Motion in a Plane
Projectile & Relative Motion

Unit 2 of the JEE Main syllabus. Two ideas carry the whole topic:

1. motion splits into independent axes
2. velocity is measured from a frame

Everything else, projectiles, river boats, rain problems, follows from those two.

The one move that matters

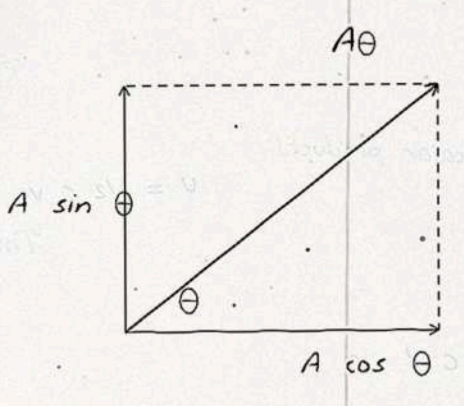
Resolve into x and y, then treat each axis as a separate 1-D problem. They share only one thing: the clock.

> Horizontal and vertical motion are independent. Time is the only bridge between them.

49 of 76 recent PYQs here are projectiles

Resolving a vector

Any vector splits into two perpendicular pieces. Those pieces do not interact.



Adding vectors

Add components, never magnitudes.

$$R_x = A_x + B_x$$
$$R_y = A_y + B_y$$

$$\boxed{R = \sqrt{R_x^2 + R_y^2}}$$

Direction: tan of the angle = Ry / Rx.

Magnitude of a sum

$$\boxed{R = \sqrt{A^2 + B^2 + 2AB\cos\theta}}$$

Same direction: $R = A + B$, the maximum.

Opposite: $R = A - B$, the minimum.

Perpendicular: $R = \sqrt{A^2 + B^2}$

Dot product

$$\boxed{A \cdot B = AB\cos\theta}$$

Gives a scalar. Zero when perpendicular. cos: along

Used for work and for projections. sin: across

Cross product

$$\boxed{A \times B = AB\sin\theta}$$

Gives a vector, perpendicular to both.

Zero when parallel. Direction by right hand.

Projection of a vector

The shadow of A along B. A PYQ favourite.

$$\boxed{proj = (A \cdot B) / B}$$

As a vector, multiply by the unit vector of B afterwards.

Unit vectors

$$\boxed{n = A / A}$$

* *

Magnitude 1, direction of A. Every unit vector satisfies $i.i = 1$ and $i.j = 0$.

Worked

$A = 3i + 4j$, $B = i$. Projection of A on B?

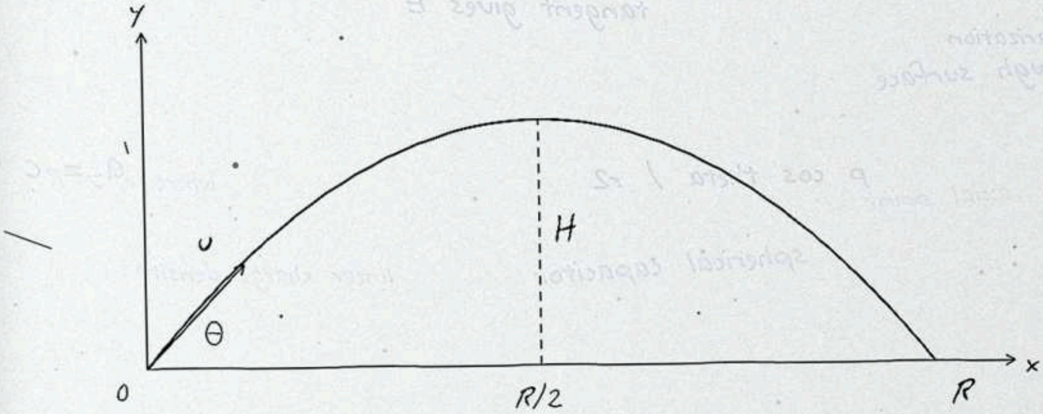
$A \cdot B = 3$, and $B = 1$.

so projection $= 3$

which is just the x-component.

Projectile motion

Thrown at an angle, then left to gravity.
Air resistance ignored, g constant.



Split the throw at once:

$u_x = u \cos\theta$ (constant)

$u_y = u \sin\theta$ (changes)

Horizontal has zero acceleration.
Vertical has $a = -g$ throughout, including at the very top.

Time, height, range

- All three come from the vertical axis.

$$\boxed{T = 2u \sin\theta / g}$$

$$\boxed{H = u^2 \sin^2\theta / 2g}$$

$$\boxed{R = u^2 \sin 2\theta / g}$$

T comes from vertical displacement zero.

H comes from vertical velocity zero.

R is $u_x \times T$, nothing more.

> At the top the vertical velocity is zero but the SPEED is not: the horizontal part never changed.

Time up = time down, for level ground only.

The path is a parabola

Eliminate t between the two axes:

$$\boxed{y = x \tan\theta - gx^2 / (2u^2 \cos^2\theta)}$$

Quadratic in x, so the path is a parabola.

Maximum range

R is largest when $\sin 2\theta = 1$.

$$\boxed{\theta = 45^\circ, \quad R_{max} = u^2 / g}$$

Complementary angles

Two angles adding to 90 degrees give the SAME range, with different times.

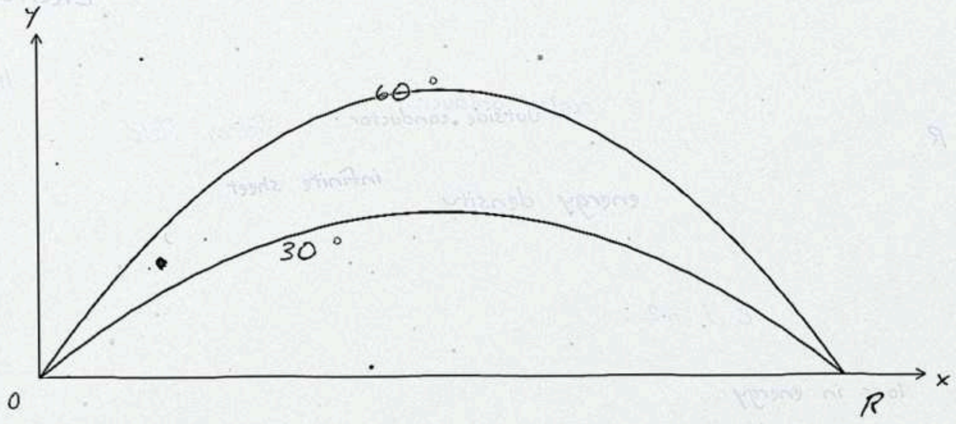
30 and 60 share a range

20 and 70 share a range

$T_1 T_2 = 2R / g$ links the two times.

The steeper angle always takes longer.

Same range, two paths



Both land at the same R. The 60 degree throw goes higher and stays up longer.

So a question giving R alone cannot fix the angle. It needs H or T as well.

~~R fixes the angle~~ is the trap.

Velocity mid-flight

Horizontal stays put, vertical decays.

$V_x = u \cos\theta$

$V_y = u \sin\theta - g\,t$

$$\boxed{V = \sqrt{V_x^2 + V_y^2}}$$

Angle with horizontal: $\tan a = vy / vx$.

Speed is symmetric

At equal heights on the way up and* the way down the SPEED is equal. The direction is mirrored, not equal.

Change in velocity

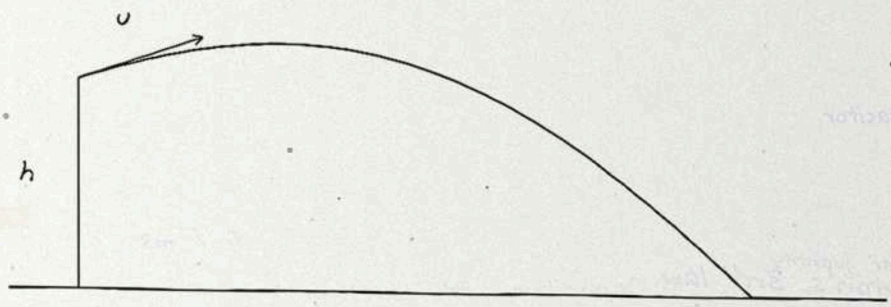
Only gravity acts, so over a whole flight:

$$\boxed{\Delta V = g\,T, \text{ straight down}}$$

Direction of the change never varies, whatever the launch angle was.

Thrown from a tower

Now the landing point is BELOW the start,
so time up no longer equals time down.



Set downward positive and solve:

$$\boxed{h = -u \sin\theta \, t + \tfrac{1}{2} g t^2}$$

Quadratic in t. Take the positive root.
Then $R = u \cos a \times t$, using that t.

Thrown horizontally

A special case: $\theta = 0$, so $u_y = 0$.

$$\boxed{t = \sqrt{2h/g}}$$

Time depends ONLY on the height. Two balls, one dropped and one thrown flat, land together.

$$\boxed{R = u\sqrt{2h/g}}$$

Landing velocity

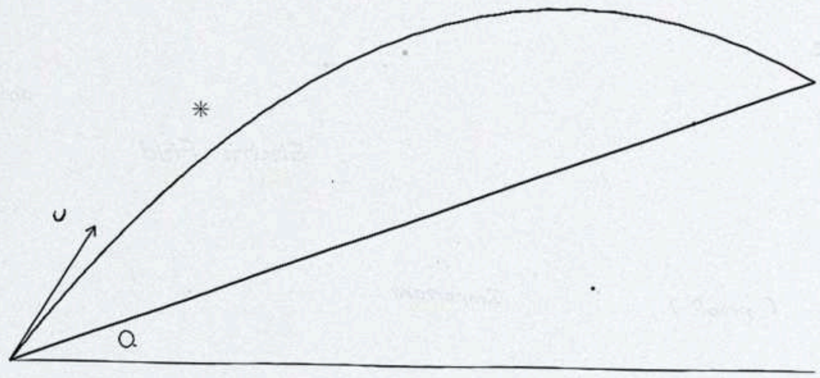
$V_y = \sqrt{2gh}$, $V_x = u$

Angle below horizontal: $\tan a = v_y / u$.

> Independence of axes is the point of this case. Throwing it faster does not keep it up any longer.

Projectile on an incline

Tilt the axes along the slope. Then BOTH axes carry an acceleration.



With axes along and across the slope:

$a_{along} = -g \sin \alpha$

$a_{across} = -g \cos \alpha$

Time of flight uses the across-axis, and the range then uses the along-axis.

* tilt the axes, do not tilt gravity

Relative velocity

Velocity is always measured FROM a frame.

$$\boxed{V_{AB} = V_A - V_B}$$ *

Read it as: velocity of A as seen by B.

Subtract as VECTORS, by components.

Two useful checks

$V_{AB} = -V_{BA}$

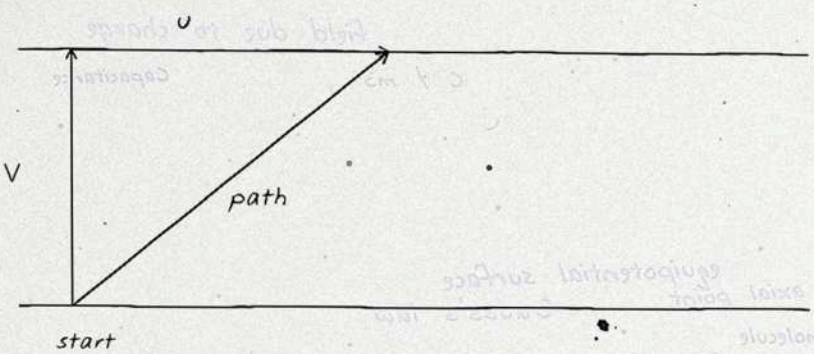
Same magnitude, opposite direction.

If two bodies have equal velocities, each sees the other at rest, however fast both are moving on the ground.

Closest approach happens when the relative position is perpendicular to the relative velocity.

River boat: shortest time

Point the boat straight across. The river carries it downstream as it goes.



Crossing time uses only the across part:

$$\boxed{t = d / V}$$

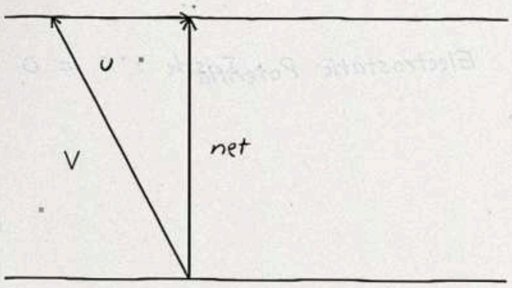
Drift downstream during that time:

$$\boxed{x = u d / V}$$

This is the FASTEST crossing, not the

River boat: shortest path

To land straight opposite, aim upstream so the current is cancelled.



Aim at angle where $V \sin\theta = u$.

$$\boxed{t = d / \sqrt{V^2 - u^2}}$$

Only possible if V is greater than u.
A slow boat cannot beat a fast river.

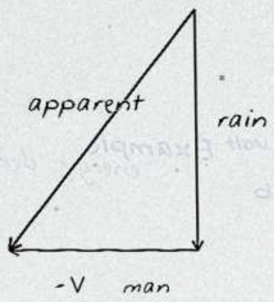
~~shortest path = shortest time~~

They are different questions.

Rain and the umbrella

Tilt the umbrella along the rain's velocity RELATIVE to you, not along the true rain.

$$\boxed{V_{rm} = V_r - V_m}$$



Walk faster and the apparent rain tilts further forward, so the umbrella leans more.

$$\tan\theta = V_{man} / V_{rain}$$

measured from the vertical.

One projectile from another

Both have the same acceleration, g down.

So the relative acceleration is ZERO.

Seen from one projectile, the other moves in a STRAIGHT LINE at constant velocity. Gravity cancels out.

*

This turns a hard two-body problem into a one-line one, and JEE has used it often.

Worked

Two stones thrown together from the same point, speeds 20 and 20, angles 30 and 60. Separation after 2 s?

Relative velocity is constant:

$20(\cos 30 - \cos 60) = 20(0.866 - 0.5)$ *

$20(\sin 30 - \sin 60) = 20(0.5 - 0.866)$

magnitude $= 20 \times 0.518 = 10.35$ m/s

separation $= 10.35 \times 2 = 20.7$ m

Solved: JEE style (1)

(1) A ball is thrown at 20 m/s at 30 degrees. Find T, H and R. Take $g = 10$.

$u_y = 10$, $u_x = 17.32$

$T = 2(10)/10 = 2$ s

$H = 100/20 = 5$ m

$R = 17.32 \times 2 = 34.6$ m

---

(2) A stone thrown flat at 15 m/s from a 20 m tower. Where does it land? $g = 10$.

$t = \text{root}(2 \times 20/10) = 2$ s

$R = 15 \times 2 = 30$ m from the base

---

(3) Two angles give the same range 40 m at $u = 20$ m/s. Find both angles. $g = 10$.

$\sin 2a = Rg/u$ ^ $2 = 400/400 = 1$

$2a = 90$, so $a = 45$ only.

This is the maximum range, so the two angles coincide.

Solved: JEE style (2)

(4) A river 100 m wide flows at 3 m/s. A boat rows at 5 m/s. Shortest crossing time and drift?

Aim straight across: $t = 100/5 = 20$ s
drift $= 3 \times 20 = 60$ m

---

(5) Same river. Now land directly opposite. Find the time taken.

across speed = root(25 - 9) = 4 m/s
$t = 100/4 = 25$ s, longer than 20 s

---

(6) Rain falls at 4 m/s vertically. A man walks at 3 m/s. At what angle should he tilt?

tan a = 3/4, so a = 37 degrees from the vertical, leaning forward.

---

(7) $A = 3i + 4j$. Find the unit vector along A and its projection on the x axis.

A = 5, unit $= 0.6i + 0.8j$, proj = 3

Common mistakes

*

1. Adding vector magnitudes
   add components

2. Speed zero at the top
   only v_y is zero

3. Time up = time down on a tower
   solve the quadratic

4. R alone fixes the angle
   two angles share R

5. Tilting gravity on an incline
   tilt the axes

6. Umbrella along the true rain
   use apparent rain

7. Shortest path = shortest time
   different questions

Quick recall

$T = 2u \sin\theta / g$

$H = u^2 \sin^2\theta / 2g$

$R = u^2 \sin 2\theta / g$

$R_{max} = u^2 / g$ at 45°

---

flat throw: $t = \sqrt{2h/g}$

incline: a along $= -g \sin a$

$V = \sqrt{V_x^2 + V_y^2}$

$V_{AB} = V_A - V_B$

boat, fastest: $t = d/V$, drift $= ud/V$

boat, straight across: $t = d / \sqrt{V^2 - u^2}$

umbrella: $\tan\theta = V_{man} / V_{rain}$

$A . B = AB \cos a$ (scalar, 0 if perp)

$A \times B = AB \sin a$ (vector, 0 if parallel)

projection of A on B $= A.B / B$

*

Relative acceleration of two projectiles is zero: each sees the other go straight.

Exam-day habits

Where this topic sits in recent papers:

Projectile motion	49 of 76
Vector arithmetic	9 of 76
Dot and cross product	7 of 76
Relative motion in 2-D	4 of 76
River boat and rain	4 of 76

Counted from the 2022 to 2026 papers, so projectiles deserve most of your practice.

Habits

Resolve before writing any equation.
Mark which way is positive, then keep it.
On a tower, expect a quadratic in t.
For two bodies, switch to a relative frame.
Check g: papers use 10 or 9.8, not both.

> Two axes, one clock. Almost every question here is that sentence applied carefully.