IIT JAM Mathematics

Sample Paper – 6

Instructions

- This paper contains **60 questions** carrying a maximum of **100 marks**. Duration: **3 hours (180 minutes)**.
- **Section A** (**Q1–Q30**, MCQ): Q1–Q10 carry **+1 mark** each; wrong response: $-\frac{1}{3}$ mark. Q11–Q30 carry **+2 marks** each; wrong response: $-\frac{2}{3}$ mark. Choose the **single best** option.
- **Section B** (**Q31–Q40**, MSQ): Each carries **+2 marks**. **No negative marking. One or more** options may be correct; full marks only if *all* correct options are selected.
- **Section C** (**Q41–Q60**, NAT): Q41–Q50 carry **+1 mark**; Q51–Q60 carry **+2 marks**. **No negative marking.** Enter the numerical value using the virtual keypad.
- **Calculator policy:** Personal calculators and electronic devices are **strictly prohibited**. A virtual scientific calculator is provided in the CBT interface.
- Sections must be attempted in order; you cannot return to a previous section.

Section A — Multiple Choice Questions (MCQ)

***Q1–Q10**: 1 mark each ($-\frac{1}{3}$ for wrong). Choose the **single** correct option.*

Q1. The comparison test states: if $0 \le a_n \le b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ converges, then:

(A) $\sum_{n=1}^{\infty} a_n$ converges

(B) $\sum_{n=1}^{\infty} a_n$ diverges

(C) $\sum_{n=1}^{\infty} a_n$ may or may not converge

(D) $\sum_{n=1}^{\infty} a_n$ converges to the same limit as $\sum b_n$

Q2. The limit comparison test states: if $a_n, b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ with $0 < L < \infty$, then:

(A) $\sum a_n$ converges iff $\sum b_n$ diverges

(B) Both series always diverge

(C) $\sum a_n$ and $\sum b_n$ have the same convergence behaviour

(D) $\sum a_n$ converges to $L \cdot \sum b_n$

Q3. The integral test asserts that $\sum_{n=1}^{\infty} f(n)$ converges if and only if:

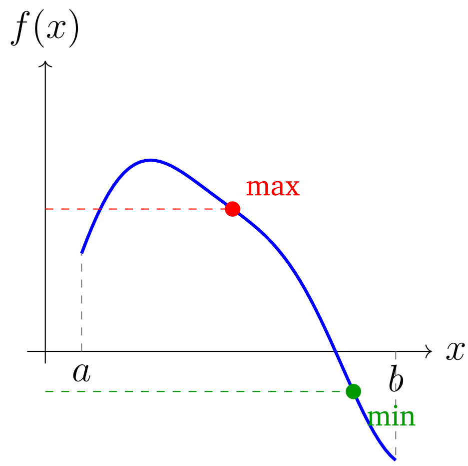
(A) $\int_1^{\infty} f(x)\,dx < \infty$ for every function f

(B) f is positive, decreasing, and continuous on $[1, \infty)$, and $\int_1^{\infty} f(x)\,dx < \infty$

(C) $\int_0^{\infty} f(x)\,dx < \infty$

(D) $f(n) \to 0$

Q4. By the Extreme Value Theorem, a continuous function f on a closed bounded interval $[a, b]$:



(A) Is differentiable on (a, b)

(B) Is monotone on $[a, b]$

(C) Has exactly one maximum value

(D) Attains its maximum and minimum values on $[a, b]$

Q5. A monotone function $f : [a, b] \to \mathbb{R}$ can have at most how many points of discontinuity?

(A) None

(B) Finitely many

(C) Countably many

(D) Exactly one

Q6. If $f : [a, b] \to \mathbb{R}$ is strictly increasing and continuous, then:

(A) f is injective and its inverse f^{-1} is continuous on $f([a, b])$

(B) f is surjective onto $\mathbb{R}$

(C) f is necessarily differentiable on (a, b)

(D) f must be a polynomial

Q7. The Intermediate Value Theorem applies to functions that are:

(A) Differentiable on (a, b)

(B) Continuous on $[a, b]$

(C) Monotone on $[a, b]$

(D) Bounded on (a, b)

Q8. For f differentiable on (a, b) with $f'(x) > 0$ for all $x \in (a, b)$, the function f is:

(A) Constant on (a, b)

(B) Periodic on (a, b)

(C) Bounded on (a, b)

(D) Strictly increasing on (a, b)

Q9. The Cayley-Hamilton theorem states that every $n \times n$ matrix A:

(A) Is diagonalizable over $\mathbb{C}$

(B) Has only real eigenvalues

(C) Satisfies its own characteristic equation $p(A) = 0$

(D) Is similar to a diagonal matrix

Q10. For the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the characteristic polynomial is $p(\lambda) = \lambda^2 + 1$. By the Cayley-Hamilton theorem, $A^2 + I$ equals:

$$A = \begin{array}{|c|c|} \hline 0 & 1 \\ \hline -1 & 0 \\ \hline \end{array} \quad A^2 = \begin{array}{|c|c|} \hline -1 & 0 \\ \hline 0 & -1 \\ \hline \end{array} \Rightarrow A^2 + I = \mathbf{0}$$

(A) The zero matrix $\mathbf{0}$
(B) The identity matrix I
(C) A
(D) $2I$

***Q11–Q30**: 2 marks each ($-\frac{2}{3}$ for wrong). Choose the **single** correct option.*

Q11. For the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ with characteristic polynomial $p(\lambda) = \lambda^2 - 5\lambda - 2$, by the Cayley-Hamilton theorem, the value of $A^2 - 5A - 2I$ is:

(A) I
(B) A
(C) $-I$
(D) The zero matrix $\mathbf{0}$

Q12. The minimal polynomial of the matrix $A = \mathrm{diag}(1, 1, 2, 2)$ is:

(A) $(\lambda - 1)^2(\lambda - 2)^2$
(B) $(\lambda - 1)(\lambda - 2)$
(C) $(\lambda - 1)(\lambda - 2)^2$
(D) $(\lambda - 1)^2(\lambda - 2)$

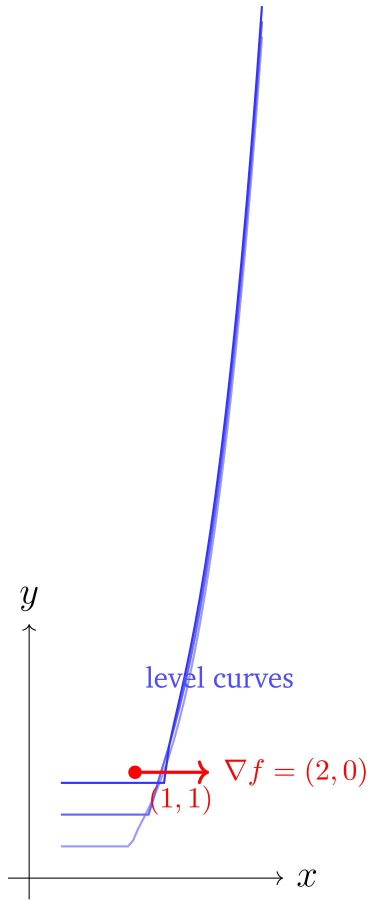
Q13. Using the method of undetermined coefficients to find a particular solution of the ODE $y'' + y = x^2$, the correct trial form for y_p is:

(A) Ax^2
(B) $Ax^2 + Bx$
(C) $Ax^2 + Bx + C$
(D) $Ax^3 + Bx^2 + Cx$

Q14. After substituting $y_p = Ax^2 + Bx + C$ into $y'' + y = x^2$ (see Q13), the particular solution is:

(A) $x^2 - 2$
(B) $x^2 + 2$
(C) $x^2 - 2x$
(D) $x^2 + 2x$

Q15. For $f(x,y) = x^2y - \frac{y^3}{3}$, the gradient ∇f at the point $(1,1)$ is:



(A) $(2,1)$
(B) $(1,0)$
(C) $(0,2)$
(D) $(2,0)$

Q16. The directional derivative of $f(x,y) = x^2 + y^2$ in the direction $\mathbf{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ at the point $(1,1)$ is:

(A) 2
(B) $2\sqrt{2}$
(C) $\sqrt{2}$
(D) 4

Q17. The equation of the tangent plane to the surface $z = x^2 + y^2$ at the point $(1,1,2)$ is:

(A) $z = 2x + 2y + 2$

(B) $z = 2x + 2y$

(C) $z = 2x + 2y - 2$

(D) $z = x + y$

Q18. For $f(x, y) = x^3 + 3xy^2 - y^3$, the value of the mixed partial derivative f_{xy} at the point $(1, 1)$ is:

(A) 6

(B) 3

(C) 9

(D) 0

Q19. Evaluate the triple integral $\int_0^1 \int_0^1 \int_0^1 (x + y + z)\, dx\, dy\, dz$:

(A) 1

(B) $\frac{1}{2}$

(C) 2

(D) $\frac{3}{2}$

Q20. The Jacobian $\frac{\partial(x, y)}{\partial(r, \theta)}$ for the polar coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$ is:

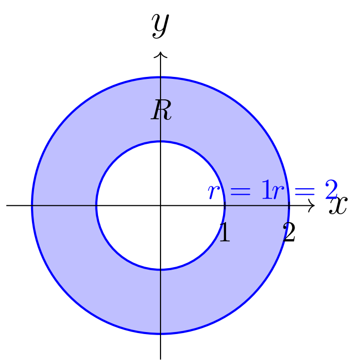
(A) r^2

(B) r

(C) $\frac{1}{r}$

(D) $\cos^2\theta - \sin^2\theta$

Q21. Using polar coordinates, evaluate $\iint_R (x^2 + y^2)\, dA$ where $R = \{(x, y) : 1 \le x^2 + y^2 \le 4\}$:



(A) 15π
(B) 30π
(C) $\frac{15\pi}{2}$
(D) $\frac{7\pi}{2}$

Q22. The permutation $(1\ 2\ 3\ 4\ 5) \in S_5$ has order:

(A) 5
(B) 2
(C) 4
(D) 3

Q23. The number of distinct cycle types (equivalently, conjugacy classes) in the symmetric group S_4 is:

(A) 4
(B) 6
(C) 3
(D) 5

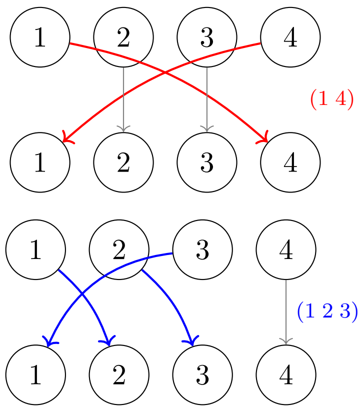
Q24. The order of the alternating group A_4 is:

(A) 24
(B) 12
(C) 8
(D) 6

Q25. In the alternating group A_4, which of the following is a **normal** subgroup?

(A) A cyclic subgroup of order 4
(B) A subgroup of order 6
(C) The Klein four-group $V = \{e,\ (12)(34),\ (13)(24),\ (14)(23)\}$
(D) The cyclic group generated by $(1\ 2\ 3)$

Q26. Compute the product of permutations $(1\ 2\ 3)(1\ 4)$ in S_4 (apply right to left):



(A) $(1\ 4\ 2\ 3)$
(B) $(1\ 2)(3\ 4)$
(C) $(1\ 3\ 4)$
(D) $(1\ 4)(2\ 3)$

Q27. In the symmetric group S_n, if $\sigma = (a_1\ a_2\ \cdots\ a_k)$ is a k-cycle and $\tau \in S_n$, then the conjugate $\tau\sigma\tau^{-1}$ equals:

(A) σ
(B) σ^{-1}
(C) τ
(D) $(\tau(a_1)\ \tau(a_2)\ \cdots\ \tau(a_k))$

Q28. The number of conjugacy classes in the symmetric group S_4 equals:

(A) 4
(B) 5
(C) 6
(D) 3

Q29. For a commutative ring R with unity and a proper ideal I, the quotient ring R/I is a field if and only if:

(A) I is a prime ideal
(B) $I = R$
(C) I is a maximal ideal
(D) I is a subring of R

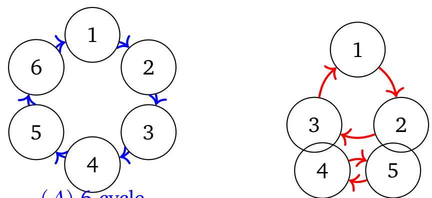
Q30. The units (invertible elements) of the polynomial ring $\mathbb{Z}[x]$ are exactly:

(A) ± 1
(B) All nonzero constant polynomials
(C) All monic polynomials
(D) The elements of $\mathbb{Z}$

Section B — Multiple Select Questions (MSQ)

***Q31–Q40**: 2 marks each. **No negative marking. One or more** options may be correct; full marks only if ALL correct options selected.*

Q31. Which of the following permutations in S_6 have **order 6**?



(A) 6-cycle (B) 3+2-cycle

(A) $(1\ 2\ 3\ 4\ 5\ 6)$ [6-cycle]
(B) $(1\ 2\ 3)(4\ 5)$ [order $\text{lcm}(3, 2) = 6$]
(C) $(1\ 2)(3\ 4)$ [order $\text{lcm}(2, 2) = 2$]
(D) $(1\ 2\ 3\ 4)(5\ 6)$ [order $\text{lcm}(4, 2) = 4$]

Q32. For a ring homomorphism $\varphi : R \to S$, which of the following **must** be TRUE?

(A) $\varphi(0_R) = 0_S$
(B) φ is surjective
(C) $\varphi(-r) = -\varphi(r)$ for all $r \in R$
(D) $\ker(\varphi)$ is an ideal of R

Q33. For the ideal $I = (2)$ (even integers) in $\mathbb{Z}$, which of the following are TRUE?

(A) I is *not* a prime ideal of $\mathbb{Z}$
(B) $\mathbb{Z}/I \cong \mathbb{Z}_2$
(C) I is a prime ideal of $\mathbb{Z}$
(D) I is a maximal ideal of $\mathbb{Z}$

Q34. For a differentiable function $f : \mathbb{R} \to \mathbb{R}$, which conditions **guarantee** that f is constant?

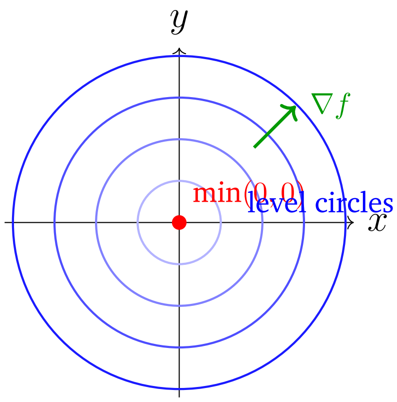
(A) $f'(x) = 0$ for all $x \in \mathbb{R}$

(B) f is differentiable everywhere and $f' \equiv 0$

(C) f is bounded above and below

(D) f is uniformly continuous on $\mathbb{R}$

Q35. For the function $f(x, y) = x^2 + y^2$ on $\mathbb{R}^2$, which of the following are TRUE?



(A) f achieves its minimum value 0 at $(0, 0)$

(B) f has a local maximum at $(0, 0)$

(C) The level curves of f are circles centred at the origin

(D) $\nabla f = (2x, 2y)$ vanishes only at the origin

Q36. For the alternating group A_4, which of the following are TRUE?

(A) A_4 has a subgroup of order 6

(B) A_4 has the Klein four-group $V_4 = \{e, (12)(34), (13)(24), (14)(23)\}$ as a normal subgroup

(C) $|A_4| = 12$

(D) A_4 is abelian

Q37. For a polynomial $p(x) = a_n x^n + \cdots + a_0$ with integer coefficients, which of the following are TRUE?

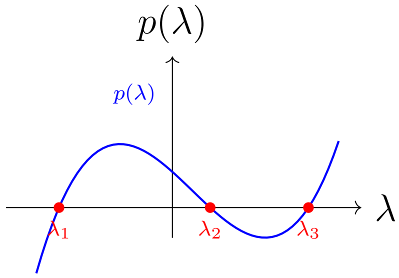
(A) If p/q (in lowest terms) is a rational root, then $p \mid a_0$ and $q \mid a_n$ (rational root theorem)

(B) If p is monic ($a_n = 1$), then any rational root of p is an integer

(C) Every polynomial of degree ≥ 1 has a real root

(D) If p is irreducible over $\mathbb{Q}$ and $z \in \mathbb{C}$ is a root, then the complex conjugate $\bar{z}$ is also a root

Q38. For a 3×3 matrix A with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ (counted with multiplicity), which of the following are TRUE?



(A) $\operatorname{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$

(B) $\det(A) = \lambda_1 + \lambda_2 + \lambda_3$

(C) $\det(A) = \lambda_1\lambda_2\lambda_3$

(D) $\operatorname{tr}(A) = \lambda_1\lambda_2\lambda_3$

Q39. For a gradient vector field $\mathbf{F} = \nabla f$ where $f : \mathbb{R}^3 \to \mathbb{R}$ is smooth, which of the following are TRUE?

(A) $\nabla \times \mathbf{F} = \mathbf{0}$ (curl of a gradient is zero)

(B) $\mathbf{F}$ is a conservative vector field

(C) $\nabla \cdot \mathbf{F} = 0$ always

(D) $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C

Q40. For the sequence of partial sums $S_n = \sum_{k=1}^{n} \frac{1}{k^2}$, which of the following are TRUE?

(A) $S_n \to \infty$ as $n \to \infty$

(B) $\{S_n\}$ is bounded above (in fact $S_n \leq \pi^2/6$ for all n)

(C) $\{S_n\}$ is a strictly increasing sequence

(D) $S_n = \pi^2/6$ for all sufficiently large n

Section C — Numerical Answer Type (NAT)

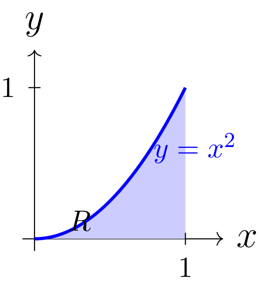
***Q41–Q50**: 1 mark each. No negative marking. Enter the exact numerical value.*

Q41. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Compute $\mathrm{tr}(A^4)$.

Q42. Find the number of distinct group homomorphisms from $\mathbb{Z}_3$ to S_3.

Q43. Evaluate the improper integral $\displaystyle\int_0^{\infty} x\, e^{-3x}\, dx$.

Q44. Evaluate $\displaystyle\int_0^1 \int_0^{x^2} y\, dy\, dx$.



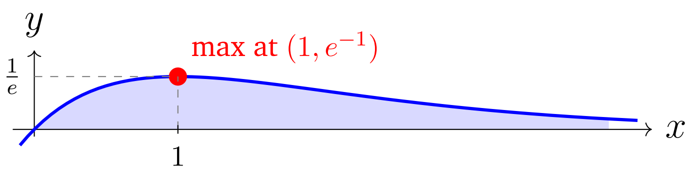
Q45. For $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, find the sum of all entries of the matrix $(A - I)^2$.

Q46. Find the number of elements of order 4 in the group $\mathbb{Z}_8 \times \mathbb{Z}_2$.

Q47. Evaluate $\displaystyle\int_0^{2\pi} \int_0^{\pi} \sin\varphi\, d\varphi\, d\theta$.

Q48. Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ be a Jordan block. If $p(\lambda) = (\lambda - 2)^3$, find the value of $\mathrm{tr}(p(A))$.

Q49. Find the maximum value of the function $f(x) = x\, e^{-x}$ on $\mathbb{R}$.



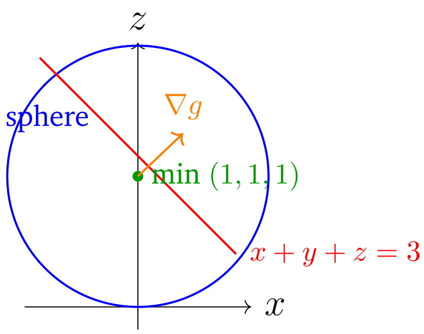
Q50. Find the number of surjective (onto) functions from the set $\{1, 2, 3, 4\}$ to the set $\{1, 2\}$.

***Q51–Q60**: 2 marks each. No negative marking. Enter the exact numerical value.*

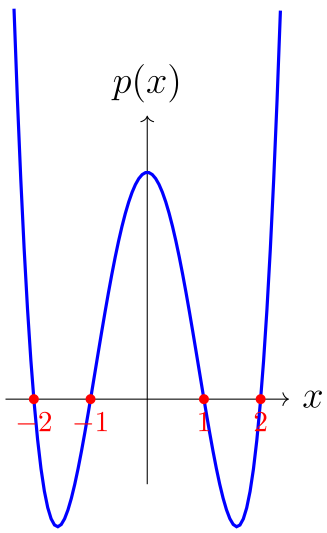
Q51. Evaluate $\displaystyle\int_0^1 x \arctan(x)\, dx$.

Q52. Let $A = \begin{pmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{pmatrix}$ be the rotation matrix by $\theta = \pi/6$. Find $\mathrm{tr}(A^2)$.

Q53. Using the method of Lagrange multipliers, find the minimum value of $f(x, y, z) = x^2+y^2+z^2$ subject to the constraint $g(x, y, z) = x+y+z-3 = 0$.



Q54. Find the number of distinct real roots of the polynomial $p(x) = x^4-5x^2+4$.

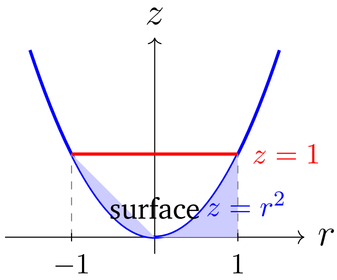


Q55. Evaluate $\displaystyle\int_0^{\pi} x \cos x\, dx$.

Q56. Find the volume of the region $E = \{(x, y, z) : -1 \le x \le 1,\ 0 \le y \le 1 - x^2,\ 0 \le z \le y\}$.

Q57. How many integers n with $1 \le n \le 100$ are divisible by **neither** 3 **nor** 7?

Q58. Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ lying above the disk $x^2 + y^2 \leq 1$.



Q59. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$. Find the sum of all eigenvalues of A (counted with multiplicity).

Q60. The sequence $\{a_n\}$ satisfies $a_1 = 1$ and $a_{n+1} = \frac{a_n}{2} + 1$ for all $n \geq 1$. Find $\lim_{n\to\infty} a_n$.

Solutions

IIT JAM Mathematics – Sample Paper 6

Section A Solutions — Q1–Q10 (1 mark each)

Q1.

Solution

Concept: The Direct Comparison Test states that if $0 \le a_n \le b_n$ for all $n \ge N$ and $\sum b_n$ converges, then $\sum a_n$ also converges. (Conversely, if $\sum a_n$ diverges and $a_n \le b_n$, then $\sum b_n$ diverges.)

Analysis of options:
(A) $\sum a_n$ converges: TRUE by the comparison test.
(B) $\sum a_n$ diverges: FALSE — convergence of $\sum b_n$ implies convergence of the smaller $\sum a_n$.
(C) $\sum a_n$ may or may not converge: FALSE — the conclusion is definitive.
(D) They converge to the same limit: FALSE — even if both converge, their sums are generally different.

Answer: (A) ← *Go back to Q1*

Q2.

Solution

Concept: The Limit Comparison Test: if $a_n, b_n > 0$ and $\lim_{n\to\infty}(a_n/b_n) = L$ with $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ *both converge or both diverge*.

Analysis of options:
(A) Converges iff the other diverges: FALSE — they have the *same* behaviour.
(B) Both always diverge: FALSE — both may converge (e.g., $a_n = b_n = 1/n^2$).
(C) Same convergence behaviour: TRUE — this is the exact statement of the theorem.
(D) $\sum a_n$ converges to $L \cdot \sum b_n$: FALSE — the theorem gives behaviour, not the exact value.

Answer: (C) ← *Go back to Q2*

Q3.

Solution

Concept: The Integral Test: suppose f is positive, continuous, and *decreasing* on $[1, \infty)$. Then $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

Analysis of options:

(A) For every function f: FALSE — f must satisfy the three conditions (positive, continuous, decreasing).
(B) This is the correct, full statement of the theorem: TRUE.
(C) $\int_0^\infty$ instead of $\int_1^\infty$: FALSE — the lower limit should be 1 (or a finite starting point), and the three conditions on f are still required.
(D) $f(n) \to 0$ alone: FALSE — this is only a necessary condition for convergence, not sufficient.

Answer: (B) ← *Go back to Q3*

Q4.

Solution

Concept: The Extreme Value Theorem (EVT): if $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded and attains both its supremum (maximum) and infimum (minimum) on $[a, b]$.

Analysis of options:
(A) f is differentiable: FALSE — the EVT says nothing about differentiability.
(B) f is monotone: FALSE — a continuous function need not be monotone.
(C) Exactly one maximum: FALSE — there may be multiple points where the maximum is attained.
(D) Attains its maximum and minimum: TRUE — this is precisely the conclusion of the EVT.

Answer: (D) ← *Go back to Q4*

Q5.

Solution

Concept: A monotone function on $[a, b]$ can only have *jump discontinuities*. The key theorem: the set of discontinuities of a monotone function is *at most countable*.

Reason: Each jump discontinuity of a monotone increasing function f corresponds to an interval $(f(x^-), f(x^+))$ of positive length. These intervals are pairwise disjoint, and each contains a rational number. Since $\mathbb{Q}$ is countable, there can be at most countably many such jumps.

Why (B) is not the best answer: The set of discontinuities can be *infinite* (countably infinite), so "finitely many" is too restrictive as an upper bound.

Correct answer: (C) — countably many discontinuities.

Answer: (C) ← *Go back to Q5*

Q6.

Solution

Concept: If $f : [a, b] \to \mathbb{R}$ is strictly increasing and continuous, then:

- f is injective (no two inputs give the same output).
- f is a homeomorphism onto $f([a, b]) = [f(a), f(b)]$ — in particular, f^{-1} is continuous on $f([a, b])$.

Analysis of options:
(A) Injective with continuous inverse: TRUE by the theorem above.
(B) Surjective onto $\mathbb{R}$: FALSE — f maps into the compact interval $[f(a), f(b)]$.
(C) Differentiable: FALSE — a strictly increasing continuous function need not be differentiable (e.g., $f(x) = x^{1/3}$ at 0).
(D) Must be polynomial: FALSE — there are infinitely many non-polynomial examples.

Answer: (A) ← *Go back to Q6*

Q7.

Solution

Concept: The Intermediate Value Theorem (IVT): if $f : [a, b] \to \mathbb{R}$ is *continuous* and $f(a) \neq f(b)$, then f takes every value between $f(a)$ and $f(b)$.

The key hypothesis is continuity on the closed interval $[a, b]$.

Analysis of options:
(A) Differentiable on (a, b): FALSE — differentiability is not required; mere continuity on $[a, b]$ suffices.
(B) Continuous on $[a, b]$: TRUE — this is the correct hypothesis.
(C) Monotone on $[a, b]$: FALSE — the IVT does not require monotonicity.
(D) Bounded on (a, b): FALSE — every continuous function on a closed bounded interval is automatically bounded, but this is a consequence, not the IVP hypothesis.

Answer: (B) ← *Go back to Q7*

Q8.

Solution

Concept: By the Mean Value Theorem, if $f'(x) > 0$ for all $x \in (a, b)$, then for any $x_1 < x_2$ in (a, b):

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0$$

for some $c \in (x_1, x_2)$. Hence $f(x_2) > f(x_1)$, so f is strictly increasing.

Analysis of options:
(A) Constant: FALSE — a constant function has $f' = 0$, not $f' > 0$.

(B) Periodic: FALSE — a strictly increasing function cannot be periodic.
(C) Bounded: FALSE — e.g., $f(x) = x$ on $\mathbb{R}$ has $f' > 0$ but is unbounded.
(D) Strictly increasing: TRUE.

Answer: (D) ← *Go back to Q8*

Q9.

Solution

Concept: The Cayley-Hamilton theorem states: if $p(\lambda) = \det(\lambda I - A)$ is the characteristic polynomial of A, then $p(A) = 0$ (the zero matrix). In other words, every square matrix satisfies its own characteristic equation.

Analysis of options:
(A) Diagonalizable: FALSE — a matrix satisfies its characteristic equation regardless of whether it is diagonalizable.
(B) Has only real eigenvalues: FALSE — complex eigenvalues are possible.
(C) Satisfies its own characteristic equation: TRUE — this is the Cayley-Hamilton theorem.
(D) Similar to a diagonal matrix: FALSE — non-diagonalizable matrices (e.g., Jordan blocks) also satisfy Cayley-Hamilton.

Answer: (C) ← *Go back to Q9*

Q10.

Solution

Computation: For $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, characteristic polynomial: $p(\lambda) = \det(\lambda I - A) = \lambda^2 + 1$.

Direct verification:

$$A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I.$$

Therefore $A^2 + I = -I + I = \mathbf{0}$.

By Cayley-Hamilton: Since $p(A) = A^2 + I = \mathbf{0}$, the answer is the zero matrix.

Analysis of options:
(A) Zero matrix: TRUE. (B) I: FALSE. (C) A: FALSE. (D) $2I$: FALSE.

Answer: (A) ← *Go back to Q10*

Section A Solutions — Q11–Q30 (2 marks each)

Q11.

Solution

Concept: By the Cayley-Hamilton theorem, every matrix satisfies its own characteristic polynomial.

Step 1: Characteristic polynomial of $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$:

$$p(\lambda) = \det(\lambda I - A) = \det\begin{pmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{pmatrix} = (\lambda-1)(\lambda-4)-(-2)(-3) = \lambda^2-5\lambda+4-6 = \lambda^2-5\lambda-2.$$

Step 2: By Cayley-Hamilton, $p(A) = A^2 - 5A - 2I = \mathbf{0}$.

Why other options fail: $A^2 - 5A - 2I = \mathbf{0}$, not I, A, or $-I$.

Answer: (D) ← *Go back to Q11*

Q12.

Solution

Concept: The minimal polynomial $m(\lambda)$ is the monic polynomial of least degree that annihilates A. For a diagonal matrix, $m(\lambda) = \prod_i(\lambda - \lambda_i)$ where the product runs over the *distinct* eigenvalues only (not with multiplicity, provided each eigenvalue's eigenspace has full dimension).

Step 1: Distinct eigenvalues of $A = \mathrm{diag}(1, 1, 2, 2)$ are $\{1, 2\}$.

Step 2: Since A is diagonal (hence diagonalizable), each eigenspace has full dimension. Therefore the minimal polynomial has each eigenvalue appearing exactly once:

$$m(\lambda) = (\lambda - 1)(\lambda - 2).$$

Verification: $(A - I)(A - 2I)$ applied to any basis vector e_i: if eigenvalue 1, $(A - I)e_i = 0$; if eigenvalue 2, $(A - 2I)e_i = 0$. So $(A - I)(A - 2I) = \mathbf{0}$. The degree-2 polynomial is minimal since $A - I \neq \mathbf{0}$ and $A - 2I \neq \mathbf{0}$.

Answer: (B) ← *Go back to Q12*

Q13.

Solution

Concept: Method of undetermined coefficients: when the right-hand side is a polynomial of degree k, the trial solution is a general polynomial of degree k (provided the homogeneous equation's complementary functions do not contain any polynomial terms).

Step 1: Homogeneous equation $y'' + y = 0$ has solutions $\cos x$ and $\sin x$ — neither is a polynomial. So no resonance with polynomial trial solutions.

Step 2: Right-hand side x^2 is a polynomial of degree 2. Trial form: $y_p = Ax^2 + Bx + C$ (all terms up to degree 2).

Why (A) is insufficient: Ax^2 misses the Bx and C terms that arise when substituting.

Why (D) is wrong: $Ax^3 + Bx^2 + Cx$ is of degree 3, more than needed; this form is used only when resonance requires degree elevation.

Answer: (C) ← Go back to Q13

Q14.

Solution

Step 1: Substitute $y_p = Ax^2 + Bx + C$ into $y'' + y = x^2$:

$$y_p'' = 2A, \qquad y_p'' + y_p = 2A + Ax^2 + Bx + C = x^2.$$

Step 2: Equate coefficients:

$$\begin{aligned} x^2 &: \quad A = 1, \\ x^1 &: \quad B = 0, \\ x^0 &: \quad 2A + C = 0 \;\Rightarrow\; C = -2. \end{aligned}$$

Step 3: $y_p = x^2 - 2$.

Verification: $y_p'' = 2$; $y_p'' + y_p = 2 + (x^2 - 2) = x^2$. ✓

Answer: (A) ← Go back to Q14

Q15.

Solution

Step 1: Compute partial derivatives of $f(x, y) = x^2 y - \dfrac{y^3}{3}$:

$$f_x = 2xy, \qquad f_y = x^2 - y^2.$$

Step 2: At $(1, 1)$:

$$\nabla f\big|_{(1,1)} = (f_x(1,1),\ f_y(1,1)) = (2 \cdot 1 \cdot 1,\ 1^2 - 1^2) = (2,\ 0).$$

Analysis of options:
(A) $(2, 1)$: incorrect y-component. (B) $(1, 0)$: incorrect x-component.
(C) $(0, 2)$: components reversed and wrong. (D) $(2, 0)$: correct.

Answer: (D) ← *Go back to Q15*

Q16.

Solution

Step 1: $\nabla f = (2x, 2y)$. At $(1, 1)$: $\nabla f = (2, 2)$.

Step 2: Direction $\mathbf{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is a unit vector.

Step 3: Directional derivative:

$$D_{\mathbf{u}} f(1, 1) = \nabla f \cdot \mathbf{u} = (2, 2) \cdot \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

Answer: (B) ← *Go back to Q16*

Q17.

Solution

Step 1: Write $F(x, y, z) = z - x^2 - y^2 = 0$. Partial derivatives at $(1, 1, 2)$:

$$F_x = -2x = -2, \quad F_y = -2y = -2, \quad F_z = 1.$$

Step 2: Tangent plane:

$$F_x(x - 1) + F_y(y - 1) + F_z(z - 2) = 0 \;\Rightarrow\; -2(x - 1) - 2(y - 1) + (z - 2) = 0.$$

$$z - 2 = 2(x - 1) + 2(y - 1) = 2x + 2y - 4 \;\Rightarrow\; z = 2x + 2y - 2.$$

Verification: At $(1, 1)$: $z = 2 + 2 - 2 = 2$. ✓

Answer: (C) ← *Go back to Q17*

Q18.

Solution

Step 1: $f(x, y) = x^3 + 3xy^2 - y^3$.

$$f_x = 3x^2 + 3y^2.$$

Step 2: Differentiate f_x with respect to y:

$$f_{xy} = \frac{\partial}{\partial y}(3x^2 + 3y^2) = 6y.$$

Step 3: At $(1, 1)$: $f_{xy}(1, 1) = 6(1) = 6$.

Answer: (A) ← *Go back to Q18*

Q19.

Solution

Step 1: By symmetry in x, y, z:

$$\int_0^1\int_0^1\int_0^1 (x+y+z)\,dx\,dy\,dz = 3\int_0^1\int_0^1\int_0^1 x\,dx\,dy\,dz.$$

Step 2:

$$\int_0^1\int_0^1\int_0^1 x\,dx\,dy\,dz = \int_0^1 x\,dx \cdot \int_0^1 dy \cdot \int_0^1 dz = \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}.$$

Step 3: Total $= 3 \times \frac{1}{2} = \frac{3}{2}$.

Answer: (D) ← *Go back to Q19*

Q20.

Solution

Step 1: With $x = r\cos\theta$, $y = r\sin\theta$, compute the Jacobian:

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix}\partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta\end{pmatrix} = \det\begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix}.$$

Step 2:

$$J = r\cos^2\theta - (-r\sin^2\theta) = r\cos^2\theta + r\sin^2\theta = r.$$

This is why $dx\,dy = r\,dr\,d\theta$ in polar coordinates.

Answer: (B) ← *Go back to Q20*

Q21.

Solution

Step 1: In polar coordinates, $x^2 + y^2 = r^2$, and $dA = r\,dr\,d\theta$. The region $1 \leq x^2 + y^2 \leq 4$ becomes $1 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$.

Step 2:

$$\iint_R (x^2 + y^2)\,dA = \int_0^{2\pi} \int_1^2 r^2 \cdot r\,dr\,d\theta = \int_0^{2\pi} d\theta \cdot \int_1^2 r^3\,dr.$$

Step 3:

$$\int_0^{2\pi} d\theta = 2\pi, \qquad \int_1^2 r^3\,dr = \left[\frac{r^4}{4}\right]_1^2 = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}.$$

Step 4: Total $= 2\pi \cdot \dfrac{15}{4} = \dfrac{15\pi}{2}$.

Answer: (C) ← *Go back to Q21*

Q22.

Solution

Concept: The order of a k-cycle in S_n is k.

Step 1: $(1\ 2\ 3\ 4\ 5)$ is a 5-cycle.

Step 2: Its order is 5.

Verification: $(1\ 2\ 3\ 4\ 5)^5 = e$ (identity), and no smaller positive power gives the identity.

Answer: (A) ← *Go back to Q22*

Solution

Concept: Conjugacy classes in S_n correspond to cycle types, which correspond to partitions of n.

Partitions of 4:

Q23.

(a) (4): one 4-cycle, e.g., $(1\ 2\ 3\ 4)$

(b) $(3, 1)$: one 3-cycle and one fixed point, e.g., $(1\ 2\ 3)$

(c) $(2, 2)$: two 2-cycles, e.g., $(1\ 2)(3\ 4)$

(d) $(2, 1, 1)$: one 2-cycle and two fixed points, e.g., $(1\ 2)$

(e) $(1, 1, 1, 1)$: identity

Total: 5 partitions $= 5$ cycle types $= 5$ conjugacy classes.

Answer: (D) ← *Go back to Q23*

Q24.

Solution

Concept: The alternating group A_n consists of all even permutations in S_n, and $|A_n| = n!/2$.

Step 1: $|A_4| = 4!/2 = 24/2 = 12$.

Remark: The elements of A_4 are: identity (1), eight 3-cycles (8), and three double transpositions (3): total $1 + 8 + 3 = 12$. ✓

Answer: (B) ← *Go back to Q24*

Q25.

Solution

Claim: $V = \{e, (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of A_4.

Step 1: V has order 4 and is closed under products (easy to check). All elements of V are in A_4 (products of two transpositions are even permutations). So $V \leq A_4$.

Step 2: Normality: V is the union of conjugacy classes $\{e\}$ and $\{(12)(34), (13)(24), (14)(23)\}$ (all double transpositions in A_4 form one conjugacy class of size 3). A subgroup is normal iff it is a union of conjugacy classes. So $V \trianglelefteq A_4$.

Why others fail:
(A) No cyclic subgroup of order 4 exists in A_4 (elements have orders 1, 2, 3 only).
(B) By Lagrange, if a subgroup of order 6 existed, index would be 2 and it'd be normal. But A_4 has no subgroup of order 6 (this is a standard result).
(D) $\langle(123)\rangle = \{e, (123), (132)\}$ has order 3, not a normal subgroup (it is conjugate to other 3-element subgroups).

Answer: (C) ← *Go back to Q25*

Q26.

Solution

Convention: Apply the *rightmost* permutation first: $(1\ 2\ 3)(1\ 4)$ means first apply $(1\ 4)$, then $(1\ 2\ 3)$.

Trace each element:

$$1 \xrightarrow{(1\ 4)} 4 \xrightarrow{(1\ 2\ 3)} 4 \qquad \Rightarrow 1 \mapsto 4$$
$$2 \xrightarrow{(1\ 4)} 2 \xrightarrow{(1\ 2\ 3)} 3 \qquad \Rightarrow 2 \mapsto 3$$
$$3 \xrightarrow{(1\ 4)} 3 \xrightarrow{(1\ 2\ 3)} 1 \qquad \Rightarrow 3 \mapsto 1$$
$$4 \xrightarrow{(1\ 4)} 1 \xrightarrow{(1\ 2\ 3)} 2 \qquad \Rightarrow 4 \mapsto 2$$

Result: $1 \to 4 \to 2 \to 3 \to 1$, which is the 4-cycle $(1\ 4\ 2\ 3)$.

Answer: (A) ← *Go back to Q26*

Q27.

Solution

Concept: Conjugation in S_n: if $\sigma = (a_1\ a_2\ \cdots\ a_k)$ and $\tau \in S_n$, then

$$\tau\sigma\tau^{-1} = (\tau(a_1)\ \tau(a_2)\ \cdots\ \tau(a_k)).$$

Conjugation preserves cycle type but relabels the elements in each cycle.

Proof idea: For any x, $\tau\sigma\tau^{-1}(\tau(a_i)) = \tau(\sigma(a_i)) = \tau(a_{i+1})$, showing the cycle follows the τ-image of each a_i.

Analysis:
(A) σ: FALSE in general (true only if τ commutes with σ).
(B) σ^{-1}: FALSE.
(C) τ: FALSE.
(D) $(\tau(a_1)\ \tau(a_2)\ \cdots\ \tau(a_k))$: TRUE.

Answer: (D) ← *Go back to Q27*

Q28.

Solution

Concept: The number of conjugacy classes in S_n equals the number of partitions of n (as established in Q23 for S_4).

Partitions of 4: (4), $(3, 1)$, $(2, 2)$, $(2, 1, 1)$, $(1, 1, 1, 1)$ — five partitions.

Therefore the number of conjugacy classes in S_4 is **5**.

Answer: (B) ← *Go back to Q28*

Q29.

Solution

Theorem: Let R be a commutative ring with unity and I a proper ideal. Then R/I is a field if and only if I is a *maximal ideal*.

Proof sketch: ($\Rightarrow$) If R/I is a field, it has no proper nonzero ideals. Ideals of R/I correspond to ideals of R containing I. So I is maximal. ($\Leftarrow$) If I is maximal, then R/I has no ideals properly between $\{0\}$ and R/I, making it a field.

Related fact: R/I is an integral domain $\Leftrightarrow$ I is a prime ideal.

Analysis:
(A) Prime ideal gives integral domain, not necessarily a field.
(B) $I = R$ gives $R/I = \{0\}$, the trivial ring, not a field.
(C) Maximal ideal: correct.
(D) I is a subring: not sufficient.

Answer: (C) ← *Go back to Q29*

Q30.

Solution

Claim: The units of $\mathbb{Z}[x]$ are exactly ± 1.

Proof: Suppose $f(x)$ and $g(x)$ are inverses in $\mathbb{Z}[x]$, so $f(x)g(x) = 1$. Taking degrees: $\deg(f) + \deg(g) = \deg(1) = 0$. Since degrees are non-negative integers, $\deg(f) = \deg(g) = 0$. So $f = a$ and $g = b$ are integer constants with $ab = 1$ in $\mathbb{Z}$. The only solutions in $\mathbb{Z}$ are $a = b = 1$ or $a = b = -1$.

Analysis of options:
(A) ± 1: correct.
(B) All nonzero constants: FALSE — for example, 2 has no inverse in $\mathbb{Z}[x]$ since $2 \cdot (1/2) = 1$ but $1/2 \notin \mathbb{Z}[x]$.
(C) Monic polynomials: FALSE — x is monic but not a unit.
(D) All of $\mathbb{Z}$: FALSE — only ± 1 from $\mathbb{Z}$ are units.

Answer: (A) ← *Go back to Q30*

Section B Solutions — Q31–Q40 (MSQ, 2 marks each)

Q31.

Solution

Concept: The order of a permutation equals the least common multiple (lcm) of its cycle lengths.

(A) $(1\ 2\ 3\ 4\ 5\ 6)$: a single 6-cycle. Order $= 6$. **TRUE.**

(B) $(1\ 2\ 3)(4\ 5)$: cycles of length 3 and 2. Order $= \text{lcm}(3, 2) = 6$. **TRUE.**

(C) $(1\ 2)(3\ 4)$: cycles of length 2 and 2. Order $= \text{lcm}(2, 2) = 2$. FALSE.

(D) $(1\ 2\ 3\ 4)(5\ 6)$: cycles of length 4 and 2. Order $= \text{lcm}(4, 2) = 4$. FALSE.

Correct options: A and B.

Answer: (AB) ← *Go back to Q31*

Q32.

Solution

Properties of a ring homomorphism $\varphi : R \to S$:

(A) $\varphi(0_R) = 0_S$: TRUE. $\varphi(0_R) = \varphi(0_R + 0_R) = \varphi(0_R) + \varphi(0_R)$, so $\varphi(0_R) = 0_S$.

(B) φ is surjective: FALSE. Not every homomorphism is surjective (e.g., the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is not surjective).

(C) $\varphi(-r) = -\varphi(r)$: TRUE. $\varphi(r) + \varphi(-r) = \varphi(r + (-r)) = \varphi(0_R) = 0_S$, so $\varphi(-r) = -\varphi(r)$.

(D) $\ker(\varphi)$ is an ideal of R: TRUE. For $a, b \in \ker(\varphi)$ and $r \in R$: $\varphi(ra) = \varphi(r)\varphi(a) = \varphi(r) \cdot 0 = 0$, so $ra \in \ker(\varphi)$. Similarly for ar.

Correct options: A, C, D.

Answer: (ACD) *← Go back to Q32*

Q33.

Solution

Analysis of ideal $I = (2)$ in $\mathbb{Z}$:

(A) I is NOT a prime ideal: FALSE. (2) IS a prime ideal.

(B) $\mathbb{Z}/I \cong \mathbb{Z}_2$: TRUE. The cosets of (2) in $\mathbb{Z}$ are $\{\text{even}\}$ and $\{\text{odd}\}$, giving $\mathbb{Z}/I \cong \mathbb{Z}_2$.

(C) I is a prime ideal: TRUE. $\mathbb{Z}/(2) \cong \mathbb{Z}_2$ is an integral domain (in fact a field), so (2) is prime.

(D) I is a maximal ideal: TRUE. $\mathbb{Z}/(2) \cong \mathbb{Z}_2$ is a field, so (2) is maximal.

Correct options: B, C, D.

Answer: (BCD) *← Go back to Q33*

Q34.

Solution

Analysis:

(A) $f'(x) = 0$ for all $x \in \mathbb{R}$: TRUE. By the MVT, for any $x_1 < x_2$, $f(x_2) - f(x_1) = f'(c)(x_2 - x_1) = 0$. So f is constant.

(B) f differentiable and $f' \equiv 0$: TRUE. This is equivalent to (A) — a restatement.

(C) f is bounded: FALSE. A bounded function need not be constant (e.g., $\sin x$).

(D) f is uniformly continuous: FALSE. Many non-constant functions are uniformly continuous (e.g., $f(x) = x$ on $\mathbb{R}$).

Correct options: A and B.

Answer: (AB) ← *Go back to Q34*

Q35.

Solution

Analysis of $f(x, y) = x^2 + y^2$:

(A) f achieves its minimum value 0 at $(0, 0)$: TRUE. $f \geq 0$ for all (x, y) and $f(0, 0) = 0$.

(B) f has a local maximum at $(0, 0)$: FALSE. $(0, 0)$ is a global *minimum*, not a maximum. $D = f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} = 2 > 0$ confirm a local (and global) minimum.

(C) Level curves are circles centred at origin: TRUE. $f(x, y) = c$ gives $x^2 + y^2 = c$ (for $c > 0$), which are circles with centre at the origin.

(D) $\nabla f = (2x, 2y)$ vanishes only at origin: TRUE. $\nabla f = \mathbf{0} \Leftrightarrow 2x = 0$ and $2y = 0 \Leftrightarrow (x, y) = (0, 0)$.

Correct options: A, C, D.

Answer: (ACD) ← *Go back to Q35*

Q36.

Solution

Analysis of A_4:

(A) A_4 has a subgroup of order 6: FALSE. This is a classical result: A_4 has no subgroup of order 6 (which would have index 2 and be normal, but exhaustive search of elements of A_4 shows no such subgroup exists).

(B) A_4 has the Klein four-group V_4 as a normal subgroup: TRUE. $V_4 = \{e, (12)(34), (13)(24), (14)(23)\} \trianglelefteq A_4$ (shown in Q25).

(C) $|A_4| = 12$: TRUE. $|A_4| = 4!/2 = 12$.

(D) A_4 is abelian: FALSE. For example, $(123)(12)(34) = (134) \neq (134)$... In fact $(123)(12)(34)$ and $(12)(34)(123)$ differ, confirming non-commutativity.

Correct options: B and C.

Answer: (BC) ← *Go back to Q36*

Q37.

Solution

Analysis:

(A) Rational root theorem: if p/q in lowest terms is a root, then $p \mid a_0$ and $q \mid a_n$. TRUE — this is the rational root theorem.

(B) Monic polynomial has only integer rational roots: TRUE. If $a_n = 1$, then $q \mid 1$, so $q = \pm 1$ and $p/q = \pm p$ is an integer.

(C) Every polynomial of degree ≥ 1 has a real root: FALSE. The polynomial $x^2 + 1$ has no real roots (its roots are $\pm i$).

(D) If p is irreducible over $\mathbb{Q}$ and z is a complex root, then $\bar{z}$ is also a root: TRUE. Polynomials with rational coefficients have complex roots appearing in conjugate pairs (since the minimal polynomial over $\mathbb{Q}$ of z divides $p(x)$, and it has $\bar{z}$ as a root by complex conjugate root theorem).

Correct options: A, B, D.

Answer: (ABD) ← *Go back to Q37*

Q38.

Solution

Concept: The characteristic polynomial of a 3×3 matrix A is

$$p(\lambda) = \det(\lambda I - A) = \lambda^3 - (\operatorname{tr} A)\lambda^2 + \cdots + (-1)^3 \det A.$$

The roots are $\lambda_1, \lambda_2, \lambda_3$, so by Vieta's formulas:

$$\lambda_1 + \lambda_2 + \lambda_3 = \operatorname{tr}(A), \qquad \lambda_1\lambda_2\lambda_3 = \det(A).$$

(A) $\operatorname{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$: TRUE.

(B) $\det(A) = \lambda_1 + \lambda_2 + \lambda_3$: FALSE — this is $\operatorname{tr}(A)$, not $\det(A)$.

(C) $\det(A) = \lambda_1\lambda_2\lambda_3$: TRUE.

(D) $\operatorname{tr}(A) = \lambda_1\lambda_2\lambda_3$: FALSE — this is $\det(A)$.

Correct options: A and C.

Answer: (AC) ← *Go back to Q38*

Q39.

Solution

Let $\mathbf{F} = \nabla f$ for smooth $f : \mathbb{R}^3 \to \mathbb{R}$:

(A) $\nabla \times \mathbf{F} = \mathbf{0}$: TRUE. The curl of any gradient is identically zero: $\nabla \times (\nabla f) = \mathbf{0}$.

(B) $\mathbf{F}$ is conservative: TRUE. By definition, $\mathbf{F}$ is conservative since it equals the gradient of a scalar potential f.

(C) $\nabla \cdot \mathbf{F} = 0$ always: FALSE. $\nabla \cdot (\nabla f) = \Delta f$ (the Laplacian), which is zero only if

f is harmonic. Not true in general.

(D) $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve: TRUE. For a conservative field, the line integral depends only on endpoints; hence any closed curve gives zero.

Correct options: A, B, D.

Answer: (ABD) ← *Go back to Q39*

Q40.

Solution

Analysis of $S_n = \sum_{k=1}^{n} 1/k^2$:

(A) $S_n \to \infty$: FALSE. The series $\sum 1/k^2$ converges (p-series with $p = 2 > 1$), so S_n converges to $\pi^2/6$.

(B) $\{S_n\}$ is bounded above: TRUE. Since the series converges to $\pi^2/6$, all partial sums satisfy $S_n \leq \pi^2/6$.

(C) $\{S_n\}$ is strictly increasing: TRUE. Each term $1/(n+1)^2 > 0$ is added at each step, so $S_{n+1} = S_n + 1/(n+1)^2 > S_n$.

(D) $S_n = \pi^2/6$ for all large n: FALSE. S_n is always strictly less than $\pi^2/6$ for finite n; it is only in the limit that $S_n \to \pi^2/6$.

Correct options: B and C.

Answer: (BC) ← *Go back to Q40*

Section C Solutions — Q41–Q50 (NAT, 1 mark each)

Q41.

Solution

Step 1: $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is diagonal with eigenvalues 1 and -1.

Step 2: $A^4 = \begin{pmatrix} 1^4 & 0 \\ 0 & (-1)^4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$

Step 3: $\mathrm{tr}(A^4) = 1 + 1 = 2.$

Answer: (2) ← *Go back to Q41*

Q42.

Solution

Concept: A group homomorphism $\varphi : \mathbb{Z}_3 \to S_3$ is determined by $\varphi(1)$. Since $\varphi(1)$ must have order dividing $|\mathbb{Z}_3| = 3$, we need $\operatorname{ord}(\varphi(1)) \mid 3$.

Elements of S_3 with order dividing 3:

- Order 1: e
- Order 3: (123), (132)

Total: 3 choices for $\varphi(1)$, giving 3 distinct homomorphisms.

Answer: (3) ← *Go back to Q42*

Q43.

Solution

Step 1: Recall $\int_0^\infty x\,e^{-ax}\,dx = \frac{1}{a^2}$ for $a > 0$.

Step 2: With $a = 3$: $\int_0^\infty x\,e^{-3x}\,dx = \frac{1}{3^2} = \frac{1}{9}$.

Verification by parts: Let $u = x$, $dv = e^{-3x}dx$, so $v = -\frac{1}{3}e^{-3x}$:

$$\left[-\frac{x}{3}e^{-3x}\right]_0^\infty + \int_0^\infty \frac{1}{3}e^{-3x}\,dx = 0 + \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9}. \quad \checkmark$$

Answer: (1/9) ← *Go back to Q43*

Q44.

Solution

Step 1: Inner integral over y from 0 to x^2:

$$\int_0^{x^2} y\,dy = \left[\frac{y^2}{2}\right]_0^{x^2} = \frac{x^4}{2}.$$

Step 2: Outer integral over x from 0 to 1:

$$\int_0^1 \frac{x^4}{2}\,dx = \frac{1}{2}\left[\frac{x^5}{5}\right]_0^1 = \frac{1}{2}\cdot\frac{1}{5} = \frac{1}{10}.$$

Answer: (1/10) ← *Go back to Q44*

Q45.

Solution

Step 1: $A - I = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Step 2: $(A - I)^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}$.

Step 3: Sum of all entries of $\mathbf{0}$ is 0.

Remark: This confirms that $A - I$ is nilpotent of index 2 (a Jordan nilpotent block).

Answer: (0) ← *Go back to Q45*

Q46.

Solution

Step 1: For $(a, b) \in \mathbb{Z}_8 \times \mathbb{Z}_2$, $\operatorname{ord}(a, b) = \operatorname{lcm}(\operatorname{ord}(a), \operatorname{ord}(b))$.

Step 2: We need $\operatorname{lcm}(\operatorname{ord}(a), \operatorname{ord}(b)) = 4$.

Elements of order 4 in $\mathbb{Z}_8$: $a \in \{2, 6\}$ (since $\gcd(2, 8) = 2$, order$= 8/2 = 4$; $\gcd(6, 8) = 2$, order$= 4$). So 2 elements.

Elements of $\mathbb{Z}_2$: orders are 1 (for $b = 0$) and 2 (for $b = 1$). $\mathbb{Z}_2$ has no element of order 4.

Cases giving lcm $= 4$:

- $\operatorname{ord}(a) = 4$, $\operatorname{ord}(b) = 1$: $a \in \{2, 6\}$, $b = 0$ — gives $2 \times 1 = 2$ elements.
- $\operatorname{ord}(a) = 4$, $\operatorname{ord}(b) = 2$: $a \in \{2, 6\}$, $b = 1$ — gives $2 \times 1 = 2$ elements.
- $\operatorname{ord}(a) \in \{1, 2\}$, $\operatorname{ord}(b) = 4$: impossible since $\mathbb{Z}_2$ has no order-4 element.

Total: $2 + 2 = 4$ elements.

Answer: (4) ← *Go back to Q46*

Q47.

Solution

Step 1: The integral separates:

$$\int_0^{2\pi} \int_0^{\pi} \sin\varphi \, d\varphi \, d\theta = \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\pi} \sin\varphi \, d\varphi \right).$$

Step 2:

$$\int_0^{2\pi} d\theta = 2\pi, \qquad \int_0^{\pi} \sin\varphi \, d\varphi = \left[-\cos\varphi \right]_0^{\pi} = -\cos\pi + \cos 0 = 1 + 1 = 2.$$

Step 3: $2\pi \times 2 = 4\pi$.

Remark: This integral arises as part of the surface area element when computing the area of a unit sphere: $\iint \sin\varphi \, d\varphi \, d\theta = 4\pi$.

Answer: (4π) ← *Go back to Q47*

Q48.

Solution

Step 1: $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ is a 3×3 Jordan block with eigenvalue 2.

Step 2: $A - 2I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. This is the standard nilpotent Jordan block N of size 3.

Step 3: $N^3 = \mathbf{0}$ (a $k \times k$ Jordan block satisfies $N^k = \mathbf{0}$).

Step 4: $p(A) = (A - 2I)^3 = N^3 = \mathbf{0}$.

Step 5: $\mathrm{tr}(p(A)) = \mathrm{tr}(\mathbf{0}) = 0$.

Answer: (0) ← *Go back to Q48*

Q49.

Solution

Step 1: Differentiate $f(x) = xe^{-x}$:

$$f'(x) = e^{-x} + x(-e^{-x}) = e^{-x}(1 - x).$$

Step 2: Critical point: $f'(x) = 0 \Rightarrow e^{-x}(1 - x) = 0 \Rightarrow x = 1$ (since $e^{-x} \neq 0$).

Step 3: Second derivative test: $f''(x) = e^{-x}(x - 2)$. At $x = 1$: $f''(1) = e^{-1}(-1) = -e^{-1} < 0$, confirming a local maximum.

Step 4: Maximum value: $f(1) = 1 \cdot e^{-1} = \frac{1}{e}$.

Global maximum: Since $f(x) \to 0$ as $x \to \pm\infty$ (for $x < 0$, $f < 0$), the global maximum over $\mathbb{R}$ is $1/e$.

Answer: (1/e) ← *Go back to Q49*

Q50.

Solution

Step 1: Total functions from $\{1,2,3,4\}$ to $\{1,2\}$: $2^4 = 16$.

Step 2: Non-surjective functions (missing at least one element of codomain):

- Constant function $f \equiv 1$: 1 function.
- Constant function $f \equiv 2$: 1 function.

Total non-surjective: 2.

Step 3: Surjective functions $= 16 - 2 = 14$.

Alternative (inclusion-exclusion): $\sum_{k=0}^{2}(-1)^k\binom{2}{k}(2-k)^4 = 1\cdot16 - 2\cdot1 + 1\cdot0 = 14$.

Answer: (14) ← *Go back to Q50*

Section C Solutions — Q51–Q60 (NAT, 2 marks each)

Q51.

Solution

Step 1: Integration by parts with $u = \arctan x$, $dv = x\,dx$:

$$du = \frac{1}{1+x^2}\,dx, \quad v = \frac{x^2}{2}.$$

Step 2:

$$\int_0^1 x\arctan x\,dx = \left[\frac{x^2}{2}\arctan x\right]_0^1 - \int_0^1 \frac{x^2}{2(1+x^2)}\,dx = \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}\,dx.$$

Step 3: Simplify the remaining integral:

$$\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}, \quad \text{so} \quad \int_0^1 \frac{x^2}{1+x^2}\,dx = [x - \arctan x]_0^1 = 1 - \frac{\pi}{4}.$$

Step 4:

$$\int_0^1 x\arctan x\,dx = \frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right) = \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2}.$$

Answer: ($\pi/4 - 1/2$) ← *Go back to Q51*

Q52.

Solution

Step 1: The rotation matrix by angle θ has $\mathrm{tr}(A) = 2\cos\theta$.

Step 2: A^2 is the rotation matrix by 2θ. Hence $\mathrm{tr}(A^2) = 2\cos(2\theta)$.

Step 3: With $\theta = \pi/6$: $2\theta = \pi/3$, so

$$\mathrm{tr}(A^2) = 2\cos\left(\frac{\pi}{3}\right) = 2 \cdot \frac{1}{2} = 1.$$

Answer: (1) ← *Go back to Q52*

Q53.

Solution

Step 1: Lagrange conditions: $\nabla f = \lambda \nabla g$, i.e., $(2x, 2y, 2z) = \lambda(1, 1, 1)$.

Step 2: $2x = 2y = 2z = \lambda$, so $x = y = z$.

Step 3: Substituting into $x + y + z = 3$: $3x = 3 \Rightarrow x = 1$. So $(x, y, z) = (1, 1, 1)$.

Step 4: Minimum value $f(1, 1, 1) = 1 + 1 + 1 = 3$.

Geometric interpretation: $(1, 1, 1)$ is the point on the plane $x + y + z = 3$ closest to the origin, at distance $\sqrt{3}$, so $f_{\min} = (\sqrt{3})^2 = 3$.

Answer: (3) ← *Go back to Q53*

Q54.

Solution

Step 1: Factor: $p(x) = x^4 - 5x^2 + 4$. Let $u = x^2$:

$$u^2 - 5u + 4 = (u - 1)(u - 4) = (x^2 - 1)(x^2 - 4).$$

Step 2: Further factor:

$$p(x) = (x - 1)(x + 1)(x - 2)(x + 2).$$

Step 3: The four real roots are $x = -2, -1, 1, 2$ — all distinct.

Number of distinct real roots $= 4$.

Answer: (4) ← *Go back to Q54*

Q55.

Solution

Step 1: Integration by parts with $u = x$, $dv = \cos x\,dx$:

$$du = dx, \quad v = \sin x.$$

Step 2:

$$\int_0^{\pi} x\cos x\,dx = [x\sin x]_0^{\pi} - \int_0^{\pi} \sin x\,dx.$$

Step 3: $[x\sin x]_0^{\pi} = \pi\sin\pi - 0\sin 0 = 0.$

Step 4:

$$-\int_0^{\pi} \sin x\,dx = -\big[-\cos x\big]_0^{\pi} = -\big(-\cos\pi + \cos 0\big) = -(1+1) = -2.$$

Answer: $\int_0^{\pi} x\cos x\,dx = 0 + (-2) = -2.$

Answer: (-2) ← *Go back to Q55*

Q56.

Solution

Step 1: The region: $-1 \le x \le 1$, $0 \le y \le 1 - x^2$, $0 \le z \le y$.

$$V = \int_{-1}^{1}\int_0^{1-x^2}\int_0^{y} dz\,dy\,dx = \int_{-1}^{1}\int_0^{1-x^2} y\,dy\,dx = \int_{-1}^{1} \frac{(1-x^2)^2}{2}\,dx.$$

Step 2: By symmetry in x:

$$V = 2\int_0^1 \frac{(1-x^2)^2}{2}\,dx = \int_0^1 (1-x^2)^2\,dx = \int_0^1 (1-2x^2+x^4)\,dx.$$

Step 3:

$$V = \left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_0^1 = 1 - \frac{2}{3} + \frac{1}{5} = \frac{15-10+3}{15} = \frac{8}{15}.$$

Answer: (8/15) ← *Go back to Q56*

Q57.

Solution

Step 1: Count divisible by 3: $\lfloor 100/3 \rfloor = 33.$

Step 2: Count divisible by 7: $\lfloor 100/7 \rfloor = 14.$

Step 3: Count divisible by $\text{lcm}(3,7) = 21$: $\lfloor 100/21 \rfloor = 4$.

Step 4: By inclusion-exclusion, divisible by 3 or 7: $33 + 14 - 4 = 43$.

Step 5: Divisible by neither: $100 - 43 = 57$.

Answer: (57) ← *Go back to Q57*

Q58.

Solution

Step 1: Surface area formula: $\iint_D \sqrt{1 + z_x^2 + z_y^2}\, dA$ where $z = x^2+y^2$, $D : x^2+y^2 \leq 1$.

$z_x = 2x$, $z_y = 2y$, so $\sqrt{1 + 4x^2 + 4y^2} = \sqrt{1 + 4r^2}$ in polar.

Step 2: Convert to polar:

$$\text{Area} = \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2} \cdot r\, dr\, d\theta = 2\pi \int_0^1 r\sqrt{1 + 4r^2}\, dr.$$

Step 3: Substitution $u = 1 + 4r^2$, $du = 8r\, dr$:

$$\int_0^1 r\sqrt{1 + 4r^2}\, dr = \frac{1}{8} \int_1^5 \sqrt{u}\, du = \frac{1}{8} \left[\frac{2u^{3/2}}{3}\right]_1^5 = \frac{1}{12}\left(5^{3/2} - 1\right) = \frac{5\sqrt{5} - 1}{12}.$$

Step 4:

$$\text{Area} = 2\pi \cdot \frac{5\sqrt{5} - 1}{12} = \frac{\pi(5\sqrt{5} - 1)}{6}.$$

Answer: $(\pi(5\sqrt{5} - 1)/6)$ ← *Go back to Q58*

Q59.

Solution

Step 1: $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$ is upper triangular.

Step 2: Eigenvalues of an upper triangular matrix are its diagonal entries: $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = 6$.

Step 3: Sum of eigenvalues $= \text{tr}(A) = 1 + 4 + 6 = 11$.

Answer: (11) ← *Go back to Q59*

Q60.

Solution

Step 1: Assume $\lim_{n\to\infty} a_n = L$ exists. Taking limits of both sides of $a_{n+1} = \dfrac{a_n}{2} + 1$:

$$L = \frac{L}{2} + 1 \;\Rightarrow\; \frac{L}{2} = 1 \;\Rightarrow\; L = 2.$$

Step 2: Verify the limit exists: define $e_n = a_n - 2$. Then $e_{n+1} = a_{n+1} - 2 = \frac{a_n}{2} + 1 - 2 = \frac{a_n - 2}{2} = \frac{e_n}{2}$. So $e_n = (1/2)^{n-1} e_1 = (1/2)^{n-1}(a_1 - 2) = (1/2)^{n-1}(-1) \to 0$.

Step 3: Therefore $a_n = 2 + e_n \to 2$.

Answer: (2) ← *Go back to Q60*

Answer Key

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	A	2	C	3	B	4	D	5	C
6	A	7	B	8	D	9	C	10	A
11	D	12	B	13	C	14	A	15	D
16	B	17	C	18	A	19	D	20	B
21	C	22	A	23	D	24	B	25	C
26	A	27	D	28	B	29	C	30	A
31	AB	32	ACD	33	BCD	34	AB	35	ACD
36	BC	37	ABD	38	AC	39	ABD	40	BC
41	2	42	3	43	1/9	44	1/10	45	0
46	4	47	4π	48	0	49	1/e	50	14
51	$\pi/4 - 1/2$	52	1	53	3	54	4	55	-2
56	8/15	57	57	58	$\pi(5\sqrt{5} - 1)/6$	59	11	60	2