\Omega, \mathcal{F}, P)$ be a probability space, where for $A \subset \Omega, A \neq \phi, A \neq \Omega$,

$$\mathcal{F} = \{\Omega, \phi, A, A^c\}, \quad P(\Omega) = 1,\ P(\phi) = 0,\ P(A) = \frac{1}{2} = P(A^c).$$

Let $X$ and $Y$ be two random variables defined on $\Omega$ as follows:

$$X(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \in A^c, \end{cases} \quad \text{and} \quad Y(\omega) = \begin{cases} 1 & \text{if } \omega \in A^c \\ 0 & \text{if } \omega \in A. \end{cases}$$

Then which of the following statements is correct?

(A) $X$ and $Y$ have the same distribution function

(B) $X = Y$ almost everywhere

(C) $X$ and $Y$ are independent

(D) $E(XY) = 1$

**Correct Answer:** (A) $X$ and $Y$ have the same distribution function

**SOLUTION:**

**Step 1:** Build a small probability table over the two atoms.
The sample space effectively splits into two disjoint pieces with $P(A) = 1/2$, $P(A^c) = 1/2$. On $A$: $X = 1$, $Y = 0$. On $A^c$: $X = 0$, $Y = 1$. So the joint pairs $(X, Y)$ only ever take the values $(1, 0)$ with probability 1/2 and $(0, 1)$ with probability 1/2; the pairs $(1, 1)$ and $(0, 0)$ never happen.

**Step 2:** Read off the marginal law of $X$ and of $Y$.
Marginal of $X$: $P(X = 1) = 1/2$, $P(X = 0) = 1/2$. Marginal of $Y$: $P(Y = 1) = 1/2$, $P(Y = 0) = 1/2$. Both marginals are identical, so as distributions $X$ and $Y$ are identical, matching option (A).

**Step 3:** Use the table to test the other three options.
Every row in the table has $X \neq Y$. So $P(X = Y) = 0$, ruling out option (B).

**Step 4:** Test independence from the table.
Independence would require $P(X = 1, Y = 1) = 1/4$. But the table shows the pair $(1, 1)$ never occurs, so $P(X = 1, Y = 1) = 0 \neq 1/4$. Ruling out option (C).

**Step 5:** Compute $E(XY)$ straight from the table.
The product $XY$ equals 0 on $A$ and 0 on $A^c$. So $E(XY) = 0$, not 1, ruling out option (D).

Only option (A) survives every check.

$$\boxed{X \text{ and } Y \text{ have the same distribution function}}$$

**Quick Tip:** Both $X$ and $Y$ are Bernoulli(1/2) indicator variables with $Y = 1 - X$; they share the same distribution but are never equal pointwise, are not independent, and $XY = 0$ always since $A \cap A^c = \phi$.

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**41.** Let $X$ and $Y$ be independent and identically distributed geometric random variables having the following probability mass function

$$P(X = x) = p(1-p)^x, \quad x = 0, 1, 2, \ldots,$$

where $p \in (0, 1)$. Then which of the following statements is correct?

(A) $E(X|X+Y) = \dfrac{X+Y}{2}$

(B) $P(X = Y) = 1$

(C) $P(X = Y) = \dfrac{1-p}{1+p}$

(D) $E(X|X+Y) = \dfrac{X+Y+2}{2}$

**Correct Answer:** (A) $E(X|X+Y) = \dfrac{X+Y}{2}$

**SOLUTION:**

**Step 1:** Set up the conditional distribution.

$X$ and $Y$ are iid with $P(X = x) = p(1-p)^x$. We want $E(X \mid X + Y = n)$ for a fixed total $n$.

**Step 2:** Find the conditional pmf directly.

For $0 \le k \le n$,

$$P(X = k, Y = n-k) = p(1-p)^k \cdot p(1-p)^{n-k} = p^2(1-p)^n.$$

This does not depend on $k$: every split of the total $n$ has the same joint probability.

**Step 3:** Normalize to get the conditional pmf.

There are $n+1$ possible values of $k$, so

$$P(X = k \mid X + Y = n) = \frac{1}{n+1}, \quad k = 0, 1, \dots, n.$$

So given the total is $n$, $X$ is exactly uniform over $\{0, 1, \dots, n\}$.

**Step 4:** Compute the conditional mean.

$$E(X \mid X + Y = n) = \frac{0 + 1 + \cdots + n}{n+1} = \frac{n}{2}.$$

**Step 5:** Write this back in terms of $X + Y$.

$$E(X \mid X + Y) = \frac{X+Y}{2}.$$

Matches option (A), rules out (D).

**Step 6:** Rule out (B) and (C).

$X$ is uniform on $\{0, \dots, n\}$ given the sum, not always equal to $Y$, so (B) fails. Direct summation gives $P(X = Y) = p/(2-p)$, not matching (C).

**Final Answer:**

$$\boxed{E(X \mid X + Y) = \frac{X+Y}{2}}$$

**Quick Tip:** Since $X$ and $Y$ are i.i.d., swapping their labels does not change the joint distribution, so $E(X|X+Y)$ and $E(Y|X+Y)$ must be equal and together add up to $X+Y$.

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**42.** Let $X$ be a random variable with support $S = \{0, 1, 2, \ldots\}$ and

$$P(X \geq k+1 \,|\, X \geq k) = p, \quad k \in S, \;\; 0 < p < 1.$$

Then which of the following statements is correct?

(A) $P(X \geq k+m \,|\, X \geq k) = P(X \geq m)$, for all $m, k \in S$

(B) $E(X) > \mathrm{Var}(X)$

(C) $P(X \leq x) = 1 - p^x$, for all $x \in S$

(D) $P(X \leq k+m \,|\, X \geq k) = 1 - p^m$, for all $k, m \in S$

**Correct Answer:** (A) $P(X \geq k+m \,|\, X \geq k) = P(X \geq m)$, for all $m, k \in S$

**SOLUTION:**

**Step 1:** Identify the distribution from the given condition.
The constant one-step survival property $P(X \geq k+1) = p\,P(X \geq k)$ characterizes the geometric distribution.

**Step 2:** Derive the pmf from first principles.
$P(X \geq k) = p^k$, so $P(X = k) = p^k(1-p)$, $k = 0, 1, 2, \ldots$

**Step 3:** Use the known memoryless property.

$$P(X \geq k+m \,|\, X \geq k) = \frac{p^{k+m}}{p^k} = p^m = P(X \geq m).$$

TRUE, matching (A).

**Step 4:** Test (B) with the geometric formulas.
$\text{Var}(X)/E(X) = 1/(1-p) > 1$, so variance always exceeds mean. (B) FALSE.

**Step 5:** Test (C) and (D) with the exact cdf.
$P(X \le x) = 1 - p^{x+1}$, not $1 - p^x$; similarly $P(X \le k + m \mid X \ge k) = 1 - p^{m+1}$, not $1 - p^m$. Both FALSE.

**Final Answer:**

$$\boxed{P(X \ge k+m \mid X \ge k) = P(X \ge m), \text{ for all } m, k \in S}$$

**Quick Tip:** The condition $P(X \ge k+1 | X \ge k) = p$ for all $k$ forces $P(X \ge k) = p^k$; this is the discrete memoryless property.

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**43.** Let $X_1, X_2, X_3$ be a random sample from a distribution having probability mass function

$$f_\theta(x) = \begin{cases} \theta & \text{if } x = 1 \\ 1-\theta & \text{if } x = 2 \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta \in \Theta = (0, 1)$. Let $\underline{X} = (X_1, X_2, X_3)$. Then which of the following is NOT a sufficient statistic for $\theta$?

(A) $T_1(\underline{X}) = (X_1 - X_2,\ X_1 + X_2,\ X_1 + X_3)$

(B) $T_2(\underline{X}) = (X_1 + X_2,\ X_1 - X_3,\ X_2 + X_3)$

(C) $T_3(\underline{X}) = (X_1 - X_2,\ X_2 - X_3,\ X_3)$

(D) $T_4(\underline{X}) = (X_1 + X_2,\ X_3)$

**Correct Answer:** (B) $T_2(\underline{X}) = (X_1 + X_2,\ X_1 - X_3,\ X_2 + X_3)$

**SOLUTION:**

**Step 1:** Recall the sufficiency rule.

A statistic is sufficient exactly when it determines $n_1$ (count of 1s), or equivalently the sample sum.

**Step 2:** Check invertibility of each linear map.

$T_1$: coefficient matrix determinant $= -2 \neq 0$, invertible, recovers everything. $T_3$: upper triangular, determinant 1, invertible. $T_4$: sum of components gives $X_1 + X_2 + X_3$ directly.

**Step 3:** Check $T_2$.

Coefficient matrix determinant $= 0$, singular, information lost.

**Step 4:** Confirm the lost information matters.

$(1, 2, 1)$ $(n_1 = 2)$ and $(2, 1, 2)$ $(n_1 = 1)$ both give $T_2 = (3, 0, 3)$, but different likelihoods, so $T_2$ is not sufficient.

**Final Answer:**

$$\boxed{T_2(\underline{X}) = (X_1 + X_2,\ X_1 - X_3,\ X_2 + X_3)}$$

**Quick Tip:** A statistic is sufficient only if it lets you recover the count of 1s among $X_1, X_2, X_3$; check whether two samples with different counts can give the same statistic value.

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**44.** Let $X_1, X_2, \ldots, X_n$ $(n \geq 2)$ be a random sample from the probability density function $f(x)$. Consider the following hypotheses:

$$H_0 : f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}; \quad -\infty < x < \infty$$

$$H_1 : f(x) = \frac{1}{2} e^{-|x|}; \quad -\infty < x < \infty.$$

For testing $H_0$ against $H_1$, let $R$ denote the critical region based on the likelihood ratio test having level 0.05. Then, for some constant $c$, the region $R$ is

(A) $\{(x_1, x_2, \ldots, x_n) : \sum_{i=1}^n (x_i - 1)^2 > c\}$

(B) $\{(x_1, x_2, \ldots, x_n) : \sum_{i=1}^n (x_i - 1)^2 < c\}$

(C) $\{(x_1, x_2, \ldots, x_n) : \sum_{i=1}^n (|x_i| - 1)^2 > c\}$

(D) $\{(x_1, x_2, \ldots, x_n) : \sum_{i=1}^n (|x_i| - 1)^2 < c\}$

**Correct Answer:** (C) $\{(x_1, x_2, \ldots, x_n) : \sum_{i=1}^n (|x_i| - 1)^2 > c\}$

**SOLUTION:**

**Step 1:** Write down the two densities.

$H_0$: standard normal $f_0(x)$. $H_1$: standard Laplace $f_1(x)$. LRT rejects for large $\Lambda = L_1/L_0$.

**Step 2:** Form the log ratio.

$$\ln \Lambda = n \ln \frac{\sqrt{2\pi}}{2} + T, \qquad T = \sum_{i=1}^{n} \left( \frac{x_i^2}{2} - |x_i| \right).$$

**Step 3:** Substitute $y_i = |x_i|$.

$$T = \frac{1}{2} \sum_{i=1}^{n} (y_i - 1)^2 - \frac{n}{2}.$$

**Step 4:** Translate the rejection rule.

$$\sum_{i=1}^{n} (|x_i| - 1)^2 > c.$$

**Step 5:** Rule out the wrong options.

$x_i$ instead of $|x_i|$ ignores the natural transformation; reversed inequality rejects when data looks normal, backward.

**Final Answer:**

$$\boxed{R = \left\{ (x_1, \dots, x_n) : \sum_{i=1}^{n} (|x_i| - 1)^2 > c \right\}}$$

**Quick Tip:** Take the log likelihood ratio and complete the square in $|x_i|$; remember $x_i^2 = |x_i|^2$, so the exponent naturally becomes a function of $|x_i| - 1$.

**45.** Let $X_1, X_2, \ldots, X_n$ $(n > 1)$ be a random sample from the following probability density function

$$f_\beta(x) = \begin{cases} \beta e^{-x}(1 - e^{-x})^{\beta - 1} & \text{if } x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\beta > 0$ is an unknown parameter. For testing the following hypotheses,

$$H_0 : \beta = 1 \quad \text{against} \quad H_1 : \beta > 1,$$

at level $\alpha \in (0, 1)$, which of the following statements is correct?

(A) The uniformly most powerful test does not exist

(B) For some constant $a$, the critical region of the uniformly most powerful test will be of the form $C = \{(x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} x_i > a\}$

(C) For some constant $a$, the critical region of the uniformly most powerful test will be of the form $C = \{(x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} \ln(1 - e^{-x_i}) > a\}$

(D) For some constant $a$, the critical region of the uniformly most powerful test will be of the form $C = \{(x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} \ln(1 - e^{-x_i}) < a\}$

**Correct Answer:** (C) For some constant $a$, the critical region of the uniformly most powerful test will be of the form $C = \{(x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} \ln(1 - e^{-x_i}) > a\}$

**SOLUTION:**

**Step 1:** Fix a single alternative value.

Pick $\beta_1 > 1$, compare $H_0 : \beta = 1$ vs simple alternative $\beta = \beta_1$. Neyman-Pearson rejects for large $L(\beta_1)/L(1)$.

**Step 2:** Compute this ratio.

$$\frac{L(\beta_1)}{L(1)} = \beta_1^n \exp\left[(\beta_1 - 1)\sum_{i=1}^{n} \ln(1 - e^{-x_i})\right].$$

**Step 3:** Note this is increasing in $T$ for every $\beta_1 > 1$.

Same rejection rule $T > a$ is most powerful against every $\beta_1 > 1$.

**Step 4:** UMP exists.

Since the same rule works for the whole composite alternative, a UMP test exists, ruling out (A).

**Step 5:** Compare with wrong forms.

Statistic is $\sum \ln(1 - e^{-x_i})$, not $\sum x_i$; test flags large $T$ not small $T$.

**Final Answer:**

$$\boxed{C = \left\{(x_1, \ldots, x_n) : \sum_{i=1}^{n} \ln(1 - e^{-x_i}) > a\right\}}$$

**Quick Tip:** Substitute $u = 1 - e^{-x}$ to see that $f_\beta$ is a power (Beta) family in $u$; this exponential family has MLR in $\sum \ln(1 - e^{-x_i})$, and Karlin-Rubin gives the UMP test directly.

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**46.** Let $X_1, X_2, \ldots, X_5$ be random observations from a continuous distribution. Let $\theta_p$ be the $p$-th population quantile. Consider the following hypotheses

$$H_0 : \theta_{1/2} = 2.5 \quad \text{against} \quad H_1 : \theta_{1/2} > 2.5.$$

Let $X_{(r)}$ denote the $r$-th order statistic of the given observations. Then which of the following is a critical region of a level 0.05 test?

(A) $X_{(1)} > 2.5$

(B) $X_{(2)} > 2.5$

(C) $X_{(3)} > 2.5$

(D) $X_{(4)} > 2.5$

**Correct Answer:** (A) $X_{(1)} > 2.5$

**SOLUTION:**

**Step 1:** Think of this as a sign test on the median.

Under $H_0$, each of the five observations has a 50% chance of exceeding 2.5. Let $S$ count how many exceed it; $S \sim$Binomial$(5, 0.5)$.

**Step 2:** Read each option as a statement about $S$.

$X_{(1)} > 2.5$ needs $S = 5$; $X_{(2)} > 2.5$ needs $S \geq 4$; $X_{(3)} > 2.5$ needs $S \geq 3$; $X_{(4)} > 2.5$ needs $S \geq 2$.

**Step 3:** Compute the false alarm rate for option (A).

$P(S = 5) = (0.5)^5 = 1/32 \approx 0.031$, below 0.05.

**Step 4:** Show the others break the level.

$X_{(2)} > 2.5$: $P(S \geq 4) = 6/32 \approx 0.188$, too big; $X_{(3)}, X_{(4)}$ even bigger.

**Step 5:** Conclude.

$$\boxed{X_{(1)} > 2.5}$$

**Quick Tip:** Convert the order statistic event into a count of observations exceeding 2.5, then use that this count is Binomial(5, 0.5) under H0.

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**47.** Suppose that $X$ and $Y$ are independent and identically distributed $N_p(\mu, \Sigma)$ random vectors, where $\mu \in \mathbb{R}^p$ and $\Sigma$ is a positive definite matrix. Let $\chi^2_m$ denote chi-square distribution with $m$-degrees of freedom. Then which of the following statements is correct?

(A) $\frac{1}{2}(X - Y)^T\Sigma^{-1}(X - Y)$ follows $\chi^2_p$

(B) $2(X - Y)^T\Sigma(X - Y)$ follows $\chi^2_p$

(C) $\frac{1}{2}(X - Y)^T\Sigma^{-1}(X - Y)$ follows $\chi^2_{2p}$

(D) $2(X - Y)^T\Sigma(X - Y)$ follows $\chi^2_{2p}$

**Correct Answer:** (A) $\frac{1}{2}(X - Y)^T\Sigma^{-1}(X - Y)$ follows $\chi^2_p$

**SOLUTION:**

**Step 1:** Reduce to a simpler vector.

$D = X - Y \sim N_p(0, 2\Sigma)$.

**Step 2:** Whiten using square root of covariance.

$W = \frac{1}{\sqrt{2}}\Sigma^{-1/2}D \sim N_p(0, I_p)$.

**Step 3:** Sum of squares is chi-square.

$W^T W \sim \chi_p^2$.

**Step 4:** Write back in terms of D.

$W^T W = \frac{1}{2} D^T \Sigma^{-1} D$. So $\frac{1}{2}(X-Y)^T \Sigma^{-1}(X-Y) \sim \chi_p^2$.

**Step 5:** Rule out remaining options.

Plain $\Sigma$ wrong; $2p$ degrees wrong.

$$\boxed{\frac{1}{2}(X-Y)^T \Sigma^{-1}(X-Y) \sim \chi_p^2}$$

**Quick Tip:** Find the distribution of X-Y first, then use that a zero-mean normal vector times the inverse of its own covariance gives a chi-square with degrees of freedom equal to the vector's dimension.

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**48.** Consider the multiple linear regression model

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \epsilon_i, \quad i = 1, 2, \dots, 31,$$

where $\epsilon_i$ are iid $N(0,1)$ variables. The $F$-test for testing significance of regression rejects $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$ at 5% level. Given

$$F_{0.05;3,27} = 2.96, \quad F_{0.05;3,30} = 2.92, \quad F_{0.025;3,27} = 4.01, \quad F_{0.025;3,30} = 3.91.$$

Then the value of $R^2$ cannot be equal to

(A) 0.50

(B) 0.80

(C) 0.30

(D) 0.20

**Correct Answer:** (D) 0.20

**SOLUTION:**

**Step 1:** Get the two degrees of freedom right.

Numerator df $= 3$, denominator df $= 31 - 4 = 27$. Use $F_{0.05;3,27} = 2.96$.

**Step 2:** Find boundary R^2.

$$9R^2 = 2.96(1 - R^2) \Rightarrow R^2 \approx 0.2475$$

**Step 3:** Compare each option.

$0.50, 0.80, 0.30$ all above $0.2475$; only $0.20$ below.

**Step 4:** Confirm.

At $R^2 = 0.20$: $F = 2.25 < 2.96$.

$$\boxed{0.20}$$

**Quick Tip:** Write F in terms of R^2 using the correct residual degrees of freedom n-p-1=27, then check which option gives F below the critical value 2.96.

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**49.** Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{\sqrt{x^2+2y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Which of the following statements is/are correct?

(A) $f$ is continuous at $(0,0)$

(B) Partial derivatives $f_x$ and $f_y$ exist at $(0,0)$ and $f_x(0,0) = 0$, $f_y(0,0)$ $= 0$

(C) $f$ is not differentiable at $(0,0)$

(D) $f$ is differentiable

**Correct Answer:** (A) $f$ is continuous at $(0,0)$ ; (B) Partial derivatives $f_x$ and $f_y$ exist at $(0,0)$ and $f_x(0,0) = 0$, $f_y(0,0) = 0$ ; (D) $f$ is differentiable

**SOLUTION:**

**Step 1:** Polar form.

$f(x,y) = r^2 \cdot \frac{\cos^3\theta + \sin^3\theta}{\sqrt{\cos^2\theta + 2\sin^2\theta}}$, bounded by $Mr^2$.

**Step 2:** Continuity.

$|f| \leq Mr^2 \to 0$. TRUE.

**Step 3:** Partials.

Along axes both give 0. TRUE.

**Step 4:** Differentiability.

$|f(h,k)|/r \leq Mr \to 0$ uniformly in direction. Differentiable. (C) FALSE.

**Step 5:** Everywhere.

Smooth quotient elsewhere. TRUE.

$$\boxed{\text{A, B, D}}$$

**Quick Tip:** Bound |f(x,y)| by x^2 + y^2/sqrt(2) using sqrt(x^2+2y^2) >= |x| and >= sqrt(2)|y|; this single bound settles continuity, the partial derivatives, and differentiability at once.

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**50.** Consider the matrix

$$A = \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}.$$

Which of the following statements is/are correct?

(A) $A$ is never diagonalizable

(B) $A$ is diagonalizable if and only if $\lambda_1, \lambda_2, \lambda_3$ are distinct

(C) The dimension of eigenspaces corresponding to each $\lambda_i$ is $1, i = 1, 2, 3$

(D) For any $A$, there exist real numbers $\alpha$ and $\beta$ such that $A^2 + \alpha A + \beta I_3 = 0$

**Correct Answer:** (B) $A$ is diagonalizable if and only if $\lambda_1, \lambda_2, \lambda_3$ are distinct ; (C) The dimension of eigenspaces corresponding to each $\lambda_i$ is $1, i = 1, 2, 3$

**SOLUTION:**

**Step 1:** Structure.

$A$ is non-derogatory Hessenberg; eigenspace always dim 1 per eigenvalue.

**Step 2:** (C) settled.
TRUE for all cases.

**Step 3:** (B) settled.
Diagonalizable iff eigenspace dim = multiplicity, so iff all distinct. TRUE.

**Step 4:** (A) false.
Distinct case is diagonalizable.

**Step 5:** (D) test with example.
$\lambda = 0, 1, 2$: minimal poly degree 3, cannot satisfy quadratic relation. FALSE.

$$\boxed{\text{B, C}}$$

**Quick Tip:** The 1's on the superdiagonal keep rank(A - lambda I) equal to 2 for every eigenvalue lambda, distinct or repeated, so each eigenspace always has dimension 1; use that to test diagonalizability and the minimal polynomial degree.

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**51.** Consider the real symmetric matrix $A = (a_{ij})$ given by

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}.$$

Consider the set

$$S = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \sum_{j=1}^{3} \sum_{i=1}^{3} a_{ij} x_i x_j = 1 \right\}.$$

Which of the following statements is/are correct?

(A) $S$ is empty set

(B) Any line $L$ in $\mathbb{R}^3$ has at most two points from $S$

(C) $S$ is bounded

(D) $S$ is unbounded

**Correct Answer:** (B) Any line $L$ in $\mathbb{R}^3$ has at most two points from $S$ ; (D) $S$ is unbounded

**SOLUTION:**

**Step 1:** Write out the quadratic form in full.

For $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, the form is $x^T A x = 3x_1^2 + 2x_1x_2 + 2x_1x_3 + 4x_2x_3$.

**Step 2:** Complete the square instead of using eigenvalues.

Group the $x_1$ terms first: $3x_1^2 + 2x_1(x_2 + x_3) = 3\left(x_1 + \dfrac{x_2 + x_3}{3}\right)^2 - \dfrac{(x_2 + x_3)^2}{3}$. Substituting this back and then completing the square in $x_2$ gives, after simplification, three squared terms with coefficients $3, -\dfrac{1}{3}$, and $8$.

**Step 3:** Read off the signs.

Two coefficients are positive and one is negative, matching the eigenvalue signs, so A is invertible and indefinite. The level set $x^T Ax$ $= 1$ is a genuine hyperboloid, not a cone or an empty set.

**Step 4:** Rule out A and C, confirm D.

Since the form reaches 1 along a positive-coefficient direction, S is not empty, so (A) fails. Since one term is negative, we can let the matching coordinate grow freely while the equation still balances, so S extends to infinity and is unbounded. So (C) fails and (D) holds.

**Step 5:** Check line intersections directly.

On any line $x = x_0 + tv$, the condition $x^T Ax = 1$ turns into $(v^T Av)t^2$ $+ 2(x_0^T Av)t + (x_0^T Ax_0 - 1) = 0$, a quadratic in $t$. A quadratic equation never has more than two roots, so any line meets S in at most two points, giving (B).

**Final Answer:**

S is an unbounded quadric surface, and every line meets it in at most two points.

$$\boxed{\text{(B) and (D)}}$$

**Quick Tip:** Find the eigenvalues of A to see that S is an indefinite quadric (a hyperboloid), then use the fact that substituting a line into x^TAx=1 gives a quadratic equation in the line parameter.

**52.** Let $X$ be a discrete random variable with support $S = \{1, 2, 3, \ldots\}$ such that $E(X^2) < \infty$. Let $F$ be the distribution function of $X$. Then which of the following statements is/are correct?

(A) $E(X) = \sum_{n=1}^{\infty}(1 - F(n-1))$

(B) $F(X)$ has discrete uniform distribution

(C) There exists at least one such random variable $X$ such that $X$ and $\frac{1}{X}$ have the same distribution

(D) $E(X^2) = \sum_{n=1}^{\infty}(2n-1)\big(1 - F(n-1)\big)$

**Correct Answer:** (A) $E(X) = \sum_{n=1}^{\infty}(1 - F(n-1))$ ; (D) $E(X^2) = \sum_{n=1}^{\infty}(2n - 1)\big(1 - F(n-1)\big)$

**SOLUTION:**

**Step 1:** Expand E(X) directly and swap the order of summation.

By definition $E(X) = \sum_{k=1}^{\infty} kP(X = k)$. Write $k = \sum_{n=1}^{k} 1$ and swap the order of the double sum:

$$E(X) = \sum_{k=1}^{\infty}\sum_{n=1}^{k} P(X = k) = \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} P(X = k) = \sum_{n=1}^{\infty} P(X \geq n).$$

Since $P(X \geq n) = 1 - F(n-1)$, this is exactly option (A), so (A) is TRUE.

**Step 2:** Test (B) with a concrete example.

Take X geometric with $P(X = k) = p(1-p)^{k-1}$. Then $F(k) = 1 - (1-p)^k$, so $F(X) = 1 - (1-p)^X$. As X ranges over the integers, $F(X)$ takes unequally spaced values with unequal probabilities, never uniform on $(0, 1)$. So (B) fails in general.

**Step 3:** Test (C) using the support condition.

Since X must place positive probability on every integer $2, 3, 4, \ldots$, we need $P(1/X = 2) = P(X = 2) > 0$. But $1/X = 2$ needs $X = 1/2$, which is impossible for an integer-valued X. So no valid X can share a distribution with its reciprocal, and (C) is FALSE.

**Step 4:** Expand E(X^2) the same way.

Write $k^2 = \sum_{n=1}^{k}(2n-1)$, the sum of the first k odd numbers. Then

$$E(X^2) = \sum_{k=1}^{\infty}\sum_{n=1}^{k}(2n-1)P(X=k) = \sum_{n=1}^{\infty}(2n-1)\sum_{k=n}^{\infty}P(X=k) = \sum_{n=1}^{\infty}(2n-1)(1-F(n-1)).$$

This matches (D) exactly, so (D) is TRUE.

**Final Answer:**

Swapping the order of summation proves both tail sum formulas; the uniform claim and the reciprocal claim both break down.

$$\boxed{\text{(A) and (D)}}$$

**Quick Tip:** Write X, and separately X squared, as a sum of indicators 1(X greater than or equal to n), then swap the order of summation with the expectation to reach the tail sum formulas.

• • •

**53.** Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables having the following probability mass function

$$P(X_n = x) = \frac{1}{5n}\left(1 - \frac{1}{5n}\right)^x, \quad x = 0, 1, 2, \ldots; \ n \in \mathbb{N}.$$

Define $Z_n = \dfrac{X_n}{n}$, $n \in \mathbb{N}$, and let $V$ be a random variable. If $Z_n \xrightarrow{d} V$, as $n \to \infty$, then which of the following statements is/are correct?

(A) $V$ has normal distribution with mean 5 and variance 25

(B) $V$ has chi-square distribution with 5 degrees of freedom

(C) $V$ has exponential distribution with mean 5

(D) $e^{-V/5}$ has uniform distribution over $(0, 1)$

**Correct Answer:** (C) $V$ has exponential distribution with mean 5 ; (D) $e^{-V/5}$ has uniform distribution over $(0, 1)$

**SOLUTION:**

**Step 1:** Write the moment generating function of X_n.

For a geometric random variable on $\{0, 1, 2, \ldots\}$ with parameter $p_n$, the MGF is

$$M_{X_n}(s) = E(e^{sX_n}) = \frac{p_n}{1 - (1 - p_n)e^s}, \qquad e^s(1 - p_n) < 1.$$

**Step 2:** Get the MGF of Z_n=X_n/n.

$$M_{Z_n}(s) = E(e^{sX_n/n}) = \frac{p_n}{1 - (1 - p_n)e^{s/n}},$$

with $p_n = \frac{1}{5n}$.

**Step 3:** Take the limit as n grows large.

For large n, $e^{s/n} \approx 1 + \frac{s}{n}$ and $1 - p_n \approx 1 - \frac{1}{5n}$, so

$$(1 - p_n)e^{s/n} \approx 1 + \frac{s - 1/5}{n}.$$

So the denominator $1 - (1 - p_n)e^{s/n} \approx \frac{1/5 - s}{n}$, and the numerator $p_n = \frac{1}{5n}$. Dividing the two,

$$M_{Z_n}(s) \to \frac{1/5}{1/5 - s} = \frac{1}{1 - 5s},$$

which is the MGF of an Exponential distribution with mean 5, valid for $s < 1/5$.

**Step 4:** Match with the options.

So $V \sim$ Exponential(mean 5), confirming (C) and ruling out (A) normal and (B) chi-square.

**Step 5:** Apply the probability integral transform.

For any continuous random variable, plugging it into its own CDF gives a Uniform(0,1) variable. Here $F_V(v) = 1 - e^{-v/5}$, so $1 - e^{-V/5} \sim U(0, 1)$. Since reflecting a Uniform(0,1) variable about one half keeps it Uniform(0,1), $e^{-V/5} = 1 - (1 - e^{-V/5})$ is also $U(0, 1)$. So (D) holds.

**Final Answer:**

The MGF limit confirms V is exponential with mean 5, and (D) follows from the probability integral transform.

$$\boxed{\text{(C) and (D)}}$$

**Quick Tip:** Find the limiting survival function P(Zn greater than t) as n grows large; it should converge to e^(-t/5), the survival function of an exponential distribution with mean 5.

• • •

**54.** Let $\{X_k\}_{k\geq 1}$ be a sequence of independent random variables such that

$$X_{2k-1} \sim \text{Bin}(1, \theta), \quad \text{and} \quad X_{2k} \sim \text{Bin}(1, 1-\theta), \quad k = 1, 2, 3, \dots,$$

where $\theta \in (0, 1)$. Let $\{Y_k\}_{k\geq 1}$ be another sequence of independent and identically distributed random variables such that $Y_k \sim \text{Poisson}(\lambda)$, $\lambda > 0$. Define, for $n \in \mathbb{N}$,

$$S_{2n} = \sum_{k=1}^{n} (X_{2k-1} - X_{2k} + 1 - 2\theta), \quad W_n = \sum_{k=1}^{n} Y_k^2 \quad \text{and} \quad \sigma_{2n}^2 = 2n\theta(1-\theta).$$

Then which of the following statements is/are correct?

(A) $\dfrac{nS_{2n}}{\sigma_{2n}W_n} \xrightarrow{d} N(0,1)$, as $n \to \infty$

(B) $\dfrac{nS_{2n}}{\sigma_{2n}W_n} \xrightarrow{d} N\left(0, \dfrac{1}{\lambda^2(1+\lambda)^2}\right)$, as $n \to \infty$

(C) $\dfrac{nS_{2n} + \sigma_{2n}W_n}{n\sigma_{2n}} \xrightarrow{d} N(\lambda(1+\lambda),\ 1)$, as $n \to \infty$

(D) $\dfrac{nS_{2n} + \sigma_{2n}W_n}{n\sigma_{2n}} \xrightarrow{d} N\left(0, \dfrac{1}{\lambda^2(1+\lambda)^2}\right)$, as $n \to \infty$

**Correct Answer:** (B) $\dfrac{nS_{2n}}{\sigma_{2n}W_n} \xrightarrow{d} N\left(0, \dfrac{1}{\lambda^2(1+\lambda)^2}\right)$, as $n \to \infty$ ; (C) $\dfrac{nS_{2n} + \sigma_{2n}W_n}{n\sigma_{2n}} \xrightarrow{d} N(\lambda(1+\lambda),\ 1)$, as $n \to \infty$

**SOLUTION:**

**Step 1:** Simplify S_2n/sigma_2n as a standardized sum.

Write $U_k = X_{2k-1} - X_{2k} + 1 - 2\theta$. Each $U_k$ has mean 0 and, using independence of the two Bernoulli pieces, variance $2\theta(1-\theta)$. Since the $U_k$ are iid across k, the classical CLT gives

$$\frac{\sum_{k=1}^{n} U_k}{\sqrt{2n\theta(1-\theta)}} = \frac{S_{2n}}{\sigma_{2n}} \xrightarrow{d} Z,$$

where $Z \sim N(0,1)$.

**Step 2:** Simplify W_n/n as a sample mean.
$W_n/n = \frac{1}{n}\sum_{k=1}^{n} Y_k^2$ is the sample average of iid variables $Y_k^2$ with $E(Y_k^2) = \lambda(1+\lambda)$ and finite variance, since all Poisson moments are finite. The Weak Law of Large Numbers gives $W_n/n \to \lambda(1+\lambda)$ in probability; call this constant $c = \lambda(1+\lambda)$.

**Step 3:** Handle the ratio nS_2n/(sigma_2n W_n).
This equals $Z_n \cdot \frac{1}{(W_n/n)}$ where $Z_n = S_{2n}/\sigma_{2n} \to Z \sim N(0,1)$ and $\frac{1}{W_n/n}$ $\to \frac{1}{c}$ in probability. By Slutsky's theorem, a distributional limit times a probability limit gives $\frac{1}{c}Z \sim N\left(0, \frac{1}{c^2}\right) = N\left(0, \frac{1}{\lambda^2(1+\lambda)^2}\right)$. This is option (B), so it is TRUE, while (A) fails since the variance is not 1 in general.

**Step 4:** Handle the sum form.
$\frac{nS_{2n}+\sigma_{2n}W_n}{n\sigma_{2n}} = Z_n + \frac{W_n}{n} \to Z + c$ in distribution, again by Slutsky, since adding a constant probability limit to a distributional limit only shifts the mean. So the limit is $N(c,1) = N(\lambda(1+\lambda),1)$, matching (C), while (D) with mean 0 is wrong.

**Final Answer:**
Slutsky's theorem gives N(0, 1/(lambda squared (1+lambda) squared)) for the ratio and N(lambda(1+lambda), 1) for the sum.

$$\boxed{\text{(B) and (C)}}$$

**Quick Tip:** Show S_2n/sigma_2n converges to N(0,1) by the Central Limit Theorem and W_n/n converges in probability to lambda(1+lambda) by the Law of Large Numbers, then combine the two limits using Slutsky's theorem.

• • •

**55.** Let $\{N(t); t \geq 0\}$ be a homogeneous Poisson process with rate 3, and let $T_1$ denote the first arrival time. Then which of the following statements is/are correct?

(A) $E(T_1) = 3$

(B) $E(T_1) = \frac{1}{3}$

(C) $E\big(T_1 \mid N(2) = 4\big) = \frac{6}{5}$

(D) $E\big(T_1 \mid N(3) = 4\big) = \frac{3}{5}$

**Correct Answer:** (B) $E(T_1) = \frac{1}{3}$ ; (D) $E\big(T_1 \mid N(3) = 4\big) = \frac{3}{5}$

**SOLUTION:**

**Step 1:** Identify the marginal law of T1.

In a Poisson process of rate $\mu = 3$, the gaps between arrivals are iid Exponential($\mu$), so $T_1 \sim$ Exponential(3) and $E(T_1) = 1/\mu = 1/3$. This rules out (A) and confirms (B).

**Step 2:** Build the conditional density of T1 given N(t)=m.

Given $N(t) = m$, the joint law of the m arrival times matches m iid Uniform$(0, t)$ points sorted in order. The smallest of m iid Uniform$(0, t)$ points has density

$$f_{(1)}(x) = \frac{m}{t}\left(1 - \frac{x}{t}\right)^{m-1}, \qquad 0 < x < t.$$

**Step 3:** Integrate to get the mean.

$$E(T_1 \mid N(t) = m) = \int_0^t x \cdot \frac{m}{t}\left(1 - \frac{x}{t}\right)^{m-1} dx.$$

Substituting $u = 1 - x/t$ turns this into a standard Beta-type integral, and it works out to the known closed form $\dfrac{t}{m+1}$.

**Step 4:** Plug in the numbers.

For (C), with $t = 2, m = 4$: $E(T_1 \mid N(2) = 4) = \dfrac{2}{5}$, not $\dfrac{6}{5}$, so (C) is FALSE.

For (D), with $t = 3, m = 4$: $E(T_1 \mid N(3) = 4) = \dfrac{3}{5}$, which matches exactly, so (D) is TRUE.

**Final Answer:**

The minimum of the uniform order statistics gives mean t/(m+1), confirming (B) and (D).

$$\boxed{\text{(B) and (D)}}$$

**Quick Tip:** T1 is exponential with mean 1 over the rate unconditionally; given N(t)=m, T1 is the minimum of m iid Uniform(0,t) variables with mean t/(m+1).

**56.** Let $X_1, X_2, \ldots, X_n$ ($n \geq 2$) be a random sample from the following probability density function

$$f(x) = \frac{1}{2} e^{-|x-\mu|}, \quad -\infty < x < \infty,$$

where $\mu \in (-\infty, \infty)$ is an unknown parameter. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $\hat{\mu}$ denote the maximum likelihood estimator of $\mu$, whenever it exists. Then which of the following statements is/are correct?

(A) $\hat{\mu}$ will always exist and $\hat{\mu} = \bar{X}$

(B) $\hat{\mu}$ may not always exist

(C) $\hat{\mu}$ always exists but it may not be unique

(D) $\hat{\mu}$ is a consistent estimator of $\mu$

**Correct Answer:** (C) $\hat{\mu}$ always exists but it may not be unique ; (D) $\hat{\mu}$ is a consistent estimator of $\mu$

**SOLUTION:**

**Step 1:** Set up minimization.

$g(\mu) = \sum |X_i - \mu|$; MLE minimizes this.

**Step 2:** Piecewise derivative.

$g'(\mu) = \sum \text{sign}(\mu - X_i)$, zero at median.

**Step 3:** Odd/even n.

Odd: unique minimizer. Even: interval of minimizers.

**Step 4-7: Test options.**

(A) FALSE median not mean. (B) FALSE always exists. (C) TRUE exists, non-unique for even n. (D) TRUE median converges in probability to population median = $\mu$ by symmetry.

$$\boxed{\text{(C) and (D)}}$$

**Quick Tip:** Maximizing the Laplace likelihood is the same as minimizing the sum of absolute deviations, and that is solved by the sample median, not the sample mean.

• • •

**57.** Let $X$ be a single observation from a distribution having a probability density function $f_\theta$, $\theta \in \Theta$. For testing $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$, under which of the following options a uniformly most powerful test of level 0.05 exists?

(A)

$$f_\theta(x) = \begin{cases} \dfrac{1}{\theta} & \text{if } 0 < x < \theta \\ 0 & \text{otherwise} \end{cases}; \quad \Theta = (0, \infty)$$

(B)

$$f_\theta(x) = \begin{cases} e^{-(x-\theta)} & \text{if } x > \theta \\ 0 & \text{otherwise} \end{cases}; \quad \Theta = (-\infty, \infty)$$

(C)

$$f_\theta(x) = \begin{cases} \dfrac{1}{\theta} e^{-x/\theta} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}; \quad \Theta = (0, \infty)$$

(D)

$$f_\theta(x) = \begin{cases} \dfrac{1}{\theta} e^{-(x-1)/\theta} & \text{if } x > 1 \\ 0 & \text{otherwise} \end{cases}; \quad \Theta = (0, \infty)$$

**Correct Answer:** (A)

$$f_\theta(x) = \begin{cases} \frac{1}{\theta} & \text{if } 0 < x < \theta \\ 0 & \text{otherwise} \end{cases}; \quad \Theta = (0, \infty)$$

; (B)

$$f_\theta(x) = \begin{cases} e^{-(x-\theta)} & \text{if } x > \theta \\ 0 & \text{otherwise} \end{cases}; \quad \Theta = (-\infty, \infty)$$

**SOLUTION:**

**Step 1:** General obstacle.

Two-sided tests usually have no UMP; best test for each direction differs.

**Step 2:** Special structure that fixes this.

Support boundary depending on $\theta$ lets one direction be free.

**Step 3:** (A) test.

Support $(0, \theta)$, one-sided info free, UMP exists. TRUE.

**Step 4:** (B) test.

Support $(\theta, \infty)$, UMP exists. TRUE.

**Step 5:** (C),(D) test.

Support fixed regardless of $\theta$, no free direction, no UMP. FALSE both.

$$\boxed{\text{(A) and (B)}}$$

**Quick Tip:** A UMP test for a two sided hypothesis can exist only when the parameter also fixes the boundary of the support of $X$, because then one side of the alternative is detected for free.

• • •

**58.** Suppose that $X = (X_1, \dots, X_p)^T$ follows $N_p(\mu, \Sigma)$, where $\mu \in \mathbb{R}^p$ and $\Sigma$ is a positive definite matrix. Let $A$ be a $p \times p$ matrix such that $A^T A = I_p$.
Let $Y = (Y_1, \dots, Y_p)^T = AX$. Then which of the following statements is/are correct?

(A) $Y$ follows $N_p(A\mu, A^T\Sigma A)$

(B) $Y$ follows $N_p(A\mu, A\Sigma A^T)$

(C) $\text{Var}\left(\sum_{i=1}^{p} Y_i\right) = \text{Var}\left(\sum_{i=1}^{p} X_i\right)$

(D) $\sum_{i=1}^{p} \text{Var}(Y_i) = \sum_{i=1}^{p} \text{Var}(X_i)$

**Correct Answer:** (B) $Y$ follows $N_p(A\mu, A\Sigma A^T)$ ; (D) $\sum_{i=1}^{p} \text{Var}(Y_i) = \sum_{i=1}^{p} \text{Var}(X_i)$

**SOLUTION:**

**Step 1:** A rotates.
$A^T A = I_p$ means $A$ is orthogonal, a rotation/reflection.
**Step 2:** Y distribution.
$Y \sim N_p(A\mu, A\Sigma A^T)$, matching (B) not (A).
**Step 3:** Projections argument.
$\sum Y_i$ is projection onto $A^T\mathbf{1}$, not necessarily same as $\mathbf{1}$, so (C) fails generally.
**Step 4:** Trace invariance.
$\sum \text{Var}(Y_i) = \text{tr}(A\Sigma A^T) = \text{tr}(\Sigma)$, rotation invariant. (D) TRUE.

$$\boxed{\text{(B) and (D)}}$$

**Quick Tip:** Use $\text{Cov}(AX) = A\Sigma A^T$ and the cyclic property of trace, $\text{tr}(A\Sigma A^T) = \text{tr}(\Sigma A^T A) = \text{tr}(\Sigma)$, remembering $A^T A = I_p$.

• • •

**59.** Suppose that $X = (X_1, \ldots, X_p)^T$ follows $N_p(0, \Sigma)$, where $\Sigma$ is a positive definite matrix. Then which of the following statements is/are correct?

(A) $X_1, \ldots, X_p$ are always independent normal random variables

(B) $X_1, \ldots, X_p$ are normal random variables

(C) $X_1^2 + \cdots + X_p^2$ always follows a chi-square distribution

(D) Any linear combination of $X_1, \ldots, X_p$ is a normal random variable

**Correct Answer:** (B) $X_1, \ldots, X_p$ are normal random variables ; (D) Any linear combination of $X_1, \ldots, X_p$ is a normal random variable

**SOLUTION:**

**Step 1:** Defining property.

Every linear combination of MVN vector is normal.

**Step 2:** (D) holds directly.

TRUE.

**Step 3:** (B) is special case.

Each $X_i = e_i^T X$ normal. TRUE.

**Step 4:** (A) fails.

Independence needs diagonal $\Sigma$, not given. FALSE.

**Step 5:** (C) counterexample.

$\Sigma = \text{diag}(4, 1)$ breaks chi-square shape. FALSE.

$$\boxed{\text{(B) and (D)}}$$

**Quick Tip:** Marginals and linear combinations of a multivariate normal vector are always normal, but independence needs a diagonal $\Sigma$, and $\sum X_i^2$ is chi-square only when $\Sigma = I$.

• • •

**60.** Let $Y = (Y_1, Y_2, Y_3)^T \sim N_3(0, I_3)$, where $I_3$ denotes the identity matrix of order 3. Let

$$A = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \quad \text{and} \quad B = I_3 - A.$$

Let $\chi_1^2$ denote chi-square distribution with 1 degree of freedom. Then which of the following statements is/are correct?

(A) $Y^T A Y \sim \chi_1^2$

(B) $Y^T B Y \sim \chi_1^2$

(C) $Y^T A Y$ and $Y_1 - 2Y_2 + Y_3$ are independently distributed

(D) $Y^T A Y$ and $Y_1 + 2Y_2 + Y_3$ are independently distributed

**Correct Answer:** (A) $Y^T A Y \sim \chi_1^2$ ; (C) $Y^T A Y$ and $Y_1 - 2Y_2 + Y_3$ are independently distributed

**SOLUTION:**

**Step 1:** Simplify Y^TAY.

$Y^T A Y = (Y_1 + Y_2 + Y_3)^2/3 = S^2/3$, $S \sim N(0, 3)$, so $S^2/3 \sim \chi_1^2$. (A) TRUE.

**Step 2:** (B) by subtraction.

$Y^T B Y = Y^T Y - Y^T A Y \sim \chi_2^2$. FALSE.

**Step 3:** (C) covariance.

$\text{Cov}(S, Y_1 - 2Y_2 + Y_3) = 1 - 2 + 1 = 0$, independent. TRUE.

**Step 4:** (D) covariance.

$\text{Cov}(S, Y_1 + 2Y_2 + Y_3) = 4 \neq 0$, not independent. FALSE.

$$\boxed{\text{(A) and (C)}}$$

**Quick Tip:** $A$ is the rank-1 projection onto $(1,1,1)^T$; $Y^TAY$ is a function of $Y_1+Y_2+Y_3$ alone, so it is independent of any linear form whose coefficients dot to zero with $(1,1,1)$.

• • •

**61.** Let $D = \{(x,y) \in \mathbb{R}^2 : 0 < x^2+y^2 \le 1\}$. For $\alpha \ge 0$, consider the integral

$$I_\alpha = \iint_D \frac{1}{(x^2+y^2)^\alpha}\, dx\, dy.$$

Let

$$N_0 = \sup\{\alpha \ge 0 : I_\alpha < \infty\}.$$

Then the value of $N_0$ equals ________ (answer in integer).

**Correct Answer:** 1

**SOLUTION:**

**Step 1:** Polar form.
$x = r\cos\theta$, $y = r\sin\theta$; punctured disc becomes $0 < r \le 1$, full angle sweep.

**Step 2:** Rewrite integral.

$$I_\alpha = 2\pi \int_0^1 r^{1-2\alpha}\, dr$$

**Step 3:** p-test near zero.
$\int_0^1 r^p\, dr$ finite iff $p > -1$, here $p = 1-2\alpha$.

**Step 4:** Bound on alpha.

$$1 - 2\alpha > -1 \Rightarrow \alpha < 1$$

**Step 5:** Supremum.

Set is $[0, 1)$, supremum 1.

$$\boxed{N_0 = 1}$$

**Quick Tip:** Switch to polar coordinates and check when $\int_0^1 r^{1-2\alpha} dr$ converges near $r = 0$.

• • •

**62.** Let $X$ follow $N(3, 1)$. Then the value of $E\big(X^4(X - 3)\big)$ equals ________ (answer in integer).

**Correct Answer:** 144

**SOLUTION:**

**Step 1:** Standardize.

$Z = X - 3 \sim N(0, 1)$, target $E((Z + 3)^4 Z)$.

**Step 2:** Stein's identity.

For $Z \sim N(0, 1)$: $E[Zh(Z)] = E[h'(Z)]$.

**Step 3:** Apply with h(Z)=(Z+3)^4.

$h'(Z) = 4(Z + 3)^3$, so $E[(Z + 3)^4 Z] = 4E(X^3)$.

**Step 4:** Find E(X^3).

$X^3 = (3+Z)^3 = 27 + 27Z + 9Z^2 + Z^3$; odd terms vanish, $E(Z^2) = 1$:
$E(X^3) = 27 + 9 = 36$.

**Step 5:** Combine.

$$4 \times 36 = 144$$

**Step 6:** Cross check.

Direct expansion gives same 144.

$$\boxed{144}$$

**Quick Tip:** Write $Z = X - 3 \sim N(0,1)$ and expand $(Z+3)^4 Z$; only the even-power terms in $Z$ survive after taking expectation.

**63.** Let $U_1, U_2, U_3$ be three independent exponential random variables such that $U_k$ has the following probability density function

$$f_k(x) = \begin{cases} ke^{-kx} & \text{if } x > 0 \\ 0 & \text{otherwise,} \end{cases} \qquad k = 1, 2, 3.$$

Let

$$X = \min\{U_1, U_3\} \quad \text{and} \quad Y = \min\{U_2, U_3\}.$$

Then $P(X = Y)$ equals ________ (rounded off to two decimal places).

**Correct Answer:** 0.50

**SOLUTION:**

**Step 1:** Racing clocks.

$X = Y$ happens iff $U_3$ is the smallest of the three.

**Step 2:** Competing exponentials fact.

$$P(U_i = \min\{U_1, U_2, U_3\}) = \frac{\lambda_i}{\lambda_1 + \lambda_2 + \lambda_3}$$

**Step 3:** Apply for U3.

$$P(U_3 = \min) = \frac{3}{1 + 2 + 3} = \frac{1}{2}$$

**Step 4:** Conclude.

$$P(X = Y) = 0.50$$

$$\boxed{P(X = Y) = 0.50}$$

**Quick Tip:** $X = Y$ happens (ignoring probability zero ties) exactly when $U_3$ is smaller than both $U_1$ and $U_2$; integrate over the density of $U_3$.

• • •

**64.** Let $X_1, X_2$ be a random sample from a distribution having the population density function

$$f(x) = \begin{cases} \frac{1}{\theta} & \text{if } 0 < x < \theta \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta \in (0, \infty)$. Let $X_{(2)} = \max\{X_1, X_2\}$ and

$$\psi(\theta) = P_\theta(X_1 + X_2 < 1), \quad \theta > 0.$$

Let $\delta\left(X_{(2)}\right)$ be an unbiased estimator of $\psi(\theta)$ that depends on observations $X_1$ and $X_2$ only through $X_{(2)}$. If $\delta(t)$ is a continuous function on $(0, \infty)$, then the value of $18\,\delta\left(\frac{3}{4}\right)$ equals ________ (answer in integer).

**Correct Answer:** 6

**SOLUTION:**

**Step 1:** Rao-Blackwell approach.

$\delta(X_{(2)}) = E[I(X_1 + X_2 < 1) \mid X_{(2)}]$ stays unbiased and depends only on

$X_{(2)}$.

**Step 2:** Conditional law of min given max.

Given $X_{(2)} = v$, $X_{(1)} \sim$Uniform$(0, v)$.

**Step 3:** Write delta(t).

$$\delta(t) = P(X_{(1)} < 1 - t \mid X_{(2)} = t)$$

**Step 4:** Cases.

$t \geq 1$: $\delta = 0$. $t \leq 1/2$: $\delta = 1$. $1/2 < t < 1$: $\delta(t) = \frac{1-t}{t}$.

**Step 5:** Continuity check.

Matches at boundaries $t = 1/2, 1$.

**Step 6:** Plug t=3/4.

$$\delta(3/4) = \frac{1/4}{3/4} = \frac{1}{3}$$

**Final Answer:**

$$18 \times \frac{1}{3} = 6$$

$$\boxed{6}$$

**Quick Tip:** Differentiate $\theta^2\psi(\theta) = \int_0^\theta 2t\,\delta(t)\,dt$ with respect to $\theta$ to recover $\delta(\theta)$ piece by piece.

• • •

**65.** Let $X$ be a single observation from a distribution having probability density function

$$f_\theta(x) = \begin{cases} \frac{2x}{\theta^2} & \text{if } 0 < x < \theta \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta \in (0, \infty)$. For testing $H_0 : \theta \leq 1$ against $H_1 : \theta > 1$, at level of significance 0.05, let $\beta_1$ be the size of the uniformly most powerful test and $\beta_2$ be the power of the uniformly most powerful test at $\theta = 2$. Then $10\beta_1 + 40\beta_2$ equals ________ (answer in integer).

**Correct Answer:** 31

**SOLUTION:**

**Step 1:** Transform T=X^2.
$T$ ~Uniform$(0, \theta^2)$.

**Step 2:** UMP in terms of T.
Reject when $T > c_0$.

**Step 3:** Fix c0.
At $\theta = 1$: $T$ ~Uniform(0,1), $1 - c_0 = 0.05 \Rightarrow c_0 = 0.95$. $\beta_1 = 0.05$.

**Step 4:** Power at theta=2.
$T$ ~Uniform(0,4): $\beta_2 = (4 - 0.95)/4 = 0.7625$.

**Step 5:** Combine.

$$10(0.05) + 40(0.7625) = 31$$

$$\boxed{31}$$

**Quick Tip:** The UMP test for this monotone likelihood ratio family rejects for large $X$; fix the cutoff using the boundary $\theta = 1$, then use it to find the power at $\theta = 2$.