IIT JAM Mathematics

Sample Paper – 1

Instructions

- This paper contains **60 questions** carrying a maximum of **100 marks**. Duration: **3 hours (180 minutes)**.
- **Section A** (**Q1–Q30**, MCQ): Q1–Q10 carry **+1 mark** each; wrong response: $-\frac{1}{3}$ mark. Q11–Q30 carry **+2 marks** each; wrong response: $-\frac{2}{3}$ mark. Choose the **single best** option.
- **Section B** (**Q31–Q40**, MSQ): Each carries **+2 marks**. **No negative marking. One or more** options may be correct; full marks only if *all* correct options are selected.
- **Section C** (**Q41–Q60**, NAT): Q41–Q50 carry **+1 mark**; Q51–Q60 carry **+2 marks**. **No negative marking.** Enter the numerical value using the virtual keypad.
- **Calculator policy:** Personal calculators and electronic devices are **strictly prohibited**. A virtual scientific calculator is provided in the CBT interface.
- Sections must be attempted in order; you cannot return to a previous section.

Section A — Multiple Choice Questions (MCQ)

***Q1–Q10**: 1 mark each ($-\frac{1}{3}$ for wrong). Choose the **single** correct option.*

Q1. Which convergence test most directly establishes the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$?

(A) Integral test

(B) Ratio test

(C) Root test

(D) Comparison with $\displaystyle\sum \frac{1}{n^2}$

Q2. Let $f(x) = \dfrac{x^2-4}{x-2}$ for $x \neq 2$ and $f(2) = k$. For f to be continuous at $x = 2$, the value of k must be:

(A) 0

(B) 2

(C) 4

(D) -2

Q3. Find the rank of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{pmatrix}$.

1	2	3
2	4	6
1	1	1

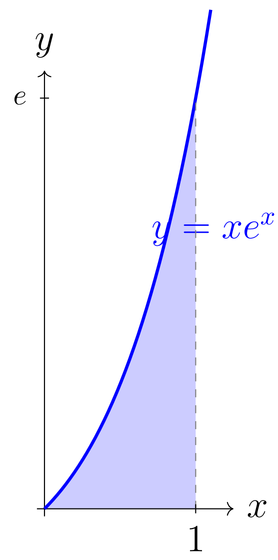
$R_2 = 2R_1$

(A) 2
(B) 3
(C) 1
(D) 0

Q4. What is the order of the element 8 in the group $(\mathbb{Z}_{12}, +)$?

(A) 2
(B) 4
(C) 8
(D) 3

Q5. Evaluate the definite integral $\int_0^1 xe^x \, dx$.



(A) e
(B) 1
(C) $e - 1$

(D) 2

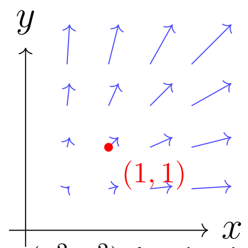
Q6. Find $\frac{d}{dx}[x^x]$ evaluated at $x = 1$.

(A) 1
(B) e
(C) 0
(D) 2

Q7. The general solution of the separable ODE $\frac{dy}{dx} = \frac{y}{x}$ is:

(A) $y = Ce^x$
(B) $y = Cx^2$
(C) $y = Cx$
(D) $y = C \ln x$

Q8. Compute $\nabla \cdot \mathbf{F}$ at the point $(1, 1, 1)$ for $\mathbf{F} = (x^2,\, y^2,\, z^2)$.



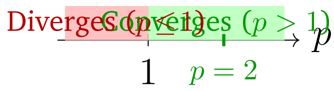
$\mathbf{F} = (x^2, y^2)$ (projection)

(A) 2
(B) 4
(C) 1
(D) 6

Q9. Evaluate $\frac{\partial^2}{\partial x\, \partial y}(x^3y^2)$ at the point $(1, 1)$.

(A) 3
(B) 6
(C) 2
(D) 12

Q10. Which of the following series **converges**?



(A) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

(B) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

(C) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

(D) $\displaystyle\sum_{n=1}^{\infty} 1$

***Q11–Q30**: 2 marks each ($-\frac{2}{3}$ for wrong). Choose the **single** correct option.*

Q11. By the Mean Value Theorem for integrals, $\displaystyle\int_0^1 e^x\,dx = e^c \cdot 1$ for some $c \in (0,1)$. In which interval does $c = \ln(e-1)$ lie?

(A) $c < 0$

(B) $0 < c < \frac{1}{2}$

(C) $\frac{1}{2} < c < 1$

(D) $c > 1$

Q12. The sequence $\left(1 + \dfrac{1}{n}\right)^n$ converges to:

(A) 1

(B) 2

(C) π

(D) e

Q13. A 3×3 real matrix A has eigenvalues 1, 2, and 3. What is $\det(A)$?

(A) 1

(B) 6

(C) 3

(D) 2

Q14. Find the nullity of the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose standard matrix is $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(A) 1
(B) 2
(C) 0
(D) 3

Q15. Which of the following is a subgroup of $(\mathbb{Z}_{12}, +)$?

(A) $\{0, 5, 10\}$
(B) $\{0, 4, 8, 2\}$
(C) $\{0, 4, 8\}$
(D) $\{0, 3, 6, 10\}$

Q16. How many group homomorphisms are there from $\mathbb{Z}_6$ to $\mathbb{Z}_4$?

(A) 6
(B) 4
(C) 3
(D) 2

Q17. Using the method of undetermined coefficients, find a particular solution y_p of

$$y'' - 3y' + 2y = e^{3x}.$$

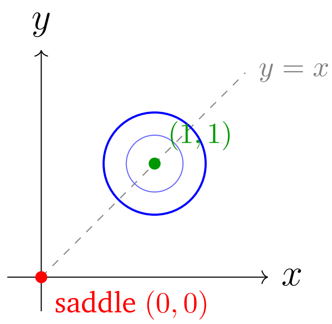
(A) $\dfrac{e^{3x}}{2}$
(B) e^{3x}
(C) $\dfrac{e^{3x}}{3}$
(D) $\dfrac{e^{3x}}{6}$

Q18. Which of the following ODEs is **NOT** exact?

(A) $(2xy)\,dx + (x^2 + 1)\,dy = 0$
(B) $(x^2y)\,dx + (xy^2)\,dy = 0$

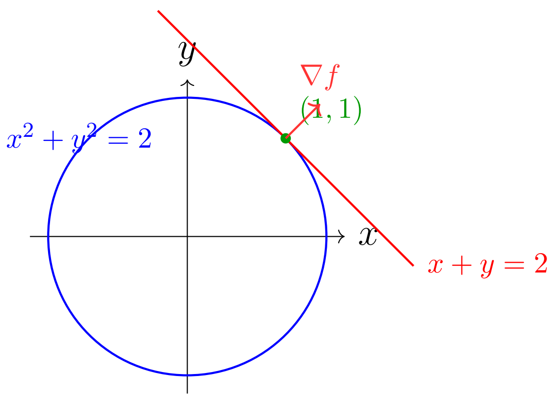
(C) $(x + y^2)\, dx + (2xy)\, dy = 0$
(D) $(e^x \sin y)\, dx + (e^x \cos y)\, dy = 0$

Q19. For $f(x, y) = x^3 + y^3 - 3xy$, classify the critical point $(1, 1)$.



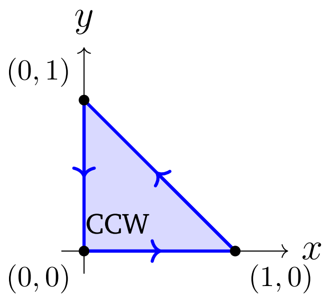
(A) Local maximum
(B) Saddle point
(C) Local minimum
(D) Cannot be determined

Q20. Using Lagrange multipliers, find the **maximum** value of $f(x, y) = x + y$ subject to the constraint $x^2 + y^2 = 2$.



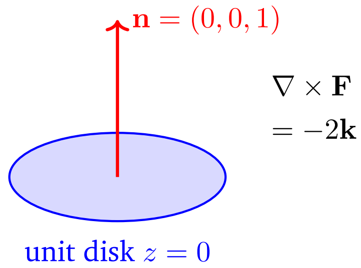
(A) 1
(B) $\sqrt{2}$
(C) $\sqrt{3}$
(D) 2

Q21. Using Green's theorem, evaluate $\oint_C (y\, dx - x\, dy)$ where C is the boundary of the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, traversed counterclockwise.



(A) -1
(B) 1
(C) -2
(D) 0

Q22. Using Stokes' theorem, compute $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ for $\mathbf{F} = (y, -x, z^2)$ over the unit disk $S : x^2 + y^2 \leq 1$, $z = 0$, with upward normal.

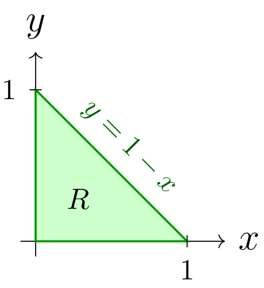


(A) $-\pi$
(B) -2π
(C) 2π
(D) 0

Q23. The value of the improper integral $\int_0^\infty e^{-x^2}\,dx$ is:

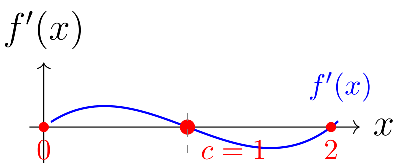
(A) 1
(B) π
(C) $\sqrt{\pi}$
(D) $\dfrac{\sqrt{\pi}}{2}$

Q24. Evaluate $\int_0^1 \int_0^{1-x} (x+y)\,dy\,dx$ over the triangular region shown.



(A) $\frac{1}{4}$
(B) $\frac{1}{6}$
(C) $\frac{1}{3}$
(D) $\frac{1}{2}$

Q25. Consider $f(x) = x^4 - 4x^3 + 4x^2$ on $[0, 2]$. Since $f(0) = f(2) = 0$, by Rolle's theorem there exists $c \in (0, 2)$ with $f'(c) = 0$. How many such values c lie in $(0, 2)$?



(A) 1
(B) 2
(C) 3
(D) 0

Q26. The Taylor series of $\sin x$ about $x = 0$ converges for:

(A) Only $x = 0$
(B) All $x \in \mathbb{R}$
(C) $|x| < 1$
(D) $|x| < \pi$

Q27. What is the dimension of the eigenspace corresponding to $\lambda = 0$ for the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$?

(A) 2

(B) 0
(C) 3
(D) 1

Q28. Which of the following is a **normal** subgroup of the symmetric group S_3?

(A) $\{e, (12)\}$
(B) $\{e, (13)\}$
(C) $A_3 = \{e, (123), (132)\}$
(D) $\{e, (23)\}$

Q29. A group homomorphism $f : G \to H$ satisfies $\ker(f) = \{e_G\}$. Then f is:

(A) Surjective
(B) Injective
(C) Bijective
(D) Neither injective nor surjective

Q30. Every Cauchy sequence converges if and only if the metric space is:

(A) Complete
(B) Compact
(C) Connected
(D) Dense

Section B — Multiple Select Questions (MSQ)

***Q31–Q40**: 2 marks each. **No negative marking. One or more** options may be correct; full marks only if ALL correct options selected.*

Q31. For which values of p does the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ **converge**?

(A) $p = 2$
(B) $p = \frac{3}{2}$
(C) $p = \frac{1}{2}$
(D) $p = 1$

Q32. At a local maximum $x = c$ of a differentiable function $f : \mathbb{R} \to \mathbb{R}$, which of the following **must** hold?

(A) $f'(c) = 0$
(B) $f''(c) > 0$
(C) $f(c) \geq f(x)$ for all x sufficiently near c
(D) $f''(c) \leq 0$ (whenever $f''(c)$ exists)

Q33. Which of the following sequences are Cauchy sequences in $\mathbb{R}$?

(A) $a_n = n$
(B) $a_n = \dfrac{1}{n}$
(C) $a_n = \dfrac{(-1)^n}{n}$
(D) $a_n = (-1)^n$

Q34. Which of the following are TRUE for any $n \times n$ real matrix A?

(A) $\text{rank}(A) + \text{nullity}(A) = n$
(B) If A is invertible, then all eigenvalues of A are nonzero
(C) $\text{rank}(A^T) \neq \text{rank}(A)$ in general
(D) If λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2

Q35. Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Which of the following are TRUE?

(A) A is diagonalizable over $\mathbb{R}$
(B) A has two distinct eigenvalues
(C) The characteristic polynomial of A is $(\lambda - 2)^2$
(D) The minimal polynomial of A is $(x - 2)^2$

Q36. A group G of order 12 **must** have:

(A) A subgroup of order 3
(B) Be abelian
(C) A subgroup of order 2
(D) Be isomorphic to $\mathbb{Z}_{12}$

Q37. For a group homomorphism $f : G \to H$, which of the following are **necessarily** TRUE?

(A) f is surjective

(B) $f(e_G) = e_H$
(C) f is injective
(D) $\ker(f)$ is a normal subgroup of G

Q38. Consider the first-order linear ODE $y' + \frac{1}{x}y = x$. Which of the following are TRUE?

(A) An integrating factor is $\mu(x) = x$
(B) The general solution is $y = \frac{x^2}{3} + \frac{C}{x}$
(C) The particular solution with $y(1) = 1$ is $y = x$
(D) An integrating factor is $\mu(x) = e^x$

Q39. Which of the following vector fields on $\mathbb{R}^3$ are **conservative**?

(A) $\mathbf{F} = (-y,\, x,\, 0)$
(B) $\mathbf{F} = (2x,\, 2y,\, 2z)$
(C) $\mathbf{F} = (yz,\, xz,\, xy)$
(D) $\mathbf{F} = (e^x,\, 0,\, 0)$

Q40. For the function $f(x, y) = x^2 - y^2$, which of the following are TRUE?

(A) $(0, 0)$ is a critical point of f
(B) $(0, 0)$ is a local minimum of f
(C) $(0, 0)$ is a saddle point of f
(D) f has no critical points

Section C — Numerical Answer Type (NAT)

***Q41–Q50**: 1 mark each. No negative marking. Enter the exact numerical value.*

Q41. Find the rank of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

$\cancel{1}$	2	3
0	$\cancel{1}$	2
0	0	$\cancel{1}$

diagonal $\neq 0$

Q42. Find the order of the element 5 in the group $(\mathbb{Z}_{20}, +)$.

Q43. Evaluate $\int_0^{\pi} \sin x \, dx$.

Q44. How many elements of order 2 are there in the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$?

Q45. Compute $\frac{\partial^2}{\partial x \, \partial y}\big(x^3y^2 + \sin(xy)\big)$ at the point $(1, 0)$.

Q46. How many critical points does $f(x, y) = x^3 - 3x + y^3 - 3y$ have?

Q47. For any smooth vector field $\mathbf{F}$ on $\mathbb{R}^3$, the value of $\nabla \cdot (\nabla \times \mathbf{F})$ is:

Q48. Evaluate $\int_0^1 x^3 \, dx$.

Q49. How many eigenvalues (counting algebraic multiplicity) does any 2×2 matrix over $\mathbb{C}$ have?

Q50. What is the order of the symmetric group S_4?

***Q51–Q60**: 2 marks each. No negative marking. Enter the exact numerical value.*

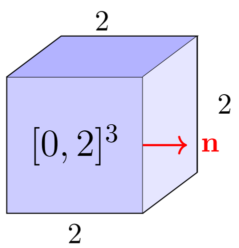
Q51. Let A be a 3×3 real matrix with eigenvalues 1, 2, and 3. Find $\operatorname{tr}(A^2)$.

Q52. How many subgroups does the group $\mathbb{Z}_{12}$ have?

Q53. Find the constant particular solution y_p of the ODE $y'' - 5y' + 6y = 6$.

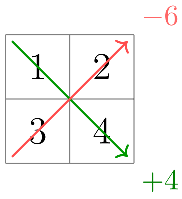
Q54. Evaluate $\int_0^1 \int_0^1 (x + y) \, dx \, dy$.

Q55. Using the Divergence Theorem, compute the outward flux of $\mathbf{F} = (x^2, y^2, z^2)$ through the surface of the cube $[0, 2]^3$.



Q56. What is the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$?

Q57. Compute $\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.



Q58. Find the minimum value of $f(x, y) = x^2 + y^2$ subject to the constraint $x + y = 1$.

Q59. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$.

Q60. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Find $\mathrm{tr}(A^4)$.

Solutions

IIT JAM Mathematics – Sample Paper 1

Section A Solutions — Q1–Q10 (1 mark each)

Q1.

Solution

Concept: Ratio Test for series $\sum a_n$: if $\lim_{n\to\infty} |a_{n+1}/a_n| = L < 1$, the series converges.

Step 1: Let $a_n = \dfrac{n!}{n^n}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!\,n^n}{(n+1)^{n+1}\,n!} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n.$$

Step 2: As $n \to \infty$, $\left(1 - \dfrac{1}{n+1}\right)^n \to e^{-1} = \dfrac{1}{e} < 1$.

Conclusion: The Ratio Test gives $L = 1/e < 1$, so the series converges. This is the most *direct* test here.

Why (A) wrong: Integral test requires integrating $x!/x^x$, which is intractable.
Why (C) wrong: Root test also works (gives same limit) but is less direct.
Why (D) wrong: Comparison with $1/n^2$ requires establishing $n!/n^n < 1/n^2$ separately.

Answer: (B) ← *Go back to Q1*

Q2.

Solution

Concept: Continuity at $x = a$ requires $\lim_{x\to a} f(x) = f(a)$.

Step 1: Simplify the limit: $\dfrac{x^2-4}{x-2} = \dfrac{(x-2)(x+2)}{x-2} = x+2$ for $x \neq 2$.

Step 2: $\lim_{x\to 2}(x+2) = 4$. For continuity, $k = f(2) = 4$.

Why (A) wrong: $k = 0$ means $f(2) = 0 \neq \lim f(x) = 4$.
Why (B) wrong: $k = 2$ is the value at $x = 0$, not at the limit.
Why (D) wrong: $k = -2$ has no connection to the limit here.

Answer: (C) ← *Go back to Q2*

Q3.

Solution

Concept: Rank = number of linearly independent rows (or columns).

Step 1: Note Row$_2$ $= 2\times$Row$_1$, so rows 1 and 2 are linearly dependent. Eliminate: $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - R_1$:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -1 & -2 \end{pmatrix}.$$

Step 2: Two nonzero rows remain (R_1 and modified R_3). Hence rank$(A) = 2$.

Why (B) wrong: Rank 3 would require all rows linearly independent — they are not.

Why (C) wrong: Rank 1 would mean all rows are multiples of one row — R_3 is not a multiple of R_1.

Why (D) wrong: Rank 0 only for the zero matrix.

Answer: (A) ← *Go back to Q3*

Q4.

Solution

Concept: The order of element a in $(\mathbb{Z}_n, +)$ is $n/\gcd(a, n)$.

Step 1: $\gcd(8, 12) = 4$.

Step 2: Order of $8 = 12/4 = 3$.

Verification: $8 + 8 + 8 = 24 \equiv 0 \pmod{12}$. ✓

Why (A) wrong: Order 2 would require $2 \times 8 = 16 \equiv 4 \neq 0$.

Why (B) wrong: Order 4 would require $4 \times 8 = 32 \equiv 8 \neq 0$.

Why (C) wrong: Order 8 would require $8 \times 8 = 64 \equiv 4 \neq 0$.

Answer: (D) ← *Go back to Q4*

Q5.

Solution

Concept: Integration by parts: $\int u\,dv = uv - \int v\,du$.

Step 1: Set $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$.

Step 2:

$$\int_0^1 xe^x\,dx = \left[xe^x\right]_0^1 - \int_0^1 e^x\,dx = e - \left[e^x\right]_0^1 = e - (e-1) = 1.$$

Why (A) wrong: $e \approx 2.718$ — the integral equals exactly 1.
Why (C) wrong: $e - 1$ is the value of $\int_0^1 e^x\,dx$, not $\int_0^1 xe^x\,dx$.
Why (D) wrong: There is no basis for the answer to be 2.

Answer: (B) ← *Go back to Q5*

Q6.

Solution

Concept: Logarithmic differentiation: write $x^x = e^{x \ln x}$.

Step 1: $\dfrac{d}{dx}[x^x] = x^x(\ln x + 1)$.

Step 2: At $x = 1$: $1^1(\ln 1 + 1) = 1 \cdot (0 + 1) = 1$.

Why (B) wrong: e would arise if the formula were evaluated incorrectly without logarithm.
Why (C) wrong: 0 is $f'(x)$ for a constant, not for x^x.
Why (D) wrong: 2 has no basis from the correct formula.

Answer: (A) ← *Go back to Q6*

Q7.

Solution

Concept: Separate variables: write $\dfrac{dy}{y} = \dfrac{dx}{x}$ and integrate.

Step 1: $\displaystyle\int \frac{dy}{y} = \int \frac{dx}{x} \Rightarrow \ln|y| = \ln|x| + \text{const}.$

Step 2: Exponentiating: $y = Cx$ for arbitrary constant C.

Why (A) wrong: $y = Ce^x$ solves $dy/dx = y$, not y/x.
Why (B) wrong: $y = Cx^2$ gives $dy/dx = 2Cx = 2y/x \neq y/x$.
Why (D) wrong: $y = C\ln x$ gives $dy/dx = C/x \neq y/x$.

Answer: (C) ← *Go back to Q7*

Q8.

Solution

Concept: $\nabla \cdot \mathbf{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$.

Step 1: $\dfrac{\partial(x^2)}{\partial x} = 2x$, $\dfrac{\partial(y^2)}{\partial y} = 2y$, $\dfrac{\partial(z^2)}{\partial z} = 2z$.

Step 2: $\nabla \cdot \mathbf{F} = 2x + 2y + 2z$. At $(1, 1, 1)$: $2 + 2 + 2 = 6$.

Why (A) wrong: 2 is one partial derivative, not the sum.
Why (B) wrong: 4 misses one component.
Why (C) wrong: 1 does not correspond to any correct partial computation.

Answer: (D) ← *Go back to Q8*

Q9.

Solution

Concept: Compute $\partial/\partial x$ first, then $\partial/\partial y$.

Step 1: $\frac{\partial}{\partial x}(x^3y^2) = 3x^2y^2$.

Step 2: $\frac{\partial}{\partial y}(3x^2y^2) = 6x^2y$.

Step 3: At $(1,1)$: $6 \times 1 \times 1 = 6$.

Why (A) wrong: 3 is $\partial/\partial x(x^3)$ evaluated, missing the y-differentiation.
Why (C) wrong: 2 is the exponent of y, not the evaluated derivative.
Why (D) wrong: 12 would arise from $\partial^2/\partial y^2(x^3y^2) = 6x^3$ at $(1,1)$.

Answer: (B) ← *Go back to Q9*

Q10.

Solution

Concept: p-series $\sum 1/n^p$ converges $\Leftrightarrow p > 1$.

Step 1: (A) $p = 2 > 1 \Rightarrow$ converges. (B) $p = 1$ (harmonic) $\Rightarrow$ diverges. (C) $p = 1/2 < 1 \Rightarrow$ diverges. (D) $p = 0 \Rightarrow a_n = 1 \not\to 0$, diverges.

Conclusion: Only (A) converges.

Why (B) wrong: The harmonic series $\sum 1/n$ is the standard example of divergence for $p = 1$.
Why (C) wrong: $p = 1/2 < 1$, so $\sum 1/\sqrt{n}$ diverges.
Why (D) wrong: $\sum 1$ diverges (terms do not tend to zero).

Answer: (A) ← *Go back to Q10*

Section A Solutions — Q11–Q30 (2 marks each)

Q11.

Solution

Concept: MVT for integrals: $\int_0^1 e^x\,dx = e^c \cdot (1-0)$ for some $c \in (0,1)$.

Step 1: $\int_0^1 e^x\,dx = e - 1$, so $e^c = e - 1$, giving $c = \ln(e-1)$.

Step 2: $e - 1 \approx 1.718$, so $c = \ln(1.718) \approx 0.541$.

Conclusion: $c \approx 0.54 \in (\frac{1}{2}, 1)$.

Why (A),(B),(D) wrong: $c \approx 0.54$ does not satisfy $c < 0$, $c \in (0, \frac{1}{2})$, or $c > 1$.

Answer: (C) ← *Go back to Q11*

Q12.

Solution

Concept: This is one of the standard definitions of the number e.

Step 1: $\ln\left(1 + \frac{1}{n}\right)^n = n \ln\left(1 + \frac{1}{n}\right) \to n \cdot \frac{1}{n} = 1$ as $n \to \infty$ (using $\ln(1 + t) \sim t$ for $t \to 0$).

Step 2: Therefore $\left(1 + \frac{1}{n}\right)^n \to e^1 = e$.

Why (A),(B),(C) wrong: 1, 2, π are all incorrect limits; the defining limit equals exactly $e \approx 2.718$.

Answer: (D) ← *Go back to Q12*

Q13.

Solution

Concept: $\det(A)$ equals the product of all eigenvalues (counted with multiplicity).

Step 1: Eigenvalues are 1, 2, 3.

Step 2: $\det(A) = 1 \times 2 \times 3 = 6$.

Why (A) wrong: 1 is just one eigenvalue.
Why (C) wrong: 3 is also one eigenvalue.
Why (D) wrong: 2 is also one eigenvalue; the product is 6.

Answer: (B) ← *Go back to Q13*

Q14.

Solution

Concept: Rank-Nullity Theorem: rank(A)+nullity$(A) = n$.

Step 1: The matrix $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is in row-echelon form with 2 pivot rows, so rank$(A) = 2$.

Step 2: Nullity $= 3 - 2 = 1$.

Verification: The null space is spanned by $(1, 0, 1)^T$. ✓

Why (B) wrong: Nullity 2 would require rank 1 — the matrix has two pivots.
Why (C) wrong: Nullity 0 would require rank 3 (full rank), but row 3 is zero.
Why (D) wrong: Nullity 3 only for the zero map.

Answer: (A) ← *Go back to Q14*

Q15.

Solution

Concept: $H \leq (\mathbb{Z}_{12}, +)$ iff H is closed under addition mod 12 and contains 0.

Check (C): $\{0, 4, 8\}$: $4 + 4 = 8 \in H$, $4 + 8 = 12 \equiv 0 \in H$, $8 + 8 = 16 \equiv 4 \in H$. Closed. Inverses: $-0 = 0$, $-4 = 8$, $-8 = 4$. Yes, this is a subgroup of order 3.

Why (A) wrong: $\{0, 5, 10\}$: $5 + 5 = 10$, $5 + 10 = 15 \equiv 3 \notin H$. Not closed.
Why (B) wrong: $\{0, 4, 8, 2\}$: $2 + 2 = 4$, $2 + 4 = 6 \notin H$. Not closed.
Why (D) wrong: $\{0, 3, 6, 10\}$: $3 + 3 = 6$, $3 + 6 = 9 \notin H$. Not closed.

Answer: (C) ← *Go back to Q15*

Q16.

Solution

Concept: A homomorphism $f : \mathbb{Z}_6 \to \mathbb{Z}_4$ is determined by $f(1)$, which must have order dividing $\gcd(6, 4) = 2$ in $\mathbb{Z}_4$.

Step 1: Elements of $\mathbb{Z}_4$ with order dividing 2: order 1 (which is 0) and order 2 (which is 2). So $f(1) \in \{0, 2\}$.

Step 2: Both choices give valid homomorphisms: $f(1) = 0$ (trivial map) and $f(1) = 2$ (maps $k \mapsto 2k \bmod 4$).

Conclusion: Exactly **2** homomorphisms.

Why (A),(B),(C) wrong: Only elements with order $\leq \gcd(6, 4) = 2$ in $\mathbb{Z}_4$ can be images of 1; there are only 2 such elements.

Answer: (D) ← *Go back to Q16*

Q17.

Solution

Concept: Since e^{3x} is not a solution of the homogeneous equation ($r = 3$ is not a root of $r^2 - 3r + 2 = 0$), try $y_p = Ae^{3x}$.

Step 1: $y_p' = 3Ae^{3x}$, $y_p'' = 9Ae^{3x}$.

Step 2: Substitute: $(9A - 9A + 2A)e^{3x} = e^{3x} \Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$.

Conclusion: $y_p = \frac{e^{3x}}{2}$.

Why (B) wrong: $A = 1$ would require $2A = 2$, but we get $2A = 1$.
Why (C) wrong: $A = 1/3$ gives $2A = 2/3 \neq 1$.
Why (D) wrong: $A = 1/6$ gives $2A = 1/3 \neq 1$.

Answer: (A) ← *Go back to Q17*

Q18.

Solution

Concept: ODE $M\,dx + N\,dy = 0$ is exact iff $\partial M/\partial y = \partial N/\partial x$.

Check (B): $M = x^2y$, $N = xy^2$. $\partial M/\partial y = x^2$, $\partial N/\partial x = y^2$. Since $x^2 \neq y^2$ in general, not exact.

Check (A): $M = 2xy$, $N = x^2 + 1$. $M_y = 2x = N_x$. Exact.
Check (C): $M = x + y^2$, $N = 2xy$. $M_y = 2y = N_x$. Exact.
Check (D): $M = e^x \sin y$, $N = e^x \cos y$. $M_y = e^x \cos y = N_x$. Exact.

Conclusion: (B) is NOT exact.

Answer: (B) ← *Go back to Q18*

Q19.

Solution

Concept: Second-derivative test for functions of two variables: $D = f_{xx}f_{yy} - f_{xy}^2$.

Step 1: $f_x = 3x^2 - 3y = 0$, $f_y = 3y^2 - 3x = 0$. At $(1,1)$: $3 - 3 = 0$ ✓.

Step 2: $f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = -3$. At $(1,1)$: $f_{xx} = 6$, $f_{yy} = 6$, $f_{xy} = -3$.

Step 3: $D = 6 \cdot 6 - (-3)^2 = 36 - 9 = 27 > 0$ and $f_{xx} = 6 > 0$, so $(1,1)$ is a **local minimum.**

Why (A) wrong: Local max requires $f_{xx} < 0$.
Why (B) wrong: Saddle requires $D < 0$.
Why (D) wrong: $D > 0$ gives a definitive conclusion.

Answer: (C) ← *Go back to Q19*

Q20.

Solution

Concept: Lagrange: $\nabla f = \lambda \nabla g$ where $g = x^2 + y^2 - 2 = 0$.

Step 1: $\nabla f = (1,1)$, $\nabla g = (2x, 2y)$. So $1 = 2\lambda x$ and $1 = 2\lambda y \Rightarrow x = y$.

Step 2: From constraint: $2x^2 = 2 \Rightarrow x = y = 1$ (maximum) or $x = y = -1$ (minimum).

Step 3: Maximum of $x + y$: at $(1,1)$, $f = 1 + 1 = 2$.

Why (A) wrong: $f = 1$ is achieved at $(1,0)$, not on the constraint circle.
Why (B) wrong: $\sqrt{2}$ is the radius of the circle, not the max of $x + y$.
Why (C) wrong: $\sqrt{3}$ has no geometric basis here.

Answer: (D) ← *Go back to Q20*

Q21.

Solution

Concept: Green's theorem: $\oint_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$.

Step 1: $P = y$, $Q = -x$. Then $Q_x - P_y = -1 - 1 = -2$.

Step 2: Area of triangle with vertices $(0,0), (1,0), (0,1)$ is $\frac{1}{2}$.

Step 3: Integral $= \iint_D (-2)\,dA = -2 \times \frac{1}{2} = -1$.

Why (B) wrong: $+1$ would require $Q_x - P_y = +2$, not -2.
Why (C) wrong: -2 would need area $= 1$.
Why (D) wrong: 0 occurs only if $Q_x = P_y$.

Answer: (A) ← *Go back to Q21*

Q22.

Solution

Concept: $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$. By Stokes or direct computation.

Step 1: $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y & -x & z^2 \end{vmatrix} = (0-0)\mathbf{i} - (0-0)\mathbf{j} + (-1-1)\mathbf{k} = -2\mathbf{k}$.

Step 2: On S (unit disk, $z = 0$), $d\mathbf{S} = \mathbf{k}\,dA$. So $(\nabla \times \mathbf{F}) \cdot d\mathbf{S} = -2\,dA$.

Step 3: $\iint_S (-2)\,dA = -2 \cdot \pi(1)^2 = -2\pi$.

Why (A) wrong: $-\pi$ would need $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = -1$.
Why (C) wrong: $+2\pi$ would need upward curl $+2\mathbf{k}$.
Why (D) wrong: 0 only if $\nabla \times \mathbf{F} = \mathbf{0}$.

Answer: (B) ← *Go back to Q22*

Q23.

Solution

Concept: Classic result computed via polar coordinates: $\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \pi$.

Step 1: Let $I = \int_0^\infty e^{-x^2}\,dx$. Then $2I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$.

Step 2: $(2I)^2 = \iint e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^\infty e^{-r^2} r\,dr\,d\theta = 2\pi \cdot \frac{1}{2} = \pi$.

Step 3: So $2I = \sqrt{\pi}$, giving $I = \dfrac{\sqrt{\pi}}{2}$.

Why (A),(B),(C) wrong: 1, π, and $\sqrt{\pi}$ are all incorrect values for the half-Gaussian integral.

Answer: (D) ← Go back to Q23

Q24.

Solution

Concept: Iterate the integral over the triangular region $0 \le x \le 1$, $0 \le y \le 1-x$.

Step 1: Inner integral: $\displaystyle\int_0^{1-x} (x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_0^{1-x} = x(1-x) + \frac{(1-x)^2}{2} = \frac{1-x^2}{2} + (x-x^2) - \frac{x^2}{2}$. More directly: $x(1-x) + \frac{(1-x)^2}{2} = (1-x)\left(x + \frac{1-x}{2}\right) = (1-x)\cdot\frac{1+x}{2} = \dfrac{1-x^2}{2}$.

Step 2: Outer integral: $\displaystyle\int_0^1 \frac{1-x^2}{2}\,dx = \frac{1}{2}\left[x - \frac{x^3}{3}\right]_0^1 = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$.

Why (A),(B),(D) wrong: Errors arise from incorrect integration limits or incorrect algebra in step 1.

Answer: (C) ← Go back to Q24

Q25.

Solution

Concept: Find all $c \in (0,2)$ with $f'(c) = 0$.

Step 1: $f(x) = x^4 - 4x^3 + 4x^2 = x^2(x-2)^2$.
$f'(x) = 4x(x-2)^2 + x^2\cdot 2(x-2) = 2x(x-2)(2x-2+x)$. More cleanly:

$$f'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2-3x+2) = 4x(x-1)(x-2).$$

Step 2: $f'(c) = 0$ at $c = 0, 1, 2$. Only $c = 1$ lies strictly in $(0,2)$.

Conclusion: Exactly **1** such value.

Why (B) wrong: $c = 0$ and $c = 2$ are the endpoints, not interior points.

Why (C) wrong: The polynomial f' has only 3 real roots total, two of which are endpoints.
Why (D) wrong: Rolle's theorem guarantees at least one; we found exactly one.

Answer: (A) ← *Go back to Q25*

Q26.

Solution

Concept: The Taylor series of $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$.

Step 1: By the ratio test on the Taylor series: $\left|\frac{x^{2k+3}/(2k+3)!}{x^{2k+1}/(2k+1)!}\right| = \frac{x^2}{(2k+3)(2k+2)} \to 0$ for any fixed x.

Step 2: The ratio $\to 0 < 1$ for all $x \in \mathbb{R}$, so the radius of convergence is $R = \infty$.

Conclusion: Converges for all $x \in \mathbb{R}$.

Why (A),(C),(D) wrong: $R = \infty$ means convergence is not limited to $x = 0$, $|x| < 1$, or $|x| < \pi$.

Answer: (B) ← *Go back to Q26*

Q27.

Solution

Concept: Eigenspace for $\lambda = 0$ is the null space of A.

Step 1: $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Row reduce: $R_2 \leftarrow R_2 - R_1$ gives $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Rank$(A) = 1$.

Step 2: Nullity$(A) = 2 - 1 = 1$. So dim(eigenspace for $\lambda = 0$)$= 1$.

Why (A) wrong: Nullity 2 would require rank 0 (zero matrix).
Why (B) wrong: Nullity 0 would require rank 2, but $\det(A) = 0$.
Why (C) wrong: Nullity cannot exceed the size of the matrix (2×2).

Answer: (D) ← *Go back to Q27*

Q28.

Solution

Concept: $H \trianglelefteq G$ iff $gHg^{-1} = H$ for all $g \in G$, equivalently iff H is a union of conjugacy classes.

Step 1: $A_3 = \{e, (123), (132)\}$ is the alternating group, which has index 2 in S_3.

Every subgroup of index 2 is normal.

Step 2: Alternatively, $|S_3 : A_3| = 2$, so $A_3 \trianglelefteq S_3$.

Why (A),(B),(D) wrong: Each is a subgroup of order 2 generated by a transposition. Transpositions are conjugate to each other in S_3, e.g., $(12)^{(13)} = (23) \notin \{e, (12)\}$, so these subgroups are *not* normal.

Answer: (C) ← *Go back to Q28*

Q29.

Solution

Concept: $f : G \to H$ is injective $\Leftrightarrow \ker(f) = \{e_G\}$.

Step 1: Suppose $\ker(f) = \{e_G\}$ and $f(a) = f(b)$. Then $f(ab^{-1}) = f(a)f(b)^{-1} = e_H$, so $ab^{-1} \in \ker(f) = \{e_G\}$, giving $a = b$.

Conclusion: f is injective.

Why (A) wrong: Surjectivity depends on the image, not the kernel; $\ker(f) = \{e\}$ does not imply f is onto.
Why (C) wrong: Bijective requires both injective and surjective; surjectivity is not given.
Why (D) wrong: f *is* injective by the argument above.

Answer: (B) ← *Go back to Q29*

Q30.

Solution

Concept: By definition, a metric space is *complete* if every Cauchy sequence in it converges to a point in the space.

Step 1: The statement "every Cauchy sequence converges" is exactly the definition of completeness.

Why (B) wrong: Compact spaces are complete, but completeness does not require compactness (e.g., $\mathbb{R}$ is complete but not compact).
Why (C) wrong: Connectedness is a topological property unrelated to Cauchy sequences.
Why (D) wrong: Dense subsets need not be complete (e.g., $\mathbb{Q} \subset \mathbb{R}$ is dense but not complete).

Answer: (A) ← *Go back to Q30*

Section B Solutions — Q31–Q40 (MSQ, 2 marks each)

Q31.

Solution

Concept: $\sum 1/n^p$ converges $\Leftrightarrow p > 1$.

(A) $p = 2 > 1$: **converges**. (B) $p = 3/2 > 1$: **converges**. (C) $p = 1/2 < 1$: diverges. (D) $p = 1$: diverges (harmonic series).

Correct options: A and B.

Answer: (AB) ← *Go back to Q31*

Q32.

Solution

Concept: Necessary and sufficient conditions at a local maximum.

(A) $f'(c) = 0$: TRUE — Fermat's theorem: every local extremum of a differentiable function has zero derivative.

(B) $f''(c) > 0$: FALSE — this is the condition for a *local minimum*. At a local max, $f''(c) \leq 0$ (if it exists).

(C) $f(c) \geq f(x)$ for x near c: TRUE — this is the definition of local maximum.

(D) $f''(c) \leq 0$ (when it exists): TRUE — necessary condition at a local max (second-order condition).

Correct options: A, C, D.

Answer: (ACD) ← *Go back to Q32*

Q33.

Solution

Concept: In $\mathbb{R}$ (complete), a sequence is Cauchy $\Leftrightarrow$ it converges.

(A) $a_n = n$: diverges to ∞, NOT Cauchy.

(B) $a_n = 1/n \to 0$: converges, hence Cauchy. ✓

(C) $a_n = (-1)^n/n \to 0$: converges (by squeeze or alternating), hence Cauchy. ✓

(D) $a_n = (-1)^n$: oscillates, does not converge, NOT Cauchy.

Correct options: B and C.

Answer: (BC) ← *Go back to Q33*

Q34.

Solution

(A) Rank+Nullity= n: TRUE — Rank-Nullity Theorem.

(B) Invertible $\Rightarrow$ all eigenvalues nonzero: TRUE — $\det(A) = \prod \lambda_i \neq 0$ iff all $\lambda_i \neq 0$.

(C) rank(A^T) $\neq$rank(A) in general: FALSE — rank(A^T) =rank(A) always.

(D) λ eigenvalue of $A \Rightarrow \lambda^2$ eigenvalue of A^2: TRUE — if $Av = \lambda v$ then $A^2 v = \lambda^2 v$.

Correct options: A, B, D.

Answer: (ABD) ← *Go back to Q34*

Q35.

Solution

Analysis of $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$**:**

(A) Diagonalizable over $\mathbb{R}$: FALSE — A is a non-trivial Jordan block; it has only one linearly independent eigenvector $(1, 0)^T$.

(B) Two distinct eigenvalues: FALSE — char. poly is $(\lambda - 2)^2$, so $\lambda = 2$ is the only eigenvalue (repeated).

(C) Char. poly is $(\lambda - 2)^2$: TRUE — $\det(A - \lambda I) = (2 - \lambda)^2$.

(D) Minimal poly is $(x-2)^2$: TRUE — since $A - 2I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$ but $(A-2I)^2 = 0$.

Correct options: C and D.

Answer: (CD) ← *Go back to Q35*

Q36.

Solution

Concept: Sylow theorems: a group of order $12 = 2^2 \cdot 3$ has Sylow 2-subgroups (order 4) and Sylow 3-subgroups (order 3).

(A) Subgroup of order 3: TRUE — Sylow's theorem guarantees a subgroup of order p^k for each prime power dividing $|G|$.

(B) G is abelian: FALSE — A_4 has order 12 and is non-abelian.

(C) Subgroup of order 2: TRUE — any Sylow 2-subgroup of order 4 contains an element of order 2, generating a subgroup of order 2.

(D) $G \cong \mathbb{Z}_{12}$: FALSE — A_4, D_6, $\mathbb{Z}_{12}$, etc., are all groups of order 12.

Correct options: A and C.

Answer: (AC) ← *Go back to Q36*

Q37.

Solution

(A) f is surjective: FALSE — not necessarily (e.g., inclusion $\mathbb{Z}_2 \hookrightarrow \mathbb{Z}_4$).

(B) $f(e_G) = e_H$: TRUE — $f(e_G) = f(e_G \cdot e_G) = f(e_G) \cdot f(e_G)$, so $f(e_G) = e_H$.

(C) f is injective: FALSE — not necessarily (e.g., trivial homomorphism $f(g) = e_H$).

(D) $\ker(f) \trianglelefteq G$: TRUE — for any $g \in G$ and $k \in \ker(f)$: $f(gkg^{-1}) = f(g)f(k)f(g)^{-1} = f(g)e_H f(g)^{-1} = e_H$, so $gkg^{-1} \in \ker(f)$.

Correct options: B and D.

Answer: (BD) ← *Go back to Q37*

Q38.

Solution

The ODE: $y' + \frac{1}{x}y = x$ is linear with $P(x) = 1/x$.

(A) Integrating factor $\mu = e^{\int(1/x)dx} = e^{\ln x} = x$: TRUE.

(B) Multiply through by $\mu = x$: $(xy)' = x^2$. Integrate: $xy = x^3/3 + C$, so $y = x^2/3 + C/x$: TRUE.

(C) $y(1) = 1$: $1 = 1/3 + C \Rightarrow C = 2/3$. So $y = x^2/3 + 2/(3x) \neq x$: FALSE.

(D) $\mu = e^x$: FALSE — the correct IF is x, not e^x.

Correct options: A and B.

Answer: (AB) ← *Go back to Q38*

Q39.

Solution

Concept: $\mathbf{F}$ is conservative $\Leftrightarrow \nabla \times \mathbf{F} = \mathbf{0}$ (on simply connected domain).

(A) $(-y, x, 0)$: $\nabla \times \mathbf{F} = (0, 0, 1 - (-1)) = (0, 0, 2) \neq \mathbf{0}$. NOT conservative.

(B) $(2x, 2y, 2z) = \nabla(x^2 + y^2 + z^2)$: conservative. ✓

(C) $(yz, xz, xy) = \nabla(xyz)$: conservative. ✓

(D) $(e^x, 0, 0) = \nabla(e^x)$: conservative. ✓

Correct options: B, C, D.

Answer: (BCD) ← *Go back to Q39*

Q40.

Solution

Analysis of $f(x, y) = x^2 - y^2$:

(A) $(0, 0)$ is a critical point: TRUE — $f_x = 2x = 0$, $f_y = -2y = 0$ at $(0, 0)$.

(B) $(0, 0)$ is a local min: FALSE — $f(x, 0) = x^2 > 0$ for $x \neq 0$ but $f(0, y) = -y^2 < 0$ for $y \neq 0$. Not a min.

(C) $(0, 0)$ is a saddle point: TRUE — $D = f_{xx}f_{yy} - f_{xy}^2 = (2)(-2) - 0 = -4 < 0$, confirming saddle.

(D) No critical points: FALSE — $(0, 0)$ is a critical point.

Correct options: A and C.

Answer: (AC) ← *Go back to Q40*

Section C Solutions — Q41–Q50 (NAT, 1 mark each)

Q41.

Solution

Concept: An upper triangular matrix with all nonzero diagonal entries has rank equal to its size.

The matrix $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ is upper triangular with diagonal entries $1, 1, 1$ — all nonzero. Hence it has 3 pivot rows and rank $= 3$.

Answer: (3) ← *Go back to Q41*

Q42.

Solution

Concept: Order of a in $(\mathbb{Z}_n, +)$ is $n/\gcd(a, n)$.

$\gcd(5, 20) = 5$. Order of $5 = 20/5 = 4$.

Verification: $5 + 5 + 5 + 5 = 20 \equiv 0 \pmod{20}$. ✓

Answer: (4) ← *Go back to Q42*

Q43.

Solution

Step 1: $\int_0^{\pi} \sin x \, dx = \left[-\cos x\right]_0^{\pi} = -\cos\pi - (-\cos 0) = -(-1) + 1 = 1 + 1 = 2.$

Answer: (2) ← *Go back to Q43*

Q44.

Solution

Concept: Every non-identity element of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has order 2 (since $2a = 0$ for all a).

Total elements $= 2^3 = 8$. Non-identity elements $= 8 - 1 = 7$. All 7 have order 2.

Answer: (7) ← *Go back to Q44*

Q45.

Solution

Step 1: Let $g(x, y) = x^3y^2 + \sin(xy)$.

$\dfrac{\partial g}{\partial x} = 3x^2y^2 + y\cos(xy)$.

Step 2: $\dfrac{\partial^2 g}{\partial y\,\partial x} = 6x^2y + \cos(xy) - xy\sin(xy)$.

Step 3: At $(1, 0)$: $6(1)^2(0) + \cos(0) - 0 = 0 + 1 - 0 = 1$.

Answer: (1) ← *Go back to Q45*

Q46.

Solution

Step 1: $f_x = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$. $f_y = 3y^2 - 3 = 0 \Rightarrow y = \pm 1$.

Step 2: Critical points: $(1, 1)$, $(1, -1)$, $(-1, 1)$, $(-1, -1)$ — exactly 4 points.

Answer: (4) ← *Go back to Q46*

Q47.

Solution

Identity: For any smooth vector field $\mathbf{F}$, $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ identically. This follows because the mixed partial derivatives commute and the terms cancel in pairs.

Answer: 0.

Answer: (0) ← *Go back to Q47*

Q48.

Solution

$$\int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4} - 0 = 0.25.$$

Answer: (0.25) ← Go back to Q48

Q49.

Solution

Concept: Every $n \times n$ matrix over $\mathbb{C}$ has exactly n eigenvalues (counting algebraic multiplicity) by the Fundamental Theorem of Algebra applied to the characteristic polynomial of degree n.

For $n = 2$: exactly **2** eigenvalues (counted with multiplicity).

Answer: (2) ← Go back to Q49

Q50.

Solution

Concept: $|S_n| = n!$ (the symmetric group on n elements has $n!$ permutations).

$|S_4| = 4! = 4 \times 3 \times 2 \times 1 = 24.$

Answer: (24) ← Go back to Q50

Section C Solutions — Q51–Q60 (NAT, 2 marks each)

Q51.

Solution

Concept: If A has eigenvalues $\lambda_1, \ldots, \lambda_n$, then A^k has eigenvalues $\lambda_1^k, \ldots, \lambda_n^k$, and $\mathrm{tr}(A^k) = \sum \lambda_i^k$.

Step 1: Eigenvalues of A: $1, 2, 3$. Eigenvalues of A^2: $1^2, 2^2, 3^2 = 1, 4, 9$.

Step 2: $\mathrm{tr}(A^2) = 1 + 4 + 9 = 14$.

Answer: (14) ← Go back to Q51

Q52.

Solution

Concept: Subgroups of $\mathbb{Z}_n$ are in bijection with divisors of n; each divisor $d \mid n$ gives a unique subgroup $\langle n/d \rangle$ of order d.

Divisors of 12: $1, 2, 3, 4, 6, 12$. Total: **6** subgroups.

Explicitly: $\{0\}$, $\{0, 6\}$, $\{0, 4, 8\}$, $\{0, 3, 6, 9\}$, $\{0, 2, 4, 6, 8, 10\}$, $\mathbb{Z}_{12}$.

Answer: (6) ← *Go back to Q52*

Q53.

Solution

Concept: For constant right-hand side and non-resonant characteristic roots, try $y_p = A$ (constant).

Step 1: $y_p = A \Rightarrow y_p' = y_p'' = 0$. Substitute: $0 - 0 + 6A = 6 \Rightarrow A = 1$.

Step 2: $y_p = 1$.

Verification: $0 - 0 + 6(1) = 6$. ✓

Answer: (1) ← *Go back to Q53*

Q54.

Solution

Step 1: Inner integral (x first): $\displaystyle\int_0^1 (x + y)\, dx = \left[\frac{x^2}{2} + xy\right]_0^1 = \frac{1}{2} + y.$

Step 2: Outer integral: $\displaystyle\int_0^1 \left(\frac{1}{2} + y\right) dy = \left[\frac{y}{2} + \frac{y^2}{2}\right]_0^1 = \frac{1}{2} + \frac{1}{2} = 1.$

Answer: (1) ← *Go back to Q54*

Q55.

Solution

Concept: Divergence Theorem: $\oiint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{F}\, dV$.

Step 1: $\nabla \cdot \mathbf{F} = \dfrac{\partial(x^2)}{\partial x} + \dfrac{\partial(y^2)}{\partial y} + \dfrac{\partial(z^2)}{\partial z} = 2x + 2y + 2z.$

Step 2:

$$\iiint_{[0,2]^3} (2x + 2y + 2z)\, dV = 2 \int_0^2 \int_0^2 \int_0^2 (x + y + z)\, dx\, dy\, dz.$$

Step 3: By symmetry, $\int_0^2 \int_0^2 \int_0^2 x\, dx\, dy\, dz = \int_0^2 x\, dx \cdot \int_0^2 dy \cdot \int_0^2 dz = 2 \cdot 2 \cdot 2 = 8$. Same for y and z.

Step 4: Total $= 2(8 + 8 + 8) = 2 \times 24 = 48$.

Answer: (48) ← *Go back to Q55*

Q56.

Solution

Concept: Ratio test for power series: $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ where $a_n = 1/n$.

Step 1: $R = \lim_{n\to\infty}\frac{1/n}{1/(n+1)} = \lim_{n\to\infty}\frac{n+1}{n} = 1$.

Conclusion: Radius of convergence $R = 1$. (The series converges for $x \in [-1, 1)$ and diverges for $|x| > 1$.)

Answer: (1) ← *Go back to Q56*

Q57.

Solution

Concept: For a 2×2 matrix, $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.

Step 1: $\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 \times 4 - 2 \times 3 = 4 - 6 = -2$.

Answer: (-2) ← *Go back to Q57*

Q58.

Solution

Concept: Lagrange multipliers or substitution. With $x+y = 1$, substitute $y = 1-x$:

Step 1: $f = x^2 + (1-x)^2 = 2x^2 - 2x + 1$. Minimize: $f' = 4x - 2 = 0 \Rightarrow x = \frac{1}{2}$, $y = \frac{1}{2}$.

Step 2: Minimum value: $f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} = 0.5$.

Answer: (0.5) ← *Go back to Q58*

Q59.

Solution

Concept: Geometric series $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for $|r| < 1$.

Step 1: $r = \frac{1}{3}$, $|r| < 1$, so the series converges.

Step 2: Sum $= \frac{1}{1-1/3} = \frac{1}{2/3} = \frac{3}{2} = 1.5$.

Answer: (1.5) ← *Go back to Q59*

Q60.

Solution

Concept: $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ is diagonal, so $A^4 = \begin{pmatrix} 1^4 & 0 \\ 0 & 2^4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 16 \end{pmatrix}$.

Step 1: $\mathrm{tr}(A^4) = 1 + 16 = 17$.

Alternatively, eigenvalues of A are 1 and 2, so $\mathrm{tr}(A^4) = 1^4 + 2^4 = 1 + 16 = 17$.

Answer: (17) ← *Go back to Q60*

Answer Key

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	B	2	C	3	A	4	D	5	B
6	A	7	C	8	D	9	B	10	A
11	C	12	D	13	B	14	A	15	C
16	D	17	A	18	B	19	C	20	D
21	A	22	B	23	D	24	C	25	A
26	B	27	D	28	C	29	B	30	A
31	AB	32	ACD	33	BC	34	ABD	35	CD
36	AC	37	BD	38	AB	39	BCD	40	AC
41	3	42	4	43	2	44	7	45	1
46	4	47	0	48	0.25	49	2	50	24
51	14	52	6	53	1	54	1	55	48
56	1	57	-2	58	0.5	59	1.5	60	17