TIFR GS Mathematics

Sample Paper – 4

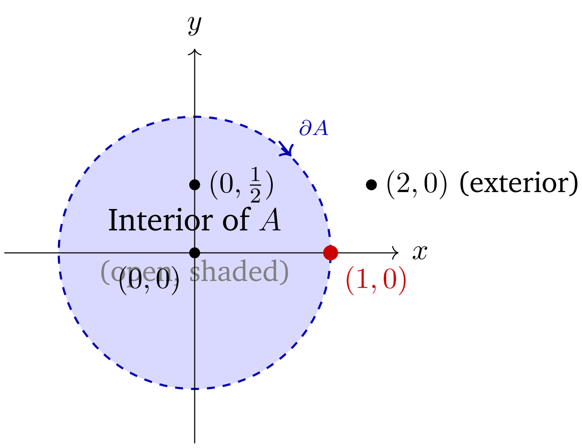
Duration: 180 Minutes **Maximum Marks: 50**

Instructions

- This paper contains **50 questions** modelled on the **TIFR GS Mathematics** entrance examination.
- **Part A (Q1–Q30):** 30 Multiple Choice Questions. Each correct answer carries **+1 mark**; each wrong answer carries $-\frac{1}{3}$ **mark**. Unattempted questions carry 0 marks.
- **Part B (Q31–Q50):** 20 True/False questions. Each correct answer carries **+1 mark**. There is **no negative marking** in Part B. Attempt all questions.
- Use of calculators, mobile phones, or any electronic gadgets is strictly prohibited. Rough work may be done on provided sheets.
- Syllabus: **BSc Mathematics (Honours) level** — Real Analysis, Linear Algebra, Abstract Algebra, Topology, Complex Analysis, Number Theory, and Probability.

Part A — Multiple Choice Questions (MCQ)

Q1. Let $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ be the open unit disk. Which of the following points belongs to the **boundary** ∂A of A?



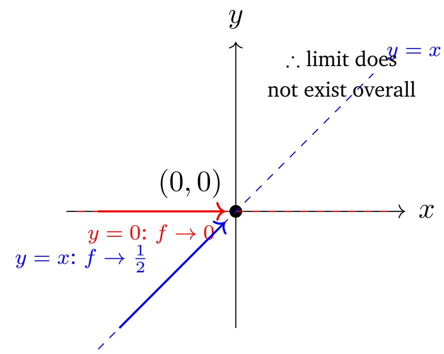
(A) $(0, 0)$

(B) $(1, 0)$

(C) $(2, 0)$

(D) $(0, \frac{1}{2})$

Q2. Consider $f(x, y) = \dfrac{xy}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$. As $(x, y) \to (0, 0)$ **along the line** $y = x$, which value does $f(x, y)$ approach?



(A) 0

(B) 1

(C) The limit does not exist (no finite value)

(D) $\dfrac{1}{2}$

Q3. Let $f(x) = x^2$ on $[0, 1]$ and let P_n be the uniform partition into n equal subintervals. The **upper Darboux sum** is $U(f, P_n) = \displaystyle\sum_{k=1}^{n} \frac{k^2}{n^2} \cdot \frac{1}{n}$. Which formula gives the exact value of $U(f, P_n)$?

(A) $\dfrac{(n+1)(2n+1)}{6n^2}$

(B) $\dfrac{(n-1)(2n-1)}{6n^2}$

(C) $\dfrac{2n+1}{6n}$

(D) $\dfrac{1}{3}$

Q4. Evaluate the integral $\int_0^{\pi} x\cos x\,dx$ using integration by parts. Which of the following is the correct value?

(A) 0

(B) 2

(C) -2

(D) π

Q5. The series $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$. By the Alternating Series Estimation Theorem, the error after n terms satisfies $|S - S_n| \leq \frac{1}{n+1}$. What is the **minimum** value of n for which this bound guarantees $|S - S_n| \leq 0.01$?

(A) $n = 100$

(B) $n = 99$

(C) $n = 50$

(D) $n = 199$

Q6. Find the radius of convergence R of the power series $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} z^n$.

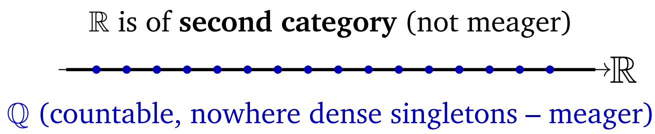
(A) $R = 4$

(B) $R = 2$

(C) $R = 1$

(D) $R = \frac{1}{4}$

Q7. The Baire Category Theorem states that a complete metric space cannot be written as a countable union of nowhere dense sets. Which of the following is a **correct consequence** for $\mathbb{R}$ with the usual topology?



(A) $\mathbb{Q}$ is a meager (first category) subset of $\mathbb{R}$

(B) $\mathbb{R} \setminus \mathbb{Q}$ is a meager subset of $\mathbb{R}$

(C) $\mathbb{R}$ cannot be expressed as a countable union of closed sets

(D) $\mathbb{Q}$ contains a non-empty open interval

Q8. Let $f = \mathbf{1}_{[0,1]}$ (the indicator function of $[0, 1]$) and define $F(x) = \int_0^x f(t)\,dt$ for $x \in \mathbb{R}$. By the Fundamental Theorem of Lebesgue Integration, $F'(x) = f(x)$ for almost every x. What is $F'(2)$?

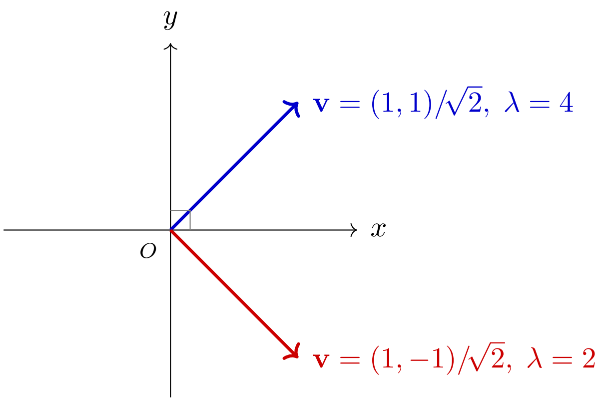
(A) 1

(B) 2

(C) 0

(D) $F'(2)$ does not exist

Q9. Let $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$. Since A is real symmetric, it has real eigenvalues and is orthogonally diagonalizable. What are the eigenvalues of A?



(A) 1 and 5

(B) 2 and 4

(C) 3 and 3 (repeated)

(D) -1 and 7

Q10. A matrix $A \in M_n(\mathbb{C})$ is called **normal** if $AA^* = A^*A$, where $A^* = \overline{A}^T$. Normal matrices are exactly those that are unitarily diagonalizable. Which of the following classes of matrices is **NOT** necessarily normal?

(A) Real symmetric matrices ($A = A^T$)

(B) Skew-Hermitian matrices ($A^* = -A$)

(C) Unitary matrices ($A^*A = I$)

(D) Upper triangular matrices

Q11. Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. Using the identities $\lambda_1 + \lambda_2 = \mathrm{tr}(A)$ and $\lambda_1\lambda_2 = \det(A)$, compute $\lambda_1^2 + \lambda_2^2$ where λ_1, λ_2 are the eigenvalues of A.

(A) 26

(B) 16

(C) -5

(D) 10

Q12. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map with matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ (using standard bases). The adjoint $T^* : \mathbb{R}^2 \to \mathbb{R}^3$ satisfies $\langle T\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, T^*\mathbf{w} \rangle$ for all $\mathbf{v}, \mathbf{w}$. What is the matrix of T^*?

(A) A (the same 2×3 matrix)

(B) A^{-1} (if it exists)

(C) $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

(D) I_3 (the 3×3 identity)

Q13. Let $A = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$ viewed as a map $\mathbb{Z}^2 \to \mathbb{Z}^2$. The order of the quotient group $\mathbb{Z}^2/A\mathbb{Z}^2$ equals $|\det A|$. What is $|\mathbb{Z}^2/A\mathbb{Z}^2|$?

(A) 4

(B) 8

(C) 16

(D) 24

Q14. For vectors $\mathbf{v}_1 = (1,0,0)$ and $\mathbf{v}_2 = (1,1,0)$ in $\mathbb{R}^3$, the Gram matrix is $G_{ij} = \langle \mathbf{v}_i, \mathbf{v}_j \rangle$. Which of the following correctly gives G and determines whether it is positive definite?

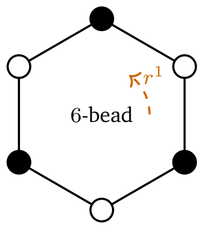
(A) $G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, positive definite

(B) $G = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, positive semidefinite (not positive definite)

(C) $G = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, positive definite

(D) $G = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, positive definite

Q15. Using Burnside's lemma with the cyclic group $\mathbb{Z}_6$ acting on binary strings of length 6 (rotations of a 6-bead necklace with 2 colours), how many **distinct** necklaces are there up to rotation?



Burnside: $\frac{1}{6}(64+2+4+8+4+2) = ?$

(A) 14

(B) 12

(C) 16

(D) 10

Q16. How many **non-isomorphic** groups of order 6 are there (up to isomorphism)?

(A) 1

(B) 3

(C) 2

(D) 4

Q17. In the Euclidean domain $\mathbb{Z}[i]$ (Gaussian integers) with norm $N(a+bi) = a^2+b^2$, divide $11+5i$ by $3+2i$ by rounding to the nearest Gaussian integer. Which pair (q,r) is the (standard) quotient and remainder satisfying $11+5i = (3+2i)q + r$ with $N(r) < N(3+2i)$?

(A) $q = 3, \quad r = 2 - i$

(B) $q = 3 - i, \quad r = 2i$

(C) $q = 4 - i, \quad r = -3$

(D) $q = 2 + i, \quad r = 5 - i$

Q18. Which of the following polynomials is **irreducible** over $\mathbb{Q}$?

(A) $x^4 - 1$

(B) $x^2 - 4$

(C) $x^3 - 3x + 2$

(D) $x^3 + 2x + 1$

Q19. Let K be the splitting field of $f(x) = x^3 - 2$ over $\mathbb{Q}$. It is known that $[K : \mathbb{Q}] = 6$ and $K = \mathbb{Q}(\sqrt[3]{2}, \omega)$ where $\omega = e^{2\pi i/3}$. What is the Galois group $\mathrm{Gal}(K/\mathbb{Q})$?

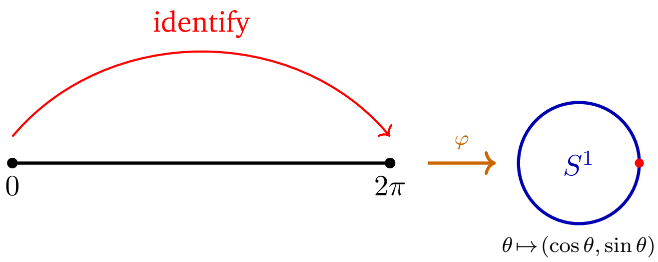
(A) S_3 (symmetric group on 3 elements, non-abelian of order 6)

(B) $\mathbb{Z}_6$ (cyclic group of order 6)

(C) $\mathbb{Z}_3$ (cyclic group of order 3)

(D) $\mathbb{Z}_2 \times \mathbb{Z}_3$

Q20. Consider the quotient space $[0, 2\pi]/\!\sim$ where $0 \sim 2\pi$, and the map $\varphi : [0, 2\pi]/\!\sim \ \to S^1$ defined by $[\theta] \mapsto (\cos\theta, \sin\theta)$. Which statement is correct?



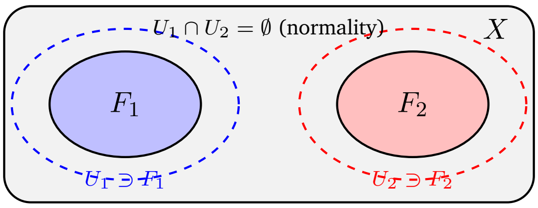
(A) φ is not injective (two points map to the same image)

(B) φ is not surjective (misses some points of S^1)

(C) φ is a homeomorphism, by the theorem that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism

(D) φ is a quotient map but not a homeomorphism

Q21. In a compact Hausdorff space X, which of the following is **NOT** necessarily true?



(A) Every singleton $\{x\}$ is a closed set

(B) Every closed subset of X is a G_δ set (countable intersection of open sets)

(C) Any two disjoint closed sets can be separated by disjoint open sets

(D) X is a normal topological space

Q22. A topological space is **first-countable** if every point has a countable neighbourhood base. Which of the following spaces is **NOT** first-countable?

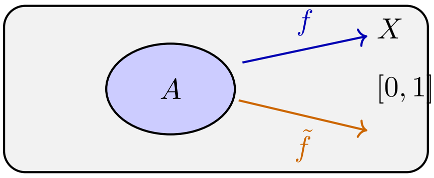
(A) Any metric space

(B) $\mathbb{R}$ with the standard (Euclidean) topology

(C) Any second-countable topological space

(D) $\mathbb{R}$ with the cocountable topology (open sets are $\emptyset$ and complements of countable sets)

Q23. The Tietze Extension Theorem states: if X is a **normal** topological space and $A \subseteq X$ is closed, then every continuous $f : A \to [0,1]$ extends to a continuous $\tilde{f} : X \to [0,1]$. Which of the following classes of spaces is **guaranteed** to be normal (and hence satisfies this theorem)?



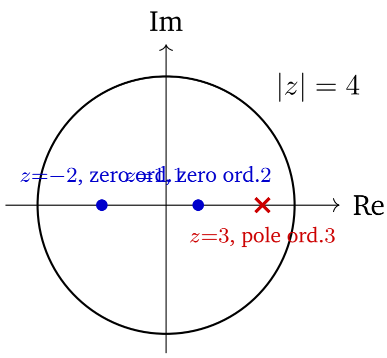
(A) Compact Hausdorff spaces

(B) All Hausdorff spaces

(C) All connected spaces

(D) All first-countable spaces

Q24. Let $f(z) = \dfrac{(z-1)^2(z+2)}{(z-3)^3}$. Inside the contour $|z| = 4$, count the **number of zeros minus number of poles** (each counted with multiplicity).



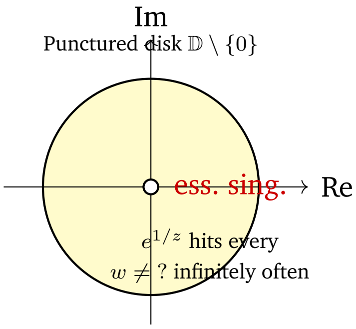
(A) 1

(B) -1

(C) 0

(D) 3

Q25. By Picard's Great Theorem, near an essential singularity a function takes every complex value with at most one exception. For $f(z) = e^{1/z}$ near $z = 0$ (an essential singularity), which value is **never** attained by f?



(A) -1

(B) 0

(C) 1

(D) e

Q26. For the closed curve $\gamma(t) = e^{2\pi i t}$, $t \in [0, 1]$ (the unit circle traversed once counterclockwise), the winding number around a point a is $n(\gamma, a) = \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z - a}$. What is $n(\gamma, a)$ for a point a with $|a| = 3$?

(A) 2

(B) 1

(C) -1

(D) 0

Q27. Using the residue theorem, evaluate $I = \displaystyle\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\, dx$. The poles of the integrand (as a function of z) in the upper half-plane are at $z = i$ and $z = 2i$. What is the value of I?

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{3}$

(C) $\frac{\pi}{4}$

(D) $\dfrac{\pi}{12}$

Q28. The Möbius function is defined by $\mu(1) = 1$, $\mu(n) = (-1)^k$ if n is a product of k distinct primes, and $\mu(n) = 0$ if $p^2 \mid n$ for some prime p. Which of the following is **correct**?

(A) $\mu(12) = -1$

(B) $\mu(6) = -1$

(C) $\mu(30) = -1$

(D) $\mu(12) = 1$

Q29. The sum-of-divisors function $\sigma(n) = \sum_{d \mid n} d$ is multiplicative. Using $\sigma(p^k) = \dfrac{p^{k+1} - 1}{p - 1}$, compute $\sigma(18)$.

(A) 21

(B) 39

(C) 26

(D) 30

Q30. Let $X \sim \text{Poisson}(\lambda)$ with $\lambda = 3$, so $P(X = k) = e^{-3} \cdot 3^k / k!$. Which of the following gives the correct value of $P(X = 2)$?

(A) $e^{-3} \cdot 3$

(B) $e^{-3} \cdot 9$

(C) $\dfrac{e^{-3}}{2}$

(D) $\dfrac{9\,e^{-3}}{2}$

Part B — True/False Questions (No Negative Marking)

Q31. State whether the following is **True** or **False**:

If $\sum a_n b_n$ converges and $\{b_n\}$ is a monotone sequence with $b_n \to 0$, then $\sum a_n$ necessarily converges.

(A) True

(B) False

Q32. State whether the following is **True** or **False**:
The finite union of nowhere dense subsets of a topological space is nowhere dense.

(A) True

(B) False

Q33. State whether the following is **True** or **False**:
Every absolutely continuous function on $[a, b]$ is Lipschitz continuous.

(A) True

(B) False

Q34. State whether the following is **True** or **False**:
If $f : (0, 1) \to \mathbb{R}$ is uniformly continuous, then $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 1^-} f(x)$ both exist **and are equal to each other**.

(A) True

(B) False

Q35. State whether the following is **True** or **False**:
If $f_n \to f$ uniformly on $[a, b]$ and each f_n is Riemann integrable on $[a, b]$, then f is Riemann integrable and $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$.

(A) True

(B) False

Q36. State whether the following is **True** or **False**:
The set of all invertible $n \times n$ real matrices forms a group under matrix multiplication (the general linear group $\mathrm{GL}(n, \mathbb{R})$).

(A) True

(B) False

Q37. State whether the following is **True** or **False**:
If a real 3×3 matrix A has a complex eigenvalue $\alpha + i\beta$ with $\beta \neq 0$, then A is diagonalizable over $\mathbb{R}$.

(A) True

(B) False

Q38. State whether the following is **True** or **False**:
For any linear map $T : V \to V$ on a finite-dimensional vector space V, $\dim(\ker T) + \dim(\operatorname{Im} T) = \dim V$.

(A) True

(B) False

Q39. State whether the following is **True** or **False**:
If V and W are finite-dimensional vector spaces with $\dim V = m$ and $\dim W = n$, then $\dim(V \otimes W) = m + n$.

(A) True

(B) False

Q40. State whether the following is **True** or **False**:
The alternating group A_n is simple for all $n \geq 5$.

(A) True

(B) False

Q41. State whether the following is **True** or **False**:
Every Noetherian commutative ring is Artinian.

(A) True

(B) False

Q42. State whether the following is **True** or **False**:
The Galois group of an irreducible polynomial of degree n over $\mathbb{Q}$ embeds as a subgroup of the symmetric group S_n.

(A) True

(B) False

Q43. State whether the following is **True** or **False**:
Every paracompact topological space is compact.

(A) True

(B) False

Q44. State whether the following is **True** or **False**:
Homotopy equivalence is an equivalence relation on the class of topological spaces.

(A) True

(B) False

Q45. State whether the following is **True** or **False**:
The fundamental group $\pi_1(S^2)$ of the 2-sphere is isomorphic to $\mathbb{Z}$.

(A) True

(B) False

Q46. State whether the following is **True** or **False**:
By the Weierstrass product theorem, for any sequence of complex numbers $\{a_n\}$ with $|a_n| \to \infty$, there exists an entire function having zeros exactly at the points a_n (with prescribed multiplicities).

(A) True

(B) False

Q47. State whether the following is **True** or **False**:

Every holomorphic bijection $f : \mathbb{D} \to \mathbb{D}$ (from the unit disk to itself) with $f(0) = 0$ must satisfy $f(z) = z$ for all $z \in \mathbb{D}$.

(A) True

(B) False

Q48. State whether the following is **True** or **False**:

By Bertrand's postulate, for every integer $n \geq 1$ there exists a prime p with $n < p \leq 2n$.

(A) True

(B) False

Q49. State whether the following is **True** or **False**:

The prime-counting function $\pi(n)$ (the number of primes $\leq n$) satisfies $\pi(n) = \left\lfloor \frac{n}{\ln n} \right\rfloor$ exactly for all sufficiently large n.

(A) True

(B) False

Q50. State whether the following is **True** or **False**:

By Kolmogorov's 0-1 law, if $X_1, X_2, \ldots$ are independent random variables and A is a tail event (depending only on the tail σ-algebra of the sequence), then $P(A) \in \{0, 1\}$.

(A) True

(B) False

Detailed Solutions

Q1.

Solution

Concept — Boundary of a set in $\mathbb{R}^2$: A point p belongs to ∂A if and only if every open neighbourhood of p intersects both A and its complement A^c.

Step 1 — Characterise A: $A = \{(x, y) : x^2 + y^2 < 1\}$ is the open unit disk. Its boundary is $\partial A = \{(x, y) : x^2 + y^2 = 1\}$, the unit circle.

Step 2 — Test each option: The point $(1, 0)$ satisfies $1^2 + 0^2 = 1$, so it lies exactly on the unit circle $\Rightarrow (1, 0) \in \partial A$.

Why other options are wrong:

- Option A: $(0, 0)$ has $0^2 + 0^2 = 0 < 1$, so it is in the *interior* of A, not on the boundary.
- Option C: $(2, 0)$ has $2^2 + 0^2 = 4 > 1$, so it is in the *exterior* of A.
- Option D: $(0, \frac{1}{2})$ has $0 + \frac{1}{4} = \frac{1}{4} < 1$, so it is in the interior of A.

Final Answer: $(1, 0) \in \partial A \Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q1

Q2.

Solution

Concept — Path limits for multivariable functions: To evaluate $f(x, y) = \dfrac{xy}{x^2 + y^2}$ along a line through the origin, substitute the parametrisation of that line.

Step 1 — Substitute $y = x$: Along the line $y = x$ (with $x \neq 0$):

$$f(x, x) = \frac{x \cdot x}{x^2 + x^2} = \frac{x^2}{2x^2} = \frac{1}{2}.$$

This is constant $1/2$ for all $x \neq 0$, so as $(x, x) \to (0, 0)$ the limit is $\frac{1}{2}$.

Why other options are wrong:

- Option A (0): This is the limit *along* $y = 0$: $f(x, 0) = 0$. Not the limit along $y = x$.
- Option B (1): Not attained along any standard straight path through the origin.

- Option C: The overall limit (over all paths) does not exist (paths give different values), but along $y = x$ specifically the limit is well-defined and equals $1/2$.

Final Answer: $f(x,x) \to \frac{1}{2} \Rightarrow \boxed{\text{D}}$

Answer: (D) Go Back to Q2

Q3.

Solution

Concept — Upper Darboux sum: For $f(x) = x^2$ on $[0,1]$ with uniform partition P_n (subintervals $[\frac{k-1}{n}, \frac{k}{n}]$), since f is increasing the supremum on each subinterval is the right endpoint value $f(\frac{k}{n}) = \frac{k^2}{n^2}$.

Step 1 — Compute the sum:

$$U(f, P_n) = \sum_{k=1}^{n} \frac{k^2}{n^2} \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}.$$

Why other options are wrong:

- Option B: $\dfrac{(n-1)(2n-1)}{6n^2}$ is the *lower* Darboux sum $L(f, P_n)$ (using left endpoints).
- Option C: $\dfrac{2n+1}{6n}$ does not follow from the formula (missing $\frac{n+1}{n}$ factor).
- Option D: $\dfrac{1}{3} = \int_0^1 x^2\,dx$ is the *limit* of $U(f, P_n)$ as $n \to \infty$, not its exact value.

Final Answer: $U(f, P_n) = \dfrac{(n+1)(2n+1)}{6n^2} \Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q3

Q4.

Solution

Concept — Integration by parts: The formula is $\int u\,dv = uv - \int v\,du$.

Step 1 — Apply IBP with $u = x$, $dv = \cos x\,dx$:

$$\int_0^{\pi} x\cos x\,dx = \left[x\sin x\right]_0^{\pi} - \int_0^{\pi} \sin x\,dx = (\pi\sin\pi - 0) - \left[-\cos x\right]_0^{\pi}.$$

Step 2 — Evaluate:

$$\pi \sin \pi = 0, \quad -\big[-\cos \pi + \cos 0\big] = -\big[1+1\big] = -2.$$

Hence $\int_0^{\pi} x \cos x \, dx = 0 + (-2) = -2.$

Why other options are wrong:

- Option A (0): Forgets the $\int v\,du$ term.
- Option B (2): Sign error — $-[\cos x]_0^{\pi} = -((-1)-1) = 2$ is a common mistake before applying the outer negative.
- Option D (π): Confuses x with π in the boundary term.

Final Answer: $\int_0^{\pi} x \cos x \, dx = -2 \Rightarrow \boxed{\text{C}}$

Answer: (C) **Go Back to Q4**

Q5.

Solution

Concept — Alternating Series Estimation: By the Leibniz theorem, for the alternating series $S = \sum(-1)^{n+1}/n$, the error bound is $|S - S_n| \le a_{n+1} = \dfrac{1}{n+1}$.

Step 1 — Solve the inequality: We need $\dfrac{1}{n+1} \le 0.01 = \dfrac{1}{100}$, i.e., $n+1 \ge 100$, so $n \ge 99$.

Step 2 — Verify minimality:

- For $n = 99$: $a_{100} = \frac{1}{100} = 0.01 \le 0.01$ ✓
- For $n = 98$: $a_{99} = \frac{1}{99} \approx 0.0101 > 0.01$ ×

The **minimum** n is **99**.

Why other options are wrong:

- Option A $(n = 100)$: $n = 100$ also works, but $n = 99$ is the *minimum*.
- Option C $(n = 50)$: $a_{51} = 1/51 \approx 0.0196 > 0.01$ — does not satisfy the bound.
- Option D $(n = 199)$: Valid but far exceeds the minimum.

Final Answer: Minimum $n = 99 \Rightarrow \boxed{\text{B}}$

Answer: (B) **Go Back to Q5**

Q6.

Solution

Concept — Radius of convergence via ratio test: For $a_n = \dfrac{(2n)!}{(n!)^2}$, compute $L = \lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$. Then $R = 1/L$.

Step 1 — Ratio:

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!\,/\,((n+1)!)^2}{(2n)!\,/\,(n!)^2} = \frac{(2n+2)(2n+1)}{(n+1)^2} = \frac{2(2n+1)}{n+1} \xrightarrow{n\to\infty} 4.$$

Step 2 — Conclude: $L = 4$, so $R = 1/L = 1/4$. The series converges for $|z| < 1/4$ and diverges for $|z| > 1/4$.

Why other options are wrong:

- Option A ($R = 4$): Confuses L with R; the radius is $1/L$, not L.
- Option B ($R = 2$): Takes $L = 2$ (the square root of the true limit), an error in simplification.
- Option C ($R = 1$): Ignores the growth of $(2n)!/(n!)^2 \sim 4^n/\sqrt{\pi n}$ (Stirling) and incorrectly applies the root test.

Final Answer: $R = \dfrac{1}{4} \Rightarrow \boxed{\text{D}}$

Answer: (D) Go Back to Q6

Q7.

Solution

Concept — Baire Category Theorem: A complete metric space (such as $\mathbb{R}$) is of *second category*; it cannot be written as a countable union of nowhere dense sets.

Step 1 — Apply to $\mathbb{Q}$: Each singleton $\{q\}$ for $q \in \mathbb{Q}$ is nowhere dense in $\mathbb{R}$ (its closure is $\{q\}$ which has empty interior). Since $\mathbb{Q} = \bigcup_{q\in\mathbb{Q}} \{q\}$ is a countable union of nowhere dense sets, $\mathbb{Q}$ is **meager** (first category) in $\mathbb{R}$.

Why other options are wrong:

- Option B: $\mathbb{R} \setminus \mathbb{Q}$ (the irrationals) is of *second category* (complement of a meager set in a Baire space is second category), not meager.
- Option C: $\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n]$ is indeed a countable union of closed sets; the theorem forbids writing $\mathbb{R}$ as a countable union of *nowhere dense* sets, not all closed sets.

- Option D: $\mathbb{Q}$ contains no nonempty open interval (since between any two rationals there is an irrational).

Final Answer: $\mathbb{Q}$ is meager (first category) in $\mathbb{R} \Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q7

Q8.

Solution

Concept — Fundamental Theorem of Lebesgue Integration: If $f \in L^1_{\text{loc}}(\mathbb{R})$ and $F(x) = \int_0^x f(t)\,dt$, then $F'(x) = f(x)$ for almost every x.

Step 1 — Compute F explicitly:

$$F(x) = \int_0^x \mathbf{1}_{[0,1]}(t)\,dt = \begin{cases} 0 & x < 0, \\ x & 0 \le x \le 1, \\ 1 & x > 1. \end{cases}$$

Step 2 — Differentiate at $x = 2$: Since $x = 2 > 1$, F is constant ($= 1$) in a neighbourhood of 2. Therefore $F'(2) = 0 = f(2) = \mathbf{1}_{[0,1]}(2)$.

Why other options are wrong:

- Option A (1): $F'(x) = 1$ only for $x \in (0,1)$, not at $x = 2$.
- Option B (2): $F(x) = x$ is only valid on $[0,1]$; there is no factor of 2 anywhere.
- Option D: $F'(2)$ does exist; F is piecewise linear and differentiable at $x = 2$ with value 0.

Final Answer: $F'(2) = 0 \Rightarrow \boxed{\text{C}}$

Answer: (C) Go Back to Q8

Q9.

Solution

Concept — Eigenvalues of a real symmetric matrix: λ is an eigenvalue iff $\det(A - \lambda I) = 0$.

Step 1 — Characteristic polynomial:

$$\det(A-\lambda I)=\det\begin{pmatrix}3-\lambda & 1\\ 1 & 3-\lambda\end{pmatrix}=(3-\lambda)^2-1=\lambda^2-6\lambda+8=(\lambda-2)(\lambda-4).$$

Step 2 — Eigenvalues: $\lambda=2$ and $\lambda=4$.

Why other options are wrong:

- Option A (1 and 5): tr $=1+5=6$ ✓but $\det=5-1=4\neq 8$; inconsistent with $\det(A)=9-1=8$.
- Option C (3 and 3): $\det=9\neq 8$; these are the diagonal entries, not the eigenvalues.
- Option D (-1 and 7): tr $=-1+7=6$ ✓but $\det=-7\neq 8$.

Final Answer: Eigenvalues are 2 and 4 $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q9

Q10.

Solution

Concept — Normal matrices: A is normal iff $AA^*=A^*A$. Symmetric, skew-Hermitian, and unitary matrices are all normal. Upper triangular matrices need NOT be normal.

Step 1 — Counterexample for upper triangular: Let $A=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ (upper triangular). Then

$$AA^*=\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad A^*A=\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

Since $AA^*\neq A^*A$, A is NOT normal.

Why other options are wrong:

- Option A: Real symmetric: $A^*=A^T=A$, so $AA^*=A^2=A^*A$. Normal.
- Option B: Skew-Hermitian: $A^*=-A$, so $AA^*=A(-A)=-A^2$ and $A^*A=(-A)A=-A^2$. Normal.
- Option C: Unitary: $A^*A=I=AA^*$ by definition. Normal.

Final Answer: Upper triangular matrices are NOT necessarily normal $\Rightarrow \boxed{\text{D}}$

Answer: (D) Go Back to Q10

Q11.

Solution

Concept — Newton's identity for power sums: $\lambda_1^2 + \lambda_2^2 = (\lambda_1 + \lambda_2)^2 - 2\lambda_1\lambda_2 = [\mathrm{tr}(A)]^2 - 2\det(A)$.

Step 1 — Compute trace and determinant:

$$\mathrm{tr}(A) = 1 + 3 = 4, \quad \det(A) = 1 \cdot 3 - 2 \cdot 4 = 3 - 8 = -5.$$

Step 2 — Apply identity:

$$\lambda_1^2 + \lambda_2^2 = 4^2 - 2(-5) = 16 + 10 = 26.$$

Why other options are wrong:

- Option B (16): Only computes $(\mathrm{tr}\, A)^2$ and forgets the $-2\det A$ correction.
- Option C (-5): Confuses $\lambda_1\lambda_2 = \det(A) = -5$ with the sum of squares.
- Option D (10): Uses $-2\det(A) = 10$ alone, forgetting to add $(\mathrm{tr}\, A)^2$.

Final Answer: $\lambda_1^2 + \lambda_2^2 = 26 \Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q11

Q12.

Solution

Concept — Matrix of the adjoint: For $T : \mathbb{R}^m \to \mathbb{R}^n$ with matrix A (with respect to standard orthonormal bases and the standard inner product), the adjoint $T^* : \mathbb{R}^n \to \mathbb{R}^m$ satisfies $\langle T\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{v}, T^*\mathbf{w}\rangle$, and its matrix is the transpose A^T.

Step 1 — Verify: $(A\mathbf{v}) \cdot \mathbf{w} = (A\mathbf{v})^T\mathbf{w} = \mathbf{v}^T A^T \mathbf{w} = \mathbf{v} \cdot (A^T\mathbf{w})$. So T^* has matrix A^T.

Why other options are wrong:

- Option A: A is a 2×3 matrix; T^* must map $\mathbb{R}^2 \to \mathbb{R}^3$, so it cannot have a 2×3 matrix.
- Option B: A^{-1} does not exist (non-square), and the adjoint is not the inverse.
- Option D: I_3 is a 3×3 matrix, but it does not depend on A at all.

Final Answer: Matrix of T^* is $A^T \Rightarrow \boxed{\text{C}}$

Answer: (C) Go Back to Q12

Q13.

Solution

Concept — Order of $\mathbb{Z}^2/A\mathbb{Z}^2$: For a $\mathbb{Z}$-linear map $A : \mathbb{Z}^n \to \mathbb{Z}^n$ with nonzero determinant, $|\mathbb{Z}^n/A\mathbb{Z}^n| = |\det(A)|$.

Step 1 — Compute determinant:

$$\det(A) = \det \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} = 2 \cdot 8 - 4 \cdot 6 = 16 - 24 = -8.$$

Step 2 — Conclude: $|\mathbb{Z}^2/A\mathbb{Z}^2| = |\det(A)| = |-8| = 8$.

Why other options are wrong:

- Option A (4): Perhaps from $\det \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix} = 0$ or a half-determinant error.
- Option C (16): Only considers $2 \cdot 8 = 16$, ignoring the $-4 \cdot 6$ term.
- Option D (24): Only considers $4 \cdot 6 = 24$, missing the full formula.

Final Answer: $|\mathbb{Z}^2/A\mathbb{Z}^2| = 8 \Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q13

Q14.

Solution

Concept — Gram matrix: $G_{ij} = \langle \mathbf{v}_i, \mathbf{v}_j \rangle$. The Gram matrix is positive definite iff the vectors are linearly independent.

Step 1 — Compute entries:

$$G_{11} = \langle (1,0,0), (1,0,0) \rangle = 1, \quad G_{12} = \langle (1,0,0), (1,1,0) \rangle = 1,$$

$$G_{21} = 1, \quad G_{22} = \langle (1,1,0), (1,1,0) \rangle = 2.$$

So $G = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

Step 2 — Positive definiteness: $\det(G) = 2 - 1 = 1 > 0$ and $\operatorname{tr}(G) = 3 > 0$ (all leading principal minors positive), so G is **positive definite**.

Why other options are wrong:

- Option A ($G = I_2$): Would require $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$, but $(1,0,0) \cdot (1,1,0) = 1 \neq 0$.
- Option B ($G = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$): Would require $\|\mathbf{v}_2\|^2 = 1$, but $\|(1,1,0)\|^2 = 2$.
- Option C ($G = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$): Would require $\|\mathbf{v}_1\|^2 = 2$, but $\|(1,0,0)\|^2 = 1$.

Final Answer: $G = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, positive definite $\Rightarrow \boxed{\text{D}}$

$\boxed{\text{Answer: (D)}}$ **Go Back to Q14**

Q15.

Solution

Concept — Burnside's lemma: The number of distinct orbits equals $\frac{1}{|G|} \sum_{g \in G} |X^g|$, where X^g is the set of elements fixed by g.

Step 1 — Fixed points for each rotation r^k of $\mathbb{Z}_6$ acting on $\{0,1\}^6$:

- r^0 (identity): all $2^6 = 64$ colourings fixed.
- r^1 (60): period must divide 1, so all 6 beads equal: $2^1 = 2$ fixed.
- r^2 (120): period divides $\gcd(2,6) = 2$, so pattern $(c_1, c_2, c_1, c_2, c_1, c_2)$: $2^2 = 4$ fixed.
- r^3 (180): period divides $\gcd(3,6) = 3$, pattern $(c_1, c_2, c_3, c_1, c_2, c_3)$: $2^3 = 8$ fixed.
- r^4: same as r^2: 4 fixed.
- r^5: same as r^1: 2 fixed.

Step 2 — Apply Burnside:

$$\text{Distinct necklaces} = \frac{64 + 2 + 4 + 8 + 4 + 2}{6} = \frac{84}{6} = 14.$$

Why other options are wrong:

- Option B (12): Arises from double-counting or omitting the r^3 contribution.
- Option C (16): Perhaps uses $|G| = 5$ or over-counts fixed points.
- Option D (10): Does not use the full sum over all group elements.

Final Answer: 14 distinct necklaces $\Rightarrow \boxed{\text{A}}$

$\boxed{\textbf{Answer: (A)}}$ **Go Back to Q15**

Q16.

Solution

Concept — Classification of groups of order 6: By Sylow's theorem, a group of order $6 = 2 \cdot 3$ has a normal Sylow 3-subgroup (since the number of Sylow 3-subgroups divides 2 and is $\equiv 1 \pmod 3$, so it equals 1). The group is either the direct product $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$ (abelian) or the non-abelian semidirect product $\mathbb{Z}_3 \rtimes \mathbb{Z}_2 \cong S_3 \cong D_3$.

Step 1 — Verify two non-isomorphic groups: $\mathbb{Z}_6$ is cyclic (abelian); S_3 is non-abelian ($\sigma\tau \neq \tau\sigma$ in general). They are not isomorphic, and no other group of order 6 exists.

Why other options are wrong:

- Option A (1): There exist two non-isomorphic groups ($\mathbb{Z}_6$ and S_3), not just one.
- Option B (3): There is no third non-isomorphic group of order 6.
- Option D (4): No groups of order 6 other than the two listed exist.

Final Answer: There are 2 non-isomorphic groups of order $6 \Rightarrow \boxed{\text{C}}$

$\boxed{\textbf{Answer: (C)}}$ **Go Back to Q16**

Q17.

Solution

Concept — Euclidean algorithm in $\mathbb{Z}[i]$: To divide a by b in $\mathbb{Z}[i]$, compute $a/b \in \mathbb{C}$, round real and imaginary parts to nearest integers to get q, then set $r = a - bq$ and verify $N(r) < N(b)$.

Step 1 — Compute a/b:

$$\frac{11+5i}{3+2i} = \frac{(11+5i)(3-2i)}{(3+2i)(3-2i)} = \frac{33-22i+15i-10i^2}{13} = \frac{43-7i}{13} \approx 3.31 - 0.538i.$$

Step 2 — Round to nearest Gaussian integer: $3.31 \to 3$, $-0.538 \to -1$, so $q = 3 - i$.

Step 3 — Verify remainder:

$$(3+2i)(3-i) = 9-3i+6i-2i^2 = 11+3i, \quad r = (11+5i)-(11+3i) = 2i.$$

$N(2i) = 4 < N(3+2i) = 13$ ✓.

Why other options are wrong:

- Option A ($q = 3, r = 2-i$): Valid ($N(2-i) = 5 < 13$) but not the standard rounding quotient.
- Option C ($q = 4-i, r = -3$): Valid ($N(-3) = 9 < 13$) but not closest to a/b.
- Option D ($q = 2+i$): $(3+2i)(2+i) = 4+7i$, $r = 7-2i$, $N = 53 > 13$. Invalid.

Final Answer: $q = 3-i,\ r = 2i \Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q17

Q18.

Solution

Concept — Irreducibility over $\mathbb{Q}$: By the Rational Root Theorem, rational roots of a monic polynomial with integer coefficients must be integers dividing the constant term. For degree ≤ 3, having no rational roots is equivalent to irreducibility.

Step 1 — Test each polynomial:

- $x^4 - 1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$: reducible.
- $x^2 - 4 = (x-2)(x+2)$: reducible.
- $x^3 - 3x + 2$: $f(1) = 1-3+2 = 0$. Has root $x = 1$; factors as $(x-1)^2(x+2)$: reducible.
- $x^3 + 2x + 1$: $f(1) = 1+2+1 = 4 \neq 0$; $f(-1) = -1-2+1 = -2 \neq 0$. No rational roots. Since degree 3, it is **irreducible** over $\mathbb{Q}$.

Why other options are wrong:

- Option A: $x^4 - 1$ factors into linear and quadratic factors over $\mathbb{Q}$.
- Option B: $x^2 - 4$ factors as a difference of squares.
- Option C: $x = 1$ is a root, so $(x-1)$ is a factor.

Final Answer: $x^3 + 2x + 1$ is irreducible over $\mathbb{Q} \Rightarrow \boxed{\text{D}}$

Answer: (D) Go Back to Q18

Q19.

Solution

Concept — Galois group of $x^3 - 2$: The splitting field of $x^3 - 2$ over $\mathbb{Q}$ is $K = \mathbb{Q}(\sqrt[3]{2}, \omega)$ where $\omega = e^{2\pi i/3}$. We have $[K : \mathbb{Q}] = [\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}]\cdot[K : \mathbb{Q}(\sqrt[3]{2})] = 3\cdot 2 = 6$.

Step 1 — Identify the Galois group: $|\mathrm{Gal}(K/\mathbb{Q})| = [K : \mathbb{Q}] = 6$. The Galois group acts faithfully on the three roots $\{\sqrt[3]{2}, \omega\sqrt[3]{2}, \omega^2\sqrt[3]{2}\}$, embedding it into S_3. Since $|S_3| = 6$, this embedding is an isomorphism: $\mathrm{Gal}(K/\mathbb{Q}) \cong S_3$.

Why other options are wrong:

- Option B ($\mathbb{Z}_6$): $\mathbb{Z}_6$ is abelian. S_3 is non-abelian (the transpositions and 3-cycle do not commute).
- Option C ($\mathbb{Z}_3$): Order 3, but $[K : \mathbb{Q}] = 6 \neq 3$.
- Option D ($\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$): Again abelian of order 6; the extension is not abelian since the complex conjugation and a non-trivial automorphism of $\mathbb{Q}(\sqrt[3]{2})$ do not commute.

Final Answer: $\mathrm{Gal}(K/\mathbb{Q}) \cong S_3 \Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q19

Q20.

Solution

Concept — Continuous bijection from compact to Hausdorff: If $f : X \to Y$ is a continuous bijection where X is compact and Y is Hausdorff, then f is a homeomorphism (its inverse is automatically continuous).

Step 1 — Verify the hypotheses:

- $[0, 2\pi]/\sim$ is compact: it is a continuous image of the compact set $[0, 2\pi]$.
- $S^1 \subset \mathbb{R}^2$ is Hausdorff.
- $\varphi : [\theta] \mapsto (\cos\theta, \sin\theta)$ is continuous and bijective (the identification exactly identifies 0 and 2π which map to the same point on S^1).

Therefore φ is a homeomorphism.

Why other options are wrong:

- Option A: φ is injective: distinct classes $[\theta_1] \neq [\theta_2]$ (with $\theta_1, \theta_2 \in [0, 2\pi)$) give distinct points on S^1.
- Option B: φ is surjective: every point $(\cos\theta, \sin\theta) \in S^1$ is in the image.

- Option D: φ is indeed a homeomorphism by the theorem stated; the inverse is also continuous.

Final Answer: φ is a homeomorphism $\Rightarrow \boxed{\text{C}}$

Answer: (C) Go Back to Q20

Q21.

Solution

Concept — Properties of compact Hausdorff spaces: These spaces are normal (T_4), so disjoint closed sets can be separated; singletons are closed; and they are Hausdorff. However, they need not be metrizable, and in non-metrizable compact Hausdorff spaces, closed sets need not be G_δ.

Step 1 — Analyse each option:

- Option A: Compact Hausdorff $\Rightarrow$ Hausdorff $\Rightarrow$ T_1 $\Rightarrow$ singletons closed. **True.**
- Option B: In a metrizable compact space every closed set F equals $\bigcap_n \{d(\cdot, F) < 1/n\}$ (a G_δ). But compact Hausdorff spaces need not be metrizable. In the Cech-Stone compactification $\beta\mathbb{N}$, points of $\beta\mathbb{N}\setminus\mathbb{N}$ are not G_δ sets. So this is **NOT necessarily true.**
- Option C: compact Hausdorff $\Rightarrow$ normal $\Rightarrow$ disjoint closed sets separated. **True.**
- Option D: compact Hausdorff $\Rightarrow$ normal. **True.**

Final Answer: Not necessarily true: every closed set is $G_\delta \Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q21

Q22.

Solution

Concept — First-countability: A space is first-countable if every point has a countable neighbourhood base. Metric spaces, standard $\mathbb{R}$, and second-countable spaces are first-countable, but the cocountable topology on $\mathbb{R}$ is not.

Step 1 — Show cocountable topology fails: Suppose $\{U_n\}_{n\geq 1}$ is a countable neighbourhood base at $x \in \mathbb{R}$. Each U_n^c is countable; let $S = \bigcup_{n=1}^{\infty} U_n^c$ (countable). Pick $z \in \mathbb{R} \setminus (S \cup \{x\})$ (possible since $\mathbb{R}$ is uncountable). Then $V = \mathbb{R} \setminus \{z\}$ is an open neighbourhood of x (its complement $\{z\}$ is countable), but $V \not\supseteq U_n$ for any n since $z \notin U_n^c$ means $z \in U_n$, so $U_n \not\subseteq V$. Contradiction.

Why other options are wrong:

- Option A: Metric spaces have countable base $\{B(x, 1/n)\}_{n\geq 1}$ at each point. First-countable.
- Option B: $\mathbb{R}$ with standard topology is a metric space. First-countable.
- Option C: Second-countable $\Rightarrow$ first-countable (restrict a countable base to a point).

Final Answer: $\mathbb{R}$ with cocountable topology is NOT first-countable $\Rightarrow$ $\boxed{\text{D}}$

$\boxed{\text{Answer: (D)}}$ Go Back to Q22

Q23.

Solution

Concept — Tietze Extension Theorem: A topological space X is normal iff every continuous function on a closed subset extends to all of X (Urysohn + Tietze).

Step 1 — Compact Hausdorff $\Rightarrow$ Normal: By a standard result (using compactness to separate disjoint closed sets by open sets), every compact Hausdorff space is normal. Hence the Tietze Extension Theorem applies.

Why other options are wrong:

- Option B (all Hausdorff): Hausdorff does not imply normal. A classic counterexample is the Moore plane (Niemytzki plane), which is Hausdorff but not normal.
- Option C (all connected): Connectedness is a topological property unrelated to normality; there exist connected non-normal spaces.
- Option D (all first-countable): First-countability does not imply normality; the Moore plane is also first-countable.

Final Answer: Compact Hausdorff spaces are guaranteed to be normal $\Rightarrow$ $\boxed{\text{A}}$

$\boxed{\text{Answer: (A)}}$ Go Back to Q23

Q24.

Solution

Concept — Argument principle: For a meromorphic function f inside a contour γ, the number of zeros minus poles (counted with multiplicity) equals $\dfrac{1}{2\pi i}\oint_\gamma \dfrac{f'(z)}{f(z)}\,dz$.

Step 1 — Locate zeros and poles inside $|z| = 4$:

- Zero at $z = 1$ (order 2): $|1| = 1 < 4$. ✓
- Zero at $z = -2$ (order 1): $|-2| = 2 < 4$. ✓
- Pole at $z = 3$ (order 3): $|3| = 3 < 4$. ✓

Total zeros $= 2 + 1 = 3$. Total poles $= 3$.

Step 2 — Net count: #zeros $-$ #poles $= 3 - 3 = 0$.

Why other options are wrong:

- Option A (1): Counts only the order-1 zero, misses order-2 zero or the pole.
- Option B (-1): Perhaps counts poles as 4 (wrong order) or zeros as 2.
- Option D (3): Counts only the zeros (3) and ignores the poles.

Final Answer: Zeros $-$ Poles $= 0 \Rightarrow \boxed{\text{C}}$

Answer: (C) Go Back to Q24

Q25.

Solution

Concept — Picard's Great Theorem: Near an essential singularity, a holomorphic function takes every complex value with at most one exception, infinitely often.

Step 1 — Identify the exceptional value: For $f(z) = e^{1/z}$, write $1/z = u + iv$. Then $f(z) = e^u(\cos v + i\sin v)$. We have $|f(z)| = e^u > 0$ for all $u \in \mathbb{R}$. Since $e^w > 0$ for all $w \in \mathbb{R}$, the modulus $|e^{1/z}|$ is always strictly positive. Hence $e^{1/z} \neq 0$ for all $z \neq 0$.

Step 2 — Confirm all other values are attained: For any $w \neq 0$, choose $1/z = \log w$ (principal branch near a point); this shows $e^{1/z}$ takes every nonzero value.

Why other options are wrong:

- Option A (-1): $e^{1/z} = -1$ when $1/z = i\pi + 2\pi ik$; this *is* attained near $z = 0$.
- Option C (1): $e^{1/z} = 1$ when $1/z = 2\pi ik$; also attained.

- Option D (e): $e^{1/z} = e$ when $1/z = 1$; also attained.

Final Answer: The exceptional value is $0 \Rightarrow \boxed{\text{B}}$

Answer: (B) **Go Back to Q25**

Q26.

Solution

Concept — Winding number and Cauchy's theorem: If a is outside the closed region bounded by γ, then $n(\gamma, a) = 0$.

Step 1 — Check location of a: γ is the unit circle $|z| = 1$. For $|a| = 3 > 1$, the point a lies *outside* the unit disk. The function $\frac{1}{z-a}$ is holomorphic on and inside γ (no singularity since a is outside). By Cauchy's theorem:

$$n(\gamma, a) = \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z-a} = 0.$$

Why other options are wrong:

- Option B (1): $n(\gamma, 0) = 1$ since 0 is *inside* the unit circle. For $|a| = 3$, the point is outside.
- Option A (2): Winding number 2 occurs when γ wraps around a twice; here it wraps once and a is outside.
- Option C (-1): Would arise if γ were traversed clockwise and a were inside.

Final Answer: $n(\gamma, a) = 0$ for $|a| = 3 \Rightarrow \boxed{\text{D}}$

Answer: (D) **Go Back to Q26**

Q27.

Solution

Concept — Residue theorem for improper integrals: $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\, dx = 2\pi i \sum_{\text{UHP}} \text{Res}(f, z_k)$.

Step 1 — Simple partial fractions: By partial fractions, $\frac{z^2}{(z^2+1)(z^2+4)} = \frac{1}{3} \cdot \frac{1}{z^2+1} - \frac{1}{3} \cdot \frac{-z^2+4}{(z^2+4)}$... Simpler: directly compute residues.

Step 2 — Residues in upper half-plane ($z = i$ and $z = 2i$):

$$\text{Res}(f, i) = \frac{-1}{(2i)(i^2+4)} = \frac{-1}{(2i)(3)} = \frac{i}{6}, \quad \text{Res}(f, 2i) = \frac{-4}{(4i^2+1)(4i)} = \frac{-4}{(-3)(4i)} = \frac{-i}{3}$$

$$\text{Sum} = \frac{i}{6} - \frac{i}{3} = -\frac{i}{6}.$$

Step 3 — Apply residue theorem and symmetry:

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \cdot \left(-\frac{i}{6}\right) = \frac{\pi}{3}.$$

Since the integrand is even, $I = \displaystyle\int_0^{\infty} f(x)\,dx = \frac{\pi}{6}$.

Why other options are wrong:

- Option B ($\pi/3$): This is $\int_{-\infty}^{\infty}$, not $\int_0^{\infty}$.
- Option C ($\pi/4$): Arises from $\int_0^{\infty} 1/(x^2+1)\,dx$, which is a different integral.
- Option D ($\pi/12$): Factor-of-2 error in one of the residues.

Final Answer: $I = \dfrac{\pi}{6} \Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q27

Q28.

Solution

Concept — Möbius function μ: $\mu(n) = 0$ if n has a squared prime factor; $\mu(n) = (-1)^k$ if n is a product of k distinct primes.

Step 1 — Evaluate each option:

- $\mu(12)$: $12 = 2^2 \cdot 3$. Since $2^2 \mid 12$, $\mu(12) = 0$. So Options A and D (claiming $\mu(12) = \pm 1$) are wrong.
- $\mu(6)$: $6 = 2 \cdot 3$ ($k = 2$ distinct primes). $\mu(6) = (-1)^2 = 1 \neq -1$. Option B is wrong.
- $\mu(30)$: $30 = 2 \cdot 3 \cdot 5$ ($k = 3$ distinct primes). $\mu(30) = (-1)^3 = -1$. **Correct.**

Why other options are wrong:

- Option A: $\mu(12) = 0$ (squared factor $4 = 2^2$), not -1.
- Option B: $\mu(6) = 1$ (two distinct prime factors), not -1.
- Option D: $\mu(12) = 0$, not 1.

Final Answer: $\mu(30) = -1 \Rightarrow \boxed{C}$

Answer: (C) Go Back to Q28

Q29.

Solution

Concept — Multiplicative function σ: $\sigma(mn) = \sigma(m)\sigma(n)$ for $\gcd(m,n) = 1$, and $\sigma(p^k) = 1 + p + \cdots + p^k = (p^{k+1} - 1)/(p-1)$.

Step 1 — Factorise and compute: $18 = 2 \cdot 3^2$.

$$\sigma(2) = 1 + 2 = 3, \quad \sigma(3^2) = 1 + 3 + 9 = 13.$$

$$\sigma(18) = \sigma(2) \cdot \sigma(3^2) = 3 \times 13 = 39.$$

Why other options are wrong:

- Option A (21): Perhaps sums divisors incorrectly as $1 + 2 + 3 + 6 + 9 = 21$ (missing divisor 18). Divisors of 18 are $1, 2, 3, 6, 9, 18$: sum $= 39$.
- Option C (26): Arises from $\sigma(13) = 14$ or a wrong factorisation.
- Option D (30): Confusion with $\sigma(p \cdot q)$ for distinct primes without using the correct formula.

Final Answer: $\sigma(18) = 39 \Rightarrow \boxed{B}$

Answer: (B) Go Back to Q29

Q30.

Solution

Concept — Poisson distribution: For $X \sim \text{Poisson}(\lambda)$, $P(X = k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$.

Step 1 — Substitute $\lambda = 3$, $k = 2$:

$$P(X = 2) = \frac{e^{-3} \cdot 3^2}{2!} = \frac{e^{-3} \cdot 9}{2} = \frac{9e^{-3}}{2}.$$

Why other options are wrong:

- Option A $(e^{-3} \cdot 3)$: Uses $k = 1$ formula $P(X = 1) = 3e^{-3}$, not $k = 2$.
- Option B $(e^{-3} \cdot 9)$: Forgets to divide by $2! = 2$.
- Option C $(e^{-3}/2)$: Sets $\lambda^k = 1$ (i.e., confuses $\lambda = 1$) and divides by $2!$.

Final Answer: $P(X = 2) = \dfrac{9e^{-3}}{2} \Rightarrow \boxed{\text{D}}$

Answer: (D) Go Back to Q30

Q31.

Solution

Concept — Dirichlet's test: If the partial sums of $\sum a_n$ are *bounded* and $\{b_n\}$ is monotone with $b_n \to 0$, then $\sum a_n b_n$ converges. The converse is **not** true.

Verdict: False — Counterexample: Take $a_n = (-1)^n$ and $b_n = \dfrac{1}{n}$.

- $\{b_n\} = \{1/n\}$ is monotone decreasing with $b_n \to 0$. ✓
- $\sum a_n b_n = \sum \dfrac{(-1)^n}{n}$ converges by the alternating series test. ✓
- But $\sum a_n = \sum (-1)^n$ **diverges** (terms do not tend to zero). ✓

So the hypotheses of the statement are satisfied, yet $\sum a_n$ diverges. The statement is **False**.

Final Answer: False $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q31

Q32.

Solution

Concept — Nowhere dense sets: A is nowhere dense iff $\text{int}(\overline{A}) = \emptyset$ (the closure has empty interior).

Verdict: True — Proof sketch (for two sets, then induction): Let A and B be nowhere dense. For any nonempty open set U:

- Since A is nowhere dense, $U \setminus \overline{A}$ is nonempty and open (as $\overline{A}$ has empty interior).
- Let $V = U \setminus \overline{A}$. Since B is nowhere dense, $V \setminus \overline{B}$ is nonempty and open, and is disjoint from both A and B.

Hence every nonempty open set U contains a nonempty open subset disjoint from $A \cup B$, proving $A \cup B$ is nowhere dense. By induction, any finite union of nowhere dense sets is nowhere dense.

Final Answer: True $\Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q32

Q33.

Solution

Concept — Absolute continuity vs Lipschitz: Lipschitz continuity ($|f(x) - f(y)| \leq K|x-y|$) implies absolute continuity (take $\delta = \varepsilon/K$). The *converse* fails: absolute continuity does not imply Lipschitz.

Verdict: False — Counterexample: Take $f(x) = \sqrt{x}$ on $[0,1]$.

- f is absolutely continuous: $f'(x) = \dfrac{1}{2\sqrt{x}} \in L^1([0,1])$ (integrable near 0).
- f is **not** Lipschitz: $f'(x) = 1/(2\sqrt{x}) \to \infty$ as $x \to 0^+$. For small $x > 0$, $\dfrac{f(x)-f(0)}{x-0} = \dfrac{1}{\sqrt{x}} \to \infty$, exceeding any finite Lipschitz constant.

Final Answer: False $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q33

Q34.

Solution

Concept — Uniform continuity on open intervals: It is a theorem that if $f : (a,b) \to \mathbb{R}$ is uniformly continuous, then both one-sided limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist (as finite real numbers). However, these two limits need not be equal.

Verdict: False — Counterexample: Take $f(x) = x$ on $(0,1)$.

- f is uniformly continuous (Lipschitz with $K = 1$). ✓
- $\lim_{x\to 0^+} f(x) = 0$ and $\lim_{x\to 1^-} f(x) = 1$.
- The two limits exist but are **not equal**: $0 \neq 1$.

The statement claims both limits exist *and are equal*, which is false.

Final Answer: False $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q34

Q35.

Solution

Concept — Uniform convergence and Riemann integration: Uniform convergence preserves Riemann integrability and commutes with the integral on a bounded closed interval.

Verdict: True — Proof sketch:

- Each f_n is Riemann integrable, so bounded. Uniform convergence gives $f_n \rightrightarrows f$, so f is bounded and the oscillation of f is controlled by that of the f_n.
- For any $\varepsilon > 0$, choose N so that $\|f_N - f\|_\infty < \varepsilon/(3(b-a))$. Then f is Riemann integrable (since f_N is, and small uniform perturbation preserves integrability).
- $\left|\int_a^b f_n - \int_a^b f\right| \leq \int_a^b |f_n - f| \leq (b-a)\|f_n - f\|_\infty \to 0.$

Final Answer: True $\Rightarrow$ $\boxed{\text{A}}$

Answer: (A) **Go Back to Q35**

Q36.

Solution

Concept — Group axioms for $\mathrm{GL}(n, \mathbb{R})$: We verify closure, associativity, identity, and inverses.

Verdict: True — Verification:

- **Closure:** If $\det(A) \neq 0$ and $\det(B) \neq 0$, then $\det(AB) = \det(A)\det(B) \neq 0$, so $AB \in \mathrm{GL}(n, \mathbb{R})$.
- **Associativity:** Matrix multiplication is associative.
- **Identity:** I_n is invertible with $\det(I_n) = 1 \neq 0$.
- **Inverses:** $\det(A) \neq 0 \Rightarrow A^{-1}$ exists and $\det(A^{-1}) = 1/\det(A) \neq 0$.

All group axioms hold.

Final Answer: True $\Rightarrow$ $\boxed{\text{A}}$

Answer: (A) **Go Back to Q36**

Q37.

Solution

Concept — Diagonalizability over $\mathbb{R}$: The characteristic polynomial of a real matrix has real coefficients, so complex eigenvalues come in conjugate pairs. A 3×3 real matrix with a conjugate pair $\alpha \pm i\beta$ ($\beta \neq 0$) has its third eigenvalue real. Such a matrix cannot be diagonalized over $\mathbb{R}$ because the conjugate pair contribute a 2×2 real Jordan block.

Verdict: False — Explanation: Let A be a real 3×3 matrix with eigenvalues $\alpha + i\beta$, $\alpha - i\beta$, and $\gamma \in \mathbb{R}$ ($\beta \neq 0$). Over $\mathbb{R}$, there are no real eigenvectors for the complex eigenvalues. The real canonical (Jordan) form of A is $\begin{pmatrix} \alpha & -\beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix}$, which is **not diagonal**. Hence A is NOT diagonalizable over $\mathbb{R}$.

Final Answer: False $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q37

Q38.

Solution

Concept — Rank-Nullity theorem: For any linear map $T : V \to W$ between finite-dimensional vector spaces, $\dim(\ker T) + \dim(\operatorname{Im} T) = \dim V$.

Verdict: True — Proof sketch: Let $\{e_1, \ldots, e_k\}$ be a basis of $\ker T$ (where $k = \dim \ker T$). Extend to a basis $\{e_1, \ldots, e_k, f_1, \ldots, f_r\}$ of V. Then $\{T(f_1), \ldots, T(f_r)\}$ is a basis of $\operatorname{Im} T$ (one shows they span $\operatorname{Im} T$ and are linearly independent). Thus $\dim(\operatorname{Im} T) = r$, and $k + r = \dim V$.

Final Answer: True $\Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q38

Q39.

Solution

Concept — Dimension of tensor product: $\dim(V \otimes W) = (\dim V) \cdot (\dim W) = mn$, not $m + n$.

Verdict: False — Explanation: If $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_n\}$ are bases of V and W, then $\{e_i \otimes f_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis of $V \otimes W$, giving mn basis elements.

Counterexample: $\dim(\mathbb{R}^2 \otimes \mathbb{R}^3) = 2 \cdot 3 = 6 \neq 2 + 3 = 5$.

The formula $\dim(V \oplus W) = m + n$ applies to the **direct sum**, not the tensor product.

Final Answer: False $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q39

Q40.

Solution

Concept — Simplicity of alternating groups: The alternating group A_n is simple (has no proper nontrivial normal subgroups) for all $n \geq 5$. This is a classical theorem in group theory.

Verdict: True — Key steps:

- For $n \geq 5$, every non-identity element of A_n is a product of 3-cycles.
- Any normal subgroup $N \trianglelefteq A_n$ containing a 3-cycle must contain all 3-cycles (since A_n is generated by 3-cycles and acts transitively by conjugation on them for $n \geq 5$), so $N = A_n$.
- Thus A_n has no proper nontrivial normal subgroups: it is simple.

(Note: A_4 is NOT simple; it has a normal subgroup of order 4.)

Final Answer: True $\Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q40

Q41.

Solution

Concept — Noetherian vs Artinian rings: Noetherian (ACC on ideals) does NOT imply Artinian (DCC on ideals).

Verdict: False — Counterexample: The ring $\mathbb{Z}$ is Noetherian: every ideal of $\mathbb{Z}$ is principal, $(n) = n\mathbb{Z}$, and every ascending chain stabilises. However $\mathbb{Z}$ is NOT Artinian: the descending chain

$$(2) \supset (4) \supset (8) \supset (16) \supset \cdots$$

(where $(2^k) = 2^k\mathbb{Z}$) is strictly descending and never stabilises.

Final Answer: False $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q41

Q42.

Solution

Concept — Galois group as permutation group: The Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$ of the splitting field K of an irreducible polynomial f of degree n acts faithfully on the n roots of f, giving an injective homomorphism $G \hookrightarrow S_n$.

Verdict: True — Argument: Let $\alpha_1, \ldots, \alpha_n$ be the roots of f in K. Every $\sigma \in G$ is a field automorphism of K fixing $\mathbb{Q}$; since $f(\sigma(\alpha_i)) = \sigma(f(\alpha_i)) = \sigma(0) = 0$, each σ permutes the roots. The resulting map $G \to S_n$ (by $\sigma \mapsto$ the permutation of roots induced by σ) is an injective group homomorphism since $\sigma = \mathrm{id}$ iff σ fixes all roots iff σ is the identity on K.

Final Answer: True $\Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q42

Q43.

Solution

Concept — Paracompact vs compact: Paracompactness (every open cover has a locally finite open refinement) is a weaker property than compactness.

Verdict: False — Counterexample: $\mathbb{R}$ with the usual topology is paracompact (it is a metric space, and by a theorem of A.H. Stone, every metric space is paracompact). However, $\mathbb{R}$ is NOT compact (the open cover $\{(-n, n) : n \in \mathbb{N}\}$ has no finite subcover).

Final Answer: False $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q43

Q44.

Solution

Concept — Homotopy equivalence: $X \simeq Y$ if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ with $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

Verdict: True — Verification of equivalence relation:

- **Reflexive:** $X \simeq X$ via $f = g = \mathrm{id}_X$, with all homotopies constant.
- **Symmetric:** If $X \simeq Y$ via (f, g), then $Y \simeq X$ via (g, f).

- **Transitive:** If $X \simeq Y$ via (f, g) and $Y \simeq Z$ via (h, k), then $X \simeq Z$ via $(h \circ f, g \circ k)$, using the composition of homotopies.

Final Answer: True $\Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q44

Q45.

Solution

Concept — Fundamental group of spheres: $\pi_1(S^n) = 0$ for $n \geq 2$ (higher-dimensional spheres are simply connected). Only $\pi_1(S^1) = \mathbb{Z}$.

Verdict: False — Explanation: The 2-sphere S^2 is simply connected: every loop in S^2 can be contracted to a point (intuitively, any loop on the surface of a ball can be shrunk to a point). Therefore $\pi_1(S^2) = 0$ (the trivial group), not $\mathbb{Z}$.

The statement confuses S^2 with S^1: it is the *circle* S^1 that has $\pi_1(S^1) = \mathbb{Z}$ (generated by the winding class).

Final Answer: False $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q45

Q46.

Solution

Concept — Weierstrass product/factorisation theorem: Given any sequence $\{a_n\}$ of nonzero complex numbers with $|a_n| \to \infty$ (and possibly finitely many repetitions for prescribed multiplicities), there exists an entire function with zeros exactly at the a_n (with prescribed multiplicities) and nowhere else.

Verdict: True — Sketch: One constructs the entire function as an infinite product $\prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right)$, where $E_p(u) = (1-u)\exp\left(u + \frac{u^2}{2} + \cdots + \frac{u^p}{p}\right)$ are Weierstrass elementary factors. The condition $|a_n| \to \infty$ ensures the product converges uniformly on compact sets, yielding an entire function with the desired zeros.

Final Answer: True $\Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q46

Q47.

Solution

Concept — Schwarz lemma: If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic with $f(0) = 0$, then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$ and $|f'(0)| \leq 1$. Equality $|f(z)| = |z|$ for some $z \neq 0$ (or $|f'(0)| = 1$) holds *if and only if* $f(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$.

Verdict: False — Explanation and Counterexample: A holomorphic bijection $f : \mathbb{D} \to \mathbb{D}$ is a Möbius transformation preserving the disk. With $f(0) = 0$, the Schwarz lemma shows $f(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$. This is NOT necessarily the identity: take $f(z) = iz$ ($\theta = \pi/2$). Then f is a holomorphic bijection of $\mathbb{D}$ with $f(0) = 0$, but $f(1/2) = i/2 \neq 1/2$.

Final Answer: False $\Rightarrow$ $\boxed{\text{B}}$

Answer: (B) Go Back to Q47

Q48.

Solution

Concept — Bertrand's postulate: For every integer $n \geq 1$, there exists at least one prime p satisfying $n < p \leq 2n$.

Verdict: True — Statement and examples:

- Proved by Chebyshev (1850) using estimates on $\pi(x)$.
- Verified easily for small n: $n = 1$: $p = 2 \in (1, 2]$; $n = 2$: $p = 3 \in (2, 4]$; $n = 3$: $p = 5 \in (3, 6]$; $n = 4$: $p = 5 \in (4, 8]$, etc.
- Erdos gave an elementary proof using properties of binomial coefficients.

The statement is exactly Bertrand's postulate, which is a proven theorem.

Final Answer: True $\Rightarrow$ $\boxed{\text{A}}$

Answer: (A) Go Back to Q48

Q49.

Solution

Concept — Prime Number Theorem (PNT): The PNT states $\pi(n) \sim \dfrac{n}{\ln n}$ as $n \to \infty$, meaning $\lim_{n\to\infty} \dfrac{\pi(n)}{n/\ln n} = 1$. This is an *asymptotic* statement only.

Verdict: False — Explanation: The PNT gives only $\pi(n) \sim n/\ln n$; it does **not** assert that $\pi(n) = \lfloor n/\ln n \rfloor$ exactly. In fact, $\pi(n) - n/\ln n$ oscillates and grows in magnitude. For example:

- $\pi(100) = 25$, but $\lfloor 100/\ln 100 \rfloor = \lfloor 21.7\ldots \rfloor = 21$.
- $\pi(1000) = 168$, but $\lfloor 1000/\ln 1000 \rfloor = \lfloor 144.7\ldots \rfloor = 144$.

No exact closed-form formula $\pi(n) = \lfloor n/\ln n \rfloor$ holds for large n; the asymptotic $\sim$ sign cannot be replaced by equality.

Final Answer: False $\Rightarrow \boxed{\text{B}}$

Answer: (B) Go Back to Q49

Q50.

Solution

Concept — Kolmogorov's 0-1 law: Let $X_1, X_2, \ldots$ be independent random variables. The *tail σ-algebra* is $\mathcal{T} = \bigcap_{n=1}^{\infty} \sigma(X_n, X_{n+1}, \ldots)$. For any event $A \in \mathcal{T}$, $P(A) \in \{0, 1\}$.

Verdict: True — Sketch:

- $A \in \mathcal{T}$ is independent of $\sigma(X_1, \ldots, X_n)$ for every n (by independence of the X_i).
- Hence A is independent of $\bigcup_n \sigma(X_1, \ldots, X_n)$, which generates $\mathcal{T}$.
- Therefore A is independent of itself: $P(A) = P(A \cap A) = P(A)^2$, forcing $P(A) \in \{0, 1\}$.

Examples of tail events: $\{$the series $\sum X_n$ converges$\}$, $\{\limsup X_n > c\}$.

Final Answer: True $\Rightarrow \boxed{\text{A}}$

Answer: (A) Go Back to Q50

Answer Key

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	B	2	D	3	A	4	C	5	B
6	D	7	A	8	C	9	B	10	D
11	A	12	C	13	B	14	D	15	A
16	C	17	B	18	D	19	A	20	C
21	B	22	D	23	A	24	C	25	B
26	D	27	A	28	C	29	B	30	D
31	B	32	A	33	B	34	B	35	A
36	A	37	B	38	A	39	B	40	A
41	B	42	A	43	B	44	A	45	B
46	A	47	B	48	A	49	B	50	A