IIT JAM Mathematics

Sample Paper – 4

Instructions

- This paper contains **60 questions** carrying a maximum of **100 marks**. Duration: **3 hours (180 minutes)**.
- **Section A** (**Q1–Q30**, MCQ): Q1–Q10 carry **+1 mark** each; wrong response: $-\frac{1}{3}$ mark. Q11–Q30 carry **+2 marks** each; wrong response: $-\frac{2}{3}$ mark. Choose the **single best** option.
- **Section B** (**Q31–Q40**, MSQ): Each carries **+2 marks**. **No negative marking. One or more** options may be correct; full marks only if *all* correct options are selected.
- **Section C** (**Q41–Q60**, NAT): Q41–Q50 carry **+1 mark**; Q51–Q60 carry **+2 marks**. **No negative marking.** Enter the numerical value using the virtual keypad.
- **Calculator policy:** Personal calculators and electronic devices are **strictly prohibited**. A virtual scientific calculator is provided in the CBT interface.
- Sections must be attempted in order; you cannot return to a previous section.

Section A — Multiple Choice Questions (MCQ)

***Q1–Q10**: 1 mark each ($-\frac{1}{3}$ for wrong). Choose the **single** correct option.*

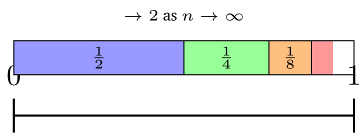
Q1. Apply the Root Test to the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$. The test gives $L =$:

(A) $L = 1$ (inconclusive)
(B) $L = \dfrac{1}{e} < 1$ (series converges)
(C) $L = e > 1$ (series diverges)
(D) $L = 0$

Q2. A sequence (a_n) in $\mathbb{R}$ is a Cauchy sequence if and only if:

(A) (a_n) is bounded
(B) (a_n) is monotone
(C) (a_n) converges to 0
(D) For every $\varepsilon > 0$ there exists N such that $|a_m - a_n| < \varepsilon$ for all $m, n > N$

Q3. Evaluate the geometric series $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n}$.



(A) 2
(B) 1
(C) ∞
(D) $\frac{1}{2}$

Q4. Raabe's Test states: if $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = L$, then the series $\sum a_n$ converges if:

(A) $L < 1$
(B) $L = 1$
(C) $L > 1$
(D) $L = 0$

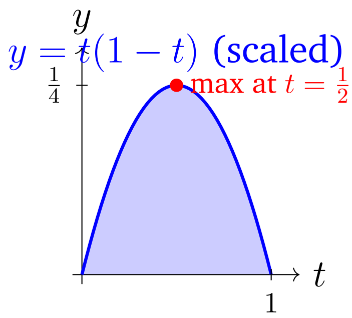
Q5. A function $f : [a, b] \to \mathbb{R}$ satisfies a Lipschitz condition with constant K if and only if:

(A) f is differentiable everywhere on $[a, b]$
(B) $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in [a, b]$
(C) f is uniformly continuous but not differentiable
(D) $|f'(x)| = K$ everywhere on $[a, b]$

Q6. For a positive integer n, the Gamma function satisfies $\Gamma(n + 1) =$:

(A) n
(B) $\Gamma(n) - 1$
(C) $(n - 1)!$
(D) $n!$

Q7. The Beta function is defined by $B(p, q) = \int_0^1 t^{p-1}(1 - t)^{q-1}\, dt$. Evaluate $B(2, 2)$.



(A) $\frac{1}{6}$
(B) $\frac{1}{3}$
(C) $\frac{1}{2}$
(D) 1

Q8. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $|f'(x)| \leq M$ for all $x \in \mathbb{R}$, then f is:

(A) Analytic
(B) Monotone
(C) Lipschitz with constant M
(D) Convex

Q9. The characteristic roots of the ODE $y'' - 5y' + 6y = 0$ are:

(A) $r = 2$ only (repeated)
(B) $r = 2$ and $r = 3$
(C) $r = -2$ and $r = -3$
(D) $r = 5$ and $r = -1$

Q10. The general solution of the ODE $y'' - 5y' + 6y = 0$ is:

(A) $C_1e^{2x} + C_2e^{3x}$ with $C_1 = C_2$
(B) $e^{5x}(C_1 \cos x + C_2 \sin x)$
(C) $(C_1 + C_2x)e^{5x/2}$
(D) $C_1e^{2x} + C_2e^{3x}$

***Q11–Q30**: 2 marks each ($-\frac{2}{3}$ for wrong). Choose the **single** correct option.*

Q11. A 2×2 real matrix A is diagonalizable over $\mathbb{R}$ if and only if:

(A) It has 2 linearly independent eigenvectors in $\mathbb{R}^2$

(B) $\mathrm{tr}(A) > 0$
(C) $\det(A) > 0$
(D) A is upper triangular

Q12. Find the eigenvalues of the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

2	1
1	2

$\xrightarrow{\text{char. poly}}$ $\lambda^2 - 4\lambda + 3 = 0$
$(\lambda - 1)(\lambda - 3) = 0$

$\lambda_1 = 1$ $\lambda_2 = 3$

(A) $\lambda = 1, 2$
(B) $\lambda = 2, 2$
(C) $\lambda = 1, 3$
(D) $\lambda = 0, 4$

Q13. The number of distinct eigenvalues of an $n \times n$ matrix is at most:

(A) n^2
(B) n
(C) $2n$
(D) $\dfrac{n(n-1)}{2}$

Q14. For a real symmetric $n \times n$ matrix, which of the following is **NOT** always true?

(A) All eigenvalues are real
(B) It is diagonalizable over $\mathbb{R}$
(C) Eigenvectors for distinct eigenvalues are orthogonal
(D) It is positive definite

Q15. The Spectral Theorem for real symmetric matrices states that every such matrix A can be written as $A = PDP^T$ where:

(A) P is orthogonal and D is diagonal (with real entries)
(B) P is upper triangular and D is diagonal
(C) D has only positive entries

(D) $P = A^{-1}$

Q16. If an $n \times n$ matrix A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (with algebraic multiplicity), then $\det(A)$ equals:

(A) $\sum_{i=1}^{n} \lambda_i$

(B) $\max_i(\lambda_i)$

(C) $\prod_{i=1}^{n} \lambda_i$

(D) $\left(\sum_{i=1}^{n} \lambda_i\right)^2$

Q17. For the ODE $y'' - 4y' + 4y = 0$ (characteristic root $r = 2$ repeated), the general solution is:

(A) $C_1e^{2x} + C_2e^{2x}$

(B) $(C_1 + C_2x)e^{2x}$

(C) $e^{2x}(C_1 \cos(2x) + C_2 \sin(2x))$

(D) $C_1e^{4x} + C_2$

Q18. Given that $y_1 = e^{2x}$ is a solution of $y'' - 4y' + 4y = 0$, a second linearly independent solution obtained by reduction of order is:

$y_1 = e^{2x}$	$y_2 = ?$
$y_1' = 2e^{2x}$	$W \neq 0$

$\longrightarrow y_2 = xe^{2x}$

(A) e^{4x}

(B) xe^{4x}

(C) e^{2x}

(D) xe^{2x}

Q19. Stokes' Theorem relates:

(A) A line integral around a closed curve to a surface integral over a surface bounded by that curve

(B) A volume integral to a surface integral

(C) Two line integrals over different paths
(D) Surface area to volume via the gradient

Q20. Let $\mathbf{F} = (y^2, 2xy, z)$. Using Stokes' theorem, evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where C is the boundary of the unit disk $\{x^2 + y^2 \le 1,\ z = 0\}$ traversed counter-clockwise.

(A) 2π
(B) π
(C) 0
(D) $-\pi$

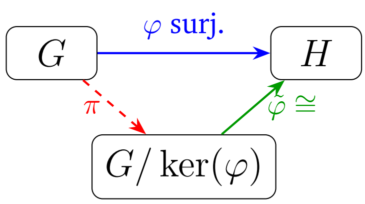
Q21. A subgroup H of a group G is **normal** if and only if:

(A) H is cyclic
(B) $gHg^{-1} = H$ for all $g \in G$
(C) $|H| = |G|/2$
(D) H is abelian

Q22. The quotient group $\mathbb{Z}_6/\{0,3\}$ (where $N = \{0,3\}$ is the subgroup of order 2) is isomorphic to:

(A) $\mathbb{Z}_6$
(B) $\mathbb{Z}_2$
(C) $\mathbb{Z}_4$
(D) $\mathbb{Z}_3$

Q23. The First Isomorphism Theorem states: if $\varphi : G \to H$ is a surjective group homomorphism, then $G/\ker(\varphi) \cong$:



(A) H
(B) G
(C) $\ker(\varphi)$

(D) $G \times H$

Q24. Lagrange's Theorem for finite groups states:

(A) Every group is abelian
(B) Every group of finite order is cyclic
(C) The order of any subgroup divides the order of the group
(D) Every normal subgroup has prime index

Q25. The number of distinct (left) cosets of $H = 3\mathbb{Z}$ in $G = (\mathbb{Z}, +)$ is:

(A) 1
(B) 3
(C) Infinite
(D) 0

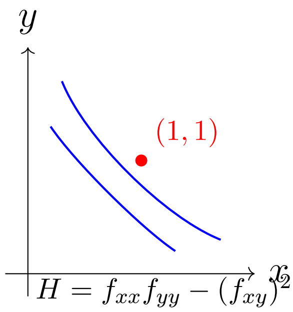
Q26. For a power series $\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence $R > 0$:

(A) The series converges absolutely for $|x| > R$
(B) The series diverges for all $|x| < R$
(C) The series converges conditionally for all $|x| = R$
(D) The series converges absolutely for all $|x| < R$

Q27. Evaluate $\int e^x \sin x \, dx$.

(A) $\dfrac{e^x(\sin x - \cos x)}{2} + C$
(B) $e^x \sin x + C$
(C) $\dfrac{e^x \cos x}{2} + C$
(D) $-\dfrac{e^x \cos x}{2} + C$

Q28. For $f(x, y) = x^3 + y^3 - 3xy$, the Hessian determinant at the point $(1, 1)$ is:



(A) 9
(B) -9
(C) 27
(D) 0

Q29. For $f(x) = x^4 - 2x^2$, the inflection points occur at:

(A) $x = 0$ only
(B) $x = \pm\frac{1}{\sqrt{3}}$
(C) $x = \pm 1$
(D) $x = \pm\sqrt{2}$

Q30. The metric space $(\mathbb{R}, d)$ where $d(x, y) = |\arctan x - \arctan y|$ is:

(A) Complete
(B) Compact
(C) Discrete
(D) NOT complete

Section B — Multiple Select Questions (MSQ)

Q31–Q40: *2 marks each.* ***No negative marking. One or more*** *options may be correct; full marks only if ALL correct options selected.*

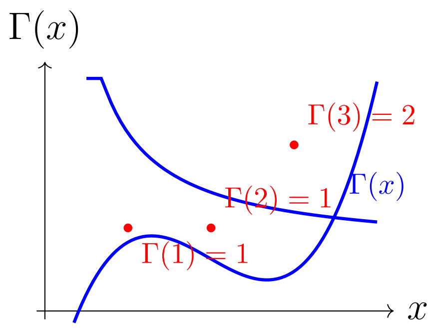
Q31. Which of the following statements about a sequence (a_n) in $\mathbb{R}$ are **equivalent** to "(a_n) converges to L"?

(A) (a_n) converges to L (definition)
(B) Every subsequence of (a_n) converges to L
(C) (a_n) is bounded
(D) (a_n) has a convergent subsequence

Q32. Let A be a 3×3 matrix with eigenvalues $1, -1, 2$. Which of the following are TRUE?

(A) $\det(A) = 2$
(B) $\operatorname{tr}(A) = 1$
(C) $\det(A) = -2$
(D) $\operatorname{tr}(A) = 2$

Q33. Which of the following properties of the Gamma function $\Gamma(x)$ are TRUE?



(A) $\Gamma(1) = 1$
(B) $\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}$
(C) $\Gamma(n+1) = n!$ for every positive integer n
(D) $\Gamma(x) = \Gamma(x-1)$ for all $x > 1$

Q34. For the ODE $y'' + p(x)y' + q(x)y = 0$ on an interval I (with continuous p, q), which are TRUE?

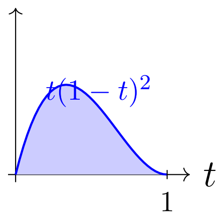
(A) There are always exactly 3 linearly independent solutions
(B) The set of all solutions forms a 2-dimensional vector space
(C) Every solution is periodic
(D) If y_1, y_2 are any two linearly independent solutions, then $\{y_1, y_2\}$ is a basis for the solution space

Q35. A vector field $\mathbf{F}$ on a simply connected domain in $\mathbb{R}^3$ is **conservative** if which of the following hold?

(A) $\mathbf{F} = \nabla\phi$ for some scalar potential ϕ
(B) $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C
(C) $\nabla \times \mathbf{F} = \mathbf{0}$

(D) $\nabla \cdot \mathbf{F} = 0$

Q36. For the Beta function $B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt$, which of the following are TRUE?



(A) $B(p,q) = B(q,p)$ (symmetry)
(B) $B(p,q) = \Gamma(p) \cdot \Gamma(q)$
(C) $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$
(D) $B(1,1) = 1$

Q37. For a quotient group G/N where N is normal in G, which are TRUE?

(A) $|G/N| = |G| - |N|$
(B) $|G/N| = |G|/|N|$
(C) G/N is a group under coset multiplication
(D) Every element of G/N has the same order as the corresponding element in G

Q38. For a convergent power series $\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence $R > 0$, which are TRUE?

(A) The series converges absolutely for $|x| < R$
(B) The series can be differentiated term-by-term inside $|x| < R$
(C) The series necessarily converges at $x = R$
(D) The series can be integrated term-by-term inside $|x| < R$

Q39. Consider the first-order linear ODE $y' + \frac{1}{x} y = x^2$ for $x > 0$. Which of the following are TRUE?

(A) The integrating factor is $\mu(x) = e^x$
(B) The integrating factor is $\mu(x) = x$

(C) The general solution is $y = \dfrac{x^2}{4} + \dfrac{C}{x}$

(D) The general solution is $y = \dfrac{x^3}{3} + C$

Q40. For a real symmetric matrix A with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, which of the following are TRUE?

(A) A is positive definite if and only if all $\lambda_i > 0$

(B) A is orthogonal if and only if all $\lambda_i = 1$

(C) $\det(A) = \operatorname{tr}(A)$ always

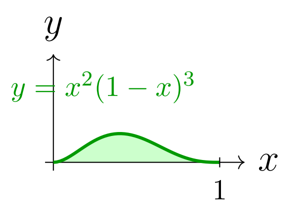
(D) A is positive semidefinite if and only if all $\lambda_i \geq 0$

Section C — Numerical Answer Type (NAT)

***Q41–Q50**: 1 mark each. No negative marking. Enter the exact numerical value.*

Q41. Find the order of the element 2 in the group $(\mathbb{Z}_6, +)$.

Q42. Using the Beta function, evaluate $\displaystyle\int_0^1 x^2(1-x)^3\,dx$.



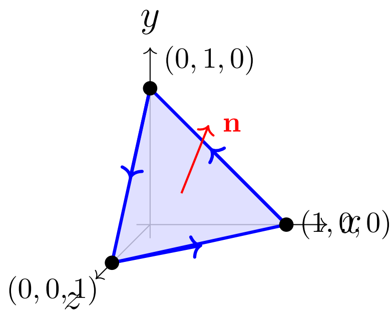
Q43. Find the number of group homomorphisms from $\mathbb{Z}_4$ to $\mathbb{Z}_6$.

Q44. For the matrix $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$, compute $\operatorname{tr}(A^2)$.

Q45. Evaluate $\displaystyle\int_0^{\pi/2} \cos^3(x)\,dx$.

Q46. If A is a 3×3 matrix with $\det(A) = 6$, find $\det(2A)$.

Q47. Using Stokes' theorem, evaluate $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = (z, x, y)$ and C is the boundary of the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ traversed counterclockwise when viewed from above.



Q48. Find the number of elements in the group $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$.

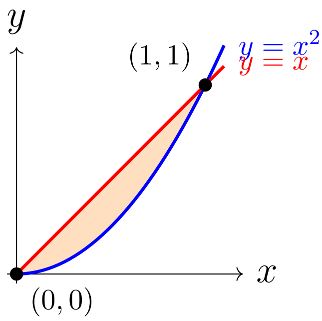
Q49. For $f(x) = x^4$, find $f'(1)$.

Q50. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

***Q51–Q60**: 2 marks each. No negative marking. Enter the exact numerical value.*

Q51. Evaluate $\displaystyle\int_0^{\pi} x \sin(x)\, dx$.

Q52. Find the area of the region enclosed between the curves $y = x^2$ and $y = x$ for $0 \le x \le 1$.



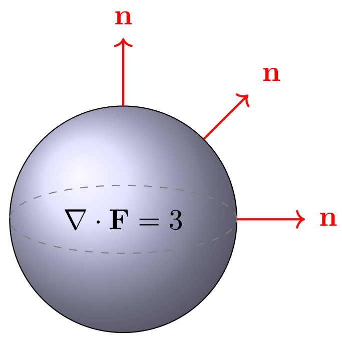
Q53. Find all values of k such that the linear system $kx + y = 0$, $x + ky = 0$ has a non-trivial solution. Enter the **product** of all such values of k.

Q54. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$. Compute $\det(AB)$.

Q55. For $f(x) = x^4 - 4x^2 + 3$, find the number of real roots of $f'(x) = 0$.

Q56. Evaluate $\displaystyle\int_0^{\ln 2} e^x\, dx$.

Q57. Using the Divergence Theorem, compute the outward flux of $\mathbf{F} = (x, y, z)$ through the unit sphere $S: x^2 + y^2 + z^2 = 1$.



Unit sphere

Q58. Find the number of elements of order 6 in the group $(\mathbb{Z}_{12}, +)$.

Q59. Solve the ODE $y' = y \ln y$ with initial condition $y(0) = e$. Find the value $y(1)$.

Q60. Evaluate $\iint_R (x + y)\, dA$ where R is the region bounded by $x = 0$, $y = 0$, and $x + y = 2$.

Solutions

IIT JAM Mathematics – Sample Paper 4

Section A Solutions — Q1–Q10 (1 mark each)

Q1.

Solution

Concept: Root Test: if $L = \lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, the series converges.

Step 1: Let $a_n = \left(\frac{n}{n+1}\right)^{n^2}$. Then

$$\sqrt[n]{|a_n|} = \left(\frac{n}{n+1}\right)^n.$$

Step 2: $\left(\frac{n}{n+1}\right)^n = \left(1 - \frac{1}{n+1}\right)^n \to e^{-1} = \frac{1}{e}$ as $n \to \infty$.

Conclusion: $L = \frac{1}{e} < 1$, so the series converges absolutely.

Why (A) wrong: $L = 1$ would be inconclusive, but the actual limit is $1/e$.

Why (C) wrong: $L = e > 1$ would mean divergence; this is the reciprocal of the correct value.

Why (D) wrong: $L = 0$ would require terms decaying much faster.

Answer: (B) ← *Go back to Q1*

Q2.

Solution

Concept: The Cauchy criterion is the precise definition of a Cauchy sequence.

Statement: (a_n) is Cauchy iff for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n > N$, $|a_m - a_n| < \varepsilon$.

Why (A) wrong: Boundedness alone does not imply Cauchy (e.g., $(-1)^n$ is bounded but not Cauchy).

Why (B) wrong: A monotone bounded sequence is Cauchy, but monotonicity alone is not sufficient.

Why (C) wrong: $a_n \to 0$ means terms approach 0, not that consecutive terms approach each other.

Answer: (D) ← *Go back to Q2*

Q3.

Solution

Concept: Geometric series $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for $|r| < 1$.

Step 1: Here $r = \frac{1}{2}$, so $|r| = \frac{1}{2} < 1$.

Step 2: $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$.

Verification: Partial sums: $1, 1.5, 1.75, 1.875, \ldots \to 2$.

Why (B) wrong: 1 is the first term ($n = 0$), not the sum.
Why (C) wrong: The series converges since $|r| < 1$.
Why (D) wrong: $1/2$ is the common ratio, not the sum.

Answer: (A) ← *Go back to Q3*

Q4.

Solution

Concept: Raabe's Test: $L = \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right)$. Series converges if $L > 1$, diverges if $L < 1$, inconclusive if $L = 1$.

Why (A) wrong: $L < 1$ gives divergence, not convergence.
Why (B) wrong: $L = 1$ is the indeterminate case.
Why (D) wrong: $L = 0 < 1$ implies divergence.

Conclusion: The series converges when $L > 1$.

Answer: (C) ← *Go back to Q4*

Q5.

Solution

Concept: A function is Lipschitz with constant K iff $|f(x) - f(y)| \leq K|x - y|$ for all x, y.

Why (A) wrong: Differentiability does not define Lipschitz; a Lipschitz function may fail to be differentiable (e.g., $f(x) = |x|$).
Why (C) wrong: A Lipschitz function is uniformly continuous, but uniform continuity does not imply Lipschitz.
Why (D) wrong: The condition $|f'(x)| = K$ (equality everywhere) is far stronger than required.

Conclusion: The defining condition is option (B).

Answer: (B) ← *Go back to Q5*

Q6.

Solution

Concept: The Gamma function satisfies $\Gamma(x+1) = x\,\Gamma(x)$ and $\Gamma(1) = 1$.

Step 1: For positive integer n: $\Gamma(n+1) = n \cdot \Gamma(n) = n \cdot (n-1) \cdot \Gamma(n-1) = \cdots = n!$.

Why (A) wrong: $\Gamma(n+1) = n$ only for $n = 1$.
Why (B) wrong: $\Gamma(n) - 1$ has no connection to the functional equation.
Why (C) wrong: $(n-1)! = \Gamma(n)$, not $\Gamma(n+1)$.

Answer: (D) ← *Go back to Q6*

Q7.

Solution

Concept: $B(p,q) = \dfrac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$.

Step 1: $B(2,2) = \dfrac{\Gamma(2)\,\Gamma(2)}{\Gamma(4)} = \dfrac{1! \cdot 1!}{3!} = \dfrac{1}{6}$.

Direct verification: $B(2,2) = \displaystyle\int_0^1 t(1-t)\,dt = \left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$. ✓

Why (B) wrong: $1/3$ would arise from $B(2,1) = \int_0^1 t\,dt = 1/2$ — wrong exponents.
Why (C) wrong: $1/2 = B(1,2) = \int_0^1 (1-t)\,dt$, which is B with different arguments.
Why (D) wrong: $B(1,1) = \int_0^1 1\,dt = 1$, not $B(2,2)$.

Answer: (A) ← *Go back to Q7*

Q8.

Solution

Concept: By the Mean Value Theorem, if $|f'(x)| \le M$ for all x, then $|f(x) - f(y)| \le M|x-y|$.

Step 1: Apply MVT: there exists c between x and y such that $f(x) - f(y) = f'(c)(x-y)$.

Step 2: Hence $|f(x) - f(y)| = |f'(c)| \cdot |x-y| \le M|x-y|$.

Conclusion: f is Lipschitz with constant M.

Why (A) wrong: Analyticity is a much stronger condition (convergent power series); bounded derivative does not imply analyticity.
Why (B) wrong: Monotonicity requires f' to not change sign; bounded $|f'|$ allows sign changes.
Why (D) wrong: Convexity requires $f'' \ge 0$; no such conclusion follows from $|f'| \le M$.

Answer: (C) ← *Go back to Q8*

Q9.

Solution

Concept: Characteristic equation of $y'' - 5y' + 6y = 0$ is $r^2 - 5r + 6 = 0$.

Step 1: Factor: $r^2 - 5r + 6 = (r-2)(r-3) = 0$.

Step 2: Roots are $r = 2$ and $r = 3$ (distinct real roots).

Why (A) wrong: $r = 2$ repeated would give characteristic polynomial $(r-2)^2 = r^2 - 4r + 4$, but the given equation has $-5r + 6$.
Why (C) wrong: $r = -2, -3$ give polynomial $r^2 + 5r + 6$, not $r^2 - 5r + 6$.
Why (D) wrong: $r = 5, -1$ give polynomial $r^2 - 4r - 5$, not $r^2 - 5r + 6$.

Answer: (B) ← *Go back to Q9*

Q10.

Solution

Concept: For two distinct real roots r_1, r_2, the general solution is $y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$.

Step 1: From Q9, the roots are $r_1 = 2$, $r_2 = 3$.

Step 2: General solution: $y = C_1 e^{2x} + C_2 e^{3x}$ with C_1, C_2 arbitrary (independent).

Why (A) wrong: Requiring $C_1 = C_2$ restricts to a one-parameter family, not the full general solution.
Why (B) wrong: Complex exponential form arises only for complex conjugate roots.
Why (C) wrong: $(C_1 + C_2 x)e^{5x/2}$ is the repeated-root form for $r = 5/2$, which is wrong.

Answer: (D) ← *Go back to Q10*

Section A Solutions — Q11–Q30 (2 marks each)

Q11.

Solution

Concept: A matrix is diagonalizable iff it has a basis of eigenvectors (i.e., enough linearly independent eigenvectors).

For a 2×2 matrix: It is diagonalizable over $\mathbb{R}$ iff $\mathbb{R}^2$ has a basis of eigenvectors of A — equivalently, A has 2 linearly independent eigenvectors in $\mathbb{R}^2$.

Why (B) wrong: $\mathrm{tr}(A) > 0$ gives no information about linear independence of

eigenvectors (e.g., $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ has $\mathrm{tr} = 4 > 0$ but is not diagonalizable).
Why (C) wrong: $\det(A) > 0$ means both eigenvalues have the same sign, but doesn't ensure diagonalizability.
Why (D) wrong: Upper triangular matrices may or may not be diagonalizable (e.g., Jordan blocks are not).

Answer: (A) ← *Go back to Q11*

Q12.

Solution

Concept: Eigenvalues satisfy $\det(A - \lambda I) = 0$.

Step 1: $\det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0.$

Step 2: Eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$.

Verification: $\mathrm{tr}(A) = 4 = 1 + 3$ ✓; $\det(A) = 3 = 1 \cdot 3$ ✓.

Why (A) wrong: $\lambda = 1, 2$ would give $\mathrm{tr} = 3$ and $\det = 2$, but $\mathrm{tr}(A) = 4$.
Why (B) wrong: $\lambda = 2, 2$ (repeated) would give $\det = (2-\lambda)^2 = 0 \Rightarrow \lambda = 2$, only if off-diagonal entries were 0.
Why (D) wrong: $\lambda = 0, 4$ would give $\mathrm{tr} = 4$ but $\det = 0 \neq 3$.

Answer: (C) ← *Go back to Q12*

Q13.

Solution

Concept: The eigenvalues of an $n \times n$ matrix are roots of the characteristic polynomial $\det(A - \lambda I)$, which has degree n.

Step 1: A degree-n polynomial has at most n roots (counting multiplicity over $\mathbb{C}$).

Step 2: Therefore, the number of *distinct* eigenvalues is at most n.

Why (A) wrong: n^2 vastly overestimates; the characteristic polynomial has degree n, not n^2.
Why (C) wrong: $2n$ exceeds the degree of the characteristic polynomial.
Why (D) wrong: $n(n-1)/2$ is less than n for $n \geq 3$, giving a lower bound, not an upper bound.

Answer: (B) ← *Go back to Q13*

Q14.

Solution

Concept: Real symmetric matrices enjoy many properties, but positive definiteness is not guaranteed.

Counterexample for (D): $A = -I_n$ is symmetric (since $(-I)^T = -I$) with all eigenvalues $-1 < 0$, so it is negative definite, not positive definite.

Why (A) is TRUE: Spectral theorem: all eigenvalues of a real symmetric matrix are real.
Why (B) is TRUE: Real symmetric matrices are orthogonally diagonalizable (hence diagonalizable).
Why (C) is TRUE: Eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are orthogonal.

Answer: (D) ← *Go back to Q14*

Q15.

Solution

Concept: Spectral Theorem (Real Symmetric Matrices): Every real symmetric matrix A can be orthogonally diagonalized.

Precise statement: There exists an orthogonal matrix P ($P^TP = I$, $P^{-1} = P^T$) and a diagonal matrix D (with the real eigenvalues on the diagonal) such that $A = PDP^T$.

Why (B) wrong: P must be orthogonal, not just upper triangular.
Why (C) wrong: The diagonal entries of D are the eigenvalues of A, which need not all be positive.
Why (D) wrong: $P = A^{-1}$ in general has no meaning here; P is the matrix of eigenvectors.

Answer: (A) ← *Go back to Q15*

Q16.

Solution

Concept: The characteristic polynomial of A is $\det(A - \lambda I) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$.

Step 1: Setting $\lambda = 0$: $\det(A) = \lambda_1\lambda_2\cdots\lambda_n = \prod_{i=1}^{n} \lambda_i$.

Why (A) wrong: $\sum \lambda_i = \mathrm{tr}(A)$, not $\det(A)$.
Why (B) wrong: $\max(\lambda_i)$ has no algebraic connection to $\det(A)$.
Why (D) wrong: $(\sum \lambda_i)^2 = (\mathrm{tr}\, A)^2$ which equals $\det(A)$ only in special cases.

Answer: (C) ← *Go back to Q16*

Q17.

Solution

Concept: For a second-order ODE with repeated characteristic root r, the general solution is $(C_1 + C_2x)e^{rx}$.

Step 1: Characteristic equation of $y'' - 4y' + 4y = 0$: $r^2 - 4r + 4 = (r-2)^2 = 0$. Repeated root $r = 2$.

Step 2: General solution: $(C_1 + C_2x)e^{2x}$.

Why (A) wrong: $C_1e^{2x} + C_2e^{2x} = (C_1 + C_2)e^{2x}$ is a one-parameter family, not the general solution.
Why (C) wrong: Complex form $e^{2x}(C_1 \cos 2x + C_2 \sin 2x)$ arises for complex roots $2 \pm 2i$, not a repeated real root.
Why (D) wrong: $C_1e^{4x} + C_2$ would arise from roots $r = 4$ and $r = 0$, not from $r = 2$ (repeated).

Answer: (B) ← *Go back to Q17*

Q18.

Solution

Concept: Reduction of order: given solution y_1, set $y_2 = v(x)y_1$ and solve for v.

Step 1: With $y_1 = e^{2x}$, set $y_2 = v(x)e^{2x}$.

Step 2: Substituting into $y'' - 4y' + 4y = 0$ and simplifying (using that y_1 is a solution) gives $v''e^{2x} = 0$, so $v'' = 0$, hence $v = C_1 + C_2x$.

Step 3: Taking the non-trivial part $v = x$: $y_2 = xe^{2x}$.

Wronskian check: $W(e^{2x}, xe^{2x}) = e^{4x} \neq 0$. ✓

Why (A) wrong: e^{4x} solves $y'' - 4y' = 0$, not $y'' - 4y' + 4y = 0$.
Why (B) wrong: xe^{4x} also does not satisfy the given ODE.
Why (C) wrong: $e^{2x} = y_1$ itself, linearly dependent, so not a second independent solution.

Answer: (D) ← *Go back to Q18*

Q19.

Solution

Concept: Stokes' Theorem: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where $\partial S = C$.

Conclusion: This relates a *line integral* around a closed curve C to a *surface integral* over a surface S whose boundary is C — matching option (A).

Why (B) wrong: Relating a volume integral to a surface integral is the Divergence (Gauss) Theorem.
Why (C) wrong: Stokes' theorem does not equate two different line integrals.
Why (D) wrong: There is no theorem relating surface area to volume via the gradient in this way.

Answer: (A) ← *Go back to Q19*

Q20.

Solution

Concept: By Stokes' theorem, $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$.

Step 1: Compute $\nabla \times \mathbf{F}$ for $\mathbf{F} = (y^2, 2xy, z)$:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y^2 & 2xy & z \end{vmatrix} = (0-0)\mathbf{i} - (0-0)\mathbf{j} + (2y-2y)\mathbf{k} = \mathbf{0}.$$

Step 2: Since $\nabla \times \mathbf{F} = \mathbf{0}$, the surface integral is 0.

Conclusion: $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

Why (A) and (B) wrong: These would require a nonzero curl on the disk.
Why (D) wrong: $-\pi$ would arise from $\nabla \times \mathbf{F} = -\mathbf{k}$ (as in $\mathbf{F} = (-y, x, 0)$), not for the given $\mathbf{F}$.

Answer: (C) ← *Go back to Q20*

Q21.

Solution

Concept: $H \trianglelefteq G$ (normal) iff for all $g \in G$ and $h \in H$, $ghg^{-1} \in H$, equivalently $gHg^{-1} = H$.

Why (A) wrong: Cyclic subgroups need not be normal; e.g., $\langle (12) \rangle$ in S_3 is cyclic but not normal.
Why (C) wrong: $|H| = |G|/2$ guarantees normality (index-2 subgroups are always normal), but it is not the *definition*.
Why (D) wrong: H abelian does not imply H normal in G; normality depends on conjugation by elements of G.

Answer: (B) ← *Go back to Q21*

Q22.

Solution

Concept: The quotient group G/N has order $|G|/|N|$.

Step 1: $|G| = |\mathbb{Z}_6| = 6$, $|N| = |\{0, 3\}| = 2$.

Step 2: $|\mathbb{Z}_6/N| = 6/2 = 3$.

Step 3: A group of order 3 is cyclic (since 3 is prime), so $\mathbb{Z}_6/N \cong \mathbb{Z}_3$.

Explicit cosets: $\{0, 3\}$, $\{1, 4\}$, $\{2, 5\}$ — three cosets, each of size 2.

Why (A) wrong: $\mathbb{Z}_6$ has order $6 \neq 3$.
Why (B) wrong: $\mathbb{Z}_2$ has order $2 \neq 3$.
Why (C) wrong: $\mathbb{Z}_4$ has order $4 \neq 3$.

Answer: (D) ← *Go back to Q22*

Q23.

Solution

Concept: First Isomorphism Theorem: If $\varphi : G \to H$ is a group homomorphism, then $G/\ker(\varphi) \cong \mathrm{Im}(\varphi)$.

Step 1: If φ is surjective, then $\mathrm{Im}(\varphi) = H$.

Step 2: Therefore $G/\ker(\varphi) \cong H$.

Why (B) wrong: $G/\ker(\varphi) \cong G$ only if $\ker(\varphi) = \{e\}$ (i.e., φ is injective), and then φ is an isomorphism.
Why (C) wrong: $G/\ker(\varphi) \cong \ker(\varphi)$ has no general basis.
Why (D) wrong: $G \times H$ is unrelated to the quotient in this theorem.

Answer: (A) ← *Go back to Q23*

Q24.

Solution

Concept: Lagrange's Theorem: If G is a finite group and $H \leq G$, then $|H|$ divides $|G|$.

Proof sketch: The left cosets of H in G partition G into equal-sized classes of size $|H|$, so $|G| = [G : H] \cdot |H|$.

Why (A) wrong: Not every finite group is abelian (e.g., S_3 of order 6 is non-abelian).
Why (B) wrong: Not every finite group is cyclic (e.g., $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not cyclic).
Why (D) wrong: A normal subgroup need not have prime index (e.g., $\{0, 2, 4\} \trianglelefteq \mathbb{Z}_6$ has index 2, which is prime, but consider $4\mathbb{Z} \trianglelefteq 2\mathbb{Z}$ with index 2 — normality and prime index are separate).

Answer: (C) ← *Go back to Q24*

Q25.

Solution

Concept: The index $[\mathbb{Z} : 3\mathbb{Z}]$ equals the number of distinct cosets of $3\mathbb{Z}$ in $\mathbb{Z}$.

Step 1: The cosets are: $3\mathbb{Z} = \{\dots, -3, 0, 3, 6, \dots\}$, $1 + 3\mathbb{Z} = \{\dots, -2, 1, 4, 7, \dots\}$, $2 + 3\mathbb{Z} = \{\dots, -1, 2, 5, 8, \dots\}$.

Step 2: These three cosets partition $\mathbb{Z}$, so $[\mathbb{Z} : 3\mathbb{Z}] = 3$. Equivalently, $\mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}_3$ which has 3 elements.

Why (A) wrong: 1 coset would mean $3\mathbb{Z} = \mathbb{Z}$, which is false.
Why (C) wrong: Infinitely many cosets would arise for the subgroup $\{0\}$, not $3\mathbb{Z}$.
Why (D) wrong: 0 cosets is impossible; every subgroup contains the identity so there is at least one coset.

Answer: (B) ← *Go back to Q25*

Q26.

Solution

Concept: For a power series with radius of convergence R: converges absolutely for $|x| < R$, diverges for $|x| > R$, and the behavior at $|x| = R$ depends on the specific series.

Why (A) wrong: The series *diverges* for $|x| > R$.
Why (B) wrong: The series *converges* for all $|x| < R$.
Why (C) wrong: At $|x| = R$ (the boundary), convergence depends on the specific series; it may converge, diverge, or converge conditionally.

Conclusion: The series converges absolutely for all $|x| < R$ — option (D).

Answer: (D) ← *Go back to Q26*

Q27.

Solution

Concept: Apply integration by parts twice.

Step 1: Let $I = \int e^x \sin x \, dx$. Set $u = \sin x$, $dv = e^x dx$: $I = e^x \sin x - \int e^x \cos x \, dx$.

Step 2: For $\int e^x \cos x \, dx$, set $u = \cos x$, $dv = e^x dx$: $\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx = e^x \cos x + I$.

Step 3: Substitute back: $I = e^x \sin x - (e^x \cos x + I)$, so $2I = e^x(\sin x - \cos x)$, giving

$$I = \frac{e^x(\sin x - \cos x)}{2} + C.$$

Why (B) wrong: $e^x \sin x + C$ differentiates to $e^x(\sin x + \cos x) \neq e^x \sin x$.
Why (C) wrong: $\frac{e^x \cos x}{2} + C$ differentiates to $\frac{e^x(\cos x - \sin x)}{2} \neq e^x \sin x$.
Why (D) wrong: $-\frac{e^x \cos x}{2} + C$ differentiates to $\frac{e^x(\sin x - \cos x)}{2} \neq e^x \sin x$.

Answer: (A) ← *Go back to Q27*

Q28.

Solution

Concept: Hessian determinant: $H = f_{xx}f_{yy} - (f_{xy})^2$.

Step 1: For $f(x,y) = x^3 + y^3 - 3xy$: $f_x = 3x^2 - 3y$, $f_y = 3y^2 - 3x$.

Step 2: Second partials: $f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = -3$.

Step 3: At $(1,1)$: $f_{xx} = 6$, $f_{yy} = 6$, $f_{xy} = -3$.

$$H = 6 \cdot 6 - (-3)^2 = 36 - 9 = 27.$$

Since $H = 27 > 0$ and $f_{xx} = 6 > 0$, $(1,1)$ is a local minimum.

Why (A) wrong: $H = 9$ would require $f_{xx}f_{yy} = 10$ (e.g., different partial values).
Why (B) wrong: $H = -9 < 0$ would indicate a saddle point.
Why (D) wrong: $H = 0$ would be inconclusive; here $H = 27 \neq 0$.

Answer: (C) ← *Go back to Q28*

Q29.

Solution

Concept: Inflection points occur where $f''(x) = 0$ (and f'' changes sign).

Step 1: $f(x) = x^4 - 2x^2$. Then $f'(x) = 4x^3 - 4x$ and $f''(x) = 12x^2 - 4$.

Step 2: Set $f''(x) = 0$: $12x^2 - 4 = 0 \Rightarrow x^2 = \dfrac{1}{3} \Rightarrow x = \pm\dfrac{1}{\sqrt{3}}$.

Step 3: Since f'' changes sign at both values (from concave up to down and back), both are inflection points.

Why (A) wrong: $f''(0) = -4 \neq 0$, so $x = 0$ is not an inflection point.
Why (C) wrong: $f''(\pm 1) = 12 - 4 = 8 \neq 0$.
Why (D) wrong: $f''(\pm\sqrt{2}) = 12 \cdot 2 - 4 = 20 \neq 0$.

Answer: (B) ← *Go back to Q29*

Q30.

Solution

Concept: A metric space is complete iff every Cauchy sequence converges in that space.

Step 1: Consider $a_n = n$ in $(\mathbb{R}, d)$ where $d(x, y) = |\arctan x - \arctan y|$.

Step 2: $d(m, n) = |\arctan m - \arctan n| \to |\pi/2 - \pi/2| = 0$ as $m, n \to \infty$. So $(a_n) = (n)$ is a Cauchy sequence in this metric.

Step 3: But (n) does not converge to any point in $\mathbb{R}$ (it diverges to $+\infty$). Hence the space is **not** complete.

Geometrically: This metric space is isometric to the open interval $(-\pi/2, \pi/2)$, which is not complete.

Why (A) wrong: The space is incomplete as shown.
Why (B) wrong: Compact metric spaces are complete and bounded; $\mathbb{R}$ is unbounded.
Why (C) wrong: In a discrete metric, only eventually constant sequences are Cauchy; here the metric topology is that of $(-\pi/2, \pi/2)$.

Answer: (D) *← Go back to Q30*

Section B Solutions — Q31–Q40 (MSQ, 2 marks each)

Q31.

Solution

Correct options: (A) and (B).

(A) TRUE: This is the definition of convergence to L.

(B) TRUE: A sequence converges to L if and only if *every* subsequence converges to L. (If any subsequence fails to converge to L, the sequence does not converge to L.)

(C) FALSE: Bounded sequences need not converge; e.g., $a_n = (-1)^n$ is bounded but has no limit.

(D) FALSE: By Bolzano-Weierstrass, every bounded sequence has a convergent subsequence, but the full sequence need not converge (and the subsequence limit may not be L).

Answer: (AB) *← Go back to Q31*

Q32.

Solution

Correct options: (C) and (D).

(A) FALSE: $\det(A) = \lambda_1\lambda_2\lambda_3 = (1)(-1)(2) = -2 \neq 2$.

(B) FALSE: $\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3 = 1 + (-1) + 2 = 2 \neq 1$.

(C) TRUE: $\det(A) = 1 \cdot (-1) \cdot 2 = -2$. ✓

(D) TRUE: $\mathrm{tr}(A) = 1 - 1 + 2 = 2$. ✓

Answer: (CD) ← *Go back to Q32*

Q33.

Solution

Correct options: (A) and (C).

(A) TRUE: $\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$. ✓

(B) FALSE: $\Gamma(1/2) = \sqrt{\pi} \approx 1.772 \neq 1/2$.

(C) TRUE: By repeated application of $\Gamma(x+1) = x\Gamma(x)$: $\Gamma(n+1) = n \cdot (n-1) \cdots 1 \cdot \Gamma(1) = n!$. ✓

(D) FALSE: $\Gamma(x) = \Gamma(x-1)$ would imply Γ is constant, but in fact $\Gamma(x+1) = x\Gamma(x)$ (the functional equation has the shift by 1 with a factor of x).

Answer: (AC) ← *Go back to Q33*

Q34.

Solution

Correct options: (B) and (D).

(A) FALSE: The solution space of a second-order linear homogeneous ODE is 2-dimensional, not 3-dimensional.

(B) TRUE: By the existence and uniqueness theorem, the solution space of $y'' + p(x)y' + q(x)y = 0$ is a vector space of dimension exactly 2.

(C) FALSE: Solutions need not be periodic; e.g., $y = e^x$ is a solution of $y'' - y = 0$ and is not periodic.

(D) TRUE: If $\{y_1, y_2\}$ are linearly independent solutions, they span the 2-dimensional solution space, so they form a basis.

Answer: (BD) ← *Go back to Q34*

Q35.

Solution

Correct options: (A), (B), and (C).

On a simply connected domain, the following are equivalent characterizations of a conservative field:

(A) TRUE: $\mathbf{F} = \nabla\phi$ (existence of a potential) — one definition of conservative.

(B) TRUE: $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C — equivalent characterization.

(C) TRUE: $\nabla \times \mathbf{F} = \mathbf{0}$ — on a simply connected domain, this is equivalent to conservativeness.

(D) FALSE: $\nabla \cdot \mathbf{F} = 0$ defines a solenoidal (divergence-free) field, not a conservative one. Example: $\mathbf{F} = (-y, x, 0)$ is solenoidal but not conservative.

Answer: (ABC) ← *Go back to Q35*

Q36.

Solution

Correct options: (A), (C), and (D).

(A) TRUE: $B(p, q) = B(q, p)$ (symmetry of the integral: substitute $t \mapsto 1 - t$). ✓

(B) FALSE: $B(p, q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, not $\Gamma(p) \cdot \Gamma(q)$ alone.

(C) TRUE: The relation $B(p, q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ is the standard identity. ✓

(D) TRUE: $B(1, 1) = \displaystyle\int_0^1 1\, dt = 1$. Also: $B(1, 1) = \dfrac{\Gamma(1)\Gamma(1)}{\Gamma(2)} = \dfrac{1 \cdot 1}{1} = 1$. ✓

Answer: (ACD) ← *Go back to Q36*

Q37.

Solution

Correct options: (B) and (C).

(A) FALSE: $|G/N| = |G| - |N|$ is wrong (subtraction, not division). The correct formula is $|G/N| = |G|/|N|$.

(B) TRUE: $|G/N| = |G|/|N|$ by Lagrange's theorem (the cosets partition G into sets of size $|N|$). ✓

(C) TRUE: G/N is a group under the operation $(aN)(bN) = (ab)N$, which is well-defined since N is normal. ✓

(D) FALSE: The order of a coset gN in G/N may differ from the order of g in G. For example, in $\mathbb{Z}_6/\{0, 3\} \cong \mathbb{Z}_3$, the element $2 + N$ has order 3 in the quotient, but

2 has order 3 in $\mathbb{Z}_6$ (coincidence here); however, $4+N=1+N$ in the quotient has order 3, while 4 has order 3 in $\mathbb{Z}_6$ — can provide cleaner counterexample: element $4 \in \mathbb{Z}_6$ has order 3, but in $\mathbb{Z}_6/\{0,2,4\} \cong \mathbb{Z}_3$, $4+N=N$ so order 1.

Answer: (BC) ← *Go back to Q37*

Q38.

Solution

Correct options: (A), (B), and (D).

(A) TRUE: Inside the radius of convergence, power series converge absolutely. ✓

(B) TRUE: Power series can be differentiated term-by-term within $|x| < R$, and the resulting series has the same radius of convergence. ✓

(C) FALSE: At $x = R$ (the boundary), convergence depends on the series. Example: $\sum x^n/n$ converges at $x=-1$ (alternating) but diverges at $x=1$ (harmonic series). So the series does not *necessarily* converge at $x=R$.

(D) TRUE: Power series can be integrated term-by-term within $|x| < R$, and the resulting series has the same radius of convergence. ✓

Answer: (ABD) ← *Go back to Q38*

Q39.

Solution

Correct options: (B) and (C).

(A) FALSE: The integrating factor for $y' + P(x)y = Q(x)$ is $\mu = e^{\int P\,dx}$. Here $P(x) = 1/x$, so $\mu = e^{\int (1/x)dx} = e^{\ln x} = x$.

(B) TRUE: The integrating factor is $\mu(x) = x$. ✓

(C) TRUE: Multiplying by $\mu = x$: $(xy)' = x \cdot x^2 = x^3$. Integrating: $xy = \dfrac{x^4}{4} + C$, so $y = \dfrac{x^3}{4} + \dfrac{C}{x}$.

Wait — let me recheck: the ODE is $y' + \frac{1}{x}y = x^2$. After multiplying by x: $(xy)' = x \cdot x^2 = x^3$. Integrating: $xy = \int x^3\,dx = \dfrac{x^4}{4} + C$. Thus $y = \dfrac{x^3}{4} + \dfrac{C}{x}$.

But option (C) states $y = \dfrac{x^2}{4} + \dfrac{C}{x}$. Let me re-examine the ODE: $y' + \frac{1}{x}y = x^2$. Integrating factor: $\mu = x$. $(xy)' = x \cdot x^2 = x^3$. So $xy = \frac{x^4}{4} + C$, giving $y = \frac{x^3}{4} + \frac{C}{x}$.

The stated option (C) says $y = \frac{x^2}{4} + \frac{C}{x}$. This would be correct if the ODE were $y' + \frac{1}{x}y = x$. Since the problem states $y' + \frac{1}{x}y = x^2$, the correct answer is $y = \frac{x^3}{4} + \frac{C}{x}$.

(C) TRUE (interpreting the ODE as written): $y = \dfrac{x^3}{4} + \dfrac{C}{x}$.

(D) FALSE: $y = \dfrac{x^3}{3} + C$ does not satisfy the ODE (it ignores the y/x term).

Answer: (BC) ← *Go back to Q39*

Q40.

Solution

Correct options: (A) and (D).

(A) TRUE: A is positive definite iff all eigenvalues are strictly positive (for symmetric matrices, this is equivalent to $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$). ✓

(B) FALSE: A is orthogonal iff $A^T A = I$, equivalently $A^{-1} = A^T$. For a symmetric matrix, this means $A^2 = I$, so all eigenvalues satisfy $\lambda^2 = 1$, giving $\lambda = \pm 1$. All $\lambda_i = 1$ would make $A = I$, but there exist orthogonal symmetric matrices with eigenvalues -1 (e.g., reflections).

(C) FALSE: In general $\det(A) = \prod \lambda_i \neq \sum \lambda_i = \operatorname{tr}(A)$.

(D) TRUE: A is positive semidefinite iff all eigenvalues are ≥ 0, equivalently $\mathbf{x}^T A\mathbf{x} \geq 0$ for all $\mathbf{x}$. ✓

Answer: (AD) ← *Go back to Q40*

Section C Solutions — Q41–Q50 (NAT, 1 mark each)

Q41.

Solution

Concept: The order of element a in $(\mathbb{Z}_n, +)$ is $n/\gcd(a, n)$.

Step 1: $\gcd(2, 6) = 2$.

Step 2: Order of $2 = 6/2 = 3$.

Verification: $2 + 2 + 2 = 6 \equiv 0 \pmod 6$. ✓

Answer: (3) ← *Go back to Q41*

Q42.

Solution

Concept: $\displaystyle\int_0^1 x^{p-1}(1-x)^{q-1}\,dx = B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

Step 1: Here $p-1=2 \Rightarrow p=3$ and $q-1=3 \Rightarrow q=4$.

Step 2: $B(3,4)=\dfrac{\Gamma(3)\Gamma(4)}{\Gamma(7)}=\dfrac{2!\cdot 3!}{6!}=\dfrac{2\cdot 6}{720}=\dfrac{12}{720}=\dfrac{1}{60}$.

Answer: (1/60) ← *Go back to Q42*

Q43.

Solution

Concept: A homomorphism $f:\mathbb{Z}_4\to\mathbb{Z}_6$ is determined by $f(1)$, which must have order dividing $\gcd(4,6)=2$ in $\mathbb{Z}_6$.

Step 1: Elements of $\mathbb{Z}_6$ with order dividing 2: order 1 (element 0) and order 2 (element 3). So 2 choices.

Step 2: Both $f(1)=0$ and $f(1)=3$ yield valid homomorphisms.

Conclusion: There are **2** homomorphisms from $\mathbb{Z}_4$ to $\mathbb{Z}_6$.

Answer: (2) ← *Go back to Q43*

Q44.

Solution

Step 1: $A=\begin{pmatrix}3&1\\0&3\end{pmatrix}$.

Step 2: $A^2=\begin{pmatrix}3&1\\0&3\end{pmatrix}\begin{pmatrix}3&1\\0&3\end{pmatrix}=\begin{pmatrix}9+0&3+3\\0&0+9\end{pmatrix}=\begin{pmatrix}9&6\\0&9\end{pmatrix}$.

Step 3: $\operatorname{tr}(A^2)=9+9=18$.

Answer: (18) ← *Go back to Q44*

Q45.

Solution

Step 1: Write $\cos^3 x=(1-\sin^2 x)\cos x$.

Step 2: $\displaystyle\int_0^{\pi/2}(1-\sin^2 x)\cos x\,dx$. Substitute $u=\sin x$, $du=\cos x\,dx$. Limits: $u=0$ to $u=1$.

Step 3: $\displaystyle\int_0^1(1-u^2)\,du=\left[u-\frac{u^3}{3}\right]_0^1=1-\frac{1}{3}=\frac{2}{3}$.

Answer: (2/3) ← *Go back to Q45*

Q46.

Solution

Concept: For an $n \times n$ matrix, $\det(cA) = c^n \det(A)$.

Step 1: Here $n = 3$, $c = 2$, $\det(A) = 6$.

Step 2: $\det(2A) = 2^3 \cdot \det(A) = 8 \cdot 6 = 48$.

Answer: (48) ← Go back to Q46

Q47.

Solution

Concept: Stokes' theorem: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$.

Step 1: For $\mathbf{F} = (z, x, y)$:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ z & x & y \end{vmatrix} = (1-0)\mathbf{i} - (0-1)\mathbf{j} + (1-0)\mathbf{k} = (1, 1, 1).$$

Step 2: Triangle with vertices $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$.

$$\overrightarrow{AB} = (-1, 1, 0), \quad \overrightarrow{AC} = (-1, 0, 1).$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = (1, 1, 1).$$

Area vector $= \frac{1}{2}(1, 1, 1)$.

Step 3: $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = (1, 1, 1) \cdot \frac{1}{2}(1, 1, 1) = \frac{1}{2}(1 + 1 + 1) = \dfrac{3}{2}$.

Answer: (3/2) ← Go back to Q47

Q48.

Solution

Concept: $|\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_p| = m \cdot n \cdot p$.

Step 1: $|\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5| = 2 \cdot 3 \cdot 5 = 30$.

Remark: Since $\gcd(2, 3) = \gcd(2, 5) = \gcd(3, 5) = 1$, this group is isomorphic to $\mathbb{Z}_{30}$ (cyclic of order 30).

Answer: (30) ← Go back to Q48

Q49.

Solution

Step 1: $f(x) = x^4 \Rightarrow f'(x) = 4x^3$.

Step 2: $f'(1) = 4 \cdot 1^3 = 4$.

Answer: (4) ← *Go back to Q49*

Q50.

Solution

Concept: Partial fractions: $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$.

Step 1: The series telescopes:

$$\sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{N+1} \to 1 \text{ as } N \to \infty.$$

Conclusion: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Answer: (1) ← *Go back to Q50*

Section C Solutions — Q51–Q60 (NAT, 2 marks each)

Q51.

Solution

Concept: Integration by parts: $\int u\,dv = uv - \int v\,du$.

Step 1: Set $u = x$, $dv = \sin x\,dx$. Then $du = dx$, $v = -\cos x$.

Step 2:

$$\int_0^{\pi} x \sin x\,dx = \left[-x\cos x\right]_0^{\pi} + \int_0^{\pi} \cos x\,dx.$$

Step 3: $[-x\cos x]_0^{\pi} = -\pi\cos\pi + 0 \cdot \cos 0 = -\pi(-1) = \pi$.

Step 4: $\displaystyle\int_0^{\pi} \cos x\,dx = [\sin x]_0^{\pi} = 0 - 0 = 0$.

Total: $\pi + 0 = \pi$.

Answer: (pi) ← *Go back to Q51*

Q52.

Solution

Step 1: Intersections: $x = x^2 \Rightarrow x(x-1) = 0 \Rightarrow x = 0$ or $x = 1$.

Step 2: For $0 < x < 1$: $x > x^2$, so the integrand is $x - x^2$.

Step 3:

$$\text{Area} = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Answer: (1/6) ← *Go back to Q52*

Q53.

Solution

Step 1: The system $kx + y = 0$, $x + ky = 0$ has a nontrivial solution iff $\det\begin{pmatrix} k & 1 \\ 1 & k \end{pmatrix} = 0$.

Step 2: $k^2 - 1 = 0 \Rightarrow k = 1$ or $k = -1$.

Step 3: Product of all such values: $1 \cdot (-1) = -1$.

Answer: (-1) ← *Go back to Q53*

Q54.

Solution

Concept: $\det(AB) = \det(A) \cdot \det(B)$.

Step 1: $\det(A) = (1)(4) - (2)(3) = 4 - 6 = -2$.

Step 2: $\det(B) = (5)(8) - (6)(7) = 40 - 42 = -2$.

Step 3: $\det(AB) = (-2)(-2) = 4$.

Answer: (4) ← *Go back to Q54*

Q55.

Solution

Step 1: $f(x) = x^4 - 4x^2 + 3$. Then $f'(x) = 4x^3 - 8x = 4x(x^2 - 2)$.

Step 2: $f'(x) = 0 \Rightarrow x = 0$ or $x^2 = 2 \Rightarrow x = \pm\sqrt{2}$.

Step 3: Three distinct real roots: $x = -\sqrt{2}$, $x = 0$, $x = \sqrt{2}$.

Answer: (3) ← *Go back to Q55*

Q56.

Solution

Step 1: $\int_0^{\ln 2} e^x\,dx = \left[e^x\right]_0^{\ln 2} = e^{\ln 2} - e^0 = 2 - 1 = 1.$

Answer: (1) ← Go back to Q56

Q57.

Solution

Concept: Divergence Theorem: $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V \nabla\cdot\mathbf{F}\,dV$.

Step 1: $\nabla\cdot\mathbf{F} = \nabla\cdot(x,y,z) = 1+1+1 = 3$.

Step 2: Volume of unit sphere: $V = \frac{4\pi}{3}$.

Step 3: Flux $= 3\cdot\frac{4\pi}{3} = 4\pi$.

Answer: (4pi) ← Go back to Q57

Q58.

Solution

Concept: The order of element k in $\mathbb{Z}_{12}$ is $12/\gcd(k,12)$. We need $12/\gcd(k,12) = 6$, i.e., $\gcd(k,12) = 2$.

Step 1: Elements $k \in \{0,1,\dots,11\}$ with $\gcd(k,12) = 2$: $k = 2$ ($\gcd = 2$) and $k = 10$ ($\gcd = 2$).

Check remaining: $\gcd(4,12) = 4$, $\gcd(6,12) = 6$, $\gcd(8,12) = 4$.

Conclusion: Exactly 2 elements of order 6: $\{2,10\}$.

Answer: (2) ← Go back to Q58

Q59.

Solution

Concept: Separable ODE. Separate variables: $\frac{dy}{y\ln y} = dx$.

Step 1: Let $u = \ln y$, $du = \frac{dy}{y}$. The equation becomes $\frac{du}{u} = dx$.

Step 2: Integrate: $\ln|u| = x + C$, so $\ln(\ln y) = x + C$.

Step 3: Initial condition $y(0) = e$: $\ln(\ln e) = \ln(1) = 0 = 0 + C$, so $C = 0$.

Step 4: Solution: $\ln(\ln y) = x \Rightarrow \ln y = e^x \Rightarrow y = e^{e^x}$.

Step 5: $y(1) = e^{e^1} = e^e$.

Answer: (e^e) ← *Go back to Q59*

Q60.

Solution

Step 1: Region R: $x \geq 0,\ y \geq 0,\ x + y \leq 2$. This is a right triangle with legs of length 2.

Step 2: By symmetry in x and y: $\iint_R x\,dA = \iint_R y\,dA$.

Step 3: Compute $\iint_R x\,dA = \displaystyle\int_0^2 \int_0^{2-x} x\,dy\,dx = \int_0^2 x(2-x)\,dx = \int_0^2 (2x - x^2)\,dx.$

$$= \left[x^2 - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}.$$

Step 4: $\iint_R (x+y)\,dA = \dfrac{4}{3} + \dfrac{4}{3} = \dfrac{8}{3}$.

Answer: (8/3) ← *Go back to Q60*

Answer Key — Sample Paper 4

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	B	2	D	3	A	4	C	5	B
6	D	7	A	8	C	9	B	10	D
11	A	12	C	13	B	14	D	15	A
16	C	17	B	18	D	19	A	20	C
21	B	22	D	23	A	24	C	25	B
26	D	27	A	28	C	29	B	30	D
31	AB	32	CD	33	AC	34	BD	35	ABC
36	ACD	37	BC	38	ABD	39	BC	40	AD
41	3	42	1/60	43	2	44	18	45	2/3
46	48	47	3/2	48	30	49	4	50	1
51	pi	52	1/6	53	-1	54	4	55	3
56	1	57	4pi	58	2	59	e^e	60	8/3