IIT JAM Mathematics

Sample Paper – 5

Instructions

- This paper contains **60 questions** carrying a maximum of **100 marks**. Duration: **3 hours (180 minutes)**.
- **Section A (Q1–Q30**, MCQ): Q1–Q10 carry **+1 mark** each; wrong response: $-\frac{1}{3}$ mark. Q11–Q30 carry **+2 marks** each; wrong response: $-\frac{2}{3}$ mark. Choose the **single best** option.
- **Section B (Q31–Q40**, MSQ): Each carries **+2 marks**. **No negative marking. One or more** options may be correct; full marks only if *all* correct options are selected.
- **Section C (Q41–Q60**, NAT): Q41–Q50 carry **+1 mark**; Q51–Q60 carry **+2 marks**. **No negative marking.** Enter the numerical value using the virtual keypad.
- **Calculator policy:** Personal calculators and electronic devices are **strictly prohibited**. A virtual scientific calculator is provided in the CBT interface.
- Sections must be attempted in order; you cannot return to a previous section.

Section A — Multiple Choice Questions (MCQ)

***Q1–Q10**: 1 mark each ($-\frac{1}{3}$ for wrong). Choose the **single** correct option.*

Q1. What is $\limsup_{n\to\infty} a_n$ for the sequence $a_n = (-1)^n + \dfrac{1}{n}$?

(A) 0

(B) -1

(C) 0.5

(D) 1

Q2. If (a_n) converges to L and (b_n) converges to M, then $\lim\limits_{n\to\infty}(a_n \cdot b_n)$ equals:

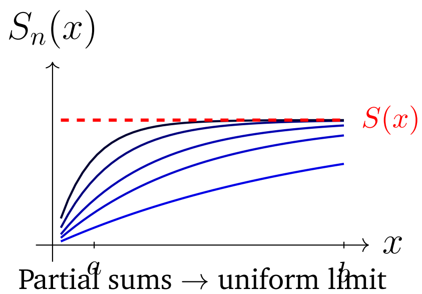
(A) $L + M$

(B) $L \cdot M$

(C) $\max(L, M)$

(D) L/M

Q3. The Weierstrass M-test states that if $|f_n(x)| \leq M_n$ for all $x \in [a, b]$ and $\sum M_n$ converges, then $\sum f_n(x)$:



Partial sums $\to$ uniform limit

(A) Diverges on $[a, b]$

(B) Converges pointwise but not uniformly on $[a, b]$

(C) Converges uniformly (and absolutely) on $[a, b]$

(D) Converges to 0 uniformly

Q4. The sequence of functions $f_n(x) = x^n$ on $[0, 1]$ converges:

(A) Pointwise to $f(x) = 0$ for $x \in [0, 1)$ and $f(1) = 1$, but not uniformly on $[0, 1]$

(B) Uniformly on $[0, 1]$

(C) In L^2 norm to $\sin x$

(D) To x uniformly

Q5. For which value of p does the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converge?

(A) $p = 0$

(B) $p = \frac{1}{2}$

(C) $p = 1$

(D) $p > 1$

Q6. Evaluate $\displaystyle\lim_{x \to 0} \frac{\sin(3x)}{\sin(5x)}$.

(A) 3

(B) $\frac{3}{5}$

(C) $\frac{5}{3}$

(D) 1

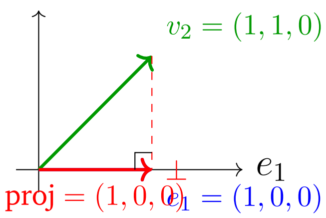
Q7. The power series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ represents:

(A) $\sin(x)$
(B) e^{-x}
(C) $\cos(x)$
(D) $\cosh(x)$

Q8. The interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ is:

(A) $-1 \leq x < 1$
(B) $-1 < x < 1$
(C) $-1 < x \leq 1$
(D) All $x \in \mathbb{R}$

Q9. In the Gram-Schmidt process applied to $\{v_1 = (1,0,0),\ v_2 = (1,1,0)\}$ in $\mathbb{R}^3$, the first orthonormal vector e_1 and the projection of v_2 onto e_1 are:



(A) $e_1 = (1,1,0)/\sqrt{2}$, proj$= (1,1,0)$
(B) $e_1 = (0,1,0)$, proj$= (0,0,0)$
(C) $e_1 = (1,0,0)$ and proj$= (1,1,0)$
(D) $e_1 = (1,0,0)$ and proj$= (1,0,0)$

Q10. The Gram-Schmidt process produces:

(A) An orthogonal matrix from any matrix
(B) An orthonormal set from a linearly independent set
(C) Eigenvalues of a matrix
(D) The inverse of a matrix

***Q11–Q30**: 2 marks each ($-\frac{2}{3}$ for wrong). Choose the **single** correct option.*

Q11. For $f_n(x) = \dfrac{nx}{1+n^2x^2}$ on $[0,1]$, which statement is TRUE?

(A) $f_n \to 0$ uniformly on $[0,1]$

(B) $f_n \to \sin(x)$ pointwise

(C) $f_n \to 0$ pointwise but NOT uniformly (since $\sup_{x\in[0,1]} |f_n(x)| = \frac{1}{2}$ for all n)

(D) f_n diverges for every $x > 0$

Q12. If $\sum f_n$ converges uniformly on $[a, b]$ and each f_n is continuous on $[a, b]$, then:

(A) $\int_a^b \sum f_n \, dx = \sum \int_a^b f_n \, dx$ (interchange is valid)

(B) $\sum f_n$ need not be continuous on $[a, b]$

(C) Convergence must be absolute for integration

(D) The result only holds when f_n are polynomials

Q13. Evaluate $\int_0^1 \int_y^1 e^{x^2} \, dx \, dy$ by reversing the order of integration.

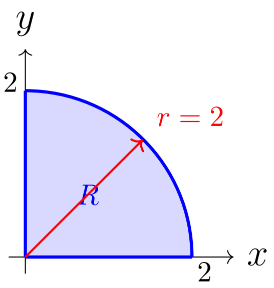
(A) $e - 1$

(B) $\frac{e+1}{2}$

(C) $\frac{e}{2}$

(D) $\frac{e-1}{2}$

Q14. The region of integration $\int_0^2 \int_0^{\sqrt{4-y^2}} f(x, y) \, dx \, dy$ expressed in polar coordinates (r, θ) has:



(A) r from 0 to 2, θ from 0 to π

(B) r from 0 to 2, θ from 0 to $\frac{\pi}{2}$

(C) r from 0 to 4, θ from 0 to 2π

(D) r from 1 to 2, θ from 0 to $\frac{\pi}{2}$

Q15. Evaluate $\displaystyle\int_0^{\pi/2}\int_0^2 r^2\,dr\,d\theta$.

(A) π
(B) 2π
(C) $\dfrac{4\pi}{3}$
(D) $\dfrac{8\pi}{3}$

Q16. The inner product (dot product) of $u = (1, 2, 3)$ and $v = (4, 5, 6)$ in $\mathbb{R}^3$ is:

(A) 32
(B) 14
(C) 45
(D) 77

Q17. For the Gram-Schmidt process applied to $\{v_1 = (3, 4),\ v_2 = (1, 0)\}$ in $\mathbb{R}^2$, the orthonormal basis vector e_1 is:

(A) $(3, 4)$
(B) $(1, 0)$
(C) $\left(\dfrac{4}{5}, -\dfrac{3}{5}\right)$
(D) $\left(\dfrac{3}{5}, \dfrac{4}{5}\right)$

Q18. If A is an $n \times n$ orthogonal matrix ($A^TA = I$), then $|\det(A)|$ equals:

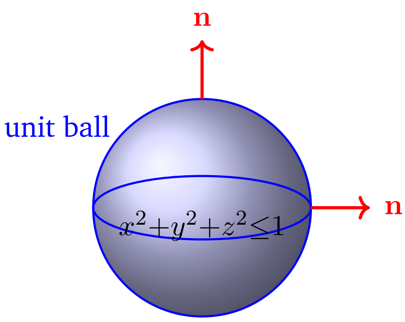
(A) 0
(B) 1
(C) n
(D) $\det(A^T)$

Q19. For a conservative vector field $\mathbf{F} = \nabla\varphi$ in $\mathbb{R}^3$, the line integral $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ from P to Q equals:

(A) 0 always
(B) $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r}$
(C) $\varphi(Q) - \varphi(P)$

(D) $\iint_S \mathbf{F} \cdot d\mathbf{S}$

Q20. Using the Divergence Theorem, compute the outward flux of $\mathbf{F} = (x^2, y^2, z^2)$ through the unit sphere $x^2 + y^2 + z^2 = 1$.



(A) 0
(B) $\frac{4\pi}{3}$
(C) 4π
(D) $\frac{8\pi}{3}$

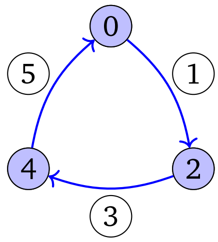
Q21. The order of the permutation $(1\ 2\ 3)(4\ 5)$ in S_5 (product of disjoint cycles of lengths 3 and 2) is:

(A) 5
(B) 2
(C) 3
(D) 6

Q22. The center $Z(S_3)$ of the symmetric group S_3 is:

(A) S_3 itself
(B) $\{e\}$ (trivial group)
(C) A_3
(D) Any subgroup of order 2

Q23. In the group $(\mathbb{Z}_6, +)$, the subgroup generated by 2 is:



$\langle 2 \rangle = \{0, 2, 4\}$

(A) $\{0, 2\}$
(B) $\{0, 3\}$
(C) $\{0, 2, 4\}$
(D) $\mathbb{Z}_6$

Q24. The kernel of the homomorphism $\varphi : \mathbb{Z}_{12} \to \mathbb{Z}_4$ defined by $\varphi(k) = k \bmod 4$ is:

(A) $\{0, 4, 8\}$
(B) $\{0, 6\}$
(C) $\{0, 3, 6, 9\}$
(D) $\{0, 2, 4, 6, 8, 10\}$

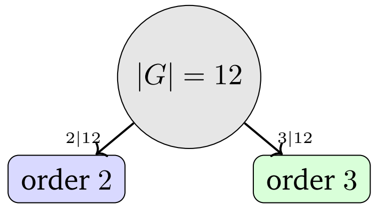
Q25. By the First Isomorphism Theorem, $\mathbb{Z}_{12}/\ker(\varphi) \cong \mathrm{im}(\varphi)$ for $\varphi : \mathbb{Z}_{12} \to \mathbb{Z}_4$ above. Thus $\mathbb{Z}_{12}/\{0, 4, 8\} \cong$:

(A) $\mathbb{Z}_6$
(B) $\mathbb{Z}_2$
(C) $\mathbb{Z}_{12}$
(D) $\mathbb{Z}_4$

Q26. The order of the permutation $(1\ 2)(3\ 4\ 5)$ in S_5 is:

(A) 2
(B) 6
(C) 3
(D) 5

Q27. By Cauchy's theorem, if p divides $|G|$, then G has an element of order p. For $|G| = 12$, which prime orders *must* elements of G possess?



(A) Only order 1

(B) Orders 2, 3, and 4

(C) Orders 2 and 3

(D) Order 12 only

Q28. By Sylow's theorem, if $p^k \mid |G|$ but $p^{k+1} \nmid |G|$, the number of Sylow p-subgroups n_p satisfies:

(A) $n_p \equiv 1 \pmod p$ and $n_p \mid |G|/p^k$

(B) $n_p = 1$ always

(C) n_p divides p^k

(D) $n_p = p$

Q29. For a group G of order 15, the number of Sylow 3-subgroups n_3 equals:

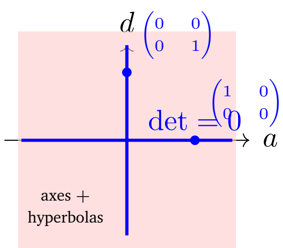
(A) 5

(B) 3

(C) 15

(D) 1

Q30. The set of all 2×2 real matrices with determinant 0 forms:



(A) A group under addition

(B) Neither a group under addition nor under multiplication

(C) A group under matrix multiplication

(D) A subspace of $M_{2\times 2}(\mathbb{R})$

Section B — Multiple Select Questions (MSQ)

Q31–Q40**: 2 marks each.* ***No negative marking. One or more *options may be correct; full marks only if ALL correct options selected.*

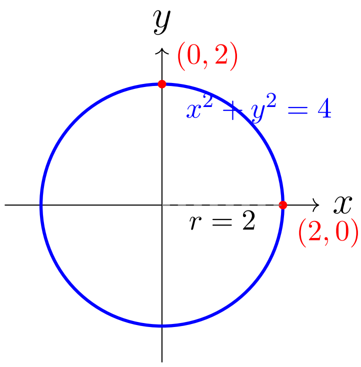
Q31. Which of the following sequences of functions converge **uniformly** on $[0, 1]$?

(A) $f_n(x) = x^n$
(B) $f_n(x) = \dfrac{x}{n}$
(C) $f_n(x) = n\,x\,(1-x)^n$
(D) $f_n(x) = \sin\left(\dfrac{x}{n}\right)$

Q32. Which of the following are TRUE for the double integral $\iint_R f(x, y)\,dA$?

(A) If $f(x, y) \geq 0$ on R, then the integral ≥ 0
(B) The integral always equals a product of two single integrals
(C) If f and g are integrable on R, then $f+g$ is integrable and $\iint(f+g) = \iint f + \iint g$
(D) $\iint_R 1\,dA =$ Area of R

Q33. For $f(x, y) = x^2 + y^2 - 4$, which statements describe the level curve $f = 0$?



(A) It is a circle of radius 2 centred at the origin
(B) Its equation is $x^2 + y^2 = 4$
(C) It is an ellipse
(D) It passes through $(1, 1)$

Q34. For a vector space V over $\mathbb{R}$, which conditions make a subset W a subspace?

(A) W is non-empty and closed under scalar multiplication only
(B) W contains the zero vector and is finite
(C) W is closed under vector addition and scalar multiplication
(D) W contains the zero vector and satisfies (C)

Q35. For the inner product space $\mathbb{R}^3$ with the standard dot product, which are TRUE?

(A) Cauchy–Schwarz inequality: $|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|$
(B) Orthogonality: $\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow |\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$
(C) The projection of $\mathbf{u}$ onto $\mathbf{v}$ always has magnitude $|\mathbf{u}|$
(D) Triangle inequality: $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$

Q36. For a group G with subgroup H, which statements are TRUE?

(A) Every subgroup H of G is normal
(B) If $[G : H] = 2$ (index 2), then H is normal in G
(C) The intersection of two normal subgroups of G is normal in G
(D) The union of two subgroups of G is always a subgroup

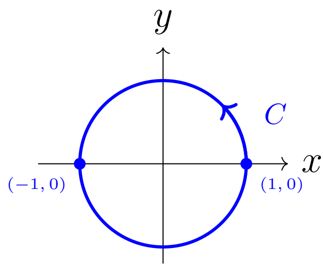
Q37. For the sequence $\{a_n\}$ defined by $a_1 = 1$, $a_{n+1} = \sqrt{a_n + 2}$, which are TRUE?

(A) The sequence is bounded above (by 2)
(B) The sequence is decreasing
(C) The sequence is increasing
(D) The sequence converges (limit $L = 2$, fixed point of $x = \sqrt{x + 2}$)

Q38. For the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, x^n}{n}$ (the series for $\ln(1 + x)$), which are TRUE?

(A) Radius of convergence is ∞
(B) It converges at $x = 1$
(C) Its sum equals $\ln(1 + x)$ for $|x| < 1$
(D) It converges at $x = -1$

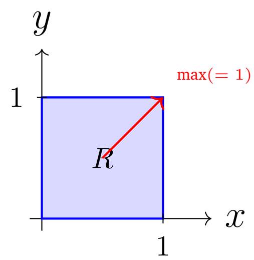
Q39. For $\mathbf{F} = (2xy,\, x^2,\, 0)$ on $\mathbb{R}^3$, which are TRUE?



Unit circle in xy-plane

(A) $\nabla \times \mathbf{F} = \mathbf{0}$, so $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed C
(B) The integral $\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ over the unit circle
(C) $\mathbf{F}$ is conservative with potential $\varphi = x^2 y$
(D) The integral $\oint_C \mathbf{F} \cdot d\mathbf{r} = \pi$ over the unit circle

Q40. For $f(x, y) = xy$ on the region $R = [0, 1] \times [0, 1]$:



(A) $\iint_R xy\, dA = \dfrac{1}{2}$
(B) $\iint_R xy\, dA = \dfrac{1}{4}$
(C) f has a maximum at $(0, 0)$ on R
(D) f has maximum value 1 at $(1, 1)$ on R

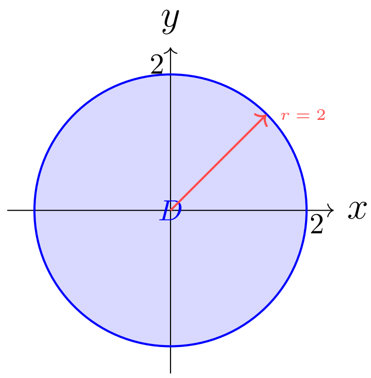
Section C — Numerical Answer Type (NAT)

***Q41–Q50**: 1 mark each. No negative marking. Enter the exact numerical value.*

Q41. Evaluate $\lim_{n\to\infty} \left(1 + \dfrac{2}{n}\right)^n$. (Enter your answer in terms of e if needed.)

Q42. For the sequence $a_1 = 2$, $a_{n+1} = \dfrac{a_n + 4/a_n}{2}$, find the limit L.

Q43. Evaluate $\displaystyle\iint_D (x^2 + y^2)\, dA$ where D is the disk $x^2 + y^2 \le 4$.



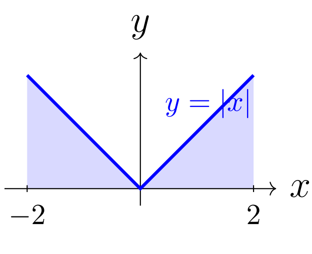
Q44. How many elements of order 3 are there in the group $(\mathbb{Z}_9, +)$?

Q45. Find the sum $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$.

Q46. Evaluate $\displaystyle\int_{-\pi}^{\pi} \sin^2(x)\,dx$.

Q47. Find the rank of the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Q48. Evaluate $\displaystyle\int_{-2}^{2} |x|\,dx$.



Q49. For $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, compute $\mathrm{tr}(A^4)$.

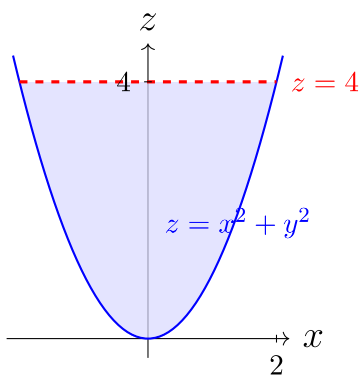
Q50. How many surjective group homomorphisms are there from $\mathbb{Z}_6$ to $\mathbb{Z}_2$?

***Q51–Q60**: 2 marks each. No negative marking. Enter the exact numerical value.*

Q51. Evaluate $\displaystyle\iint_R xy\,dA$ where R is the region $0 \le x \le 1$, $0 \le y \le x$.

Q52. If A is a 2×2 matrix with $\det(A) = 5$, find $\det(3A)$.

Q53. Find the volume of the solid bounded below by $z = x^2 + y^2$ and above by $z = 4$.



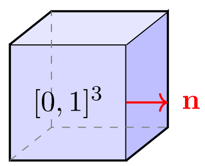
Q54. For the IVP $y' = 2x + 1$, $y(0) = 1$, find $y(2)$.

Q55. How many elements of S_5 have cycle type $(2, 3)$ (one disjoint 2-cycle and one 3-cycle)?

Q56. For $f(x) = \dfrac{\sin x}{x}$ $(x \neq 0)$ and $f(0) = 1$, find $f'(0)$.

Q57. Evaluate $\displaystyle\int_0^1 x^2 e^x \, dx$.

Q58. Using the Divergence Theorem, compute $\oiint_S \mathbf{F} \cdot d\mathbf{S}$ for $\mathbf{F} = (xy, yz, zx)$ over the surface of the unit cube $[0, 1]^3$.



Q59. The matrix $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is diagonal. What is the degree of its minimal polynomial?

Q60. Find the number of group homomorphisms from $\mathbb{Z}_8$ to $\mathbb{Z}_{12}$.

Solutions

IIT JAM Mathematics – Sample Paper 5

Section A Solutions — Q1–Q10 (1 mark each)

Q1.

Solution

Concept: $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{k\geq n} a_k$, the largest cluster point.

Step 1: Write $a_n = (-1)^n + 1/n$. For even n: $a_n = 1 + 1/n \to 1$ from above. For odd n: $a_n = -1 + 1/n \to -1$ from above.

Step 2: The sequence has two cluster points: 1 (from even subsequence) and -1 (from odd subsequence).

Conclusion: $\limsup a_n = \max\{1, -1\} = 1$.

Why (A) wrong: 0 is not a cluster point of this sequence.
Why (B) wrong: -1 is $\liminf$, not $\limsup$.
Why (C) wrong: 0.5 is the arithmetic mean of the two cluster points, not the $\limsup$.

Answer: (D) *← Go back to Q1*

Q2.

Solution

Concept: Limit laws: if $a_n \to L$ and $b_n \to M$, then $a_n b_n \to LM$.

Step 1: This is one of the fundamental limit laws for sequences.

Step 2: The product of limits equals the limit of products: $\lim(a_n b_n) = (\lim a_n)(\lim b_n) = L \cdot M$.

Why (A) wrong: $L + M$ is the limit of $a_n + b_n$, not $a_n b_n$.
Why (C) wrong: $\max(L, M)$ is not a limit law.
Why (D) wrong: L/M is the limit of a_n/b_n (when $M \neq 0$), not $a_n b_n$.

Answer: (B) *← Go back to Q2*

Q3.

Solution

Concept: Weierstrass M-test: absolute domination by a convergent numerical series implies uniform and absolute convergence.

Step 1: We have $|f_n(x)| \leq M_n$ for all $x \in [a, b]$ and $\sum M_n < \infty$.

Step 2: Both absolute convergence ($\sum |f_n(x)|$ converges) and uniform convergence (partial sums are uniformly Cauchy) follow.

Conclusion: $\sum f_n(x)$ converges uniformly and absolutely on $[a, b]$. Answer: (C).

Why (A) wrong: The M-test implies convergence, not divergence.
Why (B) wrong: The M-test yields *uniform* convergence, not merely pointwise.
Why (D) wrong: The series converges to the function $f = \sum f_n$, not necessarily to 0.

Answer: (C) ← *Go back to Q3*

Q4.

Solution

Concept: Pointwise limit of $f_n(x) = x^n$ on $[0, 1]$.

Step 1: For $0 \leq x < 1$: $\lim_{n\to\infty} x^n = 0$ (geometric decay).

Step 2: For $x = 1$: $\lim_{n\to\infty} 1^n = 1$.

Step 3: Convergence is *not* uniform on $[0, 1]$: the pointwise limit is discontinuous at $x = 1$, but each f_n is continuous — uniform convergence would preserve continuity. Alternatively, $\sup_{x\in[0,1]} |f_n(x) - f(x)| = 1 \not\to 0$.

Why (B) wrong: Not uniform — the sup of $|f_n - f|$ over $[0, 1]$ equals 1 for all n.
Why (C) wrong: The limit function is $f(x) = 0$ (a.e.), not $\sin x$.
Why (D) wrong: $f_n(x) \to x$ only if $x = 0$ or $x = 1$; for other x the limit is 0.

Answer: (A) ← *Go back to Q4*

Q5.

Solution

Concept: Cauchy condensation / integral test. Let $f(x) = 1/(x(\ln x)^p)$.

Step 1: $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^p}$. Substitute $u = \ln x$, $du = dx/x$:

$$\int_{\ln 2}^\infty u^{-p}\, du.$$

Step 2: This integral converges iff $p > 1$.

Conclusion: $\sum_{n=2}^\infty \frac{1}{n(\ln n)^p}$ converges if and only if $p > 1$.

Why (A),(B),(C) wrong: For $p \leq 1$ the integral (and hence the series) diverges.

Answer: (D) ← *Go back to Q5*

Q6.

Solution

Concept: Use $\lim_{u\to 0}\sin(u)/u = 1$.

Step 1: Write

$$\frac{\sin(3x)}{\sin(5x)} = \frac{\sin(3x)}{3x}\cdot\frac{5x}{\sin(5x)}\cdot\frac{3}{5}.$$

Step 2: As $x \to 0$: $\dfrac{\sin(3x)}{3x} \to 1$ and $\dfrac{5x}{\sin(5x)} \to 1$.

Conclusion: $\displaystyle\lim_{x\to 0}\frac{\sin(3x)}{\sin(5x)} = 1\cdot 1\cdot\frac{3}{5} = \frac{3}{5}$.

Why (A) wrong: 3 would be the limit if the denominator approached 1.
Why (C) wrong: $5/3$ is the reciprocal — arising from confusing numerator and denominator.
Why (D) wrong: 1 would hold if both arguments were identical.

Answer: (B) ← *Go back to Q6*

Q7.

Solution

Concept: Standard Maclaurin series.

Step 1: Recall $\displaystyle\cos(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

Step 2: This matches the given series exactly.

Why (A) wrong: $\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$ (odd powers).
Why (B) wrong: $e^{-x} = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}$ (all powers, no $(2n)!$).
Why (D) wrong: $\cosh x = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$ (no alternating sign).

Answer: (C) ← *Go back to Q7*

Q8.

Solution

Concept: Ratio test gives radius $R = 1$. Check endpoints separately.

Step 1: Radius of convergence: $R = \lim_{n\to\infty}|n/(n+1)| = 1$.

Step 2: At $x = 1$: $\sum 1/n$ (harmonic series) $\to$ **diverges**.

Step 3: At $x = -1$: $\sum(-1)^n/n$ (alternating harmonic series) $\to$ **converges** (alternating series test).

Conclusion: Interval of convergence is $[-1, 1)$, i.e., $-1 \le x < 1$.

Why (B) wrong: $x = -1$ is included (alternating harmonic converges).

Why (C) wrong: $x = 1$ is excluded (harmonic series diverges).
Why (D) wrong: Series diverges for $|x| > 1$.

Answer: (A) ← *Go back to Q8*

Q9.

Solution

Concept: Gram-Schmidt: $e_1 = v_1/\|v_1\|$; then project v_2 onto e_1.

Step 1: $e_1 = \dfrac{v_1}{\|v_1\|} = \dfrac{(1,0,0)}{1} = (1,0,0)$.

Step 2: Projection of $v_2 = (1,1,0)$ onto $e_1 = (1,0,0)$:

$$\mathrm{proj}_{e_1}(v_2) = (v_2 \cdot e_1)\, e_1 = (1)\,(1,0,0) = (1,0,0).$$

Conclusion: $e_1 = (1,0,0)$ and the projection is $(1,0,0)$. Answer (D).

Why (A) wrong: $(1,1,0)/\sqrt{2}$ would be e_1 if $v_1 = (1,1,0)$, not $(1,0,0)$.
Why (B) wrong: $e_1 = (0,1,0)$ would arise from a different starting vector.
Why (C) wrong: The projection equals $(1,0,0)$, not $v_2 = (1,1,0)$ itself.

Answer: (D) ← *Go back to Q9*

Q10.

Solution

Concept: Gram-Schmidt orthogonalization.

Step 1: The process takes a linearly independent set and produces a set of mutually orthogonal unit vectors (orthonormal set) spanning the same subspace.

Conclusion: It produces an *orthonormal set* from a linearly independent set. Answer (B).

Why (A) wrong: An orthogonal matrix requires the column vectors to be an orthonormal basis for the whole space — Gram-Schmidt does not produce this from an arbitrary matrix.
Why (C) wrong: Eigenvalues are found by solving $\det(A - \lambda I) = 0$, not via Gram-Schmidt.
Why (D) wrong: The inverse is computed separately (e.g., row reduction or adjugate formula).

Answer: (B) ← *Go back to Q10*

Section A Solutions — Q11–Q30 (2 marks each)

Q11.

Solution

Concept: Check pointwise limit and then test for uniformity via $\sup|f_n - f|$.

Step 1 (pointwise): Fix $x > 0$. Then $\dfrac{nx}{1+n^2x^2} = \dfrac{x/n}{1/n^2+x^2} \to \dfrac{0}{x^2} = 0$. Also $f_n(0) = 0$. So $f_n \to 0$ pointwise on $[0,1]$.

Step 2 (not uniform): Find the maximum of f_n. Set $\dfrac{d}{dx}\left(\dfrac{nx}{1+n^2x^2}\right) = 0$:

$$\frac{n(1+n^2x^2) - nx \cdot 2n^2x}{(1+n^2x^2)^2} = \frac{n(1-n^2x^2)}{(1+n^2x^2)^2} = 0 \implies x = \frac{1}{n}.$$

Step 3: $f_n(1/n) = \dfrac{n\cdot(1/n)}{1+1} = \dfrac{1}{2}$ for every n.

Conclusion: $\sup_{x\in[0,1]}|f_n(x)| = \dfrac{1}{2} \not\to 0$, so convergence is *not* uniform.

Why (A) wrong: Uniform convergence fails since the sup stays at $1/2$.
Why (B) wrong: The pointwise limit is 0, not $\sin x$.
Why (D) wrong: $f_n(x) \to 0$ for each fixed x, so it does not diverge.

Answer: (C) ← *Go back to Q11*

Q12.

Solution

Concept: Uniform convergence justifies term-by-term integration.

Step 1: Since $\sum f_n \rightrightarrows S$ uniformly on $[a,b]$ and each f_n is continuous, the limit S is also continuous.

Step 2: By the theorem on integration of uniformly convergent series:

$$\int_a^b S(x)\,dx = \int_a^b \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty}\int_a^b f_n(x)\,dx.$$

Conclusion: Interchange of sum and integral is valid. Answer (A).

Why (B) wrong: $S = \sum f_n$ is continuous (uniform limit of continuous functions).
Why (C) wrong: Absolute convergence is not required; uniform convergence suffices.
Why (D) wrong: The theorem applies to any continuous f_n, not just polynomials.

Answer: (A) ← *Go back to Q12*

Q13.

Solution

Concept: Reversing the order of integration in $\int_0^1 \int_y^1 e^{x^2}\,dx\,dy$.

Step 1: The region is $\{(x, y) : 0 \le y \le 1,\ y \le x \le 1\} = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le x\}$.

Step 2: Reverse order:

$$\int_0^1 \int_0^x e^{x^2}\,dy\,dx = \int_0^1 x\,e^{x^2}\,dx.$$

Step 3: Let $u = x^2$, $du = 2x\,dx$:

$$\int_0^1 x\,e^{x^2}\,dx = \frac{1}{2}\int_0^1 e^u\,du = \frac{1}{2}\left[e^u\right]_0^1 = \frac{e-1}{2}.$$

Conclusion: Answer (D): $\dfrac{e-1}{2}$.

Why (A) wrong: $e - 1 = \int_0^1 e^x\,dx$, not the answer here.
Why (B) wrong: $(e + 1)/2$ has the wrong sign.
Why (C) wrong: $e/2$ omits the $-1/2$ term.

Answer: (D) *← Go back to Q13*

Q14.

Solution

Concept: Convert Cartesian region to polar.

Step 1: The inner integral is over $0 \le x \le \sqrt{4 - y^2}$, i.e., $x^2 + y^2 \le 4$, $x \ge 0$. The outer integral is $0 \le y \le 2$, i.e., $y \ge 0$.

Step 2: Combined: $x^2 + y^2 \le 4$, $x \ge 0$, $y \ge 0$ — the quarter disk of radius 2 in the first quadrant.

Step 3: In polar: $0 \le r \le 2$, $0 \le \theta \le \pi/2$.

Why (A) wrong: $\theta \in [0, \pi]$ would be a half-disk (including negative x).
Why (C) wrong: $r \in [0, 4]$ and full circle gives a much larger region.
Why (D) wrong: r starts at 0 (the region includes the origin).

Answer: (B) *← Go back to Q14*

Q15.

Solution

Concept: Separate the iterated integral.

Step 1:

$$\int_0^{\pi/2}\int_0^2 r^2\,dr\,d\theta = \left(\int_0^{\pi/2} d\theta\right)\left(\int_0^2 r^2\,dr\right).$$

Step 2: $\int_0^{\pi/2} d\theta = \frac{\pi}{2}$ and $\int_0^2 r^2\,dr = \left[\frac{r^3}{3}\right]_0^2 = \frac{8}{3}$.

Step 3: Product $= \frac{\pi}{2}\cdot\frac{8}{3} = \frac{4\pi}{3}$.

Why (A) wrong: π would arise from $(\pi/2)\cdot 2$, not $(\pi/2)\cdot(8/3)$.

Why (B) wrong: 2π requires a full $[0, 2\pi]$ angular range.

Why (D) wrong: $8\pi/3$ would require θ from 0 to 2π.

Answer: (C) ← *Go back to Q15*

Q16.

Solution

Concept: $u\cdot v = \sum_i u_i v_i$.

Step 1: $(1)(4) + (2)(5) + (3)(6) = 4 + 10 + 18 = 32$.

Conclusion: Answer (A): 32.

Why (B) wrong: $14 = 1^2 + 2^2 + 3^2 = \|u\|^2$, not the inner product.

Why (C) wrong: $45 = 4^2 + 5^2 + 6^2 - 30$ is not a natural computation here.

Why (D) wrong: $77 = 4^2 + 5^2 + 6^2 = \|v\|^2 + 2$.

Answer: (A) ← *Go back to Q16*

Q17.

Solution

Concept: $e_1 = v_1/\|v_1\|$.

Step 1: $v_1 = (3,4)$, $\|v_1\| = \sqrt{9+16} = 5$.

Step 2: $e_1 = \frac{(3,4)}{5} = \left(\frac{3}{5}, \frac{4}{5}\right)$.

Why (A) wrong: $(3,4)$ is not normalised.

Why (B) wrong: $(1,0)$ is the second basis vector v_2, not e_1.

Why (C) wrong: $(4/5, -3/5)$ is perpendicular to e_1 — it would be (part of) e_2.

Answer: (D) ← *Go back to Q17*

Q18.

Solution

Concept: Properties of orthogonal matrices.

Step 1: $A^TA = I \Rightarrow \det(A^TA) = \det(I) = 1$.

Step 2: $\det(A^T)\det(A) = (\det A)^2 = 1 \Rightarrow \det(A) = \pm 1$.

Conclusion: $|\det(A)| = 1$. Answer (B).

Why (A) wrong: $|\det| = 0$ only for singular matrices; orthogonal matrices are invertible.
Why (C) wrong: n is the size, not the absolute determinant value.
Why (D) wrong: $\det(A^T) = \det(A)$ (determinant of transpose equals determinant), not a new value.

Answer: (B) ← *Go back to Q18*

Q19.

Solution

Concept: Fundamental theorem of line integrals for conservative fields.

Step 1: If $\mathbf{F} = \nabla\varphi$ (conservative) with potential φ, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(Q) - \varphi(P)$$

regardless of the path from P to Q.

Why (A) wrong: The integral is zero only for *closed* paths, not arbitrary paths.
Why (B) wrong: $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for closed curves; the formula here is for $P \to Q$.
Why (D) wrong: $\iint_S \mathbf{F} \cdot d\mathbf{S}$ is a surface integral (Stokes/divergence context).

Answer: (C) ← *Go back to Q19*

Q20.

Solution

Concept: Divergence Theorem: $\oiint_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{F}\, dV$.

Step 1: $\nabla \cdot \mathbf{F} = \dfrac{\partial x^2}{\partial x} + \dfrac{\partial y^2}{\partial y} + \dfrac{\partial z^2}{\partial z} = 2x + 2y + 2z$.

Step 2: By symmetry of the unit ball B (centred at origin):

$$\iiint_B x\, dV = \iiint_B y\, dV = \iiint_B z\, dV = 0$$

(each integrand is odd on a symmetric domain).

Step 3: Flux $= 2\iiint_B (x + y + z)\, dV = 2(0 + 0 + 0) = 0$.

Why (B),(C),(D) wrong: Non-zero answers ignore the symmetry cancellation.

Answer: (A) ← *Go back to Q20*

Q21.

Solution

Concept: Order of a product of disjoint cycles is the lcm of their lengths.

Step 1: $(1\ 2\ 3)$ has order 3; $(4\ 5)$ has order 2. They are disjoint.

Step 2: $\text{lcm}(3, 2) = 6$.

Verification: Applying the permutation 6 times returns each element to its start.

Why (A) wrong: $\text{lcm}(3, 2) \neq 5$.
Why (B) wrong: Order 2 would require both cycles to square to the identity; $(1\ 2\ 3)^2 \neq e$.
Why (C) wrong: Order 3 would require $(4\ 5)^3 = e$; but $(4\ 5)^3 = (4\ 5) \neq e$.

Answer: (D) ← *Go back to Q21*

Q22.

Solution

Concept: Center $Z(G) = \{g \in G : gx = xg\ \forall x \in G\}$.

Step 1: S_3 is non-abelian (e.g., $(12)(123) \neq (123)(12)$).

Step 2: Check each non-identity element: each transposition or 3-cycle fails to commute with some other element.

Conclusion: $Z(S_3) = \{e\}$. Answer (B).

Why (A) wrong: $Z(G) = G$ only if G is abelian; S_3 is not.
Why (C) wrong: $A_3 = \{e, (123), (132)\}$ is normal but not central (A_3 does not commute with transpositions).
Why (D) wrong: Subgroups of order 2 contain transpositions, which do not commute with all elements.

Answer: (B) ← *Go back to Q22*

Q23.

Solution

Concept: In $(\mathbb{Z}_n, +)$, $\langle k \rangle = \{0, k, 2k, \ldots\} \pmod n$.

Step 1: Compute multiples of 2 in $\mathbb{Z}_6$: $0, 2, 4, 6 \equiv 0, \ldots$

Step 2: $\langle 2 \rangle = \{0, 2, 4\}$, a subgroup of order $3 = 6/\gcd(6, 2) = 6/2 = 3$. ✓

Why (A) wrong: $\{0, 2\}$ is not closed ($2 + 2 = 4 \notin \{0, 2\}$).
Why (B) wrong: $\{0, 3\} = \langle 3 \rangle$, generated by 3, not 2.
Why (D) wrong: $\langle 1 \rangle = \mathbb{Z}_6$; the generator 2 has order $3 < 6$, so it generates a proper subgroup.

Answer: (C) ← *Go back to Q23*

Q24.

Solution

Concept: $\ker(\varphi) = \{k \in \mathbb{Z}_{12} : \varphi(k) = 0\} = \{k : k \equiv 0 \pmod 4 \text{ in } \mathbb{Z}_{12}\}$.

Step 1: Elements $k \in \{0, 1, \ldots, 11\}$ with $4 \mid k$: $k = 0, 4, 8$.

Conclusion: $\ker(\varphi) = \{0, 4, 8\}$, of order $3 = 12/4$. Answer (A).

Why (B) wrong: $\{0, 6\}$ is the kernel of $k \mapsto k \bmod 6$ (or $k \bmod 2$ scaled), not $k \bmod 4$.
Why (C) wrong: $\{0, 3, 6, 9\} = \ker(k \mapsto k \bmod 3)$.
Why (D) wrong: $\{0, 2, 4, 6, 8, 10\}$ has order 6, corresponding to a different map.

Answer: (A) ← *Go back to Q24*

Q25.

Solution

Concept: First Isomorphism Theorem: $G/\ker(\varphi) \cong \mathrm{im}(\varphi)$.

Step 1: $\varphi : \mathbb{Z}_{12} \to \mathbb{Z}_4$ is $\varphi(k) = k \bmod 4$. $\mathrm{im}(\varphi) = \mathbb{Z}_4$ (surjective).

Step 2: $|\mathbb{Z}_{12}/\{0, 4, 8\}| = 12/3 = 4 = |\mathbb{Z}_4|$.

Conclusion: $\mathbb{Z}_{12}/\{0, 4, 8\} \cong \mathbb{Z}_4$. Answer (D).

Why (A) wrong: $|\mathbb{Z}_6| = 6 \neq 4$.
Why (B) wrong: $|\mathbb{Z}_2| = 2 \neq 4$.
Why (C) wrong: $\mathbb{Z}_{12}$ itself has order $12 \neq 4$.

Answer: (D) ← *Go back to Q25*

Q26.

Solution

Concept: Order of disjoint cycle product = lcm of cycle lengths.

Step 1: $(1\ 2)$ has order 2; $(3\ 4\ 5)$ has order 3.

Step 2: $\mathrm{lcm}(2, 3) = 6$. Answer (B).

Why (A) wrong: $(3\ 4\ 5)^2 \neq e$, so order > 2.

Why (C) wrong: $(1\ 2)^3 = (1\ 2) \neq e$, so order > 3.
Why (D) wrong: $\text{lcm}(2, 3) = 6 \neq 5$.

Answer: (B) ← *Go back to Q26*

Q27.

Solution

Concept: Cauchy's theorem: if prime $p \mid |G|$, then $\exists\, g \in G$ with $\text{ord}(g) = p$.

Step 1: $|G| = 12 = 2^2 \cdot 3$. Primes dividing 12: 2 and 3.

Step 2: By Cauchy: G must contain elements of order 2 and of order 3.

Conclusion: Both orders 2 and 3 must occur. Answer (C).

Why (A) wrong: Only the identity has order 1; Cauchy gives us more.
Why (B) wrong: Order 4 need not exist (Cauchy only guarantees prime orders).
Why (D) wrong: Order 12 would require G to be cyclic, which is not guaranteed.

Answer: (C) ← *Go back to Q27*

Q28.

Solution

Concept: Sylow's Third Theorem.

Step 1: If $p^k \| \, |G|$ (i.e., $p^k \mid |G|$ but $p^{k+1} \nmid |G|$), the number n_p of Sylow p-subgroups satisfies:

- $n_p \equiv 1 \pmod p$
- $n_p \mid |G|/p^k$ (i.e., n_p divides the index of a Sylow p-subgroup)

Conclusion: Answer (A).

Why (B) wrong: $n_p = 1$ is possible but not guaranteed; it need not always hold.
Why (C) wrong: $n_p \mid p^k$ is not the correct divisibility condition.
Why (D) wrong: $n_p = p$ contradicts $n_p \equiv 1 \pmod p$ unless $p = 1$.

Answer: (A) ← *Go back to Q28*

Q29.

Solution

Concept: Apply Sylow's Third Theorem to $|G| = 15 = 3 \cdot 5$.

Step 1: For $p = 3$: $n_3 \equiv 1 \pmod 3$ and $n_3 \mid 5$. So $n_3 \in \{1\}$ (only 1 satisfies both).

Step 2: Similarly for $p = 5$: $n_5 \equiv 1 \pmod 5$ and $n_5 \mid 3$. So $n_5 = 1$.

Conclusion: $n_3 = 1$. There is exactly one (normal) Sylow 3-subgroup. Answer (D).

Why (A) wrong: $n_3 = 5$ fails the condition $n_3 \equiv 1 \pmod 3$ ($5 \equiv 2 \pmod 3$).
Why (B) wrong: $n_3 = 3$ fails $3 \equiv 0 \pmod 3 \neq 1$.
Why (C) wrong: $n_3 = 15$ fails $n_3 \mid 5$.

Answer: (D) ← *Go back to Q29*

Q30.

Solution

Concept: Check closure properties for groups and subspaces.

Step 1 (not group under addition): Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Both have $\det = 0$. But $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has $\det = 1 \neq 0$. *Not closed under addition.*

Step 2 (not group under multiplication): The zero matrix has $\det = 0$; any product with the identity would give the original matrix, but the identity I has $\det = 1 \neq 0$, so I is not in the set. No multiplicative identity.

Step 3 (not a subspace): Step 1 shows the set is not closed under addition, so it is not a subspace.

Conclusion: The set is neither a group under $+$ nor under $\times$, and not a subspace. Answer (B).

Answer: (B) ← *Go back to Q30*

Section B Solutions — Q31–Q40 (MSQ)

Q31.

Solution

Check each option for uniform convergence on $[0, 1]$:

(A) $f_n(x) = x^n$: pointwise limit is $f = \mathbf{1}_{\{1\}}$; $\sup |f_n - f| = 1 \not\to 0$. *Not uniform.*

(B) $f_n(x) = x/n$: $\sup_{[0,1]} |x/n| = 1/n \to 0$. **Uniform.**

(C) $f_n(x) = nx(1-x)^n$: maximum at $x = 1/(n+1)$, value $\approx n \cdot \frac{1}{n+1} \cdot e^{-1} \to e^{-1} \neq 0$. *Not uniform.*

(D) $f_n(x) = \sin(x/n)$: $|\sin(x/n)| \leq |x/n| \leq 1/n \to 0$ uniformly. **Uniform.**

Answer: (B) and (D).

Answer: (BD) ← *Go back to Q31*

Q32.

Solution

Check each option for double integrals:

(A) $f \geq 0 \Rightarrow \iint f\,dA \geq 0$. TRUE (monotonicity of the integral).

(B) FALSE. $\iint_R f\,dA = \int_a^b \int_c^d f\,dy\,dx$ equals a product only when $f(x,y) = g(x)h(y)$ (Fubini for product functions), not in general.

(C) Linearity: $\iint(f+g) = \iint f + \iint g$. TRUE.

(D) $\iint_R 1\,dA = \text{Area}(R)$. TRUE (definition of area via integration).

Answer: (A), (C), (D).

Answer: (ACD) ← *Go back to Q32*

Q33.

Solution

Level curve $f(x,y) = x^2 + y^2 - 4 = 0$, i.e., $x^2 + y^2 = 4$:

(A) This is a circle of radius 2 centred at the origin. TRUE.

(B) The equation is $x^2 + y^2 = 4$. TRUE (same statement).

(C) An ellipse has the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a \neq b$; here $a = b = 2$, so it is a circle, not an ellipse. FALSE.

(D) $(1,1)$: $1 + 1 = 2 \neq 4$. FALSE — the curve does not pass through $(1,1)$.

Answer: (A) and (B).

Answer: (AB) ← *Go back to Q33*

Q34.

Solution

Subspace criteria — $W \subseteq V$ is a subspace iff: closed under $+$, closed under scalar multiplication, and contains $\mathbf{0}$.

(A) Closure under scalar multiplication alone is insufficient (e.g., $\{(x,0) : x > 0\}$ is closed under positive scalars but not addition). FALSE in general.

(B) Finite subsets containing $\mathbf{0}$: e.g., $\{0, v\}$ for $v \neq 0$ — not closed under addition ($v + v = 2v$ may not be in the set). FALSE.

(C) Closed under addition and scalar multiplication is equivalent to being a subspace (combined with non-emptiness). TRUE.

(D) Contains $\mathbf{0}$ (automatic from (C) by taking scalar 0) plus (C). TRUE — this is the standard definition.

Answer: (C) and (D).

Answer: (CD) ← Go back to Q34

Q35.

Solution

Standard inner product space facts:

(A) Cauchy–Schwarz: $|u \cdot v| \leq \|u\|\|v\|$. TRUE (fundamental inequality).

(B) Pythagorean theorem: $u \cdot v = 0 \Rightarrow \|u+v\|^2 = \|u\|^2+2(u \cdot v)+\|v\|^2 = \|u\|^2+\|v\|^2$. TRUE.

(C) The projection of u onto v has magnitude $\frac{|u \cdot v|}{\|v\|}$, which equals $\|u\|$ only when u and v are parallel. FALSE in general.

(D) Triangle inequality: $\|u+v\| \leq \|u\|+\|v\|$. TRUE (follows from Cauchy–Schwarz).

Answer: (A), (B), (D).

Answer: (ABD) ← Go back to Q35

Q36.

Solution

Subgroup and normality properties:

(A) FALSE. Non-normal subgroups exist (e.g., $\langle (12) \rangle$ in S_3).

(B) If $[G : H] = 2$, then H is normal: for any $g \in G$, $G = H \cup gH = H \cup Hg$ (only two cosets), so $gH = Hg$. TRUE.

(C) If $N_1, N_2 \trianglelefteq G$, then for any $g \in G$: $g(N_1 \cap N_2)g^{-1} = gN_1g^{-1} \cap gN_2g^{-1} = N_1 \cap N_2$. TRUE.

(D) FALSE. $H_1 \cup H_2$ is a subgroup iff one contains the other. Counterexample in $\mathbb{Z}$: $2\mathbb{Z} \cup 3\mathbb{Z}$ is not closed ($2 + 3 = 5 \notin 2\mathbb{Z} \cup 3\mathbb{Z}$).

Answer: (B) and (C).

Answer: (BC) ← Go back to Q36

Q37.

Solution

Sequence $a_1 = 1$, $a_{n+1} = \sqrt{a_n + 2}$**:**

Fixed points: $L = \sqrt{L+2} \Rightarrow L^2 = L + 2 \Rightarrow L = 2$ (positive root).

(A) Bounded above by 2: Prove by induction. $a_1 = 1 \leq 2$. If $a_n \leq 2$, then $a_{n+1} = \sqrt{a_n + 2} \leq \sqrt{4} = 2$. TRUE.

(B) Decreasing: FALSE. $a_1 = 1 < a_2 = \sqrt{3} \approx 1.73$.

(C) Increasing: Prove by induction. $a_1 = 1 < \sqrt{3} = a_2$. If $a_n < a_{n+1}$, then $a_n + 2 <$

$a_{n+1} + 2$, so $a_{n+1} = \sqrt{a_n + 2} < \sqrt{a_{n+1} + 2} = a_{n+2}$. TRUE.

(D) Converges: Bounded above and increasing $\Rightarrow$ converges (Monotone Convergence Theorem); limit $= 2$. TRUE.

Answer: (A), (C), (D).

Answer: (ACD) ← *Go back to Q37*

Q38.

Solution

Power series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$ **(series for** $\ln(1+x)$**):**

Radius of convergence: $R = \lim_{n\to\infty} \dfrac{|a_n|}{|a_{n+1}|} = \lim \dfrac{n+1}{n} = 1$.

(A) FALSE. $R = 1$, not ∞.

(B) At $x = 1$: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$ (alternating harmonic series). Converges. TRUE.

(C) $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$ for $|x| < 1$ (standard Maclaurin series). TRUE.

(D) At $x = -1$: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n} = -\sum \frac{1}{n}$. Diverges (harmonic). FALSE.

Answer: (B) and (C).

Answer: (BC) ← *Go back to Q38*

Q39.

Solution

Field F $= (2xy, x^2, 0)$**:**

Compute curl: $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 2xy & x^2 & 0 \end{vmatrix} = \mathbf{i}(0-0) - \mathbf{j}(0-0) + \mathbf{k}(2x - 2x) = \mathbf{0}$.

(A) $\nabla \times \mathbf{F} = \mathbf{0}$, so $\mathbf{F}$ is irrotational $\Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any simple closed C. TRUE.

(B) FALSE (follows from (A): integral is 0, not 2π).

(C) Potential: $\varphi = x^2 y$ satisfies $\nabla\varphi = (2xy, x^2, 0) = \mathbf{F}$. TRUE.

(D) FALSE (same as (B)).

Answer: (A) and (C).

Answer: (AC) ← *Go back to Q39*

Q40.

Solution

$f(x,y) = xy$ **on** $R = [0,1]\times[0,1]$**:**

Compute the integral:

$$\iint_R xy\,dA = \int_0^1\int_0^1 xy\,dy\,dx = \int_0^1 x\cdot\frac{1}{2}\,dx = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}.$$

(A) FALSE. $\iint xy\,dA = 1/4 \neq 1/2$.

(B) TRUE. $\iint xy\,dA = 1/4$.

(C) FALSE. $f(0,0) = 0$ is the *minimum* on R, not the maximum.

(D) $f(1,1) = 1 \geq f(x,y) = xy$ for all $(x,y) \in [0,1]^2$ (since $x, y \leq 1$). Maximum is 1. TRUE.

Answer: (B) and (D).

Answer: (BD) ← *Go back to Q40*

Section C Solutions — Q41–Q50 (NAT, 1 mark)

Q41.

Solution

Concept: $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a$.

Step: Here $a = 2$, so the limit is e^2.

Answer: (e^2) ← *Go back to Q41*

Q42.

Solution

Concept: Fixed-point iteration for $x = \frac{x+4/x}{2}$.

Step 1: At a fixed point L: $L = \dfrac{L+4/L}{2} \Rightarrow 2L = L + 4/L \Rightarrow L^2 = 4 \Rightarrow L = 2$ (positive).

Step 2: The sequence is monotone and bounded (Babylonian method for $\sqrt{4}$), so it converges to $L = 2$.

Answer: (2) ← *Go back to Q42*

Q43.

Solution

Concept: Convert to polar coordinates.

Step 1: $x^2 + y^2 = r^2$, $dA = r\,dr\,d\theta$.

Step 2:

$$\iint_D (x^2+y^2)\,dA = \int_0^{2\pi}\int_0^2 r^2 \cdot r\,dr\,d\theta = \int_0^{2\pi} d\theta \cdot \int_0^2 r^3\,dr = 2\pi \cdot \left[\frac{r^4}{4}\right]_0^2 = 2\pi \cdot 4 = 8\pi.$$

Answer: (8π) ← *Go back to Q43*

Q44.

Solution

Concept: Order of k in $(\mathbb{Z}_9, +)$ is $9/\gcd(9,k)$.

Step 1: Order 3 requires $9/\gcd(9,k) = 3$, i.e., $\gcd(9,k) = 3$: elements $k = 3, 6$.

Step 2: Count: 2 elements.

Answer: (2) ← *Go back to Q44*

Q45.

Solution

Concept: Geometric series $\sum_{n=0}^{\infty} r^n = \dfrac{1}{1-r}$ for $|r| < 1$.

Step: $r = 2/3$, sum $= \dfrac{1}{1-2/3} = \dfrac{1}{1/3} = 3$.

Answer: (3) ← *Go back to Q45*

Q46.

Solution

Concept: $\sin^2 x = \dfrac{1-\cos(2x)}{2}$.

Step:

$$\int_{-\pi}^{\pi} \sin^2(x)\,dx = \int_{-\pi}^{\pi} \frac{1-\cos(2x)}{2}\,dx = \frac{1}{2}\left[x - \tfrac{\sin(2x)}{2}\right]_{-\pi}^{\pi} = \frac{1}{2}(2\pi - 0) = \pi.$$

Answer: (π) ← *Go back to Q46*

Q47.

Solution

Concept: Rank = number of nonzero rows in echelon form.

Step: The matrix is already in (upper) row echelon form. The nonzero rows are rows 1, 2, and 3. Row 4 is all zeros. Rank $= 3$.

Answer: (3) ← *Go back to Q47*

Q48.

Solution

Concept: $|x|$ is even; by symmetry $\int_{-2}^{2} |x|\,dx = 2\int_0^2 x\,dx$.

Step: $2\int_0^2 x\,dx = 2\left[\frac{x^2}{2}\right]_0^2 = 2 \cdot 2 = 4$.

Answer: (4) ← *Go back to Q48*

Q49.

Solution

Concept: Powers of the rotation matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Step 1: $A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$.

Step 2: $A^4 = (A^2)^2 = (-I)^2 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Step 3: $\mathrm{tr}(A^4) = \mathrm{tr}(I) = 1 + 1 = 2$.

Answer: (2) ← *Go back to Q49*

Q50.

Solution

Concept: A homomorphism $\varphi : \mathbb{Z}_6 \to \mathbb{Z}_2$ is determined by $\varphi(1)$.

Step 1: $\varphi(1)$ must have order dividing $\gcd(6, 2) = 2$ in $\mathbb{Z}_2$. Possible values: 0 or 1.

Step 2: $\varphi(1) = 0$ gives the trivial map (not surjective). $\varphi(1) = 1$ gives $\varphi(k) = k \bmod 2$ — surjective.

Conclusion: Exactly 1 surjective homomorphism.

Answer: (1) ← *Go back to Q50*

Section C Solutions — Q51–Q60 (NAT, 2 marks)

Q51.

Solution

Step 1:

$$\int_0^1 \int_0^x xy\,dy\,dx = \int_0^1 x \cdot \frac{x^2}{2}\,dx = \int_0^1 \frac{x^3}{2}\,dx = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.$$

Answer: (1/8) ← *Go back to Q51*

Q52.

Solution

Concept: $\det(cA) = c^n \det(A)$ for $n \times n$ matrix.

Step: $n = 2$, $c = 3$: $\det(3A) = 3^2 \det(A) = 9 \times 5 = 45$.

Answer: (45) ← *Go back to Q52*

Q53.

Solution

Concept: Use cylindrical coordinates.

Step 1: The intersection $z = r^2 = 4$ gives $r = 2$. Volume $= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r\,dz\,dr\,d\theta$.

Step 2: Inner integral: $\int_{r^2}^4 dz = 4 - r^2$.

Step 3:

$$\int_0^{2\pi} d\theta \cdot \int_0^2 r(4 - r^2)\,dr = 2\pi \left[2r^2 - \frac{r^4}{4}\right]_0^2 = 2\pi\,(8 - 4) = 8\pi.$$

Answer: (8π) ← *Go back to Q53*

Q54.

Solution

Step 1: Integrate $y' = 2x + 1$: $y = x^2 + x + C$.

Step 2: $y(0) = C = 1$, so $y = x^2 + x + 1$.

Step 3: $y(2) = 4 + 2 + 1 = 7$.

Answer: (7) ← *Go back to Q54*

Q55.

Solution

Concept: Count permutations in S_5 with cycle type $(2,3)$.

Step 1: Choose 2 elements from $\{1,2,3,4,5\}$ for the 2-cycle: $\binom{5}{2} = 10$ ways.

Step 2: Arrange the remaining 3 elements in a 3-cycle: $(3-1)! = 2$ ways.

Conclusion: Total $= 10 \times 2 = 20$.

Answer: (20) ← *Go back to Q55*

Q56.

Solution

Concept: Compute $f'(0) = \lim_{h\to 0} \dfrac{f(h)-f(0)}{h} = \lim_{h\to 0} \dfrac{\sin h/h - 1}{h}$.

Step 1: Taylor series: $\sin h = h - h^3/6 + \cdots$, so $\dfrac{\sin h}{h} = 1 - \dfrac{h^2}{6} + \cdots$

Step 2: $\dfrac{\sin h/h - 1}{h} = \dfrac{-h^2/6 + \cdots}{h} = -\dfrac{h}{6} + \cdots \to 0$ as $h \to 0$.

Conclusion: $f'(0) = 0$.

Answer: (0) ← *Go back to Q56*

Q57.

Solution

Step 1: Integration by parts with $u = x^2$, $dv = e^x\,dx$:

$$\int_0^1 x^2 e^x\,dx = \left[x^2 e^x\right]_0^1 - \int_0^1 2xe^x\,dx = e - 2\int_0^1 xe^x\,dx.$$

Step 2: $\int_0^1 xe^x\,dx = \left[xe^x\right]_0^1 - \int_0^1 e^x\,dx = e - (e-1) = 1$.

Step 3: $\int_0^1 x^2 e^x\,dx = e - 2(1) = e - 2$.

Answer: (e-2) ← *Go back to Q57*

Q58.

Solution

Concept: Divergence Theorem: $\oiint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_V \nabla\cdot\mathbf{F}\,dV$.

Step 1: $\nabla\cdot(xy, yz, zx) = y + z + x$.

Step 2:

$$\iiint_{[0,1]^3} (x+y+z)\,dV = \int_0^1\int_0^1\int_0^1 (x+y+z)\,dx\,dy\,dz.$$

By symmetry: $\iiint x\,dV = \iiint y\,dV = \iiint z\,dV = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$.

Step 3: Total $= 3 \times \dfrac{1}{2} = \dfrac{3}{2}$.

Answer: (3/2) ← *Go back to Q58*

Q59.

Solution

Concept: Minimal polynomial of a diagonal matrix with distinct eigenvalues.

Step 1: A has eigenvalues $\lambda_1 = 2$ (algebraic multiplicity 2) and $\lambda_2 = 3$ (multiplicity 1). Since A is diagonal (hence diagonalizable), the minimal polynomial has each eigenvalue appearing exactly once.

Step 2: Minimal polynomial $= (t-2)(t-3)$, degree 2.

Answer: (2) ← *Go back to Q59*

Q60.

Solution

Concept: $|\operatorname{Hom}(\mathbb{Z}_m, \mathbb{Z}_n)| = \gcd(m, n)$.

Step 1: $\gcd(8, 12) = 4$.

Step 2: There are exactly 4 group homomorphisms from $\mathbb{Z}_8$ to $\mathbb{Z}_{12}$. Each is determined by the image of the generator 1, which must be an element of $\mathbb{Z}_{12}$ whose order divides 8. Elements of $\mathbb{Z}_{12}$ with order dividing 8: order must divide $\gcd(8, 12) = 4$. These are 0 (order 1), 3 (order 4), 6 (order 2), 9 (order 4) — exactly 4 elements.

Answer: (4) ← *Go back to Q60*

Answer Key

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	D	2	B	3	C	4	A	5	D
6	B	7	C	8	A	9	D	10	B
11	C	12	A	13	D	14	B	15	C
16	A	17	D	18	B	19	C	20	A
21	D	22	B	23	C	24	A	25	D
26	B	27	C	28	A	29	D	30	B
31	BD	32	ACD	33	AB	34	CD	35	ABD
36	BC	37	ACD	38	BC	39	AC	40	BD
41	e^2	42	2	43	8π	44	2	45	3
46	π	47	3	48	4	49	2	50	1
51	1/8	52	45	53	8π	54	7	55	20
56	0	57	e-2	58	3/2	59	2	60	4