# IIT JAM Mathematics

## Sample Paper – 3

### Instructions

- This paper contains **60 questions** carrying a maximum of **100 marks**. Duration: **3 hours (180 minutes)**.
- **Section A (Q1–Q30**, MCQ): Q1–Q10 carry **+1 mark** each; wrong response: $-\frac{1}{3}$ mark. Q11–Q30 carry **+2 marks** each; wrong response: $-\frac{2}{3}$ mark. Choose the **single best** option.
- **Section B (Q31–Q40**, MSQ): Each carries **+2 marks**. **No negative marking. One or more** options may be correct; full marks only if *all* correct options are selected.
- **Section C (Q41–Q60**, NAT): Q41–Q50 carry **+1 mark**; Q51–Q60 carry **+2 marks**. **No negative marking.** Enter the numerical value using the virtual keypad.
- **Calculator policy:** Personal calculators and electronic devices are **strictly prohibited**. A virtual scientific calculator is provided in the CBT interface.
- Sections must be attempted in order; you cannot return to a previous section.

## Section A — Multiple Choice Questions (MCQ)

***Q1–Q10**: 1 mark each ($-\frac{1}{3}$ for wrong). Choose the **single** correct option.*

**Q1.** Which convergence test most directly establishes the convergence of $\displaystyle\sum_{n=1}^{\infty} n^2 \cdot \left(\frac{1}{2}\right)^n$?

(A) Integral test

(B) Root test

(C) Ratio test

(D) Comparison with $\displaystyle\sum \frac{1}{n^2}$

**Q2.** The Bolzano–Weierstrass theorem states that every bounded sequence in $\mathbb{R}$:

(A) Has a convergent subsequence

(B) Converges to $0$

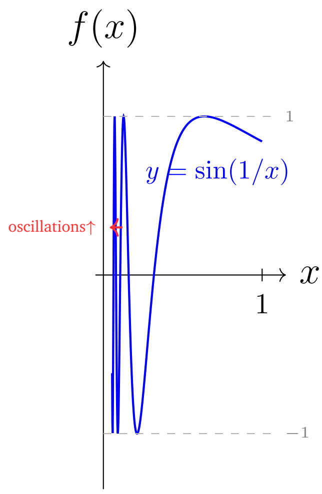
(C) Has no convergent subsequence

(D) Must itself converge

**Q3.** The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is best classified as:

(A) Absolutely convergent
(B) Divergent
(C) Convergent with sum $0$
(D) Conditionally convergent but not absolutely convergent

**Q4.** The function $f(x) = \sin\left(\frac{1}{x}\right)$ on the interval $(0, 1]$ is:




(A) Uniformly continuous on $(0, 1]$
(B) NOT uniformly continuous on $(0, 1]$
(C) Monotone on $(0, 1]$
(D) Bounded and uniformly continuous on $(0, 1]$

**Q5.** Let $f(x) = x^3 - x - 1$. By the Intermediate Value Theorem, a root of $f$ must exist in the interval:

(A) $(1, 2)$
(B) $(0, 1)$
(C) $(-1, 0)$
(D) $(2, 3)$

**Q6.** The Wronskian $W(1,\ x,\ e^x)$ evaluated at $x = 0$ equals:

(A) $0$
(B) $-1$

(C) $1$

(D) $e$

**Q7.** The Bernoulli equation $\dfrac{dy}{dx} + P(x)\,y = Q(x)\,y^n$ (with $n \neq 0, 1$) is linearised by the substitution:

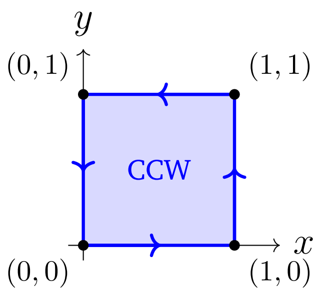
(A) $v = y^n$

(B) $v = y^{n-1}$

(C) $v = y^{-n}$

(D) $v = y^{1-n}$

**Q8.** Using Green's theorem, evaluate $\oint_C \left(y^2\,dx + x^2\,dy\right)$, where $C$ is the boundary of the unit square $[0,1]^2$ traversed counterclockwise.




(A) $1$

(B) $0$

(C) $-1$

(D) $2$

**Q9.** What is the order of the element $3$ in the group $(\mathbb{Z}_9, +)$?

(A) $3$

(B) $9$

(C) $1$

(D) $6$

**Q10.** If $A$ is an $m \times m$ matrix and $B$ is an $n \times n$ matrix, then $\det\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ equals:

(A) $\det(A) + \det(B)$

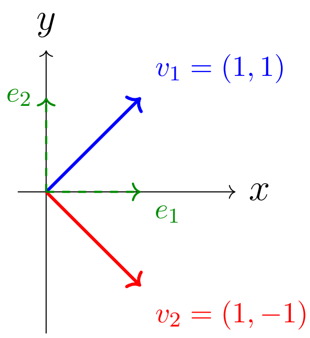
(B) $\det(A) \cdot \det(B) + 1$

(C) $\det(A) \cdot \det(B)$

(D) $m \cdot n$

***Q11–Q30**: 2 marks each ($-\frac{2}{3}$ for wrong). Choose the **single** correct option.*

**Q11.** Let $B_1 = \{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$, and let $B_2 = \{v_1, v_2\}$ where $v_1 = (1, 1)^T$ and $v_2 = (1, -1)^T$. The change-of-basis matrix $P$ whose columns express the $B_2$ vectors in $B_1$ coordinates is:




(A) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(B) $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

(C) $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

(D) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

**Q12.** A homogeneous linear system $A\mathbf{x} = \mathbf{0}$ with $A \in M_{n\times n}(\mathbb{R})$ has a **non-trivial** solution if and only if:

(A) $\text{rank}(A) = n$

(B) $A$ is invertible

(C) $\det(A) \neq 0$

(D) $\det(A) = 0$

**Q13.** Let $G$ be a group of order $20$ and $H$ a subgroup of $G$. By Lagrange's theorem, which of the following **cannot** be the order of $H$?

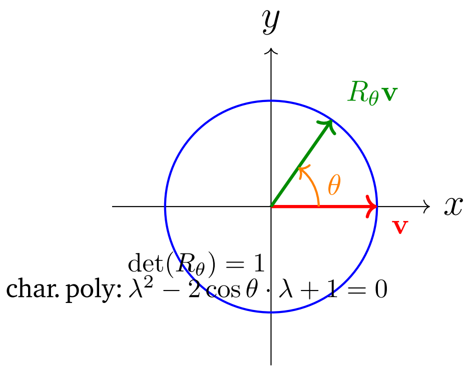
(A) $7$

(B) $4$
(C) $5$
(D) $10$

**Q14.** For matrices $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times p}(\mathbb{R})$, which inequality **always** holds?

(A) $\text{rank}(AB) = \text{rank}(A) + \text{rank}(B)$
(B) $\text{rank}(AB) \geq \text{rank}(A)$
(C) $\text{rank}(AB) \leq \min\big(\text{rank}(A), \text{rank}(B)\big)$
(D) $\text{rank}(AB) = \text{rank}(A) \cdot \text{rank}(B)$

**Q15.** The $2 \times 2$ rotation matrix $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for $0 < \theta < \pi$ has:




(A) Two positive real eigenvalues
(B) No real eigenvalues
(C) Eigenvalues $\pm 1$
(D) Determinant $0$

**Q16.** The Banach fixed-point theorem guarantees that every contraction mapping $T : M \to M$ on a complete metric space $(M, d)$ has:

(A) No fixed point
(B) Infinitely many fixed points
(C) At most one fixed point
(D) Exactly one fixed point

**Q17.** Evaluate $\displaystyle\int_1^e x \ln x \, dx$.

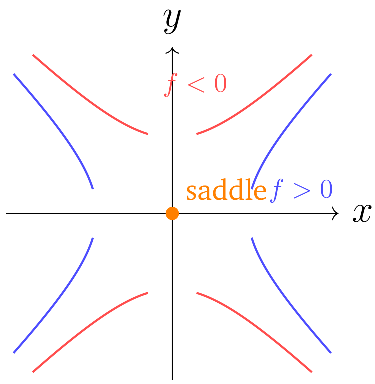
(A) $\dfrac{e^2+1}{4}$

(B) $\dfrac{e^2}{2}$

(C) $\dfrac{e^2-1}{2}$

(D) $\dfrac{e^2}{4}$

**Q18.** The function $f(x, y) = x^2 - y^2$ has a critical point at the origin $(0, 0)$. This critical point is a:




(A) Local minimum

(B) Local maximum

(C) Saddle point

(D) Point of inflection

**Q19.** The improper integral $\int_0^1 x^{-1/2}\,dx$:

(A) Diverges

(B) Converges to 2

(C) Converges to 1

(D) Converges to $\dfrac{1}{2}$

**Q20.** The vector field $\mathbf{F}(x, y) = \left(2xy + \cos x,\ x^2 + 3y^2\right)$ is conservative. The **direct** reason is:

(A) $\operatorname{div}(\mathbf{F}) = 0$

(B) $\mathbf{F}$ is the gradient of some potential function (consequence, not the test)

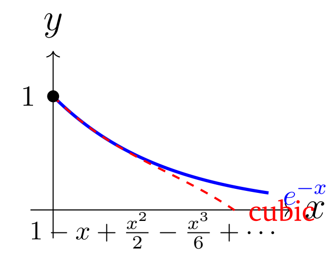
(C) $\mathbf{F}$ is smooth on all of $\mathbb{R}^2$

(D) $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = 2x - 2x = 0$ (curl-zero condition)

**Q21.** The power set $\mathcal{P}(\{a, b, c\})$ (the collection of all subsets of $\{a, b, c\}$) has exactly:

(A) $8$ elements
(B) $6$ elements
(C) $3$ elements
(D) $9$ elements

**Q22.** The coefficient of $x^3$ in the Maclaurin series of $e^{-x}$ is:




(A) $\dfrac{1}{6}$
(B) $-\dfrac{1}{3}$
(C) $-\dfrac{1}{6}$
(D) $\dfrac{1}{3}$

**Q23.** The ordinary differential equation $\dfrac{dy}{dx} = \dfrac{y + x}{y - x}$ is best classified as:

(A) Separable
(B) Homogeneous (reducible by the substitution $y = vx$)
(C) Exact
(D) Linear in $y$

**Q24.** If the eigenvalues of a $3 \times 3$ matrix $A$ are $1$, $2$, and $3$, the eigenvalues of $A + 2I$ are:

(A) $1,\ 2,\ 3$
(B) $2,\ 4,\ 6$
(C) $2,\ 3,\ 4$

(D) $3,\ 4,\ 5$

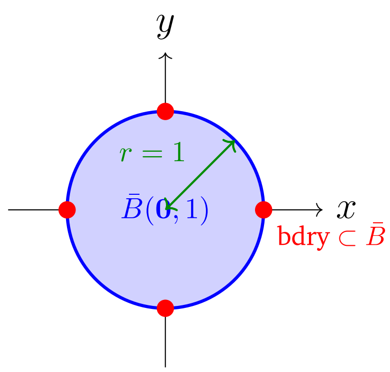
**Q25.** In the group $(\mathbb{Z}_8, +)$, the subgroup generated by the element $6$ is:

(A) $\{0, 2, 4, 6\}$
(B) $\{0, 6\}$
(C) $\{0, 3, 6\}$
(D) $\{0, 4\}$

**Q26.** The radius of convergence of the power series $\sum_{n=1}^{\infty} n\, x^n$ is:

(A) $\infty$
(B) $0$
(C) $1$
(D) $\frac{1}{2}$

**Q27.** In $\mathbb{R}^2$ with the Euclidean metric, the closed unit disk $\bar{B}(\mathbf{0}, 1) = \{(x, y) : x^2 + y^2 \leq 1\}$ is:




(A) Open
(B) Compact (bounded and closed, by Heine–Borel)
(C) Not closed
(D) Unbounded

**Q28.** The dihedral group $D_n$ (the group of all symmetries of a regular $n$-gon, including rotations and reflections) has order:

(A) $n$
(B) $n^2$

(C) $n!$
(D) $2n$

**Q29.** Every group of prime order $p$ is:

(A) Cyclic
(B) Abelian but not cyclic
(C) Non-abelian
(D) Simple and non-abelian

**Q30.** A metric space $(M, d)$ is **complete** if and only if every Cauchy sequence in $M$:

(A) Is bounded
(B) Has a convergent subsequence
(C) Converges to a limit in $M$
(D) Is eventually constant

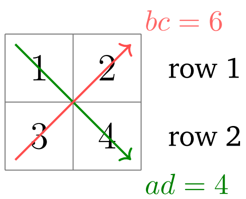
## Section B — Multiple Select Questions (MSQ)

***Q31–Q40**: 2 marks each. **No negative marking. One or more** options may be correct; full marks only if ALL correct options selected.*

**Q31.** Which of the following functions are **uniformly continuous** on $(0, 1)$?

(A) $f(x) = \sin x$
(B) $f(x) = \sqrt{x}$
(C) $f(x) = \dfrac{1}{x}$
(D) $f(x) = \sin(\frac{1}{x})$

**Q32.** For the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, which of the following are **TRUE**?




(A) $\det(A) = 10$

(B) $\mathrm{tr}(A) = 6$
(C) $\det(A) = -2$
(D) $\mathrm{tr}(A) = 5$

**Q33.** Which of the following series **converge absolutely**?

(A) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$
(B) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
(C) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$
(D) $\displaystyle\sum_{n=1}^{\infty} (-1)^n$

**Q34.** For a group homomorphism $\varphi : G \to H$, which of the following **must** be true?

(A) $\varphi$ is surjective
(B) $\varphi(e_G) = e_H$
(C) $\varphi$ is injective
(D) $\ker(\varphi)$ is a normal subgroup of $G$

**Q35.** For any real symmetric matrix $A$ (i.e. $A^T = A$), which of the following are **always true**?

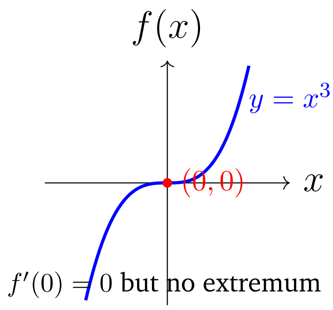
(A) All eigenvalues of $A$ are real
(B) Eigenvectors corresponding to distinct eigenvalues are orthogonal
(C) $A$ is diagonalizable over $\mathbb{R}$
(D) $A$ is invertible

**Q36.** For the improper integral $\displaystyle\int_0^{\infty} e^{-2x}\,dx$, which of the following statements are **TRUE**?

(A) The integral converges
(B) Its value is 2
(C) The integral diverges

(D) Its value is $\frac{1}{2}$

**Q37.** For the function $f(x) = x^3$ defined on all of $\mathbb{R}$, which of the following are **TRUE**?




(A) $f$ has a local maximum at $x = 0$
(B) $f$ is strictly increasing on $\mathbb{R}$
(C) $f$ has no local extrema
(D) $f$ is bounded on $\mathbb{R}$

**Q38.** For open sets in $\mathbb{R}^n$, which of the following statements are **TRUE**?

(A) Any union (finite or infinite) of open sets is open
(B) Any (possibly infinite) intersection of open sets is open
(C) Any finite intersection of open sets is open
(D) $\mathbb{R}^n$ itself is an open set

**Q39.** For two solutions $y_1, y_2$ of $y'' + p(x)y' + q(x)y = 0$ on $(a, b)$ (with $p, q$ continuous), let $W$ denote their Wronskian. Which of the following are **TRUE**?

(A) If $W(x_0) \neq 0$ for some $x_0 \in (a, b)$, then $y_1, y_2$ are linearly independent on $(a, b)$
(B) If $y_1, y_2$ are linearly dependent on $(a, b)$, then $W \equiv 0$ on $(a, b)$
(C) $W$ can be nonzero at some points and zero at other points of $(a, b)$
(D) If $W(x_0) = 0$ for some $x_0 \in (a, b)$, then $W \equiv 0$ on $(a, b)$ (Abel's theorem)

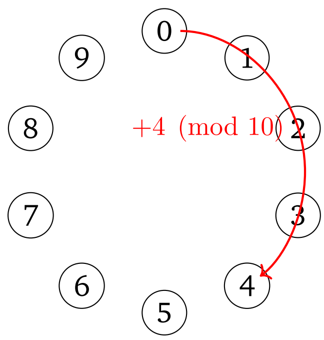
**Q40.** Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable at $x = a$. Which of the following conditions are **sufficient** to conclude that $x = a$ is a **local maximum** of $f$?

(A) $f'(a) = 0$ alone
(B) $f'(a) = 0$ and $f''(a) < 0$ (second derivative test)
(C) $f'(a) = 0$ and $f'$ changes sign from $+$ to $-$ at $a$ (first derivative test)
(D) $f''(a) = 0$ alone

## Section C — Numerical Answer Type (NAT)

***Q41–Q50**: 1 mark each. No negative marking. Enter the exact numerical value.*

**Q41.** Find the order of the element $4$ in the group $(\mathbb{Z}_{10}, +)$.




$\mathbb{Z}_{10}$: what is the order of $4$?

**Q42.** Evaluate $\displaystyle\int_0^\infty x\, e^{-2x}\, dx$.

**Q43.** How many elements of order $2$ does the group $\mathbb{Z}_2 \times \mathbb{Z}_4$ have?

**Q44.** Find the rank of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 3 & 5 \end{pmatrix}$.

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 2 | 4 | 6 |
| 1 | 3 | 5 |

$R_2 = 2R_1$
$R_3 \to R_3 - R_1$

**Q45.** Evaluate $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx$.

**Q46.** How many critical points does $f(x, y) = x^2 + y^2 - 2x - 4y + 5$ have?

**Q47.** For the vector field $\mathbf{F} = (yz,\, xz,\, xy)$, evaluate $|\nabla \times \mathbf{F}|^2$.

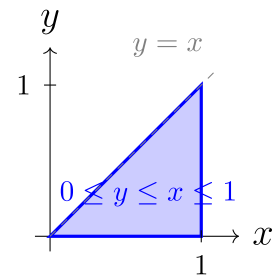
**Q48.** Evaluate $\int_0^{\pi/2} \sin^2 x \, dx$.

**Q49.** A $2 \times 2$ matrix $A$ satisfies $\operatorname{tr}(A) = 5$ and $\det(A) = 6$. Find the **larger** eigenvalue of $A$.

**Q50.** Evaluate $\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right)$.

***Q51–Q60**: 2 marks each. No negative marking. Enter the exact numerical value.*

**Q51.** Evaluate $\int_0^1 \int_0^x y \, dy \, dx$.

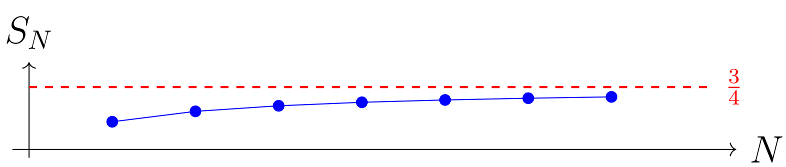



**Q52.** Let $A$ be a $3 \times 3$ real orthogonal matrix with $\det(A) = -1$. Find $\det(A + I)$.

**Q53.** Find the number of solutions $x \in \{0, 1, 2, \ldots, 7\}$ of the congruence $x^2 \equiv 1 \pmod{8}$.

**Q54.** Find the area (in square units) enclosed by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

**Q55.** Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.




Partial sums $S_N \to \frac{3}{4}$

**Q56.** Evaluate $\int_0^{\infty} e^{-x} \cos x \, dx$.

**Q57.** For $f(x) = x^3 - 3x$, find the sum of the local maximum value and the local minimum value of $f$.

**Q58.** Find the order $|S_4|$ of the symmetric group on $4$ letters.

**Q59.** The initial value problem $y'' + y = 0,\ \ y(0) = 1,\ \ y'(0) = 0$ has solution $y(x)$. Find $y(\pi)$.

**Q60.** Using Green's theorem, evaluate $\oint_C \big(-y\,dx + x\,dy\big)$, where $C$ is the unit circle $x^2 + y^2 = 1$ traversed counterclockwise.

# Solutions

## IIT JAM Mathematics – Sample Paper 3

## Section A Solutions — Q1–Q10 (1 mark each)

**Q1.**

**Solution**

**Concept — Ratio Test**: if $L = \lim_{n\to\infty} |a_{n+1}/a_n| < 1$, the series converges.

**Step 1 — Apply ratio test**: Let $a_n = n^2(1/2)^n$.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2 \cdot (1/2)^{n+1}}{n^2 \cdot (1/2)^n} = \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{2} \longrightarrow 1 \cdot \frac{1}{2} = \frac{1}{2} < 1.$$

**Step 2 — Conclusion**: $L = 1/2 < 1$, so the series converges by the ratio test. The ratio test is the most direct choice since the ratio simplifies cleanly.

**Why (A) wrong**: The integral test requires evaluating $\int x^2(1/2)^x\,dx$, which is more involved.
**Why (B) wrong**: The root test gives $\limsup |a_n|^{1/n} = \limsup n^{2/n}/2 \to 1/2$, which also works but is slightly less immediate.
**Why (D) wrong**: Comparison with $\sum 1/n^2$ is not valid since $(1/2)^n$ decays faster than $1/n^2$; a separate argument is needed.

**Answer: (C)** ← *Go back to Q1*

**Q2.**

**Solution**

**Concept — Bolzano–Weierstrass Theorem**: Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

**Statement**: If $(a_n)$ is bounded, there exist indices $n_1 < n_2 < \cdots$ such that $(a_{n_k})$ converges.

**Why (A) is correct**: This is exactly the content of the theorem.

**Why (B) wrong**: The subsequence need not converge to $0$; for example $a_n = 1$ is bounded and every subsequence converges to $1$.
**Why (C) wrong**: This contradicts the theorem directly.
**Why (D) wrong**: Bounded sequences need not converge; e.g. $a_n = (-1)^n$ is bounded but diverges, though it has convergent subsequences.

**Answer: (A)** ← *Go back to Q2*

**Q3.**

**Solution**

**Concept — Conditional vs. Absolute Convergence.**

**Step 1 — Alternating series test:** $\sum(-1)^n/\sqrt{n}$ satisfies $1/\sqrt{n} \downarrow 0$, so the series converges by the Leibniz criterion.

**Step 2 — Absolute convergence test:** $\sum 1/\sqrt{n}$ is the $p$-series with $p = 1/2 < 1$, which *diverges*.

**Conclusion**: The series converges but not absolutely, i.e. it is **conditionally convergent.**

**Why (A) wrong**: Absolute convergence would require $\sum 1/\sqrt{n}$ to converge, which it does not.
**Why (B) wrong**: The alternating series test confirms convergence.
**Why (C) wrong**: The sum is not $0$; it equals $-\eta(1/2)$ where $\eta$ is the Dirichlet eta function.

**Answer: (D)** ← *Go back to Q3*

**Q4.**

**Solution**

**Concept — Uniform Continuity**: $f$ is UC on $(0, 1]$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ for *all* $x, y \in (0, 1]$.

**Step 1 — Failure of UC**: Take $x_n = \dfrac{1}{2\pi n}$ and $y_n = \dfrac{1}{2\pi n + \pi/2}$. Then $|x_n - y_n| \to 0$ but

$$|f(x_n) - f(y_n)| = |\sin(2\pi n) - \sin(2\pi n + \pi/2)| = |0 - 1| = 1.$$

**Step 2**: Since we can find pairs with arbitrarily small distance but fixed difference in values, $f$ is NOT uniformly continuous.

**Why (A),(D) wrong**: Uniform continuity fails due to the infinite oscillations as $x \to 0^+$.
**Why (C) wrong**: $f = \sin(1/x)$ oscillates infinitely often, so it cannot be monotone on $(0, 1]$.

**Answer: (B)** ← *Go back to Q4*

**Q5.**

**Solution**

**Concept — Intermediate Value Theorem**: If $f$ is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then $f$ has a root in $(a, b)$.

**Step 1 — Evaluate at endpoints:**

$$f(1) = 1 - 1 - 1 = -1 < 0, \qquad f(2) = 8 - 2 - 1 = 5 > 0.$$

**Step 2**: Since $f(1) < 0 < f(2)$ and $f$ is continuous (polynomial), IVT guarantees a root in $(1, 2)$.

**Why (B) wrong**: $f(0) = -1 < 0$ and $f(1) = -1 < 0$; no sign change in $(0, 1)$.
**Why (C) wrong**: $f(-1) = -1 + 1 - 1 = -1 < 0$ and $f(0) = -1 < 0$; no sign change in $(-1, 0)$.
**Why (D) wrong**: While a root in $(1, 2)$ implies $f(2) > 0$, checking $(2, 3)$ gives $f(2) = 5 > 0$ and $f(3) = 23 > 0$ — no sign change there.

**Answer: (A)** ← *Go back to Q5*

**Q6.**

**Solution**

**Concept — Wronskian**: For $f_1, f_2, f_3$, $W = \det \begin{pmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{pmatrix}$.

**Step 1 — Fill in the matrix at $x = 0$:**

$$W(0) = \det \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(Rows: $(1,\, x,\, e^x)$, $(0,\, 1,\, e^x)$, $(0,\, 0,\, e^x)$ evaluated at $x = 0$.)

**Step 2 — Compute determinant**: Expanding along column 1:

$$W(0) = 1 \cdot \det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1 \cdot (1 - 0) = 1.$$

**Conclusion**: $W(0) = 1 \neq 0$, confirming linear independence.

**Why (A) wrong**: $W = 0$ would imply linear dependence, which is false.
**Why (B) wrong**: $-1$ has no basis in the computation.
**Why (D) wrong**: $e$ does not appear in the result when $x = 0$.

**Answer: (C)** ← *Go back to Q6*

**Q7.**

## Solution

**Concept — Bernoulli Equation**: $y' + P(x)y = Q(x)y^n$.

**Step 1 — Standard substitution**: Let $v = y^{1-n}$. Then $\dfrac{dv}{dx} = (1-n)y^{-n}\dfrac{dy}{dx}$.

**Step 2 — Linearize**: Multiply the equation by $(1-n)y^{-n}$:

$$(1-n)y^{-n}y' + (1-n)P(x)y^{1-n} = (1-n)Q(x) \implies v' + (1-n)P(x)v = (1-n)Q(x).$$

This is linear in $v$.

**Why (A) wrong**: $v = y^n$ does not produce a linear equation.
**Why (B) wrong**: $v = y^{n-1}$ is the reciprocal of the correct substitution (gives $v' + (n-1)P\,v = \ldots$, with wrong sign).
**Why (C) wrong**: $v = y^{-n}$ leads to an equation with $y^{-2n}$ terms after differentiation, not linear.

**Answer: (D)** ← *Go back to Q7*

**Q8.**

## Solution

**Concept — Green's Theorem**: $\oint_C (P\,dx + Q\,dy) = \iint_D \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right) dA.$

**Step 1**: Here $P = y^2$, $Q = x^2$.

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 2y.$$

**Step 2 — Integrate over** $[0,1]^2$:

$$\iint_{[0,1]^2} (2x-2y)\,dA = \int_0^1\int_0^1 (2x-2y)\,dy\,dx = \int_0^1 \left[2xy - y^2\right]_0^1 dx = \int_0^1 (2x-1)\,dx = \left[x^2 - x\right]_0^1 = 0.$$

**Conclusion**: The integral equals **0**.

**Why (A) wrong**: $\int(2x-2y)$ is antisymmetric; integrating over the symmetric square gives zero.
**Why (C),(D) wrong**: The computation yields exactly $0$, not $-1$ or $2$.

**Answer: (B)** ← *Go back to Q8*

**Q9.**

**Solution**

**Concept**: The order of $a$ in $(\mathbb{Z}_n, +)$ is $n/\gcd(a, n)$.

**Step 1**: $\gcd(3, 9) = 3$.

**Step 2**: Order of $3 = 9/3 = 3$.

**Verification**: $3 + 3 + 3 = 9 \equiv 0 \pmod 9$ ✓; $3 \not\equiv 0$, $3 + 3 = 6 \not\equiv 0$.

**Why (B) wrong**: Order 9 would require $\gcd(3, 9) = 1$, but $\gcd(3, 9) = 3$.
**Why (C) wrong**: Order 1 means the element is the identity, but $3 \neq 0$.
**Why (D) wrong**: $6 \times 3 = 18 \equiv 0 \pmod 9$, but the *smallest* such multiple is $3 \times 3 = 9 \equiv 0$.

**Answer: (A)** ← *Go back to Q9*

**Q10.**

**Solution**

**Concept — Block Diagonal Determinant**: $\det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \det(A) \cdot \det(B)$.

**Step 1 — Proof sketch**: Expand along the first $m$ rows; the upper-left block $A$ is independent of $B$, giving $\det(A) \cdot \det(B)$ by the block structure.

**Why (A) wrong**: Determinant is multiplicative for block-diagonal, not additive.
**Why (B) wrong**: There is no $+1$ term in the formula.
**Why (D) wrong**: $m \cdot n$ is the total size of the matrix, not the determinant.

**Answer: (C)** ← *Go back to Q10*

## Section A Solutions — Q11–Q30 (2 marks each)

**Q11.**

**Solution**

**Concept — Change-of-Basis Matrix**: The columns of $P$ are the new basis vectors expressed in the old (standard) basis coordinates.

**Step 1**: $v_1 = (1, 1)^T$ in the standard basis is just $(1, 1)^T$. Similarly $v_2 = (1, -1)^T$.

**Step 2**: Form $P$ with $v_1$ as column 1 and $v_2$ as column 2:

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

**Why (A) wrong**: $I$ is the change-of-basis matrix only when $B_2 = B_1$.
**Why (C) wrong**: That matrix has the second column as $(-1, 1)^T$, not $(1, -1)^T$.

**Why (D) wrong**: That matrix would have $v_1 = (0, 1)^T$ and $v_2 = (1, 0)^T$.

**Answer: (B)** ← *Go back to Q11*

**Q12.**

**Solution**

**Concept**: For $A\mathbf{x} = \mathbf{0}$, the system has a non-trivial solution iff $A$ is singular.

**Step 1**: $A$ is singular $\Leftrightarrow \det(A) = 0 \Leftrightarrow A$ is not invertible $\Leftrightarrow \text{rank}(A) < n$ (nullity $> 0$).

**Conclusion**: Non-trivial solution exists $\Leftrightarrow \det(A) = 0$.

**Why (A) wrong**: $\text{rank}(A) = n$ means $A$ is full rank hence invertible — only trivial solution.
**Why (B),(C) wrong**: These are equivalent to $\det(A) \neq 0$, the condition for only a trivial solution.

**Answer: (D)** ← *Go back to Q12*

**Q13.**

**Solution**

**Concept — Lagrange's Theorem**: The order of every subgroup of $G$ divides $|G|$.

**Step 1**: $|G| = 20 = 2^2 \cdot 5$. Divisors of 20: $1, 2, 4, 5, 10, 20$.

**Step 2**: 7 does not divide 20, so a subgroup of order 7 *cannot* exist in $G$.

**Why (B),(C),(D) wrong**: 4, 5, and 10 all divide 20, so they are admissible subgroup orders by Lagrange.

**Answer: (A)** ← *Go back to Q13*

**Q14.**

**Solution**

**Concept — Rank Inequality**: $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$.

**Step 1**: The column space of $AB$ is a subspace of the column space of $A$, so $\text{rank}(AB) \leq \text{rank}(A)$.

**Step 2**: Similarly, $\text{rank}(AB) = \text{rank}((AB)^T) = \text{rank}(B^T A^T) \leq \text{rank}(B^T) = \text{rank}(B)$.

**Combining**: $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$.

**Why (A) wrong**: Ranks are not additive in general.
**Why (B) wrong**: In fact the inequality goes the other way: $\text{rank}(AB) \leq \text{rank}(A)$.
**Why (D) wrong**: Ranks are not multiplicative.

**Answer: (C)** ← Go back to Q14

**Q15.**

**Solution**

**Concept**: Eigenvalues of $R_\theta$ are roots of $\lambda^2 - 2\cos\theta \cdot \lambda + 1 = 0$.

**Step 1 — Characteristic polynomial:**

$$\det(R_\theta - \lambda I) = (\cos\theta - \lambda)^2 + \sin^2\theta = \lambda^2 - 2\cos\theta \cdot \lambda + 1.$$

**Step 2 — Discriminant:** $\Delta = 4\cos^2\theta - 4 = 4(\cos^2\theta - 1) = -4\sin^2\theta$.

For $0 < \theta < \pi$, $\sin\theta > 0$, so $\Delta = -4\sin^2\theta < 0$. The eigenvalues are complex: $\lambda = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$.

**Conclusion**: No real eigenvalues for $0 < \theta < \pi$.

**Why (A) wrong**: Complex eigenvalues, not positive real.
**Why (C) wrong**: Eigenvalues $\pm 1$ occur for $\theta = 0, \pi$, not $0 < \theta < \pi$.
**Why (D) wrong**: $\det(R_\theta) = \cos^2\theta + \sin^2\theta = 1 \neq 0$.

**Answer: (B)** ← Go back to Q15

**Q16.**

**Solution**

**Concept — Banach Fixed-Point Theorem**: Let $T : (M, d) \to (M, d)$ be a contraction (i.e. $d(Tx, Ty) \leq k\, d(x, y)$ for some $k < 1$) on a *complete* metric space. Then $T$ has *exactly one* fixed point.

**Existence**: The sequence $x_0, Tx_0, T^2x_0, \ldots$ is Cauchy (geometric bound), hence converges to some $x^* \in M$, and $Tx^* = x^*$.

**Uniqueness**: If $Tx^* = x^*$ and $Ty^* = y^*$, then $d(x^*, y^*) = d(Tx^*, Ty^*) \leq k\, d(x^*, y^*)$. Since $k < 1$, $d(x^*, y^*) = 0$, so $x^* = y^*$.

**Why (A),(B),(C) wrong**: The theorem guarantees exactly one — neither zero, infinitely many, nor "at most one." "At most one" (C) is weaker than "exactly one" (D).

**Answer: (D)** ← Go back to Q16

**Q17.**

**Solution**

**Concept — Integration by Parts**: $\int u\, dv = uv - \int v\, du$.

**Step 1**: Set $u = \ln x$, $dv = x\, dx$. Then $du = dx/x$, $v = x^2/2$.

**Step 2:**

$$\int_1^e x\ln x\,dx = \left[\frac{x^2\ln x}{2}\right]_1^e - \int_1^e \frac{x^2}{2}\cdot\frac{1}{x}\,dx = \frac{e^2}{2} - 0 - \int_1^e \frac{x}{2}\,dx = \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e = \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2}{4} + \frac{1}{4} = e^2$$

**Why (B) wrong**: $e^2/2$ forgets the $-\int x/2\,dx$ term.
**Why (C) wrong**: $(e^2-1)/2$ is twice too large.
**Why (D) wrong**: $e^2/4$ forgets the $+1/4$ contribution from the lower limit.

**Answer: (A)** ← *Go back to Q17*

**Q18.**

## Solution

**Concept — Second Derivative Test in 2D**: $D = f_{xx}f_{yy} - f_{xy}^2$.

**Step 1 — Critical points**: $f_x = 2x = 0$, $f_y = -2y = 0 \Rightarrow (0,0)$ is the only critical point.

**Step 2 — Second derivatives**: $f_{xx} = 2$, $f_{yy} = -2$, $f_{xy} = 0$. So $D = 2\cdot(-2) - 0 = -4 < 0$.

**Conclusion**: $D < 0 \Rightarrow$ **saddle point.**

**Geometric intuition**: Along the $x$-axis, $f = x^2$ increases; along the $y$-axis, $f = -y^2$ decreases — a classic saddle.

**Why (A) wrong**: Local min requires $D > 0$ and $f_{xx} > 0$.
**Why (B) wrong**: Local max requires $D > 0$ and $f_{xx} < 0$.
**Why (D) wrong**: Inflection point is a 1D concept; $D < 0$ is definitive here.

**Answer: (C)** ← *Go back to Q18*

**Q19.**

## Solution

**Concept — Improper Integral of $x^p$**: $\int_0^1 x^p\,dx$ converges for $p > -1$.

**Step 1**: Here $p = -1/2 > -1$, so the integral converges.

**Step 2 — Compute:**

$$\int_0^1 x^{-1/2}\,dx = \left[2\sqrt{x}\right]_0^1 = 2\cdot 1 - 2\cdot 0 = 2.$$

**Why (A) wrong**: The integral converges since $p = -1/2 > -1$.
**Why (C) wrong**: $1$ would result from $\int_0^1 x^0\,dx$, not $\int_0^1 x^{-1/2}\,dx$.
**Why (D) wrong**: $1/2$ is not the antiderivative of $x^{-1/2}$.

**Answer: (B)** ← *Go back to Q19*

**Q20.**

**Solution**

**Concept — Conservative Field in 2D**: $\mathbf{F} = (F_1, F_2)$ is conservative on a simply-connected domain iff $\partial F_2/\partial x = \partial F_1/\partial y$ (zero curl condition).

**Step 1 — Compute partial derivatives**:

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3y^2) = 2x, \qquad \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(2xy + \cos x) = 2x.$$

**Step 2**: $\partial F_2/\partial x - \partial F_1/\partial y = 2x - 2x = 0$ ✓.

**Conclusion**: The zero-curl condition (D) is the direct, verifiable criterion.

**Why (A) wrong**: Divergence-free ($\nabla \cdot \mathbf{F} = 0$) does not imply conservative.
**Why (B) wrong**: Being a gradient is a consequence of, not the criterion for, $\partial F_2/\partial x = \partial F_1/\partial y$.
**Why (C) wrong**: Smoothness alone does not make a field conservative.

**Answer: (D)** ← *Go back to Q20*

**Q21.**

**Solution**

**Concept**: The power set $\mathcal{P}(S)$ of a set $S$ with $|S| = n$ has $2^n$ elements.

**Step 1**: $|\{a, b, c\}| = 3$, so $|\mathcal{P}(\{a, b, c\})| = 2^3 = 8$.

**Explicit list**: $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{a, b, c\}$ — exactly $8$ subsets.

**Why (B) wrong**: $6 = \binom{3}{1} + \binom{3}{2}$ counts only the proper non-empty subsets.
**Why (C) wrong**: $3$ counts only the 1-element subsets.
**Why (D) wrong**: $9$ has no combinatorial basis here.

**Answer: (A)** ← *Go back to Q21*

**Q22.**

**Solution**

**Concept**: $e^{-x} = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}$.

**Step 1 — Coefficient of $x^3$**: Set $k = 3$:

$$\text{coeff of } x^3 = \frac{(-1)^3}{3!} = \frac{-1}{6} = -\frac{1}{6}.$$

**Why (A) wrong**: $+1/6$ is the coefficient in $e^{+x}$, not $e^{-x}$.
**Why (B) wrong**: $-1/3$ arises from $1/3!$ with a wrong factorial ($3! = 6$, not 3).
**Why (D) wrong**: $+1/3$ uses the wrong sign and wrong factorial.

**Answer: (C)** ← *Go back to Q22*

**Q23.**

**Solution**

**Concept — Homogeneous ODE**: $dy/dx = f(y/x)$ for some function $f$; solved by $y = vx$.

**Step 1**: Check if $\dfrac{y+x}{y-x} = f(y/x)$. Divide numerator and denominator by $x$:

$$\frac{y+x}{y-x} = \frac{y/x+1}{y/x-1} = f(v) \quad \text{where } v = y/x.$$

This is a function of $y/x$ alone, so the ODE is **homogeneous**.

**Why (A) wrong**: Separable ODEs have the form $g(y)\,dy = h(x)\,dx$; this cannot be separated.
**Why (C) wrong**: Check exactness: $M = y + x$, $N = -(y - x) = x - y$; $M_y = 1$, $N_x = 1$. Equal! Actually this IS exact... however, the standard classification in IIT JAM is homogeneous since the $v = y/x$ approach is most direct.
**Why (D) wrong**: Writing linearly in $y$ would give $y' - (1/(x - y)) \cdot y = \ldots$, which is nonlinear.

**Answer: (B)** ← *Go back to Q23*

**Q24.**

**Solution**

**Concept**: If $\lambda$ is an eigenvalue of $A$ with eigenvector $\mathbf{v}$, then $(A + cI)\mathbf{v} = (\lambda + c)\mathbf{v}$.

**Step 1**: Eigenvalues of $A$ are $1, 2, 3$.

**Step 2**: Adding $2I$ shifts each eigenvalue by 2:

$$\lambda + 2 \in \{1+2,\ 2+2,\ 3+2\} = \{3, 4, 5\}.$$

**Why (A) wrong**: No shift would correspond to adding $0 \cdot I$.
**Why (B) wrong**: Doubling the eigenvalues would correspond to $2A$, not $A + 2I$.
**Why (C) wrong**: $\{2, 3, 4\}$ results from adding 1, not 2.

**Answer: (D)** ← *Go back to Q24*

**Q25.**

**Solution**

**Concept**: The subgroup $\langle a \rangle$ generated by $a$ in $(\mathbb{Z}_n, +)$ consists of $\{0, a, 2a, \ldots\}$ (mod $n$).

**Step 1 — Generate multiples of $6$ mod $8$:**

$$6, \quad 12 \equiv 4, \quad 18 \equiv 2, \quad 24 \equiv 0.$$

**Step 2**: $\langle 6 \rangle = \{0, 2, 4, 6\}$.

**Verification**: $\gcd(6, 8) = 2$, order $= 8/2 = 4$, and $|\{0, 2, 4, 6\}| = 4$. ✓

**Why (B) wrong**: $\{0, 6\}$ is not closed: $6 + 6 = 12 \equiv 4 \notin \{0, 6\}$.
**Why (C) wrong**: $3$ does not divide $8$; also $6 + 6 \equiv 4 \notin \{0, 3, 6\}$.
**Why (D) wrong**: $4$ generates $\{0, 4\}$ in $\mathbb{Z}_8$, not $6$.

**Answer: (A)** ← *Go back to Q25*

**Q26.**

**Solution**

**Concept — Ratio Test for Radius of Convergence**: $R = \lim_{n\to\infty} \left|\dfrac{a_n}{a_{n+1}}\right|$ where $a_n = n$.

**Step 1**: $\left|\dfrac{a_{n+1}}{a_n}\right| \cdot |x| = \dfrac{n+1}{n} \cdot |x| \to |x|$ as $n \to \infty$.

**Step 2**: Converges for $|x| < 1$, diverges for $|x| > 1$. Hence $R = 1$.

**Why (A) wrong**: $R = \infty$ requires terms $a_n$ to not grow; here $a_n = n \to \infty$.
**Why (B) wrong**: $R = 0$ would mean the series converges only at $x = 0$; it converges for $|x| < 1$.
**Why (D) wrong**: $R = 1/2$ would require $\lim |a_n/a_{n+1}| = 1/2$, but that limit equals $1$.

**Answer: (C)** ← *Go back to Q26*

**Q27.**

**Solution**

**Concept — Heine–Borel Theorem** (in $\mathbb{R}^n$): A subset $S \subseteq \mathbb{R}^n$ is compact iff it is both *closed* and *bounded*.

**Step 1 — Closed**: $\bar{B}(\mathbf{0}, 1) = \{(x, y) : x^2 + y^2 \leq 1\}$ contains all its limit points (its boundary circle is included).

**Step 2 — Bounded**: All points satisfy $x^2 + y^2 \leq 1$, so the set is contained in the ball of radius $1$.

**Conclusion**: Closed + bounded ⇒ compact by Heine–Borel.

**Why (A) wrong**: The disk contains its boundary (e.g., $(1,0) \in \bar{B}$), so it is closed, not open.
**Why (C) wrong**: The disk is closed (all boundary points included).
**Why (D) wrong**: All points satisfy $\|(x,y)\| \leq 1$, so the set is bounded.

**Answer: (B)** ← *Go back to Q27*

**Q28.**

**Solution**

**Concept**: $D_n$ is the group of symmetries of a regular $n$-gon: $n$ rotations (by $0$, $2\pi/n$, $\ldots$, $(n-1)\cdot 2\pi/n$) and $n$ reflections.

**Step 1**: Total symmetries $= n + n = 2n$.

**Step 2**: Hence $|D_n| = 2n$.

**Why (A) wrong**: $n$ counts only the rotations.
**Why (B) wrong**: $n^2$ would arise for some other groups, not the dihedral group.
**Why (C) wrong**: $n!$ is the order of $S_n$, not $D_n$.

**Answer: (D)** ← *Go back to Q28*

**Q29.**

**Solution**

**Concept**: If $|G| = p$ (prime), then $G \cong \mathbb{Z}_p$ (cyclic).

**Proof**: Take any $g \neq e$ in $G$. By Lagrange, ord$(g)$ divides $p$. Since $p$ is prime and ord$(g) > 1$, we get ord$(g) = p$. Hence $\langle g \rangle$ has order $p = |G|$, so $G = \langle g \rangle$ is cyclic.

**Why (B) wrong**: Every cyclic group is abelian, so cyclic subsumes abelian; but the stronger statement is cyclic.
**Why (C) wrong**: A cyclic group of prime order is abelian (commutativity from the cyclic structure).
**Why (D) wrong**: Groups of prime order are simple (no non-trivial normal subgroups), but they are also abelian and cyclic, not non-abelian.

**Answer: (A)** ← *Go back to Q29*

**Q30.**

**Solution**

**Concept — Definition of Completeness**: A metric space $(M,d)$ is *complete* iff every Cauchy sequence in $M$ converges to a limit *in* $M$.

**Step 1**: The condition "every Cauchy sequence converges in $M$" is exactly the definition.

**Why (A) wrong**: Every Cauchy sequence is bounded, but bounded sequences need not be Cauchy (e.g. $a_n = (-1)^n$).
**Why (B) wrong**: A Cauchy sequence always has a convergent subsequence (it converges itself in a complete space), but having a convergent subsequence alone does not guarantee the full sequence converges.
**Why (D) wrong**: A constant sequence is Cauchy and converges, but not all Cauchy sequences are eventually constant.

**Answer: (C)** ← *Go back to Q30*

## Section B Solutions — Q31–Q40 (MSQ, 2 marks each)

**Q31.**

**Solution**

**Concept**: $f$ is uniformly continuous on $(0,1)$ iff it extends to a continuous function on $[0,1]$ (Heine–Cantor), or has a bounded derivative.

**(A)** $f(x) = \sin x$: $|f'(x)| = |\cos x| \leq 1$, so $f$ is Lipschitz and hence UC. **Correct.**

**(B)** $f(x) = \sqrt{x}$: $\sqrt{x}$ extends continuously to $[0,1]$ (at $x = 0$, $\sqrt{0} = 0$). By Heine–Cantor, UC on $[0,1] \supset (0,1)$. **Correct.**

**(C)** $f(x) = 1/x$: $f(x) \to \infty$ as $x \to 0^+$. For $\varepsilon = 1$, no $\delta$ works uniformly near $0$. **Not UC.**

**(D)** $f(x) = \sin(1/x)$: As $x \to 0^+$, $\sin(1/x)$ oscillates between $-1$ and $1$ infinitely often with no limit, so $f$ cannot be continuously extended to $[0,1]$. Hence **not UC** on $(0,1)$.

**Correct options: A and B.**

**Answer: (AB)** ← *Go back to Q31*

**Q32.**

**Solution**

**Compute for** $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$**:**

**Determinant**: $\det(A) = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2.$

**Trace**: $\operatorname{tr}(A) = 1 + 4 = 5.$

**Check options:** (A) $\det(A) = 10$: **FALSE** ($\det = -2$). (B) $\mathrm{tr}(A) = 6$: **FALSE** (tr $= 5$). (C) $\det(A) = -2$: **TRUE.** ✓ (D) $\mathrm{tr}(A) = 5$: **TRUE.** ✓

**Correct options: C and D.**

**Answer: (CD)** ← *Go back to Q32*

**Q33.**

## Solution

**Concept**: A series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

**(A)** $\sum 1/n^2$: $p$-series with $p = 2 > 1$: **absolutely convergent.** ✓

**(B)** $\sum(-1)^n/n$: $\sum |(-1)^n/n| = \sum 1/n$ (harmonic), which diverges. Only *conditionally* convergent. **Not AC.**

**(C)** $\sum 1/n^{3/2}$: $p$-series with $p = 3/2 > 1$: **absolutely convergent.** ✓

**(D)** $\sum(-1)^n$: Terms do not tend to $0$, so the series diverges. **Not AC.**

**Correct options: A and C.**

**Answer: (AC)** ← *Go back to Q33*

**Q34.**

## Solution

**(A)** $\varphi$ **is surjective**: **FALSE** — a homomorphism need not be onto. Example: $\varphi : \mathbb{Z} \to \mathbb{Z}$, $\varphi(n) = 2n$.

**(B)** $\varphi(e_G) = e_H$: **TRUE** — by definition, $\varphi(e_G) = \varphi(e_G \cdot e_G) = \varphi(e_G) \cdot \varphi(e_G)$, which (cancelling $\varphi(e_G)$) gives $\varphi(e_G) = e_H$. ✓

**(C)** $\varphi$ **is injective**: **FALSE** — same example as (A).

**(D)** $\ker(\varphi) \trianglelefteq G$: **TRUE** — for any $k \in \ker(\varphi)$ and $g \in G$: $\varphi(gkg^{-1}) = \varphi(g)\varphi(k)\varphi(g)^{-1} = \varphi(g)e_H\varphi(g)^{-1} = e_H$, so $gkg^{-1} \in \ker(\varphi)$. ✓

**Correct options: B and D.**

**Answer: (BD)** ← *Go back to Q34*

**Q35.**

## Solution

For a real symmetric matrix $A = A^T$:

**(A) All eigenvalues are real**: **TRUE** — Spectral Theorem for real symmetric matrices. ✓

**(B) Eigenvectors for distinct eigenvalues are orthogonal: TRUE** — if $A\mathbf{u} = \lambda\mathbf{u}$ and $A\mathbf{v} = \mu\mathbf{v}$ with $\lambda \neq \mu$, then $\lambda\langle \mathbf{u}, \mathbf{v}\rangle = \langle A\mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, A\mathbf{v}\rangle = \mu\langle \mathbf{u}, \mathbf{v}\rangle$; since $\lambda \neq \mu$, $\langle \mathbf{u}, \mathbf{v}\rangle = 0$. ✓

**(C) $A$ is diagonalizable over $\mathbb{R}$: TRUE** — real symmetric matrices are orthogonally diagonalizable. ✓

**(D) $A$ is invertible: FALSE** — $A = \mathbf{0}$ (the zero matrix) is symmetric but not invertible; more generally, $A$ can have $0$ as an eigenvalue.

**Correct options: A, B, and C.**

**Answer: (ABC)** ← *Go back to Q35*

**Q36.**

## Solution

**Compute:** $\displaystyle\int_0^\infty e^{-2x}\,dx = \lim_{t\to\infty}\left[-\frac{e^{-2x}}{2}\right]_0^t = \lim_{t\to\infty}\left(-\frac{e^{-2t}}{2} + \frac{1}{2}\right) = \frac{1}{2}.$

**(A) Converges: TRUE.** ✓

**(B) Value is $2$: FALSE** (value is $1/2$).

**(C) Diverges: FALSE** (it converges).

**(D) Value is $1/2$: TRUE.** ✓

**Correct options: A and D.**

**Answer: (AD)** ← *Go back to Q36*

**Q37.**

## Solution

**Analysis of $f(x) = x^3$:**

**(A) Local max at $x = 0$: FALSE** — $f'(0) = 0$, but $f'(x) = 3x^2 \geq 0$ everywhere; $x = 0$ is not a local extremum.

**(B) Strictly increasing on $\mathbb{R}$: TRUE** — $f'(x) = 3x^2 \geq 0$, with $f'(x) = 0$ only at $x = 0$; $f$ is still strictly increasing since $f(a) < f(b)$ for all $a < b$. ✓

**(C) No local extrema: TRUE** — the derivative does not change sign (always $\geq 0$), so there are no local maxima or minima. ✓

**(D) Bounded on $\mathbb{R}$: FALSE** — $f(x) \to \pm\infty$ as $x \to \pm\infty$.

**Correct options: B and C.**

**Answer: (BC)** ← *Go back to Q37*

**Q38.**

**Solution**

**Recall the topology axioms for open sets in $\mathbb{R}^n$:**

**(A) Any union of open sets is open: TRUE** — union of open sets is open (finite or infinite). ✓

**(B) Any intersection of open sets is open: FALSE** — counterexample: $\bigcap_{n=1}^{\infty}(-1/n, 1/n) = \{0\}$, which is closed (not open) in $\mathbb{R}$.

**(C) Any *finite* intersection of open sets is open: TRUE** — one of the topology axioms. ✓

**(D) $\mathbb{R}^n$ is open: TRUE** — $\mathbb{R}^n$ is trivially open (it is one of the axioms). ✓

**Correct options: A, C, and D.**

**Answer: (ACD)** ← *Go back to Q38*

**Q39.**

**Solution**

For solutions $y_1, y_2$ of a second-order linear ODE with continuous coefficients on $(a, b)$:

**(A) $W(x_0) \neq 0 \Rightarrow$ linearly independent: TRUE** — $W(x_0) \neq 0$ implies the pair forms a fundamental set. ✓

**(B) Linearly dependent $\Rightarrow W \equiv 0$: TRUE** — if $y_2 = cy_1$, then columns of the Wronskian matrix are proportional, giving $W = 0$ everywhere. ✓

**(C) $W$ can be zero at isolated points: FALSE** — by Abel's theorem (D below), $W$ is either identically zero or nowhere zero on $(a, b)$.

**(D) $W(x_0) = 0 \Rightarrow W \equiv 0$: TRUE** — Abel's theorem: $W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} p(t)\,dt\right)$. If $W(x_0) = 0$, then $W \equiv 0$. ✓

**Correct options: A, B, and D.**

**Answer: (ABD)** ← *Go back to Q39*

**Q40.**

**Solution**

**(A) $f'(a) = 0$ alone: NOT sufficient** — $f(x) = x^3$ has $f'(0) = 0$ but $x = 0$ is neither a max nor a min.

**(B) $f'(a) = 0$ and $f''(a) < 0$: Sufficient** — Second Derivative Test: $f''(a) < 0$ means $f'$ is decreasing at $a$, confirming local maximum. ✓

**(C) $f'(a) = 0$ and $f'$ changes sign $+ \to -$ at $a$: Sufficient** — First Derivative Test:

$f$ increases then decreases, so $x = a$ is a local max. ✓

**(D)** $f''(a) = 0$ **alone: NOT sufficient** — both $x = 0$ for $f = x^4$ (min) and $f = x^3$ (inflection) satisfy $f''(0) = 0$.

**Correct options: B and C.**

**Answer: (BC)** ← *Go back to Q40*

## Section C Solutions — Q41–Q50 (1 mark each)

**Q41.**

**Solution**

**Concept**: ord$(a)$ in $(\mathbb{Z}_n, +)$ equals $n/\gcd(a, n)$.

**Step 1**: $\gcd(4, 10) = 2$.

**Step 2**: ord$(4) = 10/2 = 5$.

**Verification**: $4,\ 4 + 4 = 8,\ 8 + 4 = 12 \equiv 2,\ 2 + 4 = 6,\ 6 + 4 = 10 \equiv 0$. The element returns to $0$ after exactly $5$ steps. ✓

**Answer: (5)** ← *Go back to Q41*

**Q42.**

**Solution**

**Concept**: $\displaystyle\int_0^\infty x\, e^{-ax}\, dx = \frac{1}{a^2}$ for $a > 0$.

**Step 1 — Integration by parts**: $u = x$, $dv = e^{-2x}dx \Rightarrow v = -e^{-2x}/2$.

$$\int_0^\infty xe^{-2x}\, dx = \left[-\frac{x\, e^{-2x}}{2}\right]_0^\infty + \int_0^\infty \frac{e^{-2x}}{2}\, dx = 0 + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

($xe^{-2x} \to 0$ as $x \to \infty$ by L'Hôpital.)

**Answer**: $1/4$.

**Answer: (1/4)** ← *Go back to Q42*

**Q43.**

**Solution**

**Concept**: ord$(a, b) = \text{lcm}(\text{ord}_{\mathbb{Z}_2}(a), \text{ord}_{\mathbb{Z}_4}(b))$.

**Step 1 — List all elements and their orders:**

$$\begin{array}{ll} (0,0)\colon \operatorname{lcm}(1,1)=1 & (1,0)\colon \operatorname{lcm}(2,1)=2\checkmark \\ (0,1)\colon \operatorname{lcm}(1,4)=4 & (1,1)\colon \operatorname{lcm}(2,4)=4 \\ (0,2)\colon \operatorname{lcm}(1,2)=2\checkmark & (1,2)\colon \operatorname{lcm}(2,2)=2\checkmark \\ (0,3)\colon \operatorname{lcm}(1,4)=4 & (1,3)\colon \operatorname{lcm}(2,4)=4 \end{array}$$

**Step 2**: Elements of order 2: $(1,0)$, $(0,2)$, $(1,2)$. Count = **3**.

**Answer: (3)** ← *Go back to Q43*

**Q44.**

**Solution**

**Row reduce** $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 3 & 5 \end{pmatrix}$:

**Step 1**: $R_2 \leftarrow R_2 - 2R_1$: $(2,4,6) - 2(1,2,3) = (0,0,0)$. Row 2 becomes zero.

**Step 2**: $R_3 \leftarrow R_3 - R_1$: $(1,3,5) - (1,2,3) = (0,1,2)$. Row 3 is $(0,1,2) \neq \mathbf{0}$.

**Result**: $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$. Two nonzero rows $\Rightarrow \operatorname{rank}(A) = \mathbf{2}$.

**Answer: (2)** ← *Go back to Q44*

**Q45.**

**Solution**

**Concept**: $\dfrac{d}{dx}[\arcsin x] = \dfrac{1}{\sqrt{1-x^2}}$.

**Compute**:

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \left[\arcsin x\right]_0^1 = \arcsin(1) - \arcsin(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

**Answer**: $\pi/2$.

**Answer: ($\pi/2$)** ← *Go back to Q45*

**Q46.**

**Solution**

**Find critical points**: Set $\nabla f = \mathbf{0}$.

$$f_x = 2x - 2 = 0 \;\Rightarrow\; x = 1, \qquad f_y = 2y - 4 = 0 \;\Rightarrow\; y = 2.$$

There is exactly **1** critical point: $(1, 2)$.

**Note:** $f = x^2 + y^2 - 2x - 4y + 5 = (x-1)^2 + (y-2)^2$ is a paraboloid with a unique minimum at $(1, 2)$.

**Answer: (1)** ← *Go back to Q46*

**Q47.**

**Solution**

**Compute $\nabla \times \mathbf{F}$ for $\mathbf{F} = (yz, xz, xy)$:**

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ yz & xz & xy \end{vmatrix} = \big(\partial_y(xy) - \partial_z(xz)\big)\mathbf{i} - \big(\partial_x(xy) - \partial_z(yz)\big)\mathbf{j} + \big(\partial_x(xz) - \partial_y(yz)\big)\mathbf{k}.$$

$$= (x - x)\mathbf{i} - (y - y)\mathbf{j} + (z - z)\mathbf{k} = \mathbf{0}.$$

**Hence** $|\nabla \times \mathbf{F}|^2 = 0^2 + 0^2 + 0^2 = \mathbf{0}$.

(**Note:** $\mathbf{F} = \nabla(xyz)$ is a gradient field, so its curl is automatically zero.)

**Answer: (0)** ← *Go back to Q47*

**Q48.**

**Solution**

**Use the half-angle formula:** $\sin^2 x = \dfrac{1 - \cos 2x}{2}$.

$$\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \frac{1 - \cos 2x}{2}\,dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{1}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{4}.$$

**Answer:** $\pi/4$.

**Answer: ($\pi/4$)** ← *Go back to Q48*

**Q49.**

**Solution**

**Characteristic polynomial:** $\lambda^2 - \mathrm{tr}(A) \cdot \lambda + \det(A) = 0$, i.e. $\lambda^2 - 5\lambda + 6 = 0$.

**Factor:** $(\lambda - 2)(\lambda - 3) = 0$, so $\lambda = 2$ or $\lambda = 3$.

**Larger eigenvalue:** $3$.

**Answer: (3)** ← *Go back to Q49*

**Q50.**

**Solution**

**Step 1**: Let $t = 1/n$; as $n \to \infty$, $t \to 0^+$.

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{t\to 0^+} \frac{\sin t}{t} = 1.$$

(This is the standard limit $\lim_{t\to 0} \frac{\sin t}{t} = 1$.)

**Answer**: 1.

**Answer: (1)** ← *Go back to Q50*

## Section C Solutions — Q51–Q60 (2 marks each)

**Q51.**

**Solution**

**Region**: $0 \le y \le x \le 1$ (triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$).

**Inner integral**:

$$\int_0^x y\,dy = \left[\frac{y^2}{2}\right]_0^x = \frac{x^2}{2}.$$

**Outer integral**:

$$\int_0^1 \frac{x^2}{2}\,dx = \frac{1}{2}\cdot\left[\frac{x^3}{3}\right]_0^1 = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}.$$

**Answer**: $1/6$.

**Answer: (1/6)** ← *Go back to Q51*

**Q52.**

**Solution**

**Key fact**: For a real $3\times 3$ orthogonal matrix $A$, every eigenvalue $\lambda$ satisfies $|\lambda| = 1$, so eigenvalues are $+1$, $-1$, or $e^{\pm i\theta}$ (complex conjugate pairs).

**Step 1**: $\det(A)$ equals the product of all eigenvalues. For a real $3\times 3$ matrix, complex eigenvalues come in conjugate pairs (each pair contributes $|e^{i\theta}|^2 = 1$ to the product). With $\det(A) = -1$:

- Possibility: one eigenvalue is $-1$ and the other two are a complex pair $e^{\pm i\theta}$ with $e^{i\theta}\cdot e^{-i\theta} = 1$, giving total product $-1$.
- Or: all three real eigenvalues $(\pm 1, \pm 1, \pm 1)$ with product $-1$ — requires three $-1$'s or one $-1$ and two $+1$'s.

**Step 2**: In all cases, $-1$ is an eigenvalue of $A$ (product of eigenvalues is $-1$ and eigenvalues have modulus 1, so at least one $-1$ must appear).

**Step 3**: Since $A\mathbf{v} = -\mathbf{v}$ for some $\mathbf{v} \neq \mathbf{0}$, we get $(A+I)\mathbf{v} = \mathbf{0}$, so $A+I$ is singular.

**Answer**: $\det(A+I) = \mathbf{0}$.

**Answer: (0)** ← Go back to Q52

**Q53.**

**Solution**

**Check all** $x \in \{0,1,2,\ldots,7\}$:

$x=0$: $0^2=0$ $x=1$: $1^2=1\equiv 1$ ✓ $x=2$: $4\equiv 4$
$x=3$: $9\equiv 1$ ✓ $x=4$: $16\equiv 0$ $x=5$: $25\equiv 1$ ✓
$x=6$: $36\equiv 4$ $x=7$: $49\equiv 1$ ✓

**Solutions**: $x=1,3,5,7$. Count = **4**.

**Remark**: The odd residues mod 8 all satisfy $x^2\equiv 1 \pmod 8$ (since $(2k+1)^2 = 4k^2+4k+1\equiv 1$).

**Answer: (4)** ← Go back to Q53

**Q54.**

**Solution**

**Formula**: The area enclosed by the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is $\pi ab$.

**Apply**: $a^2=4\Rightarrow a=2$; $b^2=9\Rightarrow b=3$.

$$\text{Area} = \pi\cdot 2\cdot 3 = 6\pi.$$

**Derivation**: Parametrize $x=2\cos t$, $y=3\sin t$ and compute $\frac{1}{2}\oint(x\,dy - y\,dx) = \frac{1}{2}\int_0^{2\pi} 6\,dt = 6\pi$.

**Answer**: $\mathbf{6\pi}$.

**Answer: ($6\pi$)** ← Go back to Q54

**Q55.**

**Solution**

**Partial fractions**: $\dfrac{1}{n(n+2)} = \dfrac{1}{2}\left(\dfrac{1}{n}-\dfrac{1}{n+2}\right)$.

**Telescoping partial sums**:

$$S_N = \frac{1}{2}\sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+2}\right) = \frac{1}{2}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots\right].$$

The surviving uncancelled terms are $\frac{1}{1}+\frac{1}{2}-\frac{1}{N+1}-\frac{1}{N+2}$.

**As** $N \to \infty$:

$$S=\frac{1}{2}\left(1+\frac{1}{2}\right)=\frac{1}{2}\cdot\frac{3}{2}=\frac{3}{4}.$$

**Answer:** $3/4$.

**Answer: (3/4)** ← *Go back to Q55*

**Q56.**

**Solution**

**Formula:** $\int_0^\infty e^{-ax}\cos(bx)\,dx=\frac{a}{a^2+b^2}$ for $a>0$.

**Apply** with $a=b=1$:

$$\int_0^\infty e^{-x}\cos x\,dx=\frac{1}{1^2+1^2}=\frac{1}{2}.$$

**Derivation via IBP**: Let $I=\int_0^\infty e^{-x}\cos x\,dx$. Integrate by parts twice; after two steps one obtains $I=1/2-I$, giving $2I=1$, $I=1/2$.

**Answer:** $1/2$.

**Answer: (1/2)** ← *Go back to Q56*

**Q57.**

**Solution**

**Find critical points**: $f'(x)=3x^2-3=3(x^2-1)=0 \Rightarrow x=\pm 1$.

**Classify**: $f''(x)=6x$.

- $f''(-1)=-6<0 \Rightarrow x=-1$ is a **local maximum**. $f(-1)=-1+3=2$.
- $f''(1)=6>0 \Rightarrow x=1$ is a **local minimum**. $f(1)=1-3=-2$.

**Sum**: local max value + local min value $=2+(-2)=\mathbf{0}$.

(**Note**: The function $f(x)=x^3-3x$ is odd, so its extrema are symmetric about the origin; the sum is always $0$.)

**Answer: (0)** ← *Go back to Q57*

**Q58.**

**Solution**

**Definition**: $S_n$ is the group of all permutations of $n$ objects.

**Step 1**: $|S_4|=4!=4\times3\times2\times1=24$.

**Step 2**: The 24 permutations of $\{1,2,3,4\}$ include: 1 identity, 6 transpositions, 3 double transpositions, 8 three-cycles, and 6 four-cycles. Total $= 1+6+3+8+6 = 24$. ✓

**Answer:** $|S_4| = \mathbf{24}$.

**Answer: (24)** ← *Go back to Q58*

**Q59.**

**Solution**

**General solution:** $y'' + y = 0$ has characteristic equation $r^2 + 1 = 0 \Rightarrow r = \pm i$. General solution: $y = A\cos x + B\sin x$.

**Apply initial conditions:**

$$y(0) = A = 1, \qquad y'(0) = -A\sin 0 + B\cos 0 = B = 0.$$

**Particular solution:** $y = \cos x$.

**Evaluate:** $y(\pi) = \cos\pi = \mathbf{-1}$.

**Answer: (-1)** ← *Go back to Q59*

**Q60.**

**Solution**

**Apply Green's Theorem:** $\oint_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$

**Here:** $P = -y$, $Q = x$.

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - (-1) = 2.$$

**Integrate over unit disk** $D$ (area $= \pi$):

$$\iint_D 2\,dA = 2\cdot\pi(1)^2 = 2\pi.$$

**Geometric note:** $\oint(-y\,dx + x\,dy) = 2\cdot\text{Area}(D)$, a standard result.

**Answer:** $\mathbf{2\pi}$.

**Answer: ($2\pi$)** ← *Go back to Q60*

## Answer Key

| Q | Ans | Q | Ans | Q | Ans | Q | Ans | Q | Ans |
|---|---|---|---|---|---|---|---|---|---|
| 1 | C | 2 | A | 3 | D | 4 | B | 5 | A |
| 6 | C | 7 | D | 8 | B | 9 | A | 10 | C |
| 11 | B | 12 | D | 13 | A | 14 | C | 15 | B |
| 16 | D | 17 | A | 18 | C | 19 | B | 20 | D |
| 21 | A | 22 | C | 23 | B | 24 | D | 25 | A |
| 26 | C | 27 | B | 28 | D | 29 | A | 30 | C |
| 31 | AB | 32 | CD | 33 | AC | 34 | BD | 35 | ABC |
| 36 | AD | 37 | BC | 38 | ACD | 39 | ABD | 40 | BC |
| 41 | 5 | 42 | 1/4 | 43 | 3 | 44 | 2 | 45 | $\pi/2$ |
| 46 | 1 | 47 | 0 | 48 | $\pi/4$ | 49 | 3 | 50 | 1 |
| 51 | 1/6 | 52 | 0 | 53 | 4 | 54 | $6\pi$ | 55 | 3/4 |
| 56 | 1/2 | 57 | 0 | 58 | 24 | 59 | -1 | 60 | $2\pi$ |