TIFR GS Mathematics

Sample Paper – 2

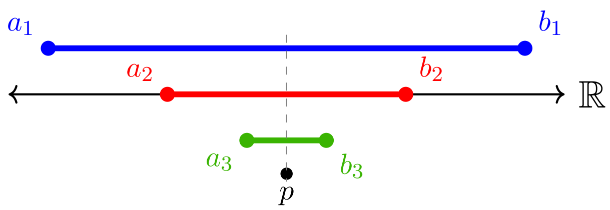
Duration: 180 Minutes **Maximum Marks: 50**

Instructions

- This paper contains **50 questions** modelled on the **TIFR GS Mathematics** entrance examination.
- **Part A (Q1–Q30):** 30 Multiple Choice Questions. Each correct answer carries **+1 mark**; each wrong answer carries $-\frac{1}{3}$ **mark**. Unattempted questions carry 0 marks.
- **Part B (Q31–Q50):** 20 True/False questions. Each correct answer carries **+1 mark**. There is **no negative marking** in Part B. Attempt all questions.
- Use of calculators, mobile phones, or any electronic gadgets is strictly prohibited. Rough work may be done on provided sheets.
- Syllabus: **BSc Mathematics (Honours) level** — Real Analysis, Linear Algebra, Abstract Algebra, Topology, Complex Analysis, Number Theory, and Probability.

Part A — Multiple Choice Questions (MCQ)

Q1. Let $[a_1, b_1] \supset [a_2, b_2] \supset \cdots$ be a sequence of nested closed bounded intervals in $\mathbb{R}$ with $b_n - a_n \to 0$ as $n \to \infty$.



Which of the following correctly describes the intersection $\bigcap_{n=1}^{\infty} [a_n, b_n]$?

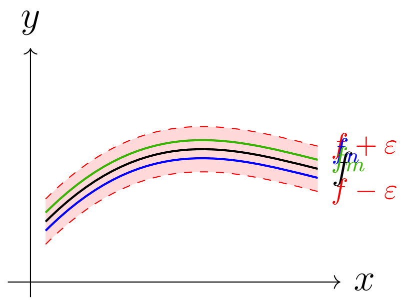
(A) The intersection is always empty.

(B) The intersection contains infinitely many points.

(C) The intersection contains exactly one point p satisfying $a_n \le p \le b_n$ for all n.

(D) The intersection is a closed interval of positive length.

Q2. Let $\{f_n\}$ be a sequence of continuous functions on $[a,b]$ converging *uniformly* to f on $[a,b]$. Each f_n lies within a band of uniform width ε around f, as shown.



Which of the following conclusions is **always** valid?

(A) f is continuous on $[a,b]$ and $\displaystyle\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$.

(B) f is continuous on $[a,b]$ and $f_n'(x) \to f'(x)$ pointwise on $[a,b]$.

(C) $\displaystyle\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$, but f need not be continuous.

(D) $f_n'(x) \to f'(x)$ uniformly on $[a,b]$.

Q3. Dini's Theorem states: *If $\{f_n\}$ is a monotone sequence of continuous functions converging pointwise to a continuous function f on a compact set K, then the convergence is uniform.* Dropping which **single** hypothesis allows the conclusion to fail? (Consider $f_n(x) = x^n$ on $(0,1)$ as a guide.)

(A) Monotonicity of the sequence $\{f_n\}$.

(B) Continuity of each f_n.

(C) Continuity of the pointwise limit function f.

(D) Compactness of the underlying set K.

Q4. The Weierstrass Approximation Theorem guarantees that every continuous function $f : [0,1] \to \mathbb{R}$ is the uniform limit of polynomials. Which of the following provides an **explicit, constructive** sequence of polynomials provably converging uniformly to f on $[0,1]$?

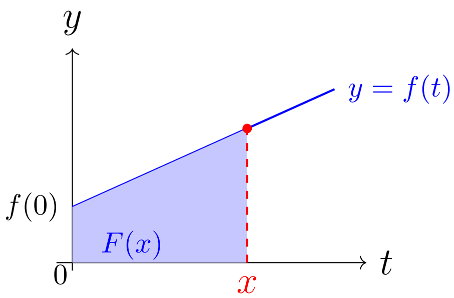
(A) Partial sums of the Fourier cosine series of f.

(B) Bernstein polynomials $B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}$.

(C) Taylor polynomials of f centered at $x = 0$.

(D) Chebyshev interpolating polynomials of f of degree n.

Q5. Let $f : [0,1] \to \mathbb{R}$ be continuous and define $F(x) = \int_0^x f(t)\,dt$. The shaded region below depicts the accumulated area from 0 to x.



Which formula correctly gives $F'(x)$ for $x \in (0,1)$?

(A) $F'(x) = F(x) - F(0)$

(B) $F'(x) = f'(x)$

(C) $F'(x) = f(x)$

(D) $F'(x) = \int_0^x f'(t)\,dt$

Q6. The Lagrange remainder in the Taylor expansion of e^x at $x = 0$ after using $n+1$ terms ($k = 0$ to $k = n$) satisfies $|R_n(1)| \le \frac{e}{(n+1)!}$. What is the **minimum** n for which this bound guarantees $|R_n(1)| < 0.01$?

(A) $n = 5$

(B) $n = 3$

(C) $n = 7$

(D) $n = 10$

Q7. The series $\sum_{n=1}^{\infty} \frac{\sin n}{n}$ converges. Which of the following tests **directly** establishes this convergence?

(A) Ratio test ($|a_{n+1}/a_n| \to L$).

(B) Root test ($|a_n|^{1/n} \to L$).

(C) Comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$.

(D) Dirichlet's test: the partial sums $\sum_{k=1}^{N} \sin k$ are bounded, and $\frac{1}{n}$ decreases monotonically to 0.

Q8. The Cantor function $f : [0,1] \to [0,1]$ (devil's staircase) is monotone non-decreasing, uniformly continuous, satisfies $f'(x) = 0$ for almost every $x \in [0,1]$, yet $f(0) = 0$ and $f(1) = 1$. Which property does f **NOT** possess?

(A) Uniform continuity on $[0,1]$.

(B) Absolute continuity on $[0,1]$.

(C) Riemann integrability on $[0,1]$.

(D) Boundedness on $[0,1]$.

Q9. Let A be a 3×3 matrix with characteristic polynomial $p_A(x) = (x-1)^2(x-2)$. The minimal polynomial $m_A(x)$ must divide $p_A(x)$ and share all its roots, as shown below.

$$\begin{tikzcd} \text{char. poly: } p_A(x) = (x-1)^2(x-2) \arrow[d, dashed, "m_A \mid p_A"] \\ \text{min. poly: } m_A(x) \text{ divides } p_A(x) \end{tikzcd}$$

Which of the following **could** be the minimal polynomial of A?

(A) $(x-2)$

(B) $(x-1)$

(C) $(x-1)(x-2)$

(D) $(x-1)^3(x-2)$

Q10. For $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, the characteristic polynomial is $p_A(\lambda) = \lambda^2 - 5\lambda - 2$. By the Cayley-Hamilton theorem, $A^2 - 5A - 2I = 0$. Which formula correctly expresses A^{-1} in terms of A and I?

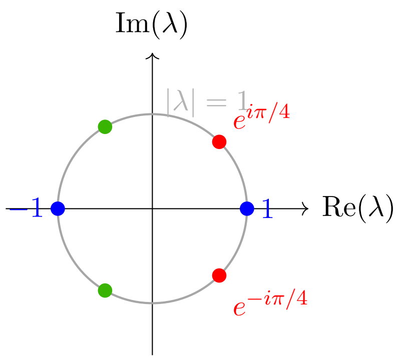
(A) $A^{-1} = \frac{1}{2}(A - 5I)$

(B) $A^{-1} = \frac{1}{5}(A^2 - 2I)$

(C) $A^{-1} = \frac{1}{2}(5I - A)$

(D) $A^{-1} = A - 5I$

Q11. Let A be an $n \times n$ real orthogonal matrix, so $A^T A = I_n$. The complex eigenvalues of any orthogonal matrix lie on the unit circle $|\lambda| = 1$ in $\mathbb{C}$, as illustrated.



Which of the following statements about A is **always true**?

(A) $\det A = 1$.

(B) All eigenvalues of A are real and positive.

(C) A is symmetric (i.e. $A = A^T$).

(D) $A^{-1} = A^T$.

Q12. A matrix N is nilpotent if $N^k = 0$ for some positive integer k; all its eigenvalues are then 0. For $N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, what is the **smallest** positive integer k for which $N^k = 0$?

(A) $k = 1$

(B) $k = 3$

(C) $k = 2$

(D) $k = 4$

Q13. Let A be a 3×3 real matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Which of the following matrices **necessarily** has the same set of eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ as A?

(A) A^{-1} (assuming A is invertible; eigenvalues become $1/\lambda_i$).

(B) $2A$ (eigenvalues become $2\lambda_i$).

(C) A^T (eigenvalues are unchanged).

(D) $A + I$ (eigenvalues become $\lambda_i + 1$).

Q14. For $n \times n$ matrices A (invertible) and B, the trace satisfies $\mathrm{tr}(XY) = \mathrm{tr}(YX)$ for any matrices X, Y of compatible size. Which of the following is **always true**?

(A) $\mathrm{tr}(ABA^{-1}) = \mathrm{tr}(B)$.

(B) $\mathrm{tr}(A^2 B) = \big(\mathrm{tr}(B)\big)^2$.

(C) $\mathrm{tr}(AB) = \mathrm{tr}(A) \cdot \mathrm{tr}(B)$.

(D) $\mathrm{tr}(A + B) = \mathrm{tr}(A) \cdot \mathrm{tr}(B)$.

Q15. Let $\phi : \mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ be the natural projection $\phi(n) = [n]_6$. By the First Isomorphism Theorem, $\mathbb{Z}/\ker\phi \cong \mathrm{Im}\,\phi$. Which option correctly identifies $\ker\phi$ and $\mathrm{Im}\,\phi$?

(A) $\ker\phi = \mathbb{Z}$ and $\mathrm{Im}\,\phi = \{0\}$.

(B) $\ker\phi = \{0\}$ and $\mathrm{Im}\,\phi \cong \mathbb{Z}$.

(C) $\ker\phi = 2\mathbb{Z}$ and $\operatorname{Im}\phi \cong \mathbb{Z}/3\mathbb{Z}$.

(D) $\ker\phi = 6\mathbb{Z}$ and $\operatorname{Im}\phi \cong \mathbb{Z}/6\mathbb{Z}$.

Q16. A group G of order 24 acts on a set X. For some $x \in X$, the stabilizer $G_x = \{g \in G : g \cdot x = x\}$ has order 4. By the Orbit-Stabilizer Theorem $|G| = |\operatorname{Orb}(x)| \cdot |G_x|$, what is $|\operatorname{Orb}(x)|$?

(A) 4

(B) 6

(C) 8

(D) 24

Q17. Which of the following **correctly** characterises $\mathbb{Z}[x]$, the ring of polynomials with integer coefficients? (Hint: consider the ideal $(2, x) = \{2f + xg : f, g \in \mathbb{Z}[x]\}$.)

(A) $\mathbb{Z}[x]$ is a Principal Ideal Domain (PID).

(B) $\mathbb{Z}[x]$ is a field.

(C) $\mathbb{Z}[x]$ is a Unique Factorization Domain (UFD) but *not* a PID.

(D) Every ideal of $\mathbb{Z}[x]$ is generated by a single element.

Q18. The Chinese Remainder Theorem for rings states $\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ when $\gcd(m, n) = 1$. With $m = 4$ and $n = 9$ (where $\gcd(4, 9) = 1$), which isomorphism is valid?

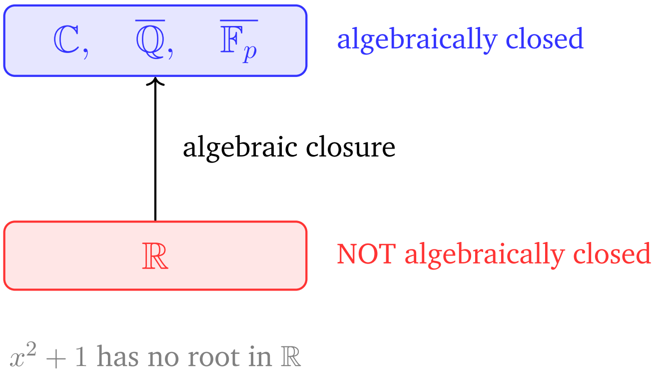
(A) $\mathbb{Z}/36\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$.

(B) $\mathbb{Z}/36\mathbb{Z} \cong \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

(C) $\mathbb{Z}/36\mathbb{Z} \cong \mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

(D) $\mathbb{Z}/36\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/18\mathbb{Z}$.

Q19. A field F is *algebraically closed* if every non-constant polynomial over F has a root in F. The diagram below contrasts algebraically closed fields (blue) with those that are not (red).



Which of the following fields is **NOT** algebraically closed?

(A) $\mathbb{C}$.

(B) $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$.

(C) $\overline{\mathbb{F}_p}$, the algebraic closure of $\mathbb{F}_p$.

(D) $\mathbb{R}$.

Q20. Let X and Y be topological spaces with projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$. The *product topology* on $X \times Y$ is characterised as:

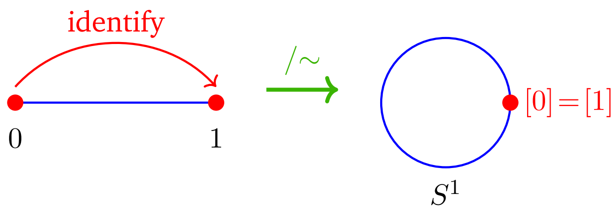
(A) The finest (largest) topology on $X \times Y$ making π_1 and π_2 continuous.

(B) The coarsest (smallest) topology on $X \times Y$ making π_1 and π_2 continuous.

(C) The discrete topology on $X \times Y$.

(D) The topology generated only by sets of the form $U \times V$ where both U and V are clopen.

Q21. Define the equivalence relation $\sim$ on $[0, 1]$ by $0 \sim 1$ (and $x \sim x$ for all x). The diagram illustrates gluing the two endpoints.



The quotient space $[0, 1]/\sim$ is homeomorphic to:

(A) The closed interval $[0, 1]$.

(B) The open interval $(0, 1)$.

(C) The unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

(D) The two-point discrete space $\{0, 1\}$.

Q22. Which of the following is the **correct statement** of the Baire Category Theorem for complete metric spaces?

(A) In a complete metric space (X, d), any countable union of nowhere dense sets has empty interior; equivalently, X cannot be expressed as a countable union of nowhere dense closed sets.

(B) Every metric space is of the second category in itself.

(C) Every open subset of a complete metric space is nowhere dense.

(D) Every complete metric space is separable (contains a countable dense subset).

Q23. Which of the following metric spaces is **NOT** complete?

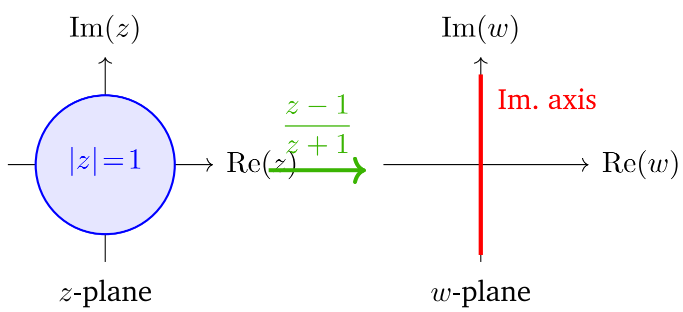
(A) $(\mathbb{R}, |\cdot|)$ with the standard Euclidean metric.

(B) $(\mathbb{Z}, d_{\text{disc}})$ with the discrete metric $d(m, n) = [m \neq n]$.

(C) $(C([0, 1]), \|\cdot\|_\infty)$, continuous functions with the sup-norm.

(D) $(\mathbb{Q}, |\cdot|)$ with the standard Euclidean metric.

Q24. The Möbius transformation $f(z) = \dfrac{z-1}{z+1}$ maps circles and lines in $\mathbb{C}$ to circles and lines. The diagram shows the transformation from the z-plane to the w-plane.



What is the image of the unit circle $|z| = 1$ under $f(z) = \dfrac{z-1}{z+1}$?

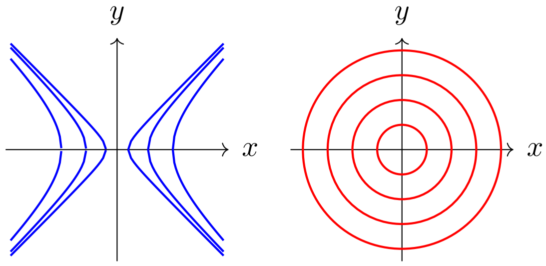
(A) The real axis $\{w \in \mathbb{C} : \mathrm{Im}(w) = 0\}$.

(B) The imaginary axis $\{w \in \mathbb{C} : \mathrm{Re}(w) = 0\}$.

(C) The unit circle $|w| = 1$.

(D) The upper half-plane $\{w \in \mathbb{C} : \mathrm{Im}(w) > 0\}$.

Q25. A function $u(x, y)$ is harmonic on a domain D if $\Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ throughout D. Level curves for $x^2 - y^2 = c$ (harmonic, left) and $x^2 + y^2 = c$ (right) are compared below.



$x^2 - y^2 = c$ (harmonic) $x^2 + y^2 = c$ (not harmonic)

Which of the following is **NOT** harmonic on all of $\mathbb{R}^2$?

(A) $u(x, y) = x^2 - y^2$

(B) $u(x, y) = e^x \cos y$

(C) $u(x, y) = x^2 + y^2$

(D) $u(x, y) = x$

Q26. A holomorphic function f on a domain D is *conformal* at $z_0 \in D$ if it preserves angles (magnitude and orientation) between smooth curves at z_0. Which of the following is the **correct characterisation**?

(A) f is conformal at z_0 if and only if $f'(z_0) \neq 0$.

(B) f is conformal at z_0 if and only if f is injective on D.

(C) Every holomorphic function is conformal everywhere on its domain.

(D) f is conformal at z_0 only if f is a Möbius transformation.

Q27. By the Hadamard factorization theorem, an entire function of finite order whose zero set is exactly $\{1, -1\}$ (each with multiplicity one) can be expressed as:

(A) $(z-1)(z+1)$ (a polynomial).

(B) e^{z^2-1} (no zeros at ± 1, since $e^w \neq 0$).

(C) $\sin(\pi z/2)$ (zeros at all even integers, not just ± 1).

(D) $(z^2-1)\, e^{g(z)}$ for some entire function $g(z)$.

Q28. Find all integers x satisfying simultaneously $x \equiv 2 \pmod 3$ and $x \equiv 3 \pmod 5$.

(A) $x \equiv 3 \pmod{15}$

(B) $x \equiv 8 \pmod{15}$

(C) $x \equiv 11 \pmod{15}$

(D) $x \equiv 14 \pmod{15}$

Q29. The Legendre symbol $\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod p$ determines whether a is a quadratic residue mod prime p. Note that the quadratic residues modulo 7 are $\{1^2, 2^2, 3^2\} \equiv \{1, 4, 2\} \pmod 7$. Which of the following is a **quadratic residue** modulo 7?

(A) 3

(B) 5

(C) 2

(D) 6

Q30. Let $X_1, \ldots, X_{100}$ be i.i.d. random variables with mean μ and $\mathrm{Var}(X_i) = 4$. Chebyshev's inequality states $P(|Z - \mathbb{E}[Z]| \geq \varepsilon) \leq \dfrac{\mathrm{Var}(Z)}{\varepsilon^2}$. Find an upper bound for $P\big(|\bar{X}_{100} - \mu| > 0.5\big)$, where $\bar{X}_{100} = \frac{1}{100}\sum_{i=1}^{100} X_i$.

(A) $\dfrac{4}{25}$

(B) $\dfrac{1}{5}$

(C) $\dfrac{4}{5}$

(D) $\dfrac{1}{25}$

Part B — True/False Questions (No Negative Marking)

Q31. State whether the following is **True** or **False**:

Every bounded sequence in $\mathbb{R}$ converges.

(A) True

(B) False

Q32. State whether the following is **True** or **False**:

The uniform limit of a sequence of uniformly continuous functions on a metric space is uniformly continuous.

(A) True

(B) False

Q33. State whether the following is **True** or **False**:

If $f_n \to f$ uniformly on $(0, 1)$ and each f_n is differentiable, then $f_n' \to f'$ pointwise on $(0, 1)$.

(A) True

(B) False

Q34. State whether the following is **True** or **False**:

Every Cauchy sequence in $\mathbb{R}$ is bounded.

(A) True

(B) False

Q35. State whether the following is **True** or **False**:

A monotone function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is continuous on $[a, b]$.

(A) True

(B) False

Q36. State whether the following is **True** or **False**:

The geometric multiplicity of any eigenvalue of a square matrix is always less than or equal to its algebraic multiplicity.

(A) True

(B) False

Q37. State whether the following is **True** or **False**:

If A is a real symmetric $n \times n$ matrix, then all its eigenvalues are positive.

(A) True

(B) False

Q38. State whether the following is **True** or **False**:

Every finite-dimensional inner product space over $\mathbb{R}$ or $\mathbb{C}$ has an orthonormal basis, which can be produced by the Gram-Schmidt process.

(A) True

(B) False

Q39. State whether the following is **True** or **False**:

If A and B are $n \times n$ matrices with $AB = 0$, then $A = 0$ or $B = 0$.

(A) True

(B) False

Q40. State whether the following is **True** or **False**:

Every subgroup of an abelian group is normal.

(A) True

(B) False

Q41. State whether the following is **True** or **False**:

Every quotient group of an infinite cyclic group is infinite.

(A) True

(B) False

Q42. State whether the following is **True** or **False**:

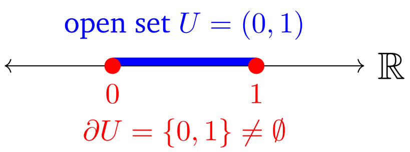
The ring of integers $\mathbb{Z}$ is a Principal Ideal Domain (PID).

(A) True

(B) False

Q43. State whether the following is **True** or **False**:

In a topological space, the boundary of every open set is empty.



(A) True

(B) False

Q44. State whether the following is **True** or **False**:

A compact subset of a Hausdorff topological space is closed.

(A) True

(B) False

Q45. State whether the following is **True** or **False**:

Every separable metric space is compact.

(A) True

(B) False

Q46. State whether the following is **True** or **False**:

If f is holomorphic on a connected domain $D \subset \mathbb{C}$ and $f'(z) = 0$ for all $z \in D$, then f is constant on D.

(A) True

(B) False

Q47. State whether the following is **True** or **False**:

A power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence $R > 0$ converges absolutely and uniformly on the entire closed disk $|z| \leq R$.

(A) True

(B) False

Q48. State whether the following is **True** or **False**:

If $\gcd(a, n) = 1$, then $a^{\phi(n)} \equiv 1 \pmod{n}$, where ϕ denotes Euler's totient function.

(A) True

(B) False

Q49. State whether the following is **True** or **False**:

Every prime $p > 2$ of the form $p = 4k + 3$ (for some non-negative integer k) can be expressed as a sum of two perfect squares.

(A) True

(B) False

Q50. State whether the following is **True** or **False**:

If X and Y are independent random variables with finite variance, then $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$.

(A) True

(B) False

Detailed Solutions

Q1.

Solution

Concept — Nested Interval Theorem: If $[a_1, b_1] \supset [a_2, b_2] \supset \cdots$ are closed bounded intervals in $\mathbb{R}$ with $b_n - a_n \to 0$, then $\bigcap_{n=1}^{\infty}[a_n, b_n]$ is a singleton.

Step 1 — Identify the unique point: Since the sequence $\{a_n\}$ is non-decreasing and bounded above by b_1, it converges to $p = \sup_n a_n$. Similarly $b_n \to p$. For every n, $a_n \le p \le b_n$, so $p \in [a_n, b_n]$ for all n, hence $p \in \bigcap[a_n, b_n]$.

Step 2 — Uniqueness: If $q \ne p$ also belonged to every $[a_n, b_n]$, then $|p - q| \le b_n - a_n \to 0$, contradicting $q \ne p$. So the intersection is exactly $\{p\}$.

Why other options are wrong:

- Option A: The intersection is non-empty by completeness of $\mathbb{R}$ (Cantor's theorem).
- Option B: Uniqueness forces exactly one point; infinitely many would violate $b_n - a_n \to 0$.
- Option D: If the intersection had positive length $\ell > 0$, then $b_n - a_n \ge \ell \not\to 0$, contradicting the hypothesis.

Final Answer: Exactly one point p with $a_n \le p \le b_n$ for all $n \Rightarrow \boxed{\text{C}}$

Answer: (C) Go Back to Q1

Q2.

Solution

Concept — Uniform Convergence Theorems: If $f_n \to f$ uniformly on $[a, b]$ and each f_n is continuous, then (i) f is continuous, and (ii) $\int_a^b f_n \to \int_a^b f$.

Step 1 — Continuity of f: Fix $x_0 \in [a, b]$ and $\varepsilon > 0$. Choose N so that $|f_n(x) - f(x)| < \varepsilon/3$ for all $n \ge N$ and all x. Since f_N is continuous, choose δ so $|f_N(x) - f_N(x_0)| < \varepsilon/3$ when $|x - x_0| < \delta$. Then $|f(x) - f(x_0)| \le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$.

Step 2 — Integral convergence: $\left|\int_a^b f_n - \int_a^b f\right| \le \int_a^b |f_n - f| \le (b-a)\sup_x |f_n(x) - f(x)| \to 0$ by uniform convergence.

Why other options are wrong:

- Option B: Uniform convergence does NOT imply convergence of derivatives. Example: $f_n(x) = \frac{\sin(n^2 x)}{n} \to 0$ uniformly, but $f_n'(x) = n\cos(n^2 x)$ does not converge to 0.

- Option C: Continuity of f IS guaranteed (not just the integral statement); C is incomplete.
- Option D: Derivative convergence fails in general (same example as B).

Final Answer: Both continuity and integral convergence hold $\Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q2

Q3.

Solution

Concept — Dini's Theorem and Compactness: Compactness of K is the hypothesis whose removal breaks uniform convergence.

Step 1 — Counterexample without compactness: Take $K = (0, 1)$ (open, not compact). Let $f_n(x) = x^n$. This sequence is monotone decreasing ($f_{n+1}(x) \leq f_n(x)$ for $x \in (0, 1)$), each f_n is continuous, and $f_n \to f \equiv 0$ pointwise, with f continuous. But the convergence is NOT uniform: $\sup_{x\in(0,1)} |x^n - 0| = \sup x^n \to 1 \neq 0$.

Step 2 — Verify other hypotheses hold: Monotonicity, continuity of f_n, and continuity of $f = 0$ all hold in this example; only compactness is missing.

Why other options are wrong:

- Option A: Dropping monotonicity: a counterexample exists, but the question asks for the counterexample $f_n = x^n$, which retains monotonicity.
- Option B: Each $f_n(x) = x^n$ is continuous; dropping this hypothesis is not what the example illustrates.
- Option C: The limit $f \equiv 0$ is continuous; dropping continuity of f is not what the example demonstrates.

Final Answer: Dropping compactness breaks uniform convergence $\Rightarrow \boxed{\text{D}}$

Answer: (D) Go Back to Q3

Q4.

Solution

Concept — Bernstein Polynomials: For continuous $f : [0,1] \to \mathbb{R}$, the Bernstein polynomials $B_n(f;x) = \sum_{k=0}^{n} f(k/n)\binom{n}{k}x^k(1-x)^{n-k}$ converge uniformly to f.

Step 1 — Why Bernstein polynomials work: Using the probabilistic interpretation: if $S_n \sim \mathrm{Bin}(n,x)$, then $B_n(f;x) = \mathbb{E}[f(S_n/n)]$. By the weak law of large numbers, $S_n/n \to x$ in probability, and uniform continuity of f on $[0,1]$ converts this to uniform convergence.

Step 2 — Constructive proof: For any $\varepsilon > 0$, uniform continuity gives $\delta > 0$; split $\mathbb{E}|f(S_n/n) - f(x)|$ into contributions from $|S_n/n - x| < \delta$ (small by continuity) and $|S_n/n - x| \geq \delta$ (small by Chebyshev), yielding $\|B_n(f) - f\|_\infty \to 0$.

Why other options are wrong:

- Option A: Fourier series of f need not converge uniformly (or even pointwise everywhere) for all continuous f.
- Option C: Taylor polynomials require f to be infinitely differentiable; they don't apply to general continuous f.
- Option D: Chebyshev interpolating polynomials converge for smooth f but can diverge for merely continuous f (Runge phenomenon).

Final Answer: Bernstein polynomials provide the explicit uniform approximation $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q4

Q5.

Solution

Concept — Fundamental Theorem of Calculus (Part 1): If f is continuous on $[0,1]$ and $F(x) = \int_0^x f(t)\,dt$, then $F'(x) = f(x)$ for all $x \in [0,1]$.

Step 1 — Derivative computation: By definition, $F'(x) = \lim_{h\to0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to0}\frac{1}{h}\int_x^{x+h} f(t)\,dt$. Since f is continuous, by the Mean Value Theorem for integrals there exists c_h between x and $x+h$ such that $\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c_h) \to f(x)$ as $h \to 0$.

Step 2 — Conclusion: Therefore $F'(x) = f(x)$. This is the accumulation function version of FTC.

Why other options are wrong:

- Option A: $F(x) - F(0) = F(x)$ (since $F(0) = 0$), which equals the accumulated value, not the derivative.
- Option B: $f'(x)$ is the derivative of f, not of F; this confuses the function with its antiderivative.
- Option D: $\int_0^x f'(t)\,dt = f(x) - f(0)$ by FTC Part 2, which does NOT equal $F'(x)$ in general.

Final Answer: $F'(x) = f(x)$ by FTC Part 1 $\Rightarrow \boxed{\text{C}}$

Answer: (C) Go Back to Q5

Q6.

Solution

Concept — Lagrange Remainder for e^x: The remainder after $n+1$ terms satisfies $|R_n(1)| \le \frac{e}{(n+1)!}$. We need $\frac{e}{(n+1)!} < 0.01$.

Step 1 — Test successive values:

$$n = 3: \quad \frac{e}{4!} = \frac{e}{24} \approx \frac{2.718}{24} \approx 0.1133 > 0.01.$$

$$n = 4: \quad \frac{e}{5!} = \frac{e}{120} \approx \frac{2.718}{120} \approx 0.0227 > 0.01.$$

$$n = 5: \quad \frac{e}{6!} = \frac{e}{720} \approx \frac{2.718}{720} \approx 0.00377 < 0.01. \checkmark$$

Step 2 — Conclusion: The minimum n for which the bound guarantees error below 0.01 is $n = 5$ (using terms $k = 0, 1, 2, 3, 4, 5$).

Why other options are wrong:

- Option B: $n = 3$ gives $e/24 \approx 0.113 \gg 0.01$; insufficient.
- Option C: $n = 7$ would work, but the minimum is $n = 5$; $n = 7$ is unnecessarily large.
- Option D: $n = 10$ is far more than required.

Final Answer: Minimum $n = 5$ guarantees $|R_n(1)| < 0.01 \Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q6

Q7.

Solution

Concept — Dirichlet's Test for Series: If $\{a_n\}$ is monotone decreasing to 0 and the partial sums of $\{b_n\}$ are bounded, then $\sum a_n b_n$ converges.

Step 1 — Identify the components: Write $\sum \frac{\sin n}{n} = \sum a_n b_n$ with $a_n = \frac{1}{n}$ and $b_n = \sin n$. Check: $a_n = 1/n \searrow 0$ monotonically. The partial sums $B_N = \sum_{k=1}^{N} \sin k$ are bounded: using the formula $|B_N| = \left|\frac{\sin\frac{N+1}{2}\sin\frac{N}{2}}{\sin\frac{1}{2}}\right| \le \frac{1}{|\sin(1/2)|} < \frac{1}{0.479} \approx 2.09$.

Step 2 — Apply Dirichlet's test: Since $a_n \searrow 0$ and $|B_N|$ is bounded, Dirichlet's test guarantees convergence of $\sum \frac{\sin n}{n}$.

Why other options are wrong:

- Option A: Ratio test: $|a_{n+1}/a_n| = \frac{n|\sin(n+1)|}{(n+1)|\sin n|}$ is not well-defined for all n (denominator vanishes) and tends to 1; inconclusive.
- Option B: Root test: $|a_n|^{1/n} = |\sin n|^{1/n}/n^{1/n} \to 1$ (when $\sin n \neq 0$); inconclusive.
- Option C: Comparison with $\sum 1/n$ fails since $\sum 1/n$ diverges; one cannot compare to a divergent series to prove convergence.

Final Answer: Dirichlet's test directly applies $\Rightarrow \boxed{\text{D}}$

Answer: (D) Go Back to Q7

Q8.

Solution

Concept — Absolute Continuity vs. Uniform Continuity: A function $f : [a,b] \to \mathbb{R}$ is absolutely continuous (AC) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum |f(b_k) - f(a_k)| < \varepsilon$ whenever $\sum(b_k - a_k) < \delta$. AC implies: f is differentiable a.e., and $f(x) - f(a) = \int_a^x f'(t)\,dt$.

Step 1 — The Cantor function fails AC: If the Cantor function f were absolutely continuous, then since $f' = 0$ a.e., we would have $f(1) - f(0) = \int_0^1 f'(t)\,dt = 0$, contradicting $f(1) - f(0) = 1$. Therefore f is NOT absolutely continuous.

Step 2 — The other properties hold: (A) Uniform continuity: f is continuous on the compact set $[0,1]$, hence uniformly continuous. (C) Riemann integrability: f is monotone on $[0,1]$, hence Riemann integrable. (D) Boundedness: $f : [0,1] \to [0,1]$ is bounded.

Why other options are wrong:

- Option A: Uniform continuity holds (continuous function on compact set).
- Option C: All monotone functions on $[a, b]$ are Riemann integrable (only countably many discontinuities).
- Option D: f maps into $[0, 1]$, so it is trivially bounded.

Final Answer: Cantor function is NOT absolutely continuous $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q8

Q9.

Solution

Concept — Minimal Polynomial: The minimal polynomial $m_A(x)$ of a matrix A is the monic polynomial of least degree such that $m_A(A) = 0$. It divides the characteristic polynomial $p_A(x)$ and has the same roots (over $\mathbb{C}$).

Step 1 — Constraints on $m_A(x)$: With $p_A(x) = (x-1)^2(x-2)$, the minimal polynomial must (i) divide $(x-1)^2(x-2)$, and (ii) be divisible by $(x-1)$ and $(x-2)$. The candidates are $(x-1)(x-2)$ and $(x-1)^2(x-2)$.

Step 2 — When is $(x-1)(x-2)$ possible? If the Jordan form of A has eigenvalue 1 appearing in two 1×1 Jordan blocks (geometric multiplicity $= 2$), then $(A - I)(A - 2I) = 0$, so $m_A(x) = (x-1)(x-2)$. This is consistent with the given characteristic polynomial.

Why other options are wrong:

- Option A: $(x-2)$ does not have 1 as a root, but 1 is an eigenvalue (it divides p_A), so $(A - 2I) \neq 0$ in general.
- Option B: $(x-1)$ does not have 2 as a root, but 2 is an eigenvalue, so $(A-I) \neq 0$ in general.
- Option D: $(x-1)^3(x-2)$ does not divide $(x-1)^2(x-2)$ (degree of factor $(x-1)$ exceeds that in p_A); impossible as minimal polynomial.

Final Answer: $(x-1)(x-2)$ is a valid minimal polynomial $\Rightarrow \boxed{\text{C}}$

Answer: (C) Go Back to Q9

Q10.

Solution

Concept — Cayley-Hamilton Theorem: Every square matrix satisfies its own characteristic polynomial: $p_A(A) = 0$.

Step 1 — Apply Cayley-Hamilton: $A^2 - 5A - 2I = 0$.

Step 2 — Solve for A^{-1}: Multiply both sides on the right by A^{-1} (noting $\det A = 4 - 6 = -2 \neq 0$, so A^{-1} exists):

$$A - 5I - 2A^{-1} = 0 \implies 2A^{-1} = A - 5I \implies A^{-1} = \tfrac{1}{2}(A - 5I).$$

Verification: $A - 5I = \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}$, so $\frac{1}{2}(A-5I) = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$. Direct computation: $A^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$. ✓

Why other options are wrong:

- Option B: $\frac{1}{5}(A^2 - 2I)$: compute $A^2 = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$, $A^2 - 2I = \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix}$, divided by 5 gives $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = A$, not A^{-1}.
- Option C: $\frac{1}{2}(5I - A) = \begin{pmatrix} 2 & -1 \\ -3/2 & 1/2 \end{pmatrix} = -A^{-1}$; wrong sign.
- Option D: $A - 5I = \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix} \neq A^{-1}$ (off by factor of 2).

Final Answer: $A^{-1} = \frac{1}{2}(A - 5I) \Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q10

Q11.

Solution

Concept — Orthogonal Matrices: For $A^TA = I_n$: (1) A is invertible with $A^{-1} = A^T$; (2) $\det A = \pm 1$; (3) complex eigenvalues satisfy $|\lambda| = 1$.

Step 1 — Verify Option D: From $A^TA = I$, we get $A^{-1} = A^T$ by definition of the matrix inverse. This is the defining property of an orthogonal matrix and holds for every such matrix.

Step 2 — Confirm it is always true: Left-multiplying $A^TA = I$ by $(A^T)^{-1} = A$: $A \cdot A^TA = A$, giving $A^TA = I$ confirms A^T is the left-inverse, hence the inverse.

Why other options are wrong:

- Option A: $\det A = \pm 1$; the matrix $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is orthogonal with $\det A = -1 \neq 1$.
- Option B: Eigenvalues of a real orthogonal matrix lie on $|\lambda| = 1$ in $\mathbb{C}$, so they can be non-real (e.g., $e^{i\pi/4}$) or negative real (e.g., -1); not necessarily positive.
- Option C: $A^T = A$ (symmetry) is not required; e.g., any rotation matrix R_θ with $\theta \neq 0, \pi$ is orthogonal but not symmetric.

Final Answer: $A^{-1} = A^T$ is always true for orthogonal $A \Rightarrow \boxed{\text{D}}$

Answer: (D) **Go Back to Q11**

Q12.

Solution

Concept — Nilpotent Index: The smallest k with $N^k = 0$ is called the nilpotency index. For an $n \times n$ Jordan block $J_n(0)$ with 0 on the diagonal and 1 on the superdiagonal, $J_n(0)^n = 0$ but $J_n(0)^{n-1} \neq 0$.

Step 1 — Compute powers of N:

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad N^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

Step 2 — Confirm $N^2 \neq 0$: The $(1,3)$ entry of N^2 is $1 \neq 0$, so $k = 2$ is not sufficient.

Why other options are wrong:

- Option A: $N^1 = N \neq 0$ (the $(1,2)$ entry is 1).
- Option C: $N^2 \neq 0$ (the $(1,3)$ entry is 1), so $k = 2$ does not give the zero matrix.
- Option D: $N^3 = 0$ already, so $k = 4$ is not the minimum.

Final Answer: Smallest k with $N^k = 0$ is $k = 3 \Rightarrow \boxed{\text{B}}$

Answer: (B) **Go Back to Q12**

Q13.

Solution

Concept — Eigenvalues of A and A^T: The characteristic polynomials of A and A^T are identical: $\det(A - \lambda I) = \det((A - \lambda I)^T) = \det(A^T - \lambda I)$, since $\det(M) = \det(M^T)$ for any matrix M.

Step 1 — Apply to A^T: Since $\det(A - \lambda I) = \det(A^T - \lambda I)$, the matrices A and A^T have the same characteristic polynomial and hence the same eigenvalues (with multiplicities).

Step 2 — Confirm no other matrix in the list works universally: The eigenvalue map differs for all other choices below.

Why other options are wrong:

- Option A: Eigenvalues of A^{-1} are $\{1/\lambda_i\}$, not $\{\lambda_i\}$ in general (e.g., $\lambda = 2 \implies 1/\lambda = 1/2$).
- Option B: Eigenvalues of $2A$ are $\{2\lambda_i\}$, scaled by 2.
- Option D: Eigenvalues of $A + I$ are $\{\lambda_i + 1\}$, shifted by 1.

Final Answer: A^T has the same eigenvalues as $A \Rightarrow \boxed{\text{C}}$

Answer: (C) Go Back to Q13

Q14.

Solution

Concept — Cyclic Property of Trace: $\operatorname{tr}(XY) = \operatorname{tr}(YX)$ for any compatible matrices X, Y.

Step 1 — Compute $\operatorname{tr}(ABA^{-1})$: Let $X = AB$ and $Y = A^{-1}$. Then

$$\operatorname{tr}(ABA^{-1}) = \operatorname{tr}((AB)A^{-1}) = \operatorname{tr}(A^{-1}(AB)) = \operatorname{tr}((A^{-1}A)B) = \operatorname{tr}(IB) = \operatorname{tr}(B).$$

Step 2 — Geometric meaning: tr is invariant under similarity transformation, so conjugate matrices have the same trace.

Why other options are wrong:

- Option B: $\operatorname{tr}(A^2B) = \sum_{i,j,k} A_{ij}A_{jk}B_{ki}$; this is not equal to $(\operatorname{tr} B)^2$ in general.
- Option C: Trace is not multiplicative: $\operatorname{tr}(AB) \neq \operatorname{tr}(A) \cdot \operatorname{tr}(B)$ in general (e.g., $A = B = I_n$ gives $n \neq n^2$ for $n > 1$).
- Option D: The identity $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$ (linearity), not a product.

Final Answer: $\mathrm{tr}(ABA^{-1}) = \mathrm{tr}(B)$ by the cyclic property $\Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q14

Q15.

Solution

Concept — First Isomorphism Theorem: If $\phi : G \to H$ is a group homomorphism, then $G/\ker\phi \cong \mathrm{Im}\,\phi$.

Step 1 — Identify $\ker\phi$**:** $\ker\phi = \{n \in \mathbb{Z} : \phi(n) = 0\} = \{n \in \mathbb{Z} : 6 \mid n\} = 6\mathbb{Z}$.

Step 2 — Identify $\mathrm{Im}\,\phi$**:** ϕ is surjective (every element $[k]_6 \in \mathbb{Z}/6\mathbb{Z}$ equals $\phi(k)$), so $\mathrm{Im}\,\phi = \mathbb{Z}/6\mathbb{Z}$.

Step 3 — Apply FIT: $\mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/\ker\phi \cong \mathrm{Im}\,\phi \cong \mathbb{Z}/6\mathbb{Z}$. ✓

Why other options are wrong:

- Option A: $\ker\phi = \mathbb{Z}$ would mean ϕ is the zero map; but $\phi(1) = [1]_6 \neq 0$.
- Option B: $\ker\phi = \{0\}$ would mean ϕ is injective; but $\phi(6) = 0 = \phi(0)$, so $6 \neq 0$ is in the kernel.
- Option C: $\ker\phi = 2\mathbb{Z}$ would mean $\phi(2) = 0$, but $\phi(2) = [2]_6 \neq 0$.

Final Answer: $\ker\phi = 6\mathbb{Z}$ and $\mathrm{Im}\,\phi \cong \mathbb{Z}/6\mathbb{Z} \Rightarrow \boxed{\text{D}}$

Answer: (D) Go Back to Q15

Q16.

Solution

Concept — Orbit-Stabilizer Theorem: For a group G acting on a set X and any $x \in X$: $|G| = |\mathrm{Orb}(x)| \cdot |G_x|$.

Step 1 — Apply the formula: $|G| = 24$, $|G_x| = 4$. Therefore:

$$|\mathrm{Orb}(x)| = \frac{|G|}{|G_x|} = \frac{24}{4} = 6.$$

Step 2 — Interpretation: The orbit of x under G consists of exactly 6 elements of X, corresponding to the 6 distinct cosets of G_x in G.

Why other options are wrong:

- Option A: $4 = |G_x|$; this is the size of the stabilizer, not the orbit.

- Option C: $8 = 24/3$; this would require $|G_x| = 3$, but $|G_x| = 4$.
- Option D: $24 = |G|$; this would require a trivial stabilizer $|G_x| = 1$, but $|G_x| = 4$.

Final Answer: $|\operatorname{Orb}(x)| = 24/4 = 6 \Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q16

Q17.

Solution

Concept — $\mathbb{Z}[x]$ is a UFD but not a PID: The ring $\mathbb{Z}[x]$ satisfies unique factorization (since $\mathbb{Z}$ is a UFD and R UFD $\Rightarrow$ $R[x]$ UFD), but it fails to be a PID.

Step 1 — $\mathbb{Z}[x]$ is a UFD: Since $\mathbb{Z}$ is a UFD, the Gauss Lemma and unique factorization extend to $\mathbb{Z}[x]$.

Step 2 — $\mathbb{Z}[x]$ is NOT a PID: The ideal $(2, x) = \{2f(x) + x\,g(x) : f, g \in \mathbb{Z}[x]\}$ is not principal. If $(2, x) = (h(x))$ for some $h \in \mathbb{Z}[x]$, then $h \mid 2$ (so $h \in \{\pm 1, \pm 2\}$) and $h \mid x$ (so h is a unit). But if $h = \pm 1$, then $(h) = \mathbb{Z}[x]$, meaning $1 \in (2, x)$; since elements of $(2, x)$ at $x = 0$ are multiples of 2, we get $1 = 2m$ for some integer m: contradiction.

Why other options are wrong:

- Option A: Incorrectly claims $\mathbb{Z}[x]$ is a PID (the ideal $(2, x)$ refutes this).
- Option B: $\mathbb{Z}[x]$ is not a field; 2 is not invertible in $\mathbb{Z}[x]$.
- Option D: Equivalent to A (every ideal principal $\Leftrightarrow$ PID), which is false.

Final Answer: $\mathbb{Z}[x]$ is a UFD but not a PID $\Rightarrow \boxed{\text{C}}$

Answer: (C) Go Back to Q17

Q18.

Solution

Concept — Chinese Remainder Theorem (CRT) for Rings: $\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ if and only if $\gcd(m, n) = 1$.

Step 1 — Check the condition: $m = 4$, $n = 9$, $mn = 36$. Since $\gcd(4, 9) = 1$, the CRT applies.

Step 2 — State the isomorphism: $\mathbb{Z}/36\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$. The isomorphism sends $[a]_{36} \mapsto ([a]_4, [a]_9)$.

Why other options are wrong:

- Option B: $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ has order 36 but is not cyclic (contains elements of order at most 6); it is not isomorphic to $\mathbb{Z}/36\mathbb{Z}$ (which is cyclic). Also $\gcd(6,6) = 6 \neq 1$.
- Option C: $12 \times 3 = 36$ but $\gcd(12,3) = 3 \neq 1$, so CRT does not apply; moreover $\mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ has elements of order at most $\mathrm{lcm}(12,3) = 12 < 36$.
- Option D: $\gcd(2,18) = 2 \neq 1$, so CRT does not apply here.

Final Answer: $\mathbb{Z}/36\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$ by CRT $\Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q18

Q19.

Solution

Concept — Algebraic Closure: F is algebraically closed iff every non-constant $f \in F[x]$ has a root in F (equivalently, the only irreducible polynomials in $F[x]$ are linear).

Step 1 — Identify the non-algebraically-closed field: Consider $p(x) = x^2 + 1 \in \mathbb{R}[x]$. Its discriminant is $-4 < 0$, so p has no real roots. Hence $\mathbb{R}$ is NOT algebraically closed.

Step 2 — Verify the others: $\mathbb{C}$: algebraically closed by the Fundamental Theorem of Algebra. $\overline{\mathbb{Q}}$: constructed as the algebraic closure of $\mathbb{Q}$, so algebraically closed by definition. $\overline{\mathbb{F}_p}$: similarly constructed as the algebraic closure of $\mathbb{F}_p$; algebraically closed.

Why other options are wrong:

- Option A: $\mathbb{C}$ is algebraically closed (Fundamental Theorem of Algebra).
- Option B: $\overline{\mathbb{Q}}$ is algebraically closed by construction.
- Option C: $\overline{\mathbb{F}_p}$ is algebraically closed by construction.

Final Answer: $\mathbb{R}$ is not algebraically closed ($x^2 + 1$ has no root) $\Rightarrow \boxed{\text{D}}$

Answer: (D) Go Back to Q19

Q20.

Solution

Concept — Product Topology via Universal Property: The product topology is the *initial topology* with respect to the projections; it is the *coarsest* topology making all projections continuous.

Step 1 — Coarsest, not finest: A topology $\mathcal{T}$ on $X \times Y$ makes π_1 and π_2 continuous iff it contains every set of the form $\pi_1^{-1}(U) = U \times Y$ (U open in X) and $\pi_2^{-1}(V) = X \times V$ (V open in Y). The *smallest* such topology is generated by these sub-basic sets; this is exactly the product topology.

Step 2 — Key consequence: The product topology has a sub-basis $\{U \times Y : U \text{ open in } X\} \cup \{X \times V : V \text{ open in } Y\}$, and a basis of sets $U \times V$ for open $U \subset X$, $V \subset Y$.

Why other options are wrong:

- Option A: The finest topology making π_1, π_2 continuous would be the discrete topology (which makes ALL maps continuous), far larger than needed.
- Option C: The discrete topology is in general strictly finer than the product topology.
- Option D: Restricting to clopen sets U, V is unnecessarily restrictive and mischaracterises the sub-basis.

Final Answer: Product topology is the coarsest making projections continuous $\Rightarrow$ $\boxed{\text{B}}$

Answer: (B) Go Back to Q20

Q21.

Solution

Concept — Quotient Space Homeomorphism: The quotient $[0,1]/\{0 \sim 1\}$ is homeomorphic to S^1 via the map $[t] \mapsto e^{2\pi i t}$.

Step 1 — Define the map: Let $f : [0,1] \to S^1$ be $f(t) = (\cos 2\pi t, \sin 2\pi t)$. Then f is continuous, surjective, and $f(0) = f(1) = (1,0)$, so f factors through the quotient: $\tilde{f} : [0,1]/\sim \to S^1$ by $\tilde{f}([t]) = f(t)$.

Step 2 — $\tilde{f}$ is a homeomorphism: $\tilde{f}$ is continuous (by the universal property of quotient spaces), bijective (injectivity: $f(t_1) = f(t_2)$ with $t_1, t_2 \in [0,1]$ implies $t_1 = t_2$ or $\{t_1, t_2\} = \{0, 1\}$, but $[0] = [1]$ in the quotient), and a homeomorphism since $[0,1]/\sim$ is compact and S^1 is Hausdorff.

Why other options are wrong:

- Option A: $[0, 1]$ is contractible; S^1 is not. Identifying endpoints strictly changes the topology.
- Option B: $(0, 1)$ is homeomorphic to $\mathbb{R}$, not to $[0, 1]/\sim$ (which is compact).
- Option D: $\{0, 1\}$ is a two-point discrete space; the quotient $[0, 1]/\sim$ is connected, not discrete.

Final Answer: $[0, 1]/\sim \ \cong S^1 \Rightarrow \boxed{\text{C}}$

Answer: (C) Go Back to Q21

Q22.

Solution

Concept — Baire Category Theorem (BCT): In a complete metric space (X, d), any countable intersection of dense open sets is dense; equivalently, X is not a countable union of nowhere dense sets (i.e., X is of the second category).

Step 1 — Statement A is correct: "Any countable union of nowhere dense sets has empty interior in (X, d)." This is the standard formulation of BCT: if A_n is nowhere dense (its closure has empty interior) for each n, then $\mathrm{int}(\bigcup_n A_n) = \emptyset$.

Step 2 — Application: For example, $\mathbb{Q}$ is a countable union of singletons $\{q\}$ (each nowhere dense in $\mathbb{R}$), and indeed $\mathbb{Q}$ has empty interior in $\mathbb{R}$.

Why other options are wrong:

- Option B: An incomplete metric space can be of the first category (e.g., $\mathbb{Q} = \bigcup_{q\in\mathbb{Q}}\{q\}$); BCT requires completeness.
- Option C: Open sets in a complete metric space are generally NOT nowhere dense; they have non-empty interior by definition.
- Option D: Complete metric spaces need not be separable; ℓ^∞ with the sup norm is complete but not separable.

Final Answer: BCT: countable union of nowhere dense sets has empty interior $\Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q22

Q23.

Solution

Concept — Completeness of Metric Spaces: A metric space is complete if every Cauchy sequence converges within the space.

Step 1 — $(\mathbb{Q}, |\cdot|)$ is not complete: The sequence $a_n = \sum_{k=0}^{n} \frac{1}{k!}$ is a Cauchy sequence in $\mathbb{Q}$ (it converges in $\mathbb{R}$), but its limit $e \notin \mathbb{Q}$. Alternatively, the sequence $a_1 = 1,\ a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$ (Newton's method for $\sqrt{2}$) converges to $\sqrt{2} \notin \mathbb{Q}$.

Step 2 — Verify completeness of the others: (A) $\mathbb{R}$: complete by construction. (B) $\mathbb{Z}$ with discrete metric: every Cauchy sequence is eventually constant, hence convergent. (C) $C([0,1])$ with $\|\cdot\|_\infty$: complete by the theorem that uniform Cauchy sequences of continuous functions converge uniformly to a continuous limit.

Why other options are wrong:

- Option A: $\mathbb{R}$ is the completion of $\mathbb{Q}$; it is complete.
- Option B: In a discrete metric space, $d(a_m, a_n) < 1 \Rightarrow a_m = a_n$ for large m, n; sequences stabilise.
- Option C: $(C([0,1]), \|\cdot\|_\infty)$ is a Banach space, hence complete.

Final Answer: $(\mathbb{Q}, |\cdot|)$ is not complete $\Rightarrow \boxed{\text{D}}$

Answer: (D) **Go Back to Q23**

Q24.

Solution

Concept — Möbius Transformation and Circle Images: A Möbius transformation maps circles (and lines) to circles (and lines). To find the image of $|z| = 1$, parametrise and compute.

Step 1 — Parametrize: Let $z = e^{i\theta}$ with $\theta \in [0, 2\pi) \setminus \{\pi\}$. Compute:

$$f(e^{i\theta}) = \frac{e^{i\theta} - 1}{e^{i\theta} + 1} = \frac{e^{i\theta/2}(e^{i\theta/2} - e^{-i\theta/2})}{e^{i\theta/2}(e^{i\theta/2} + e^{-i\theta/2})} = \frac{2i\sin(\theta/2)}{2\cos(\theta/2)} = i\tan\left(\frac{\theta}{2}\right).$$

Step 2 — Identify the image: For $\theta \in (0, 2\pi) \setminus \{\pi\}$, $i\tan(\theta/2)$ is purely imaginary $(\mathrm{Re} = 0)$ and ranges over all of $i\mathbb{R}$ as θ varies (with $z = -1$ being the pole, mapping to ∞). Hence the image is the imaginary axis $\{\mathrm{Re}(w) = 0\}$.

Why other options are wrong:

- Option A: The image has $\mathrm{Re}(w) = 0$, not $\mathrm{Im}(w) = 0$; it is the imaginary, not real, axis.

- Option C: $|f(e^{i\theta})| = |\tan(\theta/2)|$, which is not 1 in general; e.g., $\theta = \pi/2 \Rightarrow |f| = 1$ but $\theta = \pi/3 \Rightarrow |f| = \tan(\pi/6) = 1/\sqrt{3} \neq 1$.
- Option D: The image is the entire imaginary axis (a line), not a half-plane.

Final Answer: Image of $|z| = 1$ is the imaginary axis $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q24

Q25.

Solution

Concept — Harmonic Functions: $u(x, y)$ is harmonic on D iff $\Delta u = u_{xx} + u_{yy} = 0$ on D.

Step 1 — Test each option:

- Option A: $u = x^2 - y^2$: $u_{xx} = 2$, $u_{yy} = -2$, $\Delta u = 0$. *Harmonic.*
- Option B: $u = e^x \cos y$: Real part of e^z, holomorphic $\Rightarrow$ harmonic. ($u_{xx} = e^x \cos y = u$, $u_{yy} = -e^x \cos y = -u$, sum $= 0$.) *Harmonic.*
- Option C: $u = x^2 + y^2$: $u_{xx} = 2$, $u_{yy} = 2$, $\Delta u = 4 \neq 0$. *Not harmonic.*
- Option D: $u = x$: $u_{xx} = 0$, $u_{yy} = 0$, $\Delta u = 0$. *Harmonic.*

Step 2 — Conclusion: Only $u = x^2 + y^2$ fails the Laplace equation.

Why other options are wrong:

- Option A: $\Delta(x^2 - y^2) = 2 + (-2) = 0$; harmonic.
- Option B: $e^x \cos y = \mathrm{Re}(e^z)$ is harmonic as the real part of an entire function.
- Option D: $\Delta(x) = 0$; harmonic (in fact it is the real part of z).

Final Answer: $x^2 + y^2$ is not harmonic ($\Delta = 4 \neq 0$) $\Rightarrow \boxed{\text{C}}$

Answer: (C) Go Back to Q25

Q26.

Solution

Concept — Conformality and Non-Vanishing Derivative: A holomorphic map f is conformal at z_0 (preserves angles) if and only if $f'(z_0) \neq 0$.

Step 1 — Why $f'(z_0) \neq 0$ is the condition: The local behavior of f near z_0 is $f(z) \approx f(z_0) + f'(z_0)(z - z_0)$. If $f'(z_0) \neq 0$, then this linear approximation is a rotation-and-scaling (by $|f'(z_0)|$ and $\arg f'(z_0)$), which preserves angles. If $f'(z_0) = 0$, angles are multiplied by the order of vanishing; e.g., $f(z) = z^2$ at $z = 0$ doubles angles.

Step 2 — Example of non-conformality: $f(z) = z^2$, $f'(0) = 0$. Two rays from 0 at angle θ are mapped to rays at angle 2θ: the angle is doubled, not preserved. So f is not conformal at 0.

Why other options are wrong:

- Option B: Injectivity on D implies $f' \neq 0$ everywhere (by the open mapping theorem), but the condition for conformality at a single point is local: $f'(z_0) \neq 0$.
- Option C: $f(z) = z^2$ is holomorphic everywhere but not conformal at $z = 0$ (where $f' = 0$).
- Option D: Many non-Möbius holomorphic functions (e.g., e^z, $\sin z$) are conformal at points where their derivatives are non-zero.

Final Answer: Conformal at z_0 iff $f'(z_0) \neq 0 \Rightarrow \boxed{\text{A}}$

Answer: (A) **Go Back to Q26**

Q27.

Solution

Concept — Hadamard Factorization: An entire function f of finite order with zero set $\{a_n\}$ (each of multiplicity 1) can be written as $f(z) = e^{P(z)} \prod_n E_p(z/a_n)$, where P is a polynomial and E_p is a Weierstrass factor. For a finite zero set $\{1, -1\}$, the product is finite.

Step 1 — Build the factorization: An entire function with simple zeros at ± 1 and no other zeros must have the form $f(z) = (z-1)(z+1)\, e^{g(z)} = (z^2 - 1)\, e^{g(z)}$ for some entire function g.

Step 2 — The factor $e^{g(z)}$ is non-vanishing: Since $e^w \neq 0$ for any $w \in \mathbb{C}$, multiplying by $e^{g(z)}$ introduces no additional zeros. The polynomial $z^2 - 1$ accounts for the prescribed zeros.

Why other options are wrong:

- Option A: $(z-1)(z+1) = z^2 - 1$ is a valid choice (take $g = 0$), but option D is the general answer that includes all such entire functions.
- Option B: e^{z^2-1} is never zero (entire with no zeros), so it cannot have zeros at ± 1.
- Option C: $\sin(\pi z/2)$ has zeros at all even integers $0, \pm 2, \pm 4, \ldots$, not just ± 1.

Final Answer: $(z^2 - 1)e^{g(z)}$ for entire g is the general form $\Rightarrow \boxed{\text{D}}$

Answer: (D) Go Back to Q27

Q28.

Solution

Concept — Chinese Remainder Theorem for Integers: Solve $x \equiv a \pmod{m}$, $x \equiv b \pmod{n}$ when $\gcd(m, n) = 1$ by substitution.

Step 1 — Substitute from the first congruence: $x \equiv 2 \pmod{3} \Rightarrow x = 3k + 2$ for some integer k.

Step 2 — Substitute into the second:

$$3k + 2 \equiv 3 \pmod{5} \implies 3k \equiv 1 \pmod{5}.$$

Find the inverse of 3 mod 5: $3 \cdot 2 = 6 \equiv 1 \pmod{5}$, so $3^{-1} \equiv 2 \pmod{5}$. Thus $k \equiv 2 \pmod{5}$, i.e., $k = 5j + 2$.

Step 3 — Reconstruct x: $x = 3(5j + 2) + 2 = 15j + 8$. So $x \equiv 8 \pmod{15}$.

Verification: $8 \div 3 = 2$ remainder $2\checkmark$; $8 \div 5 = 1$ remainder $3\checkmark$.

Why other options are wrong:

- Option A: $3 \bmod 3 = 0 \neq 2$; fails the first congruence.
- Option C: $11 \bmod 3 = 2\checkmark$, but $11 \bmod 5 = 1 \neq 3$; fails the second.
- Option D: $14 \bmod 3 = 2\checkmark$, but $14 \bmod 5 = 4 \neq 3$; fails the second.

Final Answer: $x \equiv 8 \pmod{15} \Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q28

Q29.

Solution

Concept — Quadratic Residues mod p: a is a quadratic residue (QR) mod prime p iff $a \not\equiv 0$ and $a^{(p-1)/2} \equiv 1 \pmod{p}$ (Euler's criterion).

Step 1 — List QRs mod 7: The non-zero squares mod 7 are $1^2 = 1,\ 2^2 = 4,\ 3^2 = 9 \equiv 2,\ 4^2 = 16 \equiv 2,\ 5^2 = 25 \equiv 4,\ 6^2 = 36 \equiv 1 \pmod{7}$. So QR mod $7 = \{1, 2, 4\}$.

Step 2 — Check option C: 2 appears in the list (since $3^2 \equiv 2 \pmod{7}$), so 2 is a QR mod 7. Confirm via Euler: $2^{(7-1)/2} = 2^3 = 8 \equiv 1 \pmod{7}$. ✓

Why other options are wrong:

- Option A: $3^{(7-1)/2} = 3^3 = 27 \equiv 6 \equiv -1 \pmod{7}$, so 3 is a non-residue mod 7.
- Option B: $5^3 = 125 \equiv 125 - 17 \cdot 7 = 125 - 119 = 6 \equiv -1 \pmod{7}$, so 5 is a non-residue.
- Option D: $6^3 = 216 \equiv 216 - 30 \cdot 7 = 216 - 210 = 6 \equiv -1 \pmod{7}$, so 6 is a non-residue.

Final Answer: 2 is a quadratic residue mod $7 \Rightarrow \boxed{\text{C}}$

Answer: (C) Go Back to Q29

Q30.

Solution

Concept — Chebyshev's Inequality and the WLLN: $P(|Z - \mathbb{E}[Z]| \geq \varepsilon) \leq \frac{\mathrm{Var}(Z)}{\varepsilon^2}$.

Step 1 — Compute $\mathrm{Var}(\bar{X}_{100})$: Since $X_1, \ldots, X_{100}$ are i.i.d. with $\mathrm{Var}(X_i) = 4$:

$$\mathrm{Var}(\bar{X}_{100}) = \mathrm{Var}\left(\frac{1}{100}\sum_{i=1}^{100} X_i\right) = \frac{\mathrm{Var}(X_i)}{100} = \frac{4}{100} = \frac{1}{25}.$$

Step 2 — Apply Chebyshev with $\varepsilon = 0.5 = 1/2$:

$$P(|\bar{X}_{100} - \mu| > 0.5) \leq \frac{\mathrm{Var}(\bar{X}_{100})}{(0.5)^2} = \frac{1/25}{1/4} = \frac{4}{25}.$$

Why other options are wrong:

- Option B: $1/5 = 0.2$; this would come from $\varepsilon^2 = 4/100/(1/5) = 1/5$, but $\varepsilon = 0.5 \Rightarrow \varepsilon^2 = 0.25$.
- Option C: $4/5 = 0.8$; much too large; this ignores dividing variance by $n =$

100.

- Option D: $1/25 = 0.04$; this equals $\mathrm{Var}(\bar{X}_{100})$ itself, missing the division by $\varepsilon^2 = 1/4$.

Final Answer: Chebyshev bound is $\frac{4}{25} \Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q30

Q31.

Solution

Concept — Bounded vs. Convergent Sequences: Boundedness is necessary but not sufficient for convergence.

Counterexample (False): The sequence $a_n = (-1)^n$ satisfies $|a_n| = 1$ for all n (bounded), but oscillates between 1 and -1 and hence does not converge.

Correct theorem: By the Bolzano-Weierstrass theorem, every bounded sequence in $\mathbb{R}$ has a *convergent subsequence*, but not necessarily the full sequence.

Why True (option A) is wrong:

- Option A: Boundedness does not guarantee a unique limit; the sequence $(-1)^n$ is a standard counterexample.

Final Answer: False — $a_n = (-1)^n$ is bounded but not convergent $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q31

Q32.

Solution

Concept — Uniform Continuity under Uniform Limits: The uniform limit of uniformly continuous functions is uniformly continuous.

Proof sketch (True): Let $f_n \to f$ uniformly on (X, d) with each f_n uniformly continuous. Given $\varepsilon > 0$: choose N so $d_Y(f_N(x), f(x)) < \varepsilon/3$ for all x. Since f_N is uniformly continuous, choose $\delta > 0$ so $d(x, y) < \delta \Rightarrow d_Y(f_N(x), f_N(y)) < \varepsilon/3$. Then for $d(x, y) < \delta$:

$$d_Y(f(x), f(y)) \leq \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon.$$

Why False (option B) is wrong:

- Option B: There is no counterexample; the $\varepsilon/3$ argument above is valid regardless of whether the domain is compact.

Final Answer: True — uniform limit of uniformly continuous functions is uniformly continuous $\Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q32

Q33.

Solution

Concept — Uniform Convergence Does Not Imply Derivative Convergence: Uniform convergence of $f_n \to f$ does not force $f_n' \to f'$.

Counterexample (False): Define $f_n(x) = \dfrac{\sin(n^2 x)}{n}$ on $(0, 1)$.

- Uniform convergence: $\|f_n\|_\infty = 1/n \to 0$, so $f_n \to f \equiv 0$ uniformly.
- Derivative: $f_n'(x) = n\cos(n^2 x)$. For $x \in (0, 1)$, $\|f_n'\|_\infty = n \to \infty$; f_n' does not converge pointwise to $f' = 0$.

Correct theorem (for comparison): If $f_n \to f$ pointwise *and* $f_n' \to g$ uniformly, then f is differentiable and $f' = g$. Uniform convergence of f_n alone is not enough.

Why True (option A) is wrong:

- Option A: The counterexample above shows uniform convergence of f_n without derivative convergence.

Final Answer: False — $f_n(x) = \sin(n^2 x)/n$ converges uniformly but f_n' diverges $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q33

Q34.

Solution

Concept — Cauchy Sequences are Bounded: In any metric space, every Cauchy sequence is bounded.

Proof (True): Let $\{a_n\}$ be Cauchy in $\mathbb{R}$. For $\varepsilon = 1$, choose N so $|a_m - a_n| < 1$ for all $m, n \geq N$. In particular, $|a_n - a_N| < 1$ for all $n \geq N$, so $|a_n| < |a_N| + 1$. Let $M = \max\{|a_1|, \ldots, |a_{N-1}|, |a_N| + 1\}$. Then $|a_n| \leq M$ for all n.

Why False (option B) is wrong:

- Option B: The proof above leaves no room for a Cauchy sequence to be unbounded; boundedness follows from the Cauchy condition with $\varepsilon = 1$.

Final Answer: True — every Cauchy sequence in $\mathbb{R}$ is bounded $\Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q34

Q35.

Solution

Concept — Riemann Integrability of Monotone Functions: Every monotone function on a closed bounded interval is Riemann integrable, regardless of continuity.

Counterexample (False): Let $f : [0,1] \to \mathbb{R}$ be a monotone non-decreasing function with a jump discontinuity at $x = 1/2$, e.g., $f(x) = \lfloor 2x \rfloor$ (floor function, 0 on $[0, 1/2)$, 1 on $[1/2, 1]$). This is monotone and NOT continuous at $x = 1/2$, yet it is Riemann integrable (the single discontinuity is a set of measure zero).

General theorem: A monotone function on $[a, b]$ has at most countably many discontinuities. A function is Riemann integrable iff its set of discontinuities has measure zero. Countable sets have measure zero. Hence monotone $\Rightarrow$ Riemann integrable, independent of continuity.

Why True (option A) is wrong:

- Option A: Monotone functions are Riemann integrable even without continuity; the "if and only if continuous" direction fails.

Final Answer: False — monotone functions on $[a, b]$ are always Riemann integrable $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q35

Q36.

Solution

Concept — Geometric vs. Algebraic Multiplicity: For any eigenvalue λ of a square matrix A: geometric multiplicity $= \dim\ker(A - \lambda I) \leq$ algebraic multiplicity $=$ multiplicity of λ as a root of $\det(A - \lambda I)$.

Proof (True): The Jordan decomposition shows that each Jordan block $J_k(\lambda)$ of size $k \times k$ contributes 1 to the geometric multiplicity and k to the algebraic multiplicity. Summing over all blocks for λ: geom. mult. $\leq$ alg. mult.

Equality iff A is diagonalisable: The geometric and algebraic multiplicities are equal for λ iff the Jordan blocks for λ are all 1×1.

Why False (option B) is wrong:

- Option B: The inequality geo. mult. $\leq$ alg. mult. is a theorem, not something that can fail; there is no counterexample.

Final Answer: True — geometric multiplicity $\leq$ algebraic multiplicity always holds $\Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q36

Q37.

Solution

Concept — Eigenvalues of Real Symmetric Matrices: A real symmetric matrix has all real eigenvalues (Spectral Theorem), but they are NOT necessarily positive.

Counterexample (False): The matrix $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$ is real and symmetric ($A^T = A$), but its eigenvalues are -1 and -2, both negative.

Correct statement: A real symmetric matrix is positive definite (all eigenvalues > 0) iff $x^T A x > 0$ for all non-zero $x \in \mathbb{R}^n$. Symmetry alone does not ensure positivity.

Why True (option A) is wrong:

- Option A: Real symmetric $\Rightarrow$ all eigenvalues real (Spectral Theorem), but not necessarily positive; the example above refutes the stronger claim.

Final Answer: False — $\operatorname{diag}(-1, -2)$ is symmetric with negative eigenvalues $\Rightarrow$ $\boxed{\text{B}}$

Answer: (B) Go Back to Q37

Q38.

Solution

Concept — Gram-Schmidt Orthogonalisation: Given any basis of a finite-dimensional inner product space, the Gram-Schmidt process produces an orthonormal basis.

Proof (True): Let V be a finite-dimensional inner product space with basis $\{v_1, \dots, v_n\}$. Define $u_1 = v_1/\|v_1\|$ and inductively:

$$w_k = v_k - \sum_{j=1}^{k-1} \langle v_k, u_j \rangle u_j, \quad u_k = w_k/\|w_k\|.$$

The vectors $\{u_1, \dots, u_n\}$ form an orthonormal basis: they span V (same span as $\{v_1, \dots, v_n\}$) and satisfy $\langle u_i, u_j \rangle = \delta_{ij}$.

Why False (option B) is wrong:

- Option B: Gram-Schmidt is a constructive algorithm that always terminates in finitely many steps for finite-dimensional spaces; there is no obstruction.

Final Answer: True — Gram-Schmidt always produces an orthonormal basis $\Rightarrow$ $\boxed{\text{A}}$

Answer: (A) Go Back to Q38

Q39.

Solution

Concept — Zero Divisors in Matrix Rings: The ring $M_n(\mathbb{R})$ has zero divisors for $n \geq 2$; $AB = 0$ does not require $A = 0$ or $B = 0$.

Counterexample (False): Take $n = 2$ and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $A \neq 0$, $B \neq 0$, but $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$.

Remark: The statement "$AB = 0 \Rightarrow A = 0$ or $B = 0$" holds in integral domains (rings with no zero divisors), such as fields or $\mathbb{Z}$, but not in $M_n(\mathbb{R})$ for $n \geq 2$.

Why True (option A) is wrong:

- Option A: The explicit 2×2 example above shows both A and B are non-zero yet $AB = 0$.

Final Answer: False — $A = \text{diag}(1, 0)$, $B = \text{diag}(0, 1)$: $AB = 0$ but $A, B \neq 0 \Rightarrow \boxed{\text{B}}$

Answer: (B) **Go Back to Q39**

Q40.

Solution

Concept — Normal Subgroups of Abelian Groups: In an abelian group G, every subgroup is normal.

Proof (True): Let $H \leq G$ and $g \in G$, $h \in H$. Since G is abelian, $ghg^{-1} = gg^{-1}h = h \in H$. Thus $gHg^{-1} = H$ for every $g \in G$, so H is normal in G.

Remark: This is why every quotient group of an abelian group is well-defined. Normal subgroups in non-abelian groups require the conjugation condition explicitly.

Why False (option B) is wrong:

- Option B: Commutativity makes all conjugates trivial; no counterexample exists in abelian groups.

Final Answer: True — in abelian groups, $ghg^{-1} = h \in H$ for all $g, h \Rightarrow \boxed{\text{A}}$

Answer: (A) **Go Back to Q40**

Q41.

Solution

Concept — Quotients of Cyclic Groups: The infinite cyclic group $\mathbb{Z}$ has quotient groups $\mathbb{Z}/n\mathbb{Z}$ (finite, of order n) and $\mathbb{Z}/\{0\} = \mathbb{Z}$ (infinite).

Counterexample (False): The subgroup $n\mathbb{Z} = \{nk : k \in \mathbb{Z}\}$ is a normal subgroup of $\mathbb{Z}$ of index n. The quotient $\mathbb{Z}/n\mathbb{Z}$ is a finite cyclic group of order n for any $n \geq 1$. For $n = 2$: $\mathbb{Z}/2\mathbb{Z} = \{[0], [1]\}$, a finite group of order 2.

Correct statement: Every quotient of $\mathbb{Z}$ is either infinite (when $n = 0$, giving $\mathbb{Z}$

itself) or finite of order n (for $n \geq 1$).

Why True (option A) is wrong:

- Option A: $\mathbb{Z}/n\mathbb{Z}$ is a finite quotient for every $n \geq 1$, directly contradicting the claim.

Final Answer: False — $\mathbb{Z}/n\mathbb{Z}$ is a finite quotient of $\mathbb{Z} \Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q41

Q42.

Solution

Concept — $\mathbb{Z}$ is a PID: A ring is a PID if it is an integral domain in which every ideal is principal (generated by one element).

Proof (True): Let $I \subseteq \mathbb{Z}$ be any ideal. If $I = \{0\}$, then $I = (0)$. Otherwise, let $d = \min\{n \in I : n > 0\}$ (exists since I is non-empty and closed under negation). For any $m \in I$, write $m = qd + r$ with $0 \leq r < d$ by the division algorithm. Since $r = m - qd \in I$ and $0 \leq r < d$, minimality forces $r = 0$, so $m = qd \in (d)$. Hence $I = (d)$.

Why False (option B) is wrong:

- Option B: The division algorithm in $\mathbb{Z}$ guarantees every ideal is generated by its smallest positive element; this is a fundamental theorem.

Final Answer: True — every ideal of $\mathbb{Z}$ is principal $\Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q42

Q43.

Solution

Concept — Boundary of an Open Set: The boundary $\partial U = \bar{U} \setminus \operatorname{int}(U)$. For an open set U, $\operatorname{int}(U) = U$, so $\partial U = \bar{U} \setminus U$. This need not be empty.

Counterexample (False): In $\mathbb{R}$ with the standard topology, the open set $U = (0, 1)$ has closure $\bar{U} = [0, 1]$ and $\partial U = [0, 1] \setminus (0, 1) = \{0, 1\} \neq \emptyset$. As shown in the diagram, the two boundary points 0 and 1 lie outside U.

Correct statement: A set U has empty boundary iff U is both open and closed (clopen). In a connected space, only $\emptyset$ and the whole space are clopen.

Why True (option A) is wrong:

- Option A: $(0, 1)$ is open in $\mathbb{R}$ with non-empty boundary $\{0, 1\}$; this single example refutes the claim.

Final Answer: False — $\partial(0, 1) = \{0, 1\} \neq \emptyset \Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q43

Q44.

Solution

Concept — Compact Subsets of Hausdorff Spaces are Closed: In a Hausdorff space, compact subsets are automatically closed.

Proof (True): Let K be compact in Hausdorff space X. We show $X \setminus K$ is open. Fix $x \in X \setminus K$. For each $k \in K$, by Hausdorff property, there exist disjoint open sets $U_k \ni x$ and $V_k \ni k$. The sets $\{V_k : k \in K\}$ cover K; by compactness, a finite sub-cover $V_{k_1}, \ldots, V_{k_n}$ exists. Let $U = \bigcap_{i=1}^{n} U_{k_i}$ (open, contains x). Then $U \cap K \subseteq U \cap (V_{k_1} \cup \cdots \cup V_{k_n}) = \emptyset$, so $U \subseteq X \setminus K$. Hence $X \setminus K$ is open.

Why False (option B) is wrong:

- Option B: In non-Hausdorff spaces, compact sets need not be closed; but in Hausdorff spaces the separation axiom is exactly what the proof requires.

Final Answer: True — compact subsets of Hausdorff spaces are closed $\Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q44

Q45.

Solution

Concept — Separability vs. Compactness: Separability (existence of a countable dense subset) does not imply compactness.

Counterexample (False): $\mathbb{R}$ with the standard metric is separable (the rationals $\mathbb{Q}$ are countable and dense), but $\mathbb{R}$ is not compact (the open cover $\{(-n, n) : n \in \mathbb{N}\}$ has no finite sub-cover, or equivalently the sequence $a_n = n$ has no convergent subsequence in $\mathbb{R}$).

Correct implication direction: Compact metric spaces are always separable (and second-countable). But the converse fails.

Why True (option A) is wrong:

- Option A: $\mathbb{R}$ is a separable metric space that is not compact, directly disproving the claim.

Final Answer: False — $\mathbb{R}$ is separable but not compact $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q45

Q46.

Solution

Concept — Holomorphic Functions with Zero Derivative: If f is holomorphic on a connected domain D and $f'(z) = 0$ for all $z \in D$, then f is constant.

Proof (True): Write $f = u+iv$. From $f' = u_x+iv_x = 0$, we get $u_x = v_x = 0$. By the Cauchy-Riemann equations, $u_y = -v_x = 0$ and $v_y = u_x = 0$. Thus $\nabla u = \nabla v = 0$ on D. Since D is connected (and open), u and v are constant on D, so f is constant.

Why False (option B) is wrong:

- Option B: The argument via Cauchy-Riemann equations is rigorous; there is no counterexample for a connected domain.

Final Answer: True — $f' = 0$ on connected D implies f constant $\Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q46

Q47.

Solution

Concept — Convergence on the Circle of Convergence: A power series converges absolutely on $|z| < R$ and diverges for $|z| > R$, but convergence on $|z| = R$ must be examined case by case.

Counterexample (False): The power series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ has radius of convergence $R = 1$ (by the ratio test: $|a_{n+1}/a_n| = n/(n+1) \to 1$). At $z = 1$: $\sum 1/n$ is the harmonic series, which diverges. Hence the series does NOT converge at all points of $|z| = 1$.

Correct theorem: If a power series with $R > 0$ converges on the entire closed disk $|z| \leq R$, then the convergence is uniform there (Abel's theorem), but convergence on $|z| = R$ is not automatic.

Why True (option A) is wrong:

- Option A: The series $\sum z^n/n$ at $z = 1$ diverges, showing that $|z| \leq R$ convergence is not guaranteed.

Final Answer: False — $\sum z^n/n$ diverges at $z = 1$ on its circle of convergence $\Rightarrow$ $\boxed{\text{B}}$

Answer: (B) Go Back to Q47

Q48.

Solution

Concept — Euler's Theorem: If $\gcd(a, n) = 1$, then $a^{\phi(n)} \equiv 1 \pmod{n}$, where $\phi(n) = |(\mathbb{Z}/n\mathbb{Z})^\times|$.

Proof (True): The set $G = \{[k]_n : \gcd(k, n) = 1\}$ forms a group under multiplication mod n, with $|G| = \phi(n)$. Since $[a]_n \in G$, by Lagrange's theorem the order of $[a]_n$ divides $|G| = \phi(n)$. Hence $a^{\phi(n)} \equiv 1 \pmod{n}$.

Special case: When $n = p$ is prime, $\phi(p) = p - 1$ and $a^{p-1} \equiv 1 \pmod{p}$ for $\gcd(a, p) = 1$ (Fermat's Little Theorem).

Why False (option B) is wrong:

- Option B: Euler's theorem is a standard result in number theory with a complete group-theoretic proof; there is no counterexample when $\gcd(a, n) = 1$.

Final Answer: True — Euler's theorem: $a^{\phi(n)} \equiv 1 \pmod{n}$ when $\gcd(a, n) = 1 \Rightarrow$ $\boxed{\text{A}}$

Answer: (A) Go Back to Q48

Q49.

Solution

Concept — Fermat's Theorem on Sums of Two Squares: A prime p is expressible as $p = a^2 + b^2$ (with $a, b \in \mathbb{Z}$) if and only if $p = 2$ or $p \equiv 1 \pmod{4}$.

Counterexample (False): $p = 3$ is prime with $3 = 4(0) + 3$, i.e., $p \equiv 3 \pmod{4}$. We check all cases: $0^2 + 1^2 = 1$, $0^2 + 2^2 = 4$, $1^2 + 1^2 = 2$, $1^2 + 2^2 = 5 \neq 3$. No pair of squares sums to 3. Hence 3 cannot be written as a sum of two squares.

Why primes $4k + 3$ fail: Any square $a^2 \equiv 0$ or $1 \pmod{4}$. So $a^2 + b^2 \equiv 0, 1,$ or 2

$\pmod 4$, but never $\equiv 3 \pmod 4$. A prime $p \equiv 3 \pmod 4$ cannot be a sum of two squares.

Why True (option A) is wrong:

- Option A: The parity argument modulo 4 definitively rules out $p \equiv 3 \pmod 4$ as a sum of two squares.

Final Answer: False — primes $p \equiv 3 \pmod 4$ cannot be expressed as sums of two squares $\Rightarrow$ $\boxed{\text{B}}$

Answer: (B) Go Back to Q49

Q50.

Solution

Concept — Variance of Sum of Independent Random Variables: For independent X and Y with finite variance, $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$.

Proof (True): By definition, $\mathrm{Var}(X+Y) = \mathbb{E}[(X+Y)^2] - (\mathbb{E}[X+Y])^2$. Expanding:

$\mathbb{E}[(X+Y)^2] = \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2], \quad (\mathbb{E}[X+Y])^2 = (\mathbb{E}X)^2 + 2\mathbb{E}X\mathbb{E}Y + (\mathbb{E}Y)^2.$

Independence gives $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ (so $\mathrm{Cov}(X,Y) = 0$). Subtracting:

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

Important note: Without independence, $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)$, which may differ.

Why False (option B) is wrong:

- Option B: Independence implies zero covariance, making the cross term vanish; the additivity of variance is a theorem.

Final Answer: True — $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$ for independent $X, Y \Rightarrow$ $\boxed{\text{A}}$

Answer: (A) Go Back to Q50

Answer Key

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	C	2	A	3	D	4	B	5	C
6	A	7	D	8	B	9	C	10	A
11	D	12	B	13	C	14	A	15	D
16	B	17	C	18	A	19	D	20	B
21	C	22	A	23	D	24	B	25	C
26	A	27	D	28	B	29	C	30	A
31	B	32	A	33	B	34	A	35	B
36	A	37	B	38	A	39	B	40	A
41	B	42	A	43	B	44	A	45	B
46	A	47	B	48	A	49	B	50	A