TIFR GS Mathematics

Sample Paper – 1

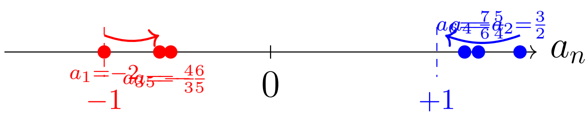
Duration: 180 Minutes **Maximum Marks: 50**

Instructions

- This paper contains **50 questions** modelled on the **TIFR GS Mathematics** entrance examination.
- **Part A (Q1–Q30):** 30 Multiple Choice Questions. Each correct answer carries **+1 mark**; each wrong answer carries $-\frac{1}{3}$ **mark**. Unattempted questions carry 0 marks.
- **Part B (Q31–Q50):** 20 True/False questions. Each correct answer carries **+1 mark**. There is **no negative marking** in Part B. Attempt all questions.
- Use of calculators, mobile phones, or any electronic gadgets is strictly prohibited. Rough work may be done on provided sheets.
- Syllabus: **BSc Mathematics (Honours) level** — Real Analysis, Linear Algebra, Abstract Algebra, Topology, Complex Analysis, Number Theory, and Probability.

Part A — Multiple Choice Questions (MCQ)

Q1. Let $a_n = (-1)^n\left(1+\frac{1}{n}\right)$ for $n \geq 1$. What are $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$?



(A) $\limsup = 1, \quad \liminf = -1$

(B) $\limsup = 2, \quad \liminf = -2$

(C) $\limsup = 1, \quad \liminf = -2$

(D) The sequence diverges to $+\infty$, so $\limsup$ does not exist

Q2. Which of the following series is **convergent**?

(A) $\sum_{n=1}^{\infty} \frac{1}{n}$

(B) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

(C) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

(D) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$

Q3. Let f be a function defined on some domain. Which of the following statements is **FALSE**?

(A) If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded.

(B) If $f : [a, b] \to \mathbb{R}$ is continuous, then f attains its maximum value on $[a, b]$.

(C) If $f : (a, b) \to \mathbb{R}$ is continuous, then f attains its maximum value on (a, b).

(D) If $f : [a, b] \to \mathbb{R}$ is continuous, then f attains its minimum value on $[a, b]$.

Q4. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in \mathbb{R}$ and some constant $K > 0$. Which of the following conclusions is **necessarily valid**?

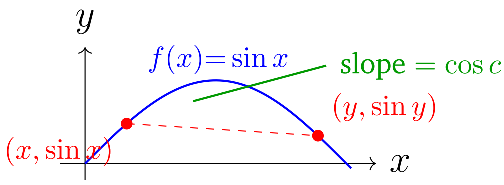
(A) f is differentiable everywhere and $|f'(x)| \leq K$.

(B) f is continuously differentiable on $\mathbb{R}$.

(C) f is analytic (has a convergent power series expansion at every point).

(D) f is uniformly continuous on $\mathbb{R}$.

Q5. Which of the following inequalities is a **direct consequence** of the Mean Value Theorem applied to $f(x) = \sin x$?



(A) $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$

(B) $|\sin x - \sin y| \leq \dfrac{|x-y|}{2}$ for all $x, y \in \mathbb{R}$

(C) $|\sin x| \leq |x|^2$ for all $x \in \mathbb{R}$

(D) $|\sin x - \sin y| \leq |x - y|^2$ for all $x, y \in \mathbb{R}$

Q6. A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if the set of its discontinuities has (Lebesgue) measure zero. Which of the following functions is **NOT** Riemann integrable on $[0, 1]$?

(A) Every monotone bounded function on $[0, 1]$

(B) The Dirichlet function $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

(C) $f(x) = \sin(1/x)$ for $x \in (0, 1]$ and $f(0) = 0$

(D) Every step function on $[0, 1]$

Q7. Let $a_n = \dfrac{n!}{n^n}$. The series $\displaystyle\sum_{n=1}^{\infty} a_n$:

(A) Diverges, because $a_n \to \infty$.

(B) Diverges by the comparison test with $\sum 1/n$.

(C) Converges, because by the ratio test $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = e^{-1} < 1$.

(D) Converges absolutely but the ratio test is inconclusive.

Q8. Consider the alternating series $S = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.

By the Leibniz alternating series test and its error bound, if S_N denotes the N-th partial sum, which statement is **correct**?

(A) The series diverges since $\sum 1/n$ diverges.

(B) $|S - S_N| \leq \dfrac{1}{N}$ and $S = \ln 2 \approx 0.693$.

(C) The series converges to $\pi/4$.

(D) $|S - S_N| \leq \dfrac{1}{N+1}$ and the error bound uses the $(N+1)$-th term.

Q9. Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$. The characteristic polynomial of A is $p(\lambda) = \lambda^2 - 5\lambda + 6$. By the Cayley-Hamilton theorem, $A^2 - 5A + 6I = 0$. Use this to compute A^3.

$$\left|\begin{array}{cc} 2 & 1 \\ \hdashline 0 & 3 \end{array}\right| \text{eigenvalues } 2, 3$$

$$\chi_A(\lambda) = (\lambda - 2)(\lambda - 3)$$

(A) $A^3 = \begin{pmatrix} 8 & 19 \\ 0 & 27 \end{pmatrix}$

(B) $A^3 = \begin{pmatrix} 8 & 7 \\ 0 & 27 \end{pmatrix}$

(C) $A^3 = \begin{pmatrix} 4 & 5 \\ 0 & 9 \end{pmatrix}$

(D) $A^3 = \begin{pmatrix} 12 & 5 \\ 0 & 27 \end{pmatrix}$

Q10. Let $M = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$ where $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$. What is $\det(M)$?

(A) 10

(B) -10

(C) 5

(D) -5

Q11. Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be a linear transformation such that $\dim(\ker T) = 3$. What is $\dim(\operatorname{Im} T)$?

(A) 0

(B) 1

(C) 2

(D) 3

Q12. Let $u, v \in \mathbb{R}^n$ with $\|u\| = 3$ and $\|v\| = 4$. Which of the following correctly states the Cauchy-Schwarz inequality and its consequence for $|u \cdot v|$?

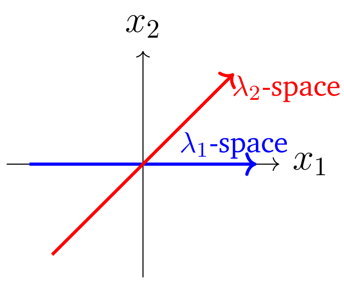
(A) $|u \cdot v| \leq \|u\| + \|v\| = 7$

(B) $|u \cdot v| \leq \|u\|^2 + \|v\|^2 = 25$

(C) $|u \cdot v| \leq \dfrac{\|u\|^2 + \|v\|^2}{2} = 12.5$

(D) $|u \cdot v| \leq \|u\| \cdot \|v\| = 12$, and equality holds iff u and v are linearly dependent

Q13. Which of the following 2×2 real matrices is **diagonalizable over** $\mathbb{R}$?



Two distinct eigenspaces $\Rightarrow$ diagonalizable

(A) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ (distinct real eigenvalues 1 and 2)

(B) $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ (repeated eigenvalue 2 with deficient eigenspace)

(C) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (eigenvalues $\pm i$, not real)

(D) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (Jordan block, not diagonalizable)

Q14. For any real $m \times n$ matrix A, the product $A^T A$ is always positive semidefinite. Under which condition is $A^T A$ positive **definite**?

(A) A is a square matrix.

(B) A has full column rank (i.e., $\text{rank}(A) = n$).

(C) A is symmetric.

(D) A has full row rank (i.e., $\text{rank}(A) = m$).

Q15. Recall that every subgroup of index 2 in a group G is normal in G. Which of the following statements does **NOT** follow from this result (or standard group theory), and may in fact be **false**?

(A) The alternating group A_n is a normal subgroup of S_n for $n \geq 2$.

(B) In the group $(\mathbb{Z}, +)$, every subgroup is normal.

(C) Every subgroup of index 3 in an arbitrary finite group G is normal in G.

(D) The subgroup $\{e, (12)(34), (13)(24), (14)(23)\}$ of order 4 is normal in S_4.

Q16. How many Sylow 3-subgroups does the symmetric group S_4 have? (Note: $|S_4| = 24 = 2^3 \cdot 3$.)

(A) 1

(B) 3

(C) 6

(D) 4

Q17. For which values of $n \geq 2$ is $\mathbb{Z}/n\mathbb{Z}$ an integral domain? For which values is it a field?

(A) $\mathbb{Z}/n\mathbb{Z}$ is an integral domain if and only if n is prime; it is a field if and only if n is prime.

(B) $\mathbb{Z}/n\mathbb{Z}$ is an integral domain for all $n \geq 2$; it is a field only when n is prime.

(C) $\mathbb{Z}/n\mathbb{Z}$ is a field for all $n \geq 2$; it is an integral domain only when n is prime.

(D) $\mathbb{Z}/n\mathbb{Z}$ is an integral domain when n is prime or a prime power; it is a field only when n is prime.

Q18. Which of the following correctly applies Eisenstein's irreducibility criterion to the polynomial $f(x) = x^4 + 4x + 2$ over $\mathbb{Q}$?

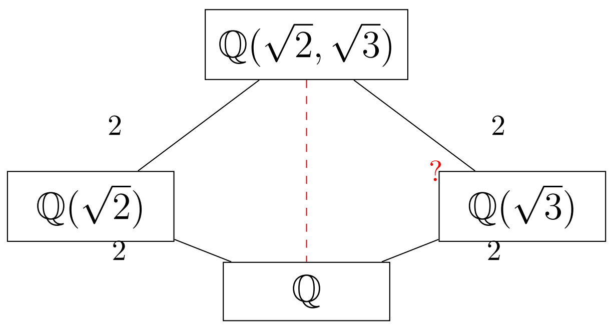
(A) Take $p = 4$: $p \mid 4$ and $p \mid 2$, but $4^2 = 16 \nmid 2$, so f is irreducible.

(B) Take $p = 2$: $p \nmid 1$ (leading coeff), $p \mid 4$ and $p \mid 2$ (all non-leading non-constant coefficients are divisible by p, as is the constant term), and $p^2 = 4 \nmid 2$; hence f is irreducible over $\mathbb{Q}$.

(C) Eisenstein does not apply here; irreducibility must be verified by other means.

(D) Take $p = 3$: $3 \nmid 1$, $3 \nmid 4$, so Eisenstein with $p = 3$ shows irreducibility.

Q19. What is the degree $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$?



(A) 2, since $\sqrt{3} \in \mathbb{Q}(\sqrt{2})$.

(B) 3, by the tower law.

(C) 4, since $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ and $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] = 2$ and these multiply.

(D) 8, since we are adjoining two algebraic numbers each of degree 2.

Q20. Which of the following statements about open and closed sets in a metric space (X, d) is **TRUE**?

(A) An arbitrary intersection of open sets is open.

(B) An arbitrary union of closed sets is closed.

(C) A set can be neither open nor closed, but never both open and closed simultaneously.

(D) The complement of a closed set is open, and the complement of an open set is closed.

Q21. In a metric space (X, d), which of the following is **TRUE** about compact sets?

(A) Every closed subset of a compact set is compact.

(B) Every compact subset of X is open.

(C) In a general topological space (not necessarily Hausdorff), every compact subset is closed.

(D) If $K \subseteq X$ is compact, then K is not necessarily bounded.

Q22. Let $f : X \to Y$ be a continuous map of topological spaces, and suppose X is connected. Which is the **best** consequence of connectedness preserved by f?

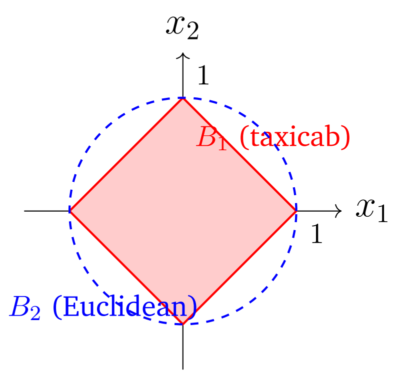
(A) f must be surjective.

(B) If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < c < f(b)$, then there exists $\xi \in (a, b)$ with $f(\xi) = c$ (Intermediate Value Theorem).

(C) $f(X)$ is compact in Y.

(D) f is a homeomorphism from X to $f(X)$.

Q23. In $\mathbb{R}^2$, let $d_1(x, 0) = |x_1| + |x_2|$ (taxicab metric) and $d_2(x, 0) = \sqrt{x_1^2 + x_2^2}$ (Euclidean metric). The unit ball $B_1 = \{x \in \mathbb{R}^2 : d_1(x, 0) \leq 1\}$ in the taxicab metric is:



(A) A circle of radius 1 centered at the origin.

(B) A square with vertices at $(\pm 1, \pm 1)$.

(C) A square (diamond) with vertices at $(\pm 1, 0)$ and $(0, \pm 1)$.

(D) An ellipse with semi-axes 1 and 2.

Q24. A function $f(z) = u(x, y) + iv(x, y)$ is holomorphic at $z_0 = x_0 + iy_0$ if and only if u and v are differentiable at (x_0, y_0) and the Cauchy-Riemann (C-R) equations $u_x = v_y$, $u_y = -v_x$ hold there. Which of the following functions **fails** to be holomorphic at the origin?

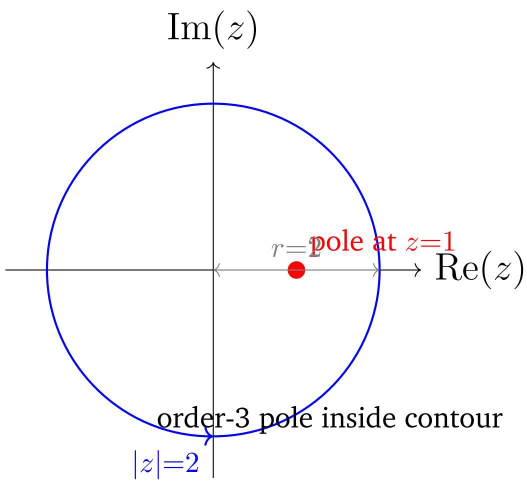
(A) $f(z) = z^2$ (so $u = x^2 - y^2$, $v = 2xy$)

(B) $f(z) = e^z$ (so $u = e^x \cos y$, $v = e^x \sin y$)

(C) $f(z) = \sin z$

(D) $f(z) = \bar{z} = x - iy$ (so $u = x$, $v = -y$; then $u_x = 1 \neq -1 = v_y$)

Q25. Compute the contour integral $I = \oint_{|z|=2} \frac{e^z}{(z-1)^3}\, dz$ where the contour $|z| = 2$ is traversed counterclockwise.



(A) $\pi i e$

(B) $2\pi i e$

(C) 0 (the pole is outside the contour)

(D) $\frac{\pi i e}{2}$

Q26. What is the radius of convergence R of the power series $\sum_{n=0}^{\infty} n!\, z^n$?

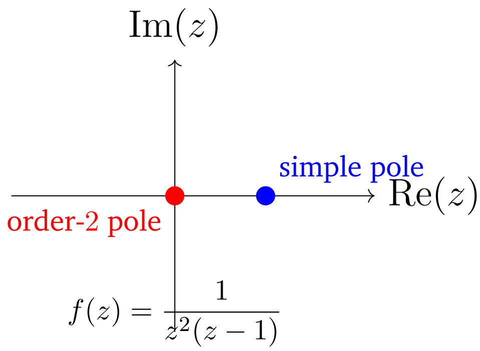
(A) $R = \infty$ (converges for all $z \in \mathbb{C}$)

(B) $R = 0$ (converges only at $z = 0$)

(C) $R = 1$

(D) $R = e$

Q27. Compute the residue of $f(z) = \dfrac{1}{z^2(z-1)}$ at its pole $z = 0$.



(A) $\mathrm{Res}(f, 0) = 1$

(B) $\mathrm{Res}(f, 0) = 0$

(C) $\mathrm{Res}(f, 0) = -1$

(D) $\mathrm{Res}(f, 0) = \dfrac{1}{2}$

Q28. Find all integer solutions (x, y) to the linear Diophantine equation $15x + 21y = 6$.

(A) No integer solutions exist, since $\gcd(15, 21) = 3$ does not divide 6.

(B) $x = 1 + 21t,\ y = -1 - 15t$ for all $t \in \mathbb{Z}$.

(C) $x = 2 + 7t,\ y = -1 - 5t$ for all $t \in \mathbb{Z}$.

(D) $x = -2 + 7t,\ y = 2 - 5t$ for all $t \in \mathbb{Z}$.

Q29. Compute $2^{100} \pmod{101}$. (Note: 101 is prime.)

(A) 1

(B) 2

(C) 100

(D) 50

Q30. Let X be a non-negative random variable with $E[X] = 5$. Using Markov's inequality, what is the best upper bound on $P(X \geq 15)$?

(A) $P(X \geq 15) \leq \frac{1}{3}$

(B) $P(X \geq 15) \leq \frac{1}{3}$, obtained via $P(X \geq a) \leq E[X]/a = 5/15$

(C) $P(X \geq 15) \leq \frac{1}{9}$

(D) $P(X \geq 15) \leq \frac{25}{225}$ by Chebyshev applied to the mean

Part B — True/False Questions (No Negative Marking)

Q31. State whether the following is **True** or **False**:
Every Cauchy sequence in $\mathbb{R}$ is convergent.

(A) True

(B) False

Q32. State whether the following is **True** or **False**:
Every continuous function $f : (0, 1) \to \mathbb{R}$ is uniformly continuous.

(A) True

(B) False

Q33. State whether the following is **True** or **False**:
Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

(A) True

(B) False

Q34. State whether the following is **True** or **False**:

If $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n^2$ must also diverge.

(A) True

(B) False

Q35. State whether the following is **True** or **False**:

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it converges.

(A) True

(B) False

Q36. State whether the following is **True** or **False**:

For $n \times n$ matrices A and B, $\mathrm{rank}(A + B) = \mathrm{rank}(A) + \mathrm{rank}(B)$ always holds.

(A) True

(B) False

Q37. State whether the following is **True** or **False**:

The trace of an $n \times n$ matrix equals the sum of its eigenvalues (counted with algebraic multiplicity).

(A) True

(B) False

Q38. State whether the following is **True** or **False**:

Every invertible $n \times n$ real matrix is diagonalizable over $\mathbb{R}$.

(A) True

(B) False

Q39. State whether the following is **True** or **False**:

For any matrices A and B for which the product AB is defined, $\mathrm{rank}(AB) \leq \min(\mathrm{rank}(A), \mathrm{rank}(B))$.

(A) True

(B) False

Q40. State whether the following is **True** or **False**:
Every group of order 4 is cyclic.

(A) True

(B) False

Q41. State whether the following is **True** or **False**:
Every subgroup of index 2 in a finite group is normal.

(A) True

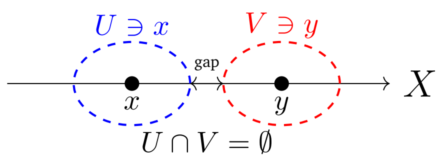
(B) False

Q42. State whether the following is **True** or **False**:
Every integral domain is a field.

(A) True

(B) False

Q43. State whether the following is **True** or **False**:
Every metric space is Hausdorff (satisfies the T_2 separation axiom).



(A) True

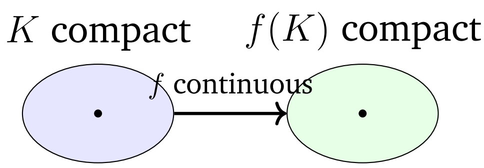
(B) False

Q44. State whether the following is **True** or **False**:
A countable union of closed subsets of $\mathbb{R}$ is always closed.

(A) True

(B) False

Q45. State whether the following is **True** or **False**:
The continuous image of a compact set is compact.



(A) True

(B) False

Q46. State whether the following is **True** or **False**:
There exists a non-constant bounded holomorphic function on all of $\mathbb{C}$.

(A) True

(B) False

Q47. State whether the following is **True** or **False**:
The zeros of a non-constant holomorphic function on a domain are isolated (they have no accumulation point within the domain).

(A) True

(B) False

Q48. State whether the following is **True** or **False**:
There are only finitely many prime numbers.

(A) True

(B) False

Q49. State whether the following is **True** or **False**:
For every prime p, $(p-1)! \equiv -1 \pmod{p}$ (Wilson's theorem).

(A) True

(B) False

Q50. State whether the following is **True** or **False**:
For any two random variables X and Y, $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$.

(A) True

(B) False

Detailed Solutions

Q1.

Solution

Concept — Limsup and liminf of oscillating sequences: For $a_n = (-1)^n(1+1/n)$, split into even and odd index subsequences. The even terms form an increasing subsequence approaching 1 from above, and the odd terms form a decreasing subsequence approaching -1 from below.

Step 1 — Even subsequence: $a_{2k} = 1 + \dfrac{1}{2k} \searrow 1$ as $k \to \infty$. These terms converge to 1.

Step 2 — Odd subsequence: $a_{2k-1} = -\left(1 + \dfrac{1}{2k-1}\right) \nearrow -1$ as $k \to \infty$. These terms converge to -1.

Step 3 — Conclusion: The set of subsequential limits is $\{-1, +1\}$. Therefore $\limsup_{n\to\infty} a_n = 1$ and $\liminf_{n\to\infty} a_n = -1$.

Why other options are wrong:

- Option B: $\limsup = 2$ would require even-indexed terms to approach 2, but $a_{2k} = 1 + 1/(2k) \to 1$, not 2.
- Option C: $\liminf = -2$ is wrong; odd-indexed terms approach -1, not -2 (note $a_1 = -2$ but it is only the first term, not the limiting value).
- Option D: a_n is bounded between -2 and 2 for all n, so it cannot diverge to $+\infty$; the limsup always exists for bounded sequences.

Final Answer: $\limsup = 1$, $\liminf = -1 \Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q1

Q2.

Solution

Concept — p-series and convergence tests: The series $\sum 1/n^p$ converges if and only if $p > 1$ (p-series test). For alternating series, the divergence test and Leibniz test apply.

Step 1 — Test each option: (A) $\sum 1/n$ is the harmonic series with $p = 1$; it diverges (proven via the integral test: $\int_1^\infty dx/x = \infty$). (B) $\sum 1/n^2$ is a p-series with $p = 2 > 1$; it converges (in fact its sum is $\pi^2/6$). (C) $\sum n/(n^2+1) \sim \sum 1/n$ by limit comparison, so it diverges. (D) For $\sum(-1)^n n/(n+1)$, the general term

$(-1)^n n/(n+1)$ does not tend to 0 in absolute value (it tends to 1), so the series diverges by the divergence test.

Step 2 — Confirm (B) via integral test: $\int_1^\infty \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_1^\infty = 1 < \infty$, confirming convergence.

Why other options are wrong:

- Option A: Harmonic series $\sum 1/n$ diverges ($p = 1$ is the boundary case).
- Option C: Limit comparison with $1/n$: $\frac{n/(n^2+1)}{1/n} = \frac{n^2}{n^2+1} \to 1 > 0$, so it diverges.
- Option D: Since $|a_n| = n/(n+1) \to 1 \neq 0$, the series fails the necessary condition for convergence.

Final Answer: $\sum 1/n^2$ converges (p-series, $p = 2 > 1$) $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q2

Q3.

Solution

Concept — Extreme Value Theorem (EVT): A continuous function on a *closed bounded* (compact) interval $[a, b]$ is bounded and attains both its maximum and minimum. This fails for open intervals.

Step 1 — Identify the false statement: Option (C) claims that a continuous $f : (a, b) \to \mathbb{R}$ on an *open* interval attains its maximum. This is FALSE. Counterexample: $f(x) = x$ on $(0, 1)$ is continuous, and $\sup f = 1$, but $f(x) < 1$ for all $x \in (0, 1)$, so the supremum is never attained.

Step 2 — Verify the true statements: (A), (B), (D) all apply to continuous functions on the closed interval $[a, b]$, where the EVT guarantees boundedness and attainment of both extremes.

Why other options are wrong:

- Option A: TRUE by EVT: $f([a, b])$ is a compact subset of $\mathbb{R}$, hence bounded.
- Option B: TRUE by EVT: f attains $\sup_{[a,b]} f$.
- Option D: TRUE by EVT: f attains $\inf_{[a,b]} f$.

Final Answer: Continuity on an open interval does NOT guarantee attaining the maximum $\Rightarrow \boxed{\text{C}}$

Answer: (C) Go Back to Q3

Q4.

Solution

Concept — Lipschitz implies uniform continuity: If $|f(x) - f(y)| \le K|x - y|$ for all x, y, then given $\varepsilon > 0$, choose $\delta = \varepsilon/K$. Then $|x - y| < \delta \Rightarrow |f(x) - f(y)| \le K\delta = \varepsilon$, uniformly in x, y.

Step 1 — Verify uniform continuity: The δ depends only on ε (not on the particular points x, y), establishing uniform continuity.

Step 2 — Show differentiability need not hold: The function $f(x) = |x|$ satisfies $|f(x) - f(y)| = ||x| - |y|| \le |x - y|$ (Lipschitz, $K = 1$) but is not differentiable at $x = 0$.

Why other options are wrong:

- Option A: Lipschitz does NOT imply differentiability; $f(x) = |x|$ is Lipschitz but non-differentiable at 0.
- Option B: Continuous differentiability (C^1) is a strictly stronger condition than Lipschitz continuity; e.g., $f(x) = |x|$ is Lipschitz but not C^1.
- Option C: Analyticity (real-analytic functions) is far stronger than Lipschitz; a Lipschitz function may fail to be representable by a power series.

Final Answer: A Lipschitz function is uniformly continuous $\Rightarrow \boxed{\text{D}}$

Answer: (D) **Go Back to Q4**

Q5.

Solution

Concept — Mean Value Theorem applied to $\sin x$: For $f(x) = \sin x$, which is differentiable on $\mathbb{R}$ with $f'(x) = \cos x$, the MVT gives: for any $x \ne y$, there exists c between x and y with $\sin x - \sin y = \cos(c)(x - y)$.

Step 1 — Apply MVT: $|\sin x - \sin y| = |\cos c| \cdot |x - y|$.

Step 2 — Use the bound on cosine: Since $|\cos c| \le 1$ for all $c \in \mathbb{R}$, we get $|\sin x - \sin y| \le 1 \cdot |x - y| = |x - y|$ for all $x, y \in \mathbb{R}$.

Step 3 — Sharpness: Near $x = 0$, $\sin x \approx x$, so $|\sin x - \sin y| \approx |x - y|$, showing the bound is tight.

Why other options are wrong:

- Option B: The bound $|x - y|/2$ is tighter than what MVT gives; MVT only yields $|\cos c| \le 1$, not $|\cos c| \le 1/2$ for all c.

- Option C: $|\sin x| \le |x|^2$ fails for large x (e.g., $x = \pi/2$ gives $1 \le (\pi/2)^2 \approx 2.47$, which holds, but for $x \approx 0.01$, $|\sin x| \approx 0.01 > 0.0001 = x^2$).
- Option D: $|\sin x - \sin y| \le |x-y|^2$ fails for small $|x-y| > 0$; for $x = \varepsilon, y = 0$, LHS $\approx \varepsilon$ while RHS $= \varepsilon^2 \ll \varepsilon$.

Final Answer: MVT gives $|\sin x - \sin y| \le |x-y|$ for all $x, y \in \mathbb{R} \Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q5

Q6.

Solution

Concept — Lebesgue's criterion for Riemann integrability: A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if its discontinuity set has Lebesgue measure zero.

Step 1 — Analyze the Dirichlet function: $f(x) = 1$ for $x \in \mathbb{Q} \cap [0,1]$ and $f(x) = 0$ for $x \in [0,1] \setminus \mathbb{Q}$. At every point $x_0 \in [0,1]$, both rationals and irrationals are dense in any neighborhood, so f oscillates between 0 and 1 near x_0. Thus f is discontinuous at *every* point of $[0,1]$, i.e., the discontinuity set is all of $[0,1]$, which has measure $1 \ne 0$.

Step 2 — Confirm other functions are Riemann integrable: Monotone functions on $[a,b]$ have at most countably many (hence measure-zero) discontinuities. Step functions are discontinuous at finitely many points. $\sin(1/x)$ extended to $f(0) = 0$ is discontinuous only at 0, a set of measure zero.

Why other options are wrong:

- Option A: Monotone bounded functions ARE Riemann integrable (Lebesgue's criterion: at most countably many discontinuities).
- Option C: $\sin(1/x)$ on $(0,1]$ extended by $f(0) = 0$ is bounded and has only one discontinuity ($x = 0$); measure zero $\Rightarrow$ Riemann integrable.
- Option D: Step functions are Riemann integrable (finitely many jump discontinuities, measure zero).

Final Answer: The Dirichlet function has discontinuity set of measure 1; it is NOT Riemann integrable $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q6

Q7.

Solution

Concept — Ratio test for series convergence: If $L = \lim_{n\to\infty} |a_{n+1}/a_n| < 1$, the series $\sum a_n$ converges absolutely.

Step 1 — Compute the ratio: With $a_n = n!/n^n$,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n = \left(1 - \frac{1}{n+1}\right)^n.$$

Step 2 — Evaluate the limit: $\lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^{n+1} \cdot \left(1 - \frac{1}{n+1}\right)^{-1} = e^{-1} \cdot 1 = e^{-1} \approx 0.368.$

Step 3 — Conclusion: Since $L = e^{-1} < 1$, the ratio test confirms convergence.

Why other options are wrong:

- Option A: By Stirling's approximation $n! \sim \sqrt{2\pi n}(n/e)^n$, so $a_n = n!/n^n \sim \sqrt{2\pi n}/e^n \to 0$; the terms go to 0, not ∞.
- Option B: The ratio test proves convergence, not divergence.
- Option D: The ratio test IS conclusive here since $L = e^{-1} \neq 1$; the test is inconclusive only when $L = 1$.

Final Answer: Ratio test: $a_{n+1}/a_n \to e^{-1} < 1$; the series converges $\Rightarrow \boxed{\text{C}}$

Answer: (C) Go Back to Q7

Q8.

Solution

Concept — Leibniz alternating series test and error bound: If $\{b_n\}$ is positive, monotone decreasing, and $b_n \to 0$, then $\sum(-1)^{n+1}b_n$ converges and the error after N terms satisfies $|S - S_N| \leq b_{N+1}$.

Step 1 — Verify conditions: $b_n = 1/n$ is positive, strictly decreasing, and $1/n \to 0$. All conditions hold.

Step 2 — Identify the sum: Using the Taylor series $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}x^n/n$ at $x = 1$: $S = \ln 2 \approx 0.693$.

Step 3 — Error bound: The error satisfies $|S - S_N| \leq b_{N+1} = \dfrac{1}{N+1}$. The bound uses the *first omitted* term, which is the $(N+1)$-th term.

Why other options are wrong:

- Option A: The series converges by Leibniz test ($b_n = 1/n \to 0$); it does NOT diverge.
- Option B: The error bound $1/N$ (using the N-th term) is weaker than the sharp bound $1/(N+1)$; the Leibniz bound uses the first *omitted* term b_{N+1}.
- Option C: $\pi/4 = \sum_{n=0}^{\infty}(-1)^n/(2n+1)$ (Leibniz formula for π), which is a different series.

Final Answer: Error bound $|S - S_N| \leq 1/(N+1)$ using the first omitted term $\Rightarrow$ $\boxed{\text{D}}$

$\boxed{\textbf{Answer: (D)}}$ Go Back to Q8

Q9.

Solution

Concept — Cayley-Hamilton theorem: Every $n \times n$ matrix A satisfies its own characteristic polynomial $\chi_A(A) = 0$.

Step 1 — Express A^2 using CH: Since $A^2 - 5A + 6I = 0$, we have $A^2 = 5A - 6I$.

Computation: $A^2 = 5\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} - 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 10-6 & 5-0 \\ 0 & 15-6 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 0 & 9 \end{pmatrix}$.

Step 2 — Compute $A^3 = A \cdot A^2$:

$$A^3 = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 0 & 9 \end{pmatrix} = \begin{pmatrix} 2\cdot 4 + 1\cdot 0 & 2\cdot 5 + 1\cdot 9 \\ 0\cdot 4 + 3\cdot 0 & 0\cdot 5 + 3\cdot 9 \end{pmatrix} = \begin{pmatrix} 8 & 19 \\ 0 & 27 \end{pmatrix}.$$

Why other options are wrong:

- Option B: Entry $(1,2)$ should be $2\cdot 5 + 1\cdot 9 = 19$, not 7.
- Option C: $\begin{pmatrix} 4 & 5 \\ 0 & 9 \end{pmatrix}$ is A^2, not A^3.
- Option D: Entry $(1,1) = 2\cdot 4 + 0 = 8$, not 12; entry $(1,2) = 19$, not 5.

Final Answer: $A^3 = \begin{pmatrix} 8 & 19 \\ 0 & 27 \end{pmatrix} \Rightarrow \boxed{\text{A}}$

$\boxed{\textbf{Answer: (A)}}$ Go Back to Q9

Q10.

Solution

Concept — Determinant of block diagonal matrices: For block diagonal $M = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$, $\det(M) = \det(B) \cdot \det(C)$.

Step 1 — Compute $\det(B)$**:** $\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2$.

Step 2 — Compute $\det(C)$**:** $\det \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = 2 \cdot 3 - 1 \cdot 1 = 6 - 1 = 5$.

Step 3 — Multiply: $\det(M) = (-2)(5) = -10$.

Why other options are wrong:

- Option A: $+10$ has the wrong sign; $\det(B) = -2$ makes the product negative.
- Option C: $5 = \det(C)$ alone; one forgot to multiply by $\det(B) = -2$.
- Option D: -5 would require one of the determinants to be $\pm 5/2$, which is not the case here.

Final Answer: $\det(M) = \det(B) \cdot \det(C) = (-2)(5) = -10 \Rightarrow \boxed{\text{B}}$

$\boxed{\textbf{Answer: (B)}}$ **Go Back to Q10**

Q11.

Solution

Concept — Rank-Nullity theorem: For a linear map $T : V \to W$ with $\dim V = n$, $\dim(\ker T) + \dim(\operatorname{Im} T) = n$.

Step 1 — Apply the theorem: $\dim(\operatorname{Im} T) = \dim(\mathbb{R}^5) - \dim(\ker T) = 5 - 3 = 2$.

Step 2 — Feasibility check: Since $T : \mathbb{R}^5 \to \mathbb{R}^3$, the image lies in $\mathbb{R}^3$, so $\dim(\operatorname{Im} T) \leq 3$. The value $2 \leq 3$ is consistent.

Why other options are wrong:

- Option A: $\dim(\operatorname{Im} T) = 0$ requires $\ker T = \mathbb{R}^5$ (zero map), meaning $\dim(\ker T) = 5 \neq 3$.
- Option B: $\dim(\operatorname{Im} T) = 1$ would require $\dim(\ker T) = 4 \neq 3$.
- Option D: $\dim(\operatorname{Im} T) = 3$ would require $\dim(\ker T) = 2 \neq 3$.

Final Answer: By rank-nullity, $\dim(\operatorname{Im} T) = 5 - 3 = 2 \Rightarrow \boxed{\text{C}}$

Answer: (C) Go Back to Q11

Q12.

Solution

Concept — Cauchy-Schwarz inequality: For $u, v \in \mathbb{R}^n$, $|u \cdot v| \leq \|u\| \cdot \|v\|$, with equality iff $u = \lambda v$ for some scalar λ (linear dependence).

Step 1 — Apply with given norms: $|u \cdot v| \leq \|u\| \cdot \|v\| = 3 \times 4 = 12$.

Step 2 — Attainability: The maximum 12 is achieved when $v = \lambda u$ (i.e., u and v are parallel), giving $u \cdot v = \lambda \|u\|^2 = 12$.

Why other options are wrong:

- Option A: $\|u\| + \|v\| = 7$ is the triangle inequality bound for $\|u + v\|$, not for $|u \cdot v|$.
- Option B: $\|u\|^2 + \|v\|^2 = 9 + 16 = 25$ appears in the parallelogram identity; it is not a bound for $|u \cdot v|$.
- Option C: $(\|u\|^2 + \|v\|^2)/2 = 12.5$ is the AM of squared norms; the Cauchy-Schwarz bound is the product $\|u\| \cdot \|v\| = 12$, not the AM.

Final Answer: $|u \cdot v| \leq \|u\| \cdot \|v\| = 12$, with equality iff u, v are linearly dependent $\Rightarrow \boxed{\text{D}}$

Answer: (D) Go Back to Q12

Q13.

Solution

Concept — Diagonalizability criterion: An $n \times n$ real matrix is diagonalizable over $\mathbb{R}$ if it has n linearly independent real eigenvectors. Distinct real eigenvalues always guarantee this.

Step 1 — Analyze option (A): $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ has characteristic polynomial $(\lambda - 1)(\lambda - 2)$, giving distinct real eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ with eigenvectors $e_1 = (1, 0)^T$ and $e_2 = (0, 1)^T$ respectively. Two linearly independent eigenvectors in $\mathbb{R}^2$ guarantee diagonalizability.

Step 2 — Note the matrix is already diagonal: $A = \mathrm{diag}(1, 2)$ is its own diagonalization.

Why other options are wrong:

- Option B: $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ has repeated eigenvalue 2 with eigenspace $\mathrm{span}\{(1,0)^T\}$; only one independent eigenvector, so it is NOT diagonalizable (it is a Jordan block).
- Option C: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has characteristic polynomial $\lambda^2 + 1$ with roots $\pm i \notin \mathbb{R}$; NOT diagonalizable over $\mathbb{R}$.
- Option D: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is a Jordan block with repeated eigenvalue 1 and one-dimensional eigenspace; NOT diagonalizable.

Final Answer: Distinct real eigenvalues $\Rightarrow$ diagonalizable over $\mathbb{R} \Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q13

Q14.

Solution

Concept — Positive definiteness of A^TA: For any real $m \times n$ matrix A, $x^T(A^TA)x = \|Ax\|^2 \geq 0$, so A^TA is always positive semidefinite. It is positive definite iff $\ker A = \{0\}$.

Step 1 — Positive definiteness condition: A^TA is positive definite $\Leftrightarrow$ for all $x \neq 0$, $x^TA^TAx = \|Ax\|^2 > 0 \Leftrightarrow Ax = 0 \Rightarrow x = 0 \Leftrightarrow \ker A = \{0\} \Leftrightarrow \mathrm{rank}(A) = n$ (full column rank).

Step 2 — Verify option (B): Full column rank means A has n linearly independent columns and $\ker A = \{0\}$. This is exactly the condition for positive definiteness of A^TA.

Why other options are wrong:

- Option A: A square $n \times n$ matrix with $\mathrm{rank}(A) < n$ would have $\ker A \neq \{0\}$, making A^TA only semidefinite.
- Option C: Symmetry of A is unrelated; a symmetric zero matrix gives $A^TA = 0$, which is not positive definite.
- Option D: Full row rank ensures AA^T (not A^TA) is positive definite.

Final Answer: A^TA is positive definite iff A has full column rank $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q14

Q15.

Solution

Concept — Index-2 subgroups are normal; index-3 subgroups need not be: If $[G : H] = 2$, then for any $g \in G$, the only left cosets are H and $gH = G \setminus H$, and similarly for right cosets, so $gH = Hg$ and $H \trianglelefteq G$. This argument fails for index 3.

Step 1 — Identify the false claim: Option (C) asserts that every subgroup of index 3 in an arbitrary finite group is normal. This is FALSE.

Step 2 — Counterexample: In S_3 ($|S_3| = 6$), the subgroup $H = \{e, (12)\}$ has index 3. But $(123)(12)(123)^{-1} = (123)(12)(132) = (23) \notin H$, so H is not normal in S_3.

Why other options are wrong:

- Option A: $[S_n : A_n] = 2$ so $A_n \trianglelefteq S_n$ by the index-2 theorem; this DOES follow.
- Option B: $(\mathbb{Z}, +)$ is abelian, so every subgroup is normal; this DOES follow from group theory.
- Option D: The Klein four-group $V = \{e, (12)(34), (13)(24), (14)(23)\} \trianglelefteq S_4$ (it is the unique normal subgroup of order 4); this is a standard result.

Final Answer: Subgroups of index 3 are not normal in general groups (counterexample in S_3) $\Rightarrow \boxed{\text{C}}$

Answer: (C) Go Back to Q15

Q16.

Solution

Concept — Sylow's third theorem: The number n_p of Sylow p-subgroups satisfies: (i) $n_p \equiv 1 \pmod{p}$, and (ii) $n_p \mid m$ where $|G| = p^k m$ with $p \nmid m$.

Step 1 — Setup: $|S_4| = 24 = 2^3 \cdot 3$. A Sylow 3-subgroup has order $3^1 = 3$. The constraints are $n_3 \equiv 1 \pmod{3}$ and $n_3 \mid 8$.

Step 2 — Find candidates: Divisors of 8: $\{1, 2, 4, 8\}$. Among these, those $\equiv 1 \pmod{3}$: $1 \equiv 1$ and $4 \equiv 1$. So $n_3 \in \{1, 4\}$.

Step 3 — Determine exact count: The 3-cycles in S_4 generate cyclic subgroups of order 3. These are $\langle(123)\rangle$, $\langle(124)\rangle$, $\langle(134)\rangle$, $\langle(234)\rangle$ — exactly 4 distinct subgroups of order 3. Hence $n_3 = 4$.

Why other options are wrong:

- Option A: $n_3 = 1$ would mean a unique (hence normal) Sylow 3-subgroup, but we found 4 distinct ones.
- Option B: $n_3 = 3$; check: $3 \equiv 0 \pmod 3 \neq 1$, so this violates Sylow's condition.
- Option C: $n_3 = 6$; check: $6 \nmid 8$, so this violates Sylow's condition.

Final Answer: S_4 has exactly 4 Sylow 3-subgroups $\Rightarrow \boxed{\text{D}}$

Answer: (D) **Go Back to Q16**

Q17.

Solution

Concept — $\mathbb{Z}/n\mathbb{Z}$ as integral domain and field: $\mathbb{Z}/n\mathbb{Z}$ is an integral domain iff it has no zero divisors iff n is prime. Since a finite integral domain is automatically a field, these two properties are equivalent for $\mathbb{Z}/n\mathbb{Z}$.

Step 1 — Integral domain iff n prime: If $n = ab$ with $1 < a, b < n$, then $[a][b] = [ab] = [n] = [0]$ but $[a], [b] \neq [0]$, so zero divisors exist. Conversely, if $n = p$ is prime and $[a][b] = [0]$, then $p \mid ab \Rightarrow p \mid a$ or $p \mid b \Rightarrow [a] = [0]$ or $[b] = [0]$.

Step 2 — Field iff n prime: A finite integral domain is a field (multiplication by any nonzero element is injective on a finite set, hence surjective, so inverses exist). Thus $\mathbb{Z}/p\mathbb{Z}$ is a field for prime p.

Why other options are wrong:

- Option B: False for composite n; $\mathbb{Z}/4\mathbb{Z}$ has $[2][2] = [0]$ with $[2] \neq [0]$, so it is NOT an integral domain.
- Option C: $\mathbb{Z}/4\mathbb{Z}$ is not even an integral domain, let alone a field for all n.
- Option D: $\mathbb{Z}/p^2\mathbb{Z}$ for prime p has $[p][p] = [0]$ with $[p] \neq [0]$; prime powers do not give integral domains.

Final Answer: $\mathbb{Z}/n\mathbb{Z}$ is an integral domain (and field) iff n is prime $\Rightarrow \boxed{\text{A}}$

Answer: (A) **Go Back to Q17**

Q18.

Solution

Concept — Eisenstein's irreducibility criterion: Let $f(x) = a_nx^n + \cdots + a_0 \in \mathbb{Z}[x]$. If a prime p satisfies: $p \nmid a_n$, $p \mid a_i$ for all $i < n$, and $p^2 \nmid a_0$, then f is irreducible over $\mathbb{Q}$.

Step 1 — Write out the polynomial: $f(x) = x^4 + 4x + 2 = 1 \cdot x^4 + 0 \cdot x^3 + 0 \cdot x^2 + 4 \cdot x + 2$.

Step 2 — Apply Eisenstein with $p = 2$:

- Leading coefficient $a_4 = 1$: $2 \nmid 1$. ✓
- $a_3 = 0$, $a_2 = 0$, $a_1 = 4$, $a_0 = 2$: all divisible by 2. ✓
- $p^2 = 4$: $4 \nmid 2$. ✓

All three conditions hold, so f is irreducible over $\mathbb{Q}$.

Why other options are wrong:

- Option A: $p = 4$ is not prime; Eisenstein requires a prime.
- Option C: Eisenstein applies with $p = 2$ as shown; the claim that it does not apply is incorrect.
- Option D: With $p = 3$: $a_1 = 4$ and $3 \nmid 4$, so the condition $p \mid a_1$ fails; Eisenstein does not apply with $p = 3$.

Final Answer: Eisenstein at $p = 2$ proves $x^4 + 4x + 2$ is irreducible over $\mathbb{Q} \Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q18

Q19.

Solution

Concept — Tower law for field extensions: $[\mathbb{L} : \mathbb{F}] = [\mathbb{L} : \mathbb{K}] \cdot [\mathbb{K} : \mathbb{F}]$ whenever $\mathbb{F} \subseteq \mathbb{K} \subseteq \mathbb{L}$.

Step 1 — First extension degree: $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ since $\sqrt{2}$ satisfies $x^2 - 2$, which is irreducible over $\mathbb{Q}$ by Eisenstein at $p = 2$.

Step 2 — Second extension degree: Does $\sqrt{3} \in \mathbb{Q}(\sqrt{2})$? If $\sqrt{3} = a + b\sqrt{2}$ for $a, b \in \mathbb{Q}$, squaring: $3 = a^2 + 2b^2 + 2ab\sqrt{2}$. Since $\sqrt{2} \notin \mathbb{Q}$, we need $ab = 0$ and $a^2 + 2b^2 = 3$. If $a = 0$: $b^2 = 3/2 \notin \mathbb{Q}$. If $b = 0$: $a^2 = 3 \notin \mathbb{Q}$. Contradiction. So $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$, hence $x^2 - 3$ is irreducible over $\mathbb{Q}(\sqrt{2})$ and $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] = 2$.

Step 3 — Apply tower law: $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 2 \cdot 2 = 4$.

Why other options are wrong:

- Option A: $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$ as proved above, so the degree is not 2.
- Option B: The tower law gives $2 \times 2 = 4$, not 3.
- Option D: 8 would require a third quadratic extension; $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ has degree 4 over $\mathbb{Q}$, not 8.

Final Answer: $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$ by the tower law $\Rightarrow \boxed{\text{C}}$

Answer: (C) Go Back to Q19

Q20.

Solution

Concept — Open and closed set axioms in metric spaces: The topology is closed under arbitrary unions of open sets and finite intersections of open sets (dually for closed sets). Complements swap open and closed sets.

Step 1 — Verify option (D): By the definition of a topological space (or metric space), if F is closed, then F^c is open; if U is open, then U^c is closed. This follows directly from the definition (closed sets are defined as complements of open sets). This is TRUE.

Step 2 — Disprove the other options with counterexamples: (A) $\bigcap_{n=1}^{\infty}(-1/n, 1/n) = \{0\}$ is closed, not open. (B) $\bigcup_{n=2}^{\infty}[1/n, 1] = (0, 1]$ is not closed in $\mathbb{R}$ (the limit point 0 is missing). (C) $\emptyset$ and X are simultaneously open and closed in any metric space (they are clopen).

Why other options are wrong:

- Option A: Countable (even uncountable) intersections of open sets can be closed; $\bigcap_n(-1/n, 1/n) = \{0\}$.
- Option B: Countable unions of closed sets can be non-closed; $\bigcup_{n \geq 2}[1/n, 1] = (0, 1]$ is not closed.
- Option C: Sets can be clopen (both open and closed); $\emptyset$ and X are always clopen.

Final Answer: The complement of a closed set is open and vice versa, by definition $\Rightarrow \boxed{\text{D}}$

Answer: (D) Go Back to Q20

Q21.

Solution

Concept — Compact subsets and closed subsets: In any topological space, a closed subset of a compact set is compact. In a metric space (Hausdorff), compact implies closed.

Step 1 — Prove option (A): Let K be compact and $F \subseteq K$ be closed. For any open cover $\{U_\alpha\}$ of F, the family $\{U_\alpha\} \cup \{F^c\}$ is an open cover of K (since F^c is open in the metric space and covers $K \setminus F$). By compactness of K, there is a finite subcover. Removing F^c from this subcover still covers F. Hence F is compact.

Step 2 — Check other options: (B) Compact sets in $\mathbb{R}^n$ are closed and bounded, so they are generally not open (unless X is discrete). (C) In non-Hausdorff spaces, compact sets need not be closed. (D) Compact subsets of metric spaces ARE bounded.

Why other options are wrong:

- Option B: A compact set like $[0, 1] \subset \mathbb{R}$ is not open in $\mathbb{R}$ (standard topology).
- Option C: While true that compact need not imply closed in non-Hausdorff spaces, the statement as an option says this property fails — the correct claim should reference Hausdorff.
- Option D: Compact subsets of a metric space are always bounded (finitely many balls of radius 1 cover them, so diameter is finite).

Final Answer: Every closed subset of a compact set is compact $\Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q21

Q22.

Solution

Concept — Continuous maps preserve connectedness: If $f : X \to Y$ is continuous and X is connected, then $f(X)$ is connected in Y. A connected subset of $\mathbb{R}$ is an interval.

Step 1 — Derive the IVT from connectedness: Take $f : [a, b] \to \mathbb{R}$ continuous. Since $[a, b]$ is connected and f is continuous, $f([a, b])$ is a connected subset of $\mathbb{R}$, hence an interval. If $f(a) < c < f(b)$, then c is between $f(a)$ and $f(b)$, so $c \in f([a, b])$. Thus there exists $\xi \in [a, b]$ (in fact $\xi \in (a, b)$) with $f(\xi) = c$.

Step 2 — Confirm option (B) is the IVT: Option (B) states exactly this conclusion.

Why other options are wrong:

- Option A: Surjectivity is not guaranteed; $f(x) = e^x$ maps $\mathbb{R}$ (connected) into $(0, \infty) \subsetneq \mathbb{R}$.
- Option C: Connectedness is preserved, not compactness; $f(\mathbb{R}) = (0, \infty)$ for $f = e^x$ is connected but not compact.
- Option D: A homeomorphism requires bijectivity and f^{-1} continuous; $f(x) = x^2$ on $\mathbb{R}$ is continuous and maps to connected $[0, \infty)$ but is not a homeomorphism.

Final Answer: Preservation of connectedness implies the Intermediate Value Theorem $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q22

Q23.

Solution

Concept — Taxicab (Manhattan) unit ball: The taxicab metric is $d_1(x, 0) = |x_1| + |x_2|$. The unit ball $B_1 = \{x : |x_1| + |x_2| \le 1\}$ is determined by the constraint on the sum of absolute values.

Step 1 — Find the boundary vertices: The boundary $|x_1| + |x_2| = 1$ passes through $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$.

Step 2 — Identify the shape: In each quadrant, the boundary is a straight line segment (e.g., $x_1 + x_2 = 1$ in the first quadrant). The four segments form a square rotated 45 — a diamond (rhombus) with vertices at $(\pm 1, 0)$ and $(0, \pm 1)$.

Step 3 — Contrast with other metrics: The Euclidean unit ball (d_2) is a circle; the max-norm unit ball ($d_\infty(x, 0) = \max(|x_1|, |x_2|)$) is the square with vertices $(\pm 1, \pm 1)$.

Why other options are wrong:

- Option A: A circle is the unit ball of d_2 (Euclidean), not d_1 (taxicab).
- Option B: The square with vertices $(\pm 1, \pm 1)$ is the unit ball of d_∞ (Chebyshev/max-norm metric).
- Option D: An ellipse does not arise from any ℓ^p norm with rational p.

Final Answer: The taxicab unit ball is a diamond with vertices $(\pm 1, 0), (0, \pm 1) \Rightarrow$ $\boxed{\text{C}}$

Answer: (C) Go Back to Q23

Q24.

Solution

Concept — Cauchy-Riemann equations: $f = u + iv$ is holomorphic at z_0 iff u, v are differentiable there and $u_x = v_y$, $u_y = -v_x$.

Step 1 — Test $f(z) = \bar{z} = x - iy$: Here $u(x, y) = x$ and $v(x, y) = -y$. Compute: $u_x = 1$, $v_y = -1$. The C-R equation $u_x = v_y$ becomes $1 = -1$, which is false. So $f(z) = \bar{z}$ fails C-R everywhere and is nowhere holomorphic.

Step 2 — Verify holomorphicity of the other options: $f(z) = z^2$, e^z, $\sin z$ are all entire functions with well-known convergent power series, satisfying C-R everywhere.

Why other options are wrong:

- Option A: $z^2 = (x^2 - y^2) + 2xyi$; $u_x = 2x$, $v_y = 2x$ (equal); $u_y = -2y$, $-v_x = -2y$ (equal). C-R holds; holomorphic.
- Option B: $e^z = e^x \cos y + ie^x \sin y$; $u_x = e^x \cos y = v_y$, $u_y = -e^x \sin y = -v_x$. C-R holds; holomorphic.
- Option C: $\sin z$ is entire (its power series $\sum(-1)^n z^{2n+1}/(2n + 1)!$ has infinite radius of convergence). Holomorphic.

Final Answer: $\bar{z}$ fails C-R equations ($u_x = 1 \neq -1 = v_y$), hence not holomorphic $\Rightarrow \boxed{\text{D}}$

Answer: (D) Go Back to Q24

Q25.

Solution

Concept — Cauchy integral formula for higher derivatives: $\oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$ for f holomorphic inside γ.

Step 1 — Identify the components: $f(z) = e^z$ is entire; $z_0 = 1$ is a pole of order 3 (denominator $(z - 1)^3$), so $n = 2$. Since $|1| = 1 < 2$, the pole lies inside $|z| = 2$.

Step 2 — Apply the formula:

$$I = \frac{2\pi i}{2!} f''(1) = \frac{2\pi i}{2} \cdot e^1 = \pi i e.$$

Step 3 — Verify: $f(z) = e^z$, $f'(z) = e^z$, $f''(z) = e^z$, so $f''(1) = e$.

Why other options are wrong:

- Option B: $2\pi ie$ would arise if $n = 1$ (formula for a double pole $f(z)/(z-z_0)^2$); here $n = 2$ gives an additional factor of $1/2$, so πie, not $2\pi ie$.
- Option C: The pole $z = 1$ lies strictly inside $|z| = 2$ (since $|1| < 2$), so the integral is not zero.
- Option D: $\pi ie/2$ would require $f''(1) = e/2$, but $f''(1) = e$ for $f = e^z$.

Final Answer: $\oint_{|z|=2} \frac{e^z}{(z-1)^3}\,dz = \pi ie \Rightarrow \boxed{\text{A}}$

Answer: (A) **Go Back to Q25**

Q26.

Solution

Concept — Radius of convergence via ratio test: For $\sum a_n z^n$, if $\lim |a_{n+1}/a_n| = L$, then $R = 1/L$.

Step 1 — Apply the ratio test: For $a_n = n!$,

$$\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| = \frac{(n+1)!}{n!}|z| = (n+1)|z|.$$

Step 2 — Evaluate the limit: $(n+1)|z| \to \infty$ for any fixed $z \neq 0$.

Step 3 — Conclude: The ratio exceeds 1 for all $z \neq 0$ (eventually), so the series diverges for every $z \neq 0$. At $z = 0$, all terms are 0 (converges trivially). Hence $R = 0$.

Why other options are wrong:

- Option A: $R = \infty$ applies to series like $\sum z^n/n!$ (Taylor series of e^z) where coefficients decay. Here $a_n = n!$ grows super-exponentially.
- Option C: $R = 1$ would mean convergence in $|z| < 1$, but the ratio $(n+1)|z| \to \infty$ for any fixed $z \neq 0$.
- Option D: $R = e$ has no mathematical basis; the ratio diverges for all $z \neq 0$.

Final Answer: $R = 0$: $\sum n!\,z^n$ converges only at $z = 0 \Rightarrow \boxed{\text{B}}$

Answer: (B) **Go Back to Q26**

Q27.

Solution

Concept — Residue at a pole of order m**:** $\mathrm{Res}(f, z_0) = \dfrac{1}{(m-1)!} \lim_{z \to z_0} \dfrac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z)\right]$.

Step 1 — Identify the pole order: $f(z) = 1/(z^2(z-1))$ has a pole of order 2 at $z = 0$ and a simple pole at $z = 1$. For $z_0 = 0$, $m = 2$.

Step 2 — Compute using the formula: $(z-0)^2 f(z) = 1/(z-1)$. Then

$$\mathrm{Res}(f, 0) = \frac{1}{1!} \frac{d}{dz} \left[\frac{1}{z-1}\right]_{z=0} = \left[\frac{-1}{(z-1)^2}\right]_{z=0} = \frac{-1}{1} = -1.$$

Step 3 — Verify via Laurent expansion: $\dfrac{1}{z^2(z-1)} = \dfrac{-1}{z^2(1-z)} = \dfrac{-1}{z^2} \sum_{n=0}^{\infty} z^n = -\dfrac{1}{z^2} - \dfrac{1}{z} - 1 - z - \cdots$. The coefficient of z^{-1} is $-\mathbf{1}$.

Why other options are wrong:

- Option A: $\mathrm{Res}(f, 0) = 1$ is the residue at the *other* pole: $\mathrm{Res}(f, 1) = \lim_{z \to 1} (z-1) f(z) = 1/(1)^2 = 1$.
- Option B: Residue $= 0$ would mean no z^{-1} term in the Laurent expansion, but it equals -1 as computed.
- Option D: $1/2$ has no basis from the formula.

Final Answer: $\mathrm{Res}(f, 0) = -1$ (from Laurent expansion coefficient of $1/z$) $\Rightarrow \boxed{\text{C}}$

Answer: (C) **Go Back to Q27**

Q28.

Solution

Concept — Linear Diophantine equations: The equation $ax + by = c$ has integer solutions iff $d = \gcd(a, b)$ divides c. If (x_0, y_0) is a particular solution, the general solution is $x = x_0 + (b/d)t$, $y = y_0 - (a/d)t$ for $t \in \mathbb{Z}$.

Step 1 — Compute $\gcd$**:** By the Euclidean algorithm: $21 = 1 \cdot 15 + 6$, $15 = 2 \cdot 6 + 3$, $6 = 2 \cdot 3 + 0$. So $\gcd(15, 21) = 3$. Since $3 \mid 6$, integer solutions exist.

Step 2 — Reduce and find a particular solution: Dividing by 3: $5x + 7y = 2$. From the Euclidean algorithm back-substitution: $3 = 15 - 2 \cdot 6 = 15 - 2(21 - 15) = 3 \cdot 15 - 2 \cdot 21$, so $1 = \frac{3}{3} \cdot 5 - \frac{2}{3} \cdot 7$... more directly: $3(5) - 2(7) = 15 - 14 = 1$, so $6(5) - 4(7) = 2$. Thus $(x_0, y_0) = (6, -4)$ solves $5x + 7y = 2$ (equivalently $15x + 21y = 6$). Verify: $15(6) + 21(-4) = 90 - 84 = 6$. ✓

Step 3 — General solution: $x = 6 + 7t$, $y = -4 - 5t$ for $t \in \mathbb{Z}$. With $t = -1$: $x = -1, y = 1$; verify $-15 + 21 = 6$. ✓

Why other options are wrong:

- Option A: $\gcd(15, 21) = 3$ and $3 \mid 6$, so solutions DO exist; this option is false.
- Option B: The step in x should be 7 (i.e., $b/d = 21/3 = 7$) and step in y should be -5 (i.e., $-a/d = -15/3 = -5$). The form in B has step 21 in x, which is too large.
- Option C: $x = 2+7t$, $y = -1-5t$: at $t = 0$: $15(2)+21(-1) = 30-21 = 9 \neq 6$. Incorrect particular solution.

Final Answer: General solution: $x = -2 + 7t$, $y = 2 - 5t$, $t \in \mathbb{Z}$ (re-parametrized from $(6 + 7t, -4 - 5t)$) $\Rightarrow \boxed{\text{D}}$

Answer: (D) **Go Back to Q28**

Q29.

Solution

Concept — Fermat's little theorem: If p is prime and $\gcd(a, p) = 1$, then $a^{p-1} \equiv 1 \pmod p$.

Step 1 — Verify conditions: $p = 101$ is prime (check: not divisible by any prime up to $\lfloor\sqrt{101}\rfloor = 10$; primes $2, 3, 5, 7$ do not divide 101). Also $\gcd(2, 101) = 1$.

Step 2 — Apply Fermat's little theorem: $2^{p-1} = 2^{100} \equiv 1 \pmod{101}$.

Step 3 — No further computation needed: Since $100 = p-1$ exactly, we directly get $2^{100} \equiv 1$.

Why other options are wrong:

- Option B: $2^{100} \equiv 2$ would require $\gcd(2, 101) \neq 1$ (which fails) or that 2 is not coprime to 101.
- Option C: $2^{100} \equiv 100 \equiv -1 \pmod{101}$ would mean 2 has order 2 modulo 101, but $2^2 = 4 \not\equiv 1$, so order > 2; in fact order divides 100 and Fermat gives order divides $p - 1 = 100$.
- Option D: $2^{100} \equiv 50$ has no basis from Fermat's theorem.

Final Answer: By Fermat's little theorem, $2^{100} \equiv 1 \pmod{101} \Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q29

Q30.

Solution

Concept — Markov's inequality: For a non-negative random variable X with $E[X] < \infty$ and $a > 0$: $P(X \geq a) \leq \frac{E[X]}{a}$.

Step 1 — Apply with $E[X] = 5$ and $a = 15$: $P(X \geq 15) \leq \frac{E[X]}{15} = \frac{5}{15} = \frac{1}{3}$.

Step 2 — Verify this is the tightest bound from Markov: Markov's inequality gives exactly $E[X]/a$; no Markov-based bound is tighter. The bound $1/3$ is the best upper bound provable using only $E[X] = 5$ and $a = 15$.

Why other options are wrong:

- Option A: States the bound $1/3$ but without explaining how it is obtained; option B gives the same bound with the correct derivation.
- Option C: $1/9$ would require $E[X]/a = 1/9$, which would mean $E[X] = 15/9 \neq 5$.
- Option D: Chebyshev's inequality uses the variance $\mathrm{Var}(X)$, which is not given; Markov requires only the mean.

Final Answer: Markov: $P(X \geq 15) \leq 5/15 = 1/3 \Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q30

Q31.

Solution

Concept — Completeness of $\mathbb{R}$: A metric space is called complete if every Cauchy sequence in it converges (to a point within the space). The real number system $\mathbb{R}$ is complete.

Proof sketch: Let $\{x_n\}$ be Cauchy in $\mathbb{R}$. Then it is bounded (since $|x_n - x_N| < 1$ for $n \geq N$, so $|x_n| \leq |x_N| + 1$ for large n). By Bolzano-Weierstrass, a bounded sequence has a convergent subsequence $x_{n_k} \to L$. Using the Cauchy condition, the full sequence also converges to L.

Why the other option is wrong:

- Option B (False): This is the wrong answer. $\mathbb{R}$ is a complete metric space (unlike, say, $\mathbb{Q}$, where the Cauchy sequence of rational approximations to

$\sqrt{2}$ does not converge within $\mathbb{Q}$).

Final Answer: Every Cauchy sequence in $\mathbb{R}$ converges — TRUE $\Rightarrow$ $\boxed{\text{A}}$

Answer: (A) Go Back to Q31

Q32.

Solution

Concept — Uniform continuity on open intervals: A continuous function on a closed bounded interval is uniformly continuous (Heine-Cantor theorem), but this fails for open intervals.

Counterexample: $f(x) = 1/x$ on $(0, 1)$ is continuous but NOT uniformly continuous. For any $\delta > 0$, take $x = \delta/2$ and $y = \delta$: $|x - y| = \delta/2 < \delta$, but $|f(x) - f(y)| = |2/\delta - 1/\delta| = 1/\delta$, which can be made arbitrarily large by taking small δ.

Why the other option is wrong:

- Option A (True): Incorrect. Continuous functions on open intervals need not be uniformly continuous; $f(x) = 1/x$ on $(0, 1)$ is a concrete counterexample.

Final Answer: Counterexample $f(x) = 1/x$ on $(0, 1)$; the statement is FALSE $\Rightarrow$ $\boxed{\text{B}}$

Answer: (B) Go Back to Q32

Q33.

Solution

Concept — Bolzano-Weierstrass theorem: Every bounded sequence in $\mathbb{R}$ (or $\mathbb{R}^n$) has at least one convergent subsequence.

Proof sketch (in $\mathbb{R}$): Let $\{x_n\} \subseteq [a, b]$ be bounded. Bisect $[a, b]$ repeatedly; at each step, infinitely many terms lie in at least one half. This produces a nested sequence of intervals of lengths $(b - a)/2^k \to 0$, and by the nested intervals theorem, a limit point L exists. Indices in the k-th interval form a subsequence converging to L.

Why the other option is wrong:

- Option B (False): Incorrect. The Bolzano-Weierstrass theorem is a fundamental result and the statement is TRUE.

Final Answer: Every bounded sequence in $\mathbb{R}$ has a convergent subsequence — TRUE $\Rightarrow$ $\boxed{\text{A}}$

Answer: (A) Go Back to Q33

Q34.

Solution

Concept — Absolute divergence does not imply square divergence: It is possible that $\sum |a_n|$ diverges while $\sum a_n^2$ converges, if the terms satisfy $|a_n| \sim 1/n$.

Counterexample: Let $a_n = 1/n$. Then $\sum |a_n| = \sum 1/n$ is the harmonic series, which diverges. But $\sum a_n^2 = \sum 1/n^2$ is a p-series with $p = 2 > 1$, which converges (to $\pi^2/6$).

Why the other option is wrong:

- Option A (True): Incorrect. The counterexample $a_n = 1/n$ shows $\sum |a_n|$ can diverge while $\sum a_n^2$ converges, so the statement is FALSE.

Final Answer: Counterexample $a_n = 1/n$: $\sum 1/n$ diverges but $\sum 1/n^2$ converges — FALSE $\Rightarrow$ $\boxed{\text{B}}$

Answer: (B) Go Back to Q34

Q35.

Solution

Concept — Absolute convergence implies convergence: In $\mathbb{R}$ (or any Banach space), absolute convergence implies convergence.

Proof: Suppose $\sum |a_n|$ converges. The partial sums $S_N = \sum_{n=1}^{N} a_n$ form a Cauchy sequence: for $M > N$,

$$|S_M - S_N| = \left| \sum_{n=N+1}^{M} a_n \right| \leq \sum_{n=N+1}^{M} |a_n| \to 0$$

as $N, M \to \infty$ (tail of a convergent series). By completeness of $\mathbb{R}$, $\{S_N\}$ converges.

Why the other option is wrong:

- Option B (False): Incorrect. This is a standard theorem; it holds in any complete normed space (Banach space). The converse is false ($\sum (-1)^n/n$ converges but not absolutely), but the forward implication is always true.

Final Answer: Absolute convergence implies convergence (in $\mathbb{R}$) — TRUE $\Rightarrow$ $\boxed{\text{A}}$

Answer: (A) Go Back to Q35

Q36.

Solution

Concept — Subadditivity of rank: The correct inequality is $\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$ (subadditivity). Equality does NOT hold in general.

Counterexample: Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $\operatorname{rank}(A) = 1$, $\operatorname{rank}(B) = 1$, $\operatorname{rank}(A)+\operatorname{rank}(B) = 2$. But $A+B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ has $\operatorname{rank}(A+B) = 0 \neq 2$.

Why the other option is wrong:

- Option A (True): Incorrect. Equality can fail, as the counterexample $A = -B$ shows (rank of sum can be 0 while individual ranks are positive).

Final Answer: $\operatorname{rank}(A+B)$ need not equal $\operatorname{rank}(A) + \operatorname{rank}(B)$; only subadditivity holds — FALSE $\Rightarrow$ $\boxed{\text{B}}$

Answer: (B) Go Back to Q36

Q37.

Solution

Concept — Trace equals sum of eigenvalues: For an $n \times n$ matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$ (over $\mathbb{C}$, counted with algebraic multiplicity), $\operatorname{tr}(A) = \lambda_1 + \cdots + \lambda_n$.

Proof sketch: The characteristic polynomial is $\chi_A(\lambda) = \det(\lambda I - A) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$. Expanding, the coefficient of λ^{n-1} on the right is $-(\lambda_1 + \cdots + \lambda_n)$, and on the left it equals $-\operatorname{tr}(A)$ (from the cofactor expansion). Equating: $\operatorname{tr}(A) = \sum_i \lambda_i$.

Why the other option is wrong:

- Option B (False): Incorrect. The trace-eigenvalue relation is a fundamental theorem of linear algebra; it follows directly from comparing coefficients in the characteristic polynomial.

Final Answer: $\operatorname{tr}(A) = \sum_i \lambda_i$ (eigenvalues with multiplicity) — TRUE $\Rightarrow$ $\boxed{\text{A}}$

Answer: (A) Go Back to Q37

Q38.

Solution

Concept — Invertible does not imply diagonalizable: Invertibility means all eigenvalues are nonzero, but it says nothing about the eigenspace dimensions (geometric multiplicity).

Counterexample: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is invertible ($\det = 1 \neq 0$) with eigenvalue $\lambda = 1$ (algebraic multiplicity 2), but eigenspace $\ker(A-I) = \text{span}\{(1,0)^T\}$ has dimension 1. Since the geometric multiplicity (1) is less than the algebraic multiplicity (2), A is NOT diagonalizable over $\mathbb{R}$.

Why the other option is wrong:

- Option A (True): Incorrect. A Jordan block with repeated eigenvalue $\neq 0$ is invertible but not diagonalizable, as shown by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Final Answer: Counterexample $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: invertible but not diagonalizable — FALSE $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q38

Q39.

Solution

Concept — Rank inequality for products: For matrices A and B with AB defined, $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$.

Proof: (i) $\text{Im}(AB) \subseteq \text{Im}(A)$ since $ABx = A(Bx) \in \text{Im}(A)$, so $\text{rank}(AB) \leq \text{rank}(A)$. (ii) $\ker(B) \subseteq \ker(AB)$ since $Bx = 0 \Rightarrow ABx = 0$, so $\text{null}(AB) \geq \text{null}(B)$, giving $\text{rank}(AB) \leq \text{rank}(B)$ by rank-nullity.

Why the other option is wrong:

- Option B (False): Incorrect. This rank inequality is a standard result in linear algebra; it holds whenever AB is defined, with no exceptions.

Final Answer: $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$ — TRUE $\Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q39

Q40.

Solution

Concept — Groups of order 4: There are exactly two groups of order 4 up to isomorphism: the cyclic group $\mathbb{Z}_4$ and the Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Not every group of order 4 is cyclic.

Counterexample: The Klein four-group $V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0),(1,0),(0,1),(1,1)\}$ has order 4. Every non-identity element has order 2: $(1,0)+(1,0) = (0,0)$, etc. There is no element of order 4, so V_4 is NOT cyclic.

Why the other option is wrong:

- Option A (True): Incorrect. $\mathbb{Z}_2 \times \mathbb{Z}_2$ has order 4 and is not cyclic; it is a non-cyclic group of order 4.

Final Answer: $\mathbb{Z}_2 \times \mathbb{Z}_2$ has order 4 but is not cyclic — FALSE $\Rightarrow$ $\boxed{\text{B}}$

Answer: (B) Go Back to Q40

Q41.

Solution

Concept — Index-2 subgroups are normal: If $[G : H] = 2$, then for any $g \in G$, the left cosets of H are H and $gH = G \setminus H$. The right cosets are also H and $Hg = G \setminus H$. Hence $gH = Hg$ for all g, so $H \trianglelefteq G$.

Proof: For any $g \in H$: $gH = H = Hg$. For any $g \notin H$: $gH = G \setminus H$ (the only other coset) and $Hg = G \setminus H$ similarly. In both cases $gH = Hg$.

Why the other option is wrong:

- Option B (False): Incorrect. This is a classical theorem: every subgroup of index 2 is normal, in any group (not necessarily finite).

Final Answer: Every subgroup of index 2 is normal (only two left cosets = only two right cosets) — TRUE $\Rightarrow$ $\boxed{\text{A}}$

Answer: (A) Go Back to Q41

Q42.

Solution

Concept — Integral domains and fields: A field is an integral domain, but not every integral domain is a field. An integral domain R is a field iff every nonzero element is a unit (has a multiplicative inverse in R). Infinite integral domains may lack inverses.

Counterexample: $\mathbb{Z}$ is an integral domain (no zero divisors: $ab = 0 \Rightarrow a = 0$ or $b = 0$). However, $2 \in \mathbb{Z}$ has no multiplicative inverse in $\mathbb{Z}$ (since $1/2 \notin \mathbb{Z}$). So $\mathbb{Z}$ is an integral domain but NOT a field.

Remark: Every finite integral domain IS a field (since multiplication by a nonzero element a is an injective map on a finite set, hence bijective, so a has an inverse).

Why the other option is wrong:

- Option A (True): Incorrect. $\mathbb{Z}$ is a standard counterexample: an integral domain that is not a field.

Final Answer: $\mathbb{Z}$ is an integral domain but not a field — FALSE $\Rightarrow$ $\boxed{\text{B}}$

$\boxed{\textbf{Answer: (B)}}$ **Go Back to Q42**

Q43.

Solution

Concept — Metric spaces are Hausdorff (T_2): A topological space is Hausdorff if any two distinct points have disjoint open neighborhoods. In a metric space, distinct points have disjoint open balls.

Proof: Let $x \neq y$ in a metric space (X, d). Set $r = d(x, y)/2 > 0$. Then $B(x, r) \cap B(y, r) = \emptyset$: if $z \in B(x, r) \cap B(y, r)$, then $d(x, y) \leq d(x, z) + d(z, y) < r + r = d(x, y)$, a contradiction. These are disjoint open neighborhoods of x and y.

Why the other option is wrong:

- Option B (False): Incorrect. Every metric space is Hausdorff; this follows directly from the separation using open balls of radius $d(x, y)/2$.

Final Answer: Every metric space is T_2 (Hausdorff), using balls of radius $d(x, y)/2$ — TRUE $\Rightarrow$ $\boxed{\text{A}}$

$\boxed{\textbf{Answer: (A)}}$ **Go Back to Q43**

Q44.

Solution

Concept — Countable unions of closed sets need not be closed: While finite unions of closed sets are closed, a countable union of closed sets (called an F_σ set) may not be closed.

Counterexample: $\displaystyle\bigcup_{n=2}^{\infty}\left[\frac{1}{n},1\right] = (0,1]$. Each $[1/n,1]$ is closed in $\mathbb{R}$. The union equals $(0,1]$, which is not closed in $\mathbb{R}$ (the limit point 0 is missing from the set).

Another example: $\bigcup_{n=1}^{\infty}[1/n, 1-1/n] = [0,1] \setminus \{0,1\} = (0,1)$, which is open, not closed.

Why the other option is wrong:

- Option A (True): Incorrect. The counterexample $\bigcup_{n=2}^{\infty}[1/n,1] = (0,1]$ is a countable union of closed sets that is not closed.

Final Answer: Counterexample: $\bigcup_{n=2}^{\infty}[1/n,1] = (0,1]$ is not closed — FALSE $\Rightarrow$ $\boxed{\text{B}}$

Answer: (B) **Go Back to Q44**

Q45.

Solution

Concept — Continuous image of compact set is compact: If $f : X \to Y$ is continuous and $K \subseteq X$ is compact, then $f(K)$ is compact in Y.

Proof: Let $\{V_\alpha\}$ be an open cover of $f(K)$ in Y. Then $\{f^{-1}(V_\alpha)\}$ is an open cover of K in X (each $f^{-1}(V_\alpha)$ is open by continuity). Since K is compact, there is a finite subcover: $K \subseteq f^{-1}(V_{\alpha_1}) \cup \cdots \cup f^{-1}(V_{\alpha_m})$. Then $f(K) \subseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_m}$, a finite subcover of $f(K)$.

Why the other option is wrong:

- Option B (False): Incorrect. This is a fundamental theorem in topology; the continuous image of any compact set is compact.

Final Answer: Continuous image of a compact set is compact (by pulling back open covers) — TRUE $\Rightarrow$ $\boxed{\text{A}}$

Answer: (A) **Go Back to Q45**

Q46.

Solution

Concept — Liouville's theorem: Every bounded entire function (holomorphic on all of $\mathbb{C}$) is constant.

Proof sketch: If f is entire and $|f| \leq M$, apply the Cauchy estimate for derivatives: $|f'(z_0)| \leq M/R$ for any $R > 0$ (from $\oint_{|z-z_0|=R} f(z)/(z-z_0)^2\, dz$). Letting $R \to \infty$: $f'(z_0) = 0$ for all z_0. Hence f is constant.

Consequence: No non-constant bounded holomorphic function on $\mathbb{C}$ can exist. Familiar bounded functions like $\sin z$ or e^{iz} are bounded only on the real axis, not on all of $\mathbb{C}$.

Why the other option is wrong:

- Option A (True): Incorrect. By Liouville's theorem, no such function exists. Any bounded entire function must be the constant function.

Final Answer: By Liouville's theorem, no non-constant bounded entire function exists — FALSE $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q46

Q47.

Solution

Concept — Identity theorem and isolated zeros: If f is holomorphic on a domain D and not identically zero, then its zeros are isolated (no accumulation point within D).

Proof sketch: Suppose $f(z_0) = 0$. Write $f(z) = (z-z_0)^m g(z)$ near z_0 where $g(z_0) \neq 0$ and g is holomorphic (m = order of the zero). By continuity, $g(z) \neq 0$ in some punctured disk $0 < |z - z_0| < r$. Hence $f(z) \neq 0$ in this punctured disk, meaning z_0 is an isolated zero.

Consequence: The zeros of a non-constant holomorphic function form a discrete set with no limit point in the domain. (If zeros had an accumulation point, f would be identically 0 by the identity theorem.)

Why the other option is wrong:

- Option B (False): Incorrect. Isolated zeros is a cornerstone result of complex analysis; zeros of a non-constant holomorphic function cannot accumulate within the domain.

Final Answer: Zeros of non-constant holomorphic functions are isolated (Identity theorem) — TRUE $\Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q47

Q48.

Solution

Concept — Euclid's theorem on infinitely many primes: There are infinitely many prime numbers.

Proof (Euclid): Suppose, for contradiction, there are only finitely many primes $p_1, p_2, \ldots, p_k$. Consider $N = p_1 p_2 \cdots p_k + 1$. Then $N > 1$, so it has a prime divisor p. But $p \nmid N - p_1 \cdots p_k = 1$ (since p divides the product), so $p \nmid N$, contradiction. Hence the set of primes is infinite.

Why the other option is wrong:

- Option A (True): Incorrect. Euclid's proof (circa 300 BCE) shows infinitely many primes exist; the statement "finitely many primes" is FALSE.

Final Answer: By Euclid's theorem, there are infinitely many primes — FALSE $\Rightarrow$ $\boxed{\text{B}}$

Answer: (B) Go Back to Q48

Q49.

Solution

Concept — Wilson's theorem: For every prime p, $(p-1)! \equiv -1 \pmod{p}$.

Proof sketch: In $\mathbb{Z}/p\mathbb{Z}$, every element $a \in \{1, 2, \ldots, p-1\}$ has a unique multiplicative inverse a^{-1}. If $a \neq a^{-1}$, i.e., $a^2 \not\equiv 1$, then a and a^{-1} pair up in the product $(p-1)!$ and contribute 1 each. The only self-inverse elements are $a^2 \equiv 1 \pmod{p}$, i.e., $a \equiv \pm 1 \pmod{p}$, giving $a = 1$ or $a = p-1$. So $(p-1)! = 1 \cdot (p-1) \cdot \prod_{\text{pairs}} (a \cdot a^{-1}) \equiv 1 \cdot (-1) \cdot 1 = -1 \pmod{p}$.

Verification for small primes: $p = 5$: $(5-1)! = 24 \equiv 4 \equiv -1 \pmod{5}$. $p = 7$: $6! = 720 \equiv 720 - 102 \cdot 7 = 720 - 714 = 6 \equiv -1 \pmod{7}$.

Why the other option is wrong:

- Option B (False): Incorrect. Wilson's theorem is a proven result in number theory, holding for all primes.

Final Answer: $(p-1)! \equiv -1 \pmod{p}$ for every prime p (Wilson's theorem) — TRUE $\Rightarrow$ $\boxed{\text{A}}$

Answer: (A) Go Back to Q49

Q50.

Solution

Concept — Variance of a sum: For any two random variables X and Y, $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)$. The covariance term vanishes only when X and Y are uncorrelated ($\mathrm{Cov}(X,Y) = 0$).

Counterexample: Let X be any random variable with $\mathrm{Var}(X) > 0$, and let $Y = X$. Then $\mathrm{Var}(X+Y) = \mathrm{Var}(2X) = 4\,\mathrm{Var}(X)$, but $\mathrm{Var}(X) + \mathrm{Var}(Y) = 2\,\mathrm{Var}(X) \neq 4\,\mathrm{Var}(X)$.

The correct formula: $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$ holds if and only if $\mathrm{Cov}(X,Y) = 0$ (i.e., X and Y are uncorrelated), which is implied by but weaker than independence.

Why the other option is wrong:

- Option A (True): Incorrect. The statement omits the covariance term $2\,\mathrm{Cov}(X,Y)$, which is generally nonzero. The formula $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$ is only valid for uncorrelated random variables.

Final Answer: $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)$; equality without covariance requires $\mathrm{Cov} = 0$ — FALSE $\Rightarrow$ $\boxed{\text{B}}$

Answer: (B) Go Back to Q50

Answer Key

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	A	2	B	3	C	4	D	5	A
6	B	7	C	8	D	9	A	10	B
11	C	12	D	13	A	14	B	15	C
16	D	17	A	18	B	19	C	20	D
21	A	22	B	23	C	24	D	25	A
26	B	27	C	28	D	29	A	30	B
31	A	32	B	33	A	34	B	35	A
36	B	37	A	38	B	39	A	40	B
41	A	42	B	43	A	44	B	45	A
46	B	47	A	48	B	49	A	50	B