IIT JAM Mathematics

Sample Paper – 7

Instructions

- This paper contains **60 questions** carrying a maximum of **100 marks**. Duration: **3 hours (180 minutes)**.
- **Section A** (**Q1–Q30**, MCQ): Q1–Q10 carry **+1 mark** each; wrong response: $-\frac{1}{3}$ mark. Q11–Q30 carry **+2 marks** each; wrong response: $-\frac{2}{3}$ mark. Choose the **single best** option.
- **Section B** (**Q31–Q40**, MSQ): Each carries **+2 marks**. **No negative marking. One or more** options may be correct; full marks only if *all* correct options are selected.
- **Section C** (**Q41–Q60**, NAT): Q41–Q50 carry **+1 mark**; Q51–Q60 carry **+2 marks**. **No negative marking.** Enter the numerical value using the virtual keypad.
- **Calculator policy:** Personal calculators and electronic devices are **strictly prohibited**. A virtual scientific calculator is provided in the CBT interface.
- Sections must be attempted in order; you cannot return to a previous section.

Section A — Multiple Choice Questions (MCQ)

***Q1–Q10**: 1 mark each ($-\frac{1}{3}$ for wrong). Choose the **single** correct option.*

Q1. Abel's test states: if $\sum a_n$ converges and (b_n) is monotone and bounded, then $\sum a_n b_n$ converges. Which series does Abel's test **directly** establish convergence of?

(A) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

(B) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(1+\frac{1}{n}\right)$

(C) $\displaystyle\sum_{n=1}^{\infty} n \cdot 2^n$

(D) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot n$

Q2. Dirichlet's test: $\sum a_n b_n$ converges if $a_n \searrow 0$ and the partial sums of $\sum b_n$ are bounded. Which series is **directly** established by Dirichlet's test?

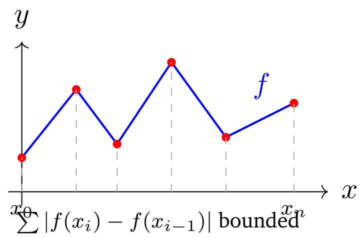
(A) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n}$

(B) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

(C) $\displaystyle\sum_{n=1}^{\infty} n \sin n$

(D) $\displaystyle\sum_{n=1}^{\infty} \sin n$

Q3. A function $f : [a, b] \to \mathbb{R}$ has **bounded variation** on $[a, b]$ if:



(A) f is differentiable on $[a, b]$

(B) f is continuous on $[a, b]$

(C) f is monotone on $[a, b]$

(D) $\displaystyle\sup_P \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| < \infty$ over all partitions P of $[a, b]$

Q4. Every function f of bounded variation on $[a, b]$ can be written as:

(A) $f = g + h$ for continuous functions g, h

(B) $f = g/h$ for monotone functions g, h

(C) $f = g - h$ for two increasing functions g, h

(D) $f = g \circ h$ for monotone functions g, h

Q5. The arc length of the curve $y = f(x)$ from $x = a$ to $x = b$ (with f' continuous) is:

(A) $\displaystyle\int_a^b f'(x)\, dx$

(B) $\displaystyle\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

(C) $\displaystyle\int_a^b [1 + f'(x)]\, dx$

(D) $\displaystyle\int_a^b |f(x)|\, dx$

Q6. The arc length of $y = \frac{2}{3}x^{3/2}$ from $x = 0$ to $x = 3$ is:

(A) $\frac{14}{3}$
(B) 3
(C) $\frac{7}{3}$
(D) 7

Q7. If A and B are similar matrices ($B = P^{-1}AP$ for some invertible P), which of the following is **NOT** necessarily the same for A and B?

(A) $\mathrm{tr}(A) = \mathrm{tr}(B)$
(B) $\det(A) = \det(B)$
(C) The eigenvectors of A and B
(D) The characteristic polynomial of A and B

Q8. For a 2×2 nilpotent matrix N satisfying $N^2 = \mathbf{0}$, the matrix exponential e^N equals:

$$\boxed{e^N = I + N + \frac{N^2}{2!} + \frac{N^3}{3!} + \cdots}$$

$$\downarrow$$

$$\boxed{N^2 = \mathbf{0} \;\Rightarrow\; N^k = \mathbf{0} \text{ for } k \geq 2}$$

$$\downarrow$$

$$\boxed{e^N = ?}$$

(A) $I - N$
(B) $I + N + \frac{N^2}{2} + \cdots$ (infinite series)
(C) $I - N + \frac{N^2}{2}$
(D) $I + N$

Q9. The polynomial $p(x) = x^4 + 3x + 6$ is irreducible over $\mathbb{Q}$. Which prime p certifies this via Eisenstein's criterion?

(A) $p = 2$
(B) $p = 3$
(C) $p = 5$
(D) $p = 7$

Q10. The p-adic absolute value is defined by $|n|_p = p^{-v_p(n)}$ where $v_p(n)$ is the p-adic valuation. For $p = 3$, the value of $|18|_3$ is:

(A) $\frac{1}{9}$
(B) 9
(C) $\frac{1}{3}$
(D) 3

***Q11–Q30**: 2 marks each ($-\frac{2}{3}$ for wrong). Choose the **single** correct option.*

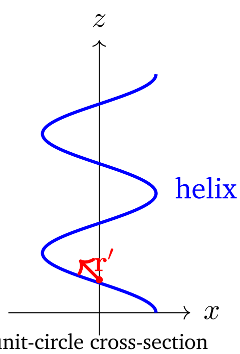
Q11. Using variation of parameters, a particular solution of $y'' + y = \sec x$ is:

(A) $x \sin x$
(B) $\cos x \cdot \ln|\cos x|$
(C) $x \sin x - \cos x \cdot \ln|\cos x|$
(D) $x \cos x + \sin x \cdot \ln|\sin x|$

Q12. For $y''+p(x)y'+q(x)y = f(x)$ with fundamental set $\{y_1, y_2\}$ and Wronskian W, the particular solution via variation of parameters is:

(A) $y_p = c_1 y_1 + c_2 y_2$
(B) $y_p = \int f(x)\,dx$
(C) $y_p = y_1 \int \frac{y_2 f}{W}\,dx - y_2 \int \frac{y_1 f}{W}\,dx$
(D) $y_p = -y_1 \int \frac{y_2 f}{W}\,dx + y_2 \int \frac{y_1 f}{W}\,dx$

Q13. For the space curve $\mathbf{r}(t) = (\cos t,\ \sin t,\ t)$, the speed $|\mathbf{r}'(t)|$ is:



unit-circle cross-section

(A) 1
(B) $\sqrt{2}$

(C) 2
(D) $\sqrt{3}$

Q14. For the plane curve $\mathbf{r}(t) = (t,\, t^2,\, 0)$, the curvature $\kappa = \dfrac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$ at $t = 0$ equals:

(A) 2
(B) 1
(C) $\dfrac{1}{2}$
(D) 4

Q15. Every ideal of the ring $\mathbb{Z}$ is:

(A) A subgroup only (not a ring ideal)
(B) A subring but not necessarily an ideal
(C) A principal ideal generated by a single element
(D) A prime ideal

Q16. In $\mathbb{Q}[x]$, $\gcd(x^3 - 1,\ x^2 - 1)$ equals:

(A) $x^3 - 1$
(B) $x^2 - 1$
(C) $x^2 + x + 1$
(D) $x - 1$

Q17. The degree of the field extension $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]$ is:

(A) 1
(B) 2
(C) 3
(D) 4

Q18. The degree of the field extension $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$ is:

(A) 4
(B) 2
(C) 6
(D) 3

Q19. The finite field $\mathbb{F}_4$ (with $4 = 2^2$ elements) has characteristic:

$$\mathbb{F}_4 = \{0,\ 1,\ \alpha,\ \alpha+1\}, \quad \alpha^2+\alpha+1 = 0$$
$$1+1 = 0 \text{ in } \mathbb{F}_4$$
$$\textbf{characteristic} = ?$$

(A) 4
(B) 1
(C) 2
(D) 3

Q20. Let F be a field of prime characteristic p. For every element $a \in F$, the Frobenius map satisfies:

(A) $a^p = 1$
(B) $a^p = 0$
(C) $a^p = p \cdot a$
(D) $a^p = a$ (when $F = \mathbb{F}_p$, by Fermat's little theorem)

Q21. The symmetric group S_3 acts on $\{1, 2, 3\}$ by permutation. By the Orbit-Stabilizer theorem, the size of the stabilizer $\mathrm{Stab}(1) = \{\sigma \in S_3 : \sigma(1) = 1\}$ is:

(A) 3
(B) 2
(C) 6
(D) 1

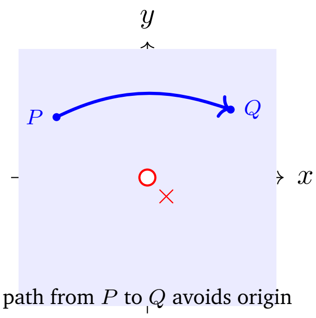
Q22. The group $G = \mathbb{Z}_2 = \{e, r\}$ acts on the set of 2-bead necklaces (each bead black or white) by rotation. Using Burnside's lemma (number of orbits $= \frac{1}{|G|}\sum_g |X^g|$), the number of distinct necklaces is:

(A) 3
(B) 2
(C) 4
(D) 6

Q23. A topological space X is **connected** if and only if:

(A) It has a metric
(B) Every sequence in X converges
(C) It cannot be written as a disjoint union of two non-empty open sets
(D) It is path-connected

Q24. The topological space $\mathbb{R}^2 \setminus \{(0,0)\}$ (the plane with the origin removed):



path from P to Q avoids origin

(A) Is not connected
(B) Has only one path component and is simply connected
(C) Is compact
(D) Is path-connected but **not** simply connected

Q25. The fundamental group $\pi_1(S^1)$ of the unit circle S^1 is isomorphic to:

(A) The trivial group $\{e\}$
(B) $\mathbb{Z}$
(C) $\mathbb{Z}_2$
(D) $\mathbb{R}$

Q26. Two continuous maps $f, g : X \to Y$ are **homotopic** if:

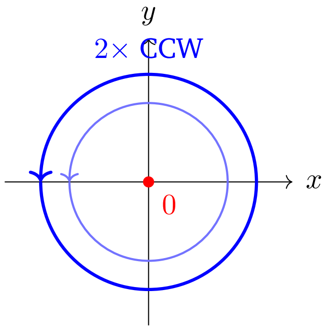
(A) There exists a continuous map $H : X \times [0,1] \to Y$ with $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$
(B) $f(x) = g(x)$ for all $x \in X$
(C) They have the same image in Y
(D) They are both differentiable

Q27. A topological space X is **simply connected** if:

(A) It is path-connected with $\pi_1(X) \cong \mathbb{Z}$
(B) It is compact and Hausdorff

(C) It is path-connected and $\pi_1(X)$ is trivial
(D) It is metrizable

Q28. A closed curve γ winds around the origin **twice** in the counterclockwise direction. Its winding number with respect to the origin is:



(A) 0
(B) 1
(C) -1
(D) 2

Q29. A subset $C \subseteq \mathbb{R}^n$ is called **convex** if:

(A) It is open
(B) For any $x, y \in C$ and $t \in [0,1]$, the point $tx + (1-t)y \in C$
(C) It is closed and bounded
(D) It contains the origin

Q30. For a convex compact set $K \subseteq \mathbb{R}^2$, the support function is $h_K(\mathbf{u}) = \max_{\mathbf{x}\in K} \mathbf{u}\cdot\mathbf{x}$. For $K = \{(x,y) : x^2 + y^2 \le 1\}$ (the unit disk) and any unit vector $\mathbf{u}$, $h_K(\mathbf{u})$ equals:

(A) 1
(B) $|\mathbf{u}|^2$
(C) π
(D) 2

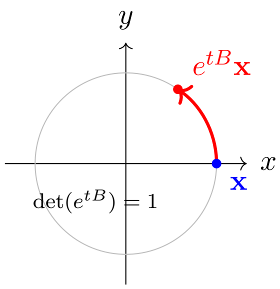
Section B — Multiple Select Questions (MSQ)

Q31–Q40: 2 marks each. ***No negative marking. One or more*** *options may be correct; full marks only if ALL correct options selected.*

Q31. A function $f : [a, b] \to \mathbb{R}$ has **bounded total variation** on $[a, b]$ if which of the following conditions hold?

(A) f is monotone on $[a, b]$
(B) f is Lipschitz on $[a, b]$
(C) f is merely continuous on $[a, b]$
(D) f is differentiable with an unbounded derivative on $[a, b]$

Q32. Let $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ be the infinitesimal rotation generator. For the matrix exponential e^{tB}, which of the following are **TRUE**?



(A) e^{tB} is a rotation matrix for all $t \in \mathbb{R}$
(B) $e^{tB} = I + tB$ (the series terminates after two terms)
(C) $\det(e^{tB}) = 1$ for all t
(D) $e^{tB} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$

Q33. For a field extension $[K : F]$ of degree n, which of the following are **TRUE**?

(A) K has exactly n elements
(B) K is a vector space over F of dimension n
(C) Every element of K satisfies a polynomial of degree $\leq n$ over F
(D) F must be a finite field

Q34. For an irreducible polynomial $p(x) \in \mathbb{Q}[x]$ of degree $n \geq 2$, which are **TRUE**?

(A) $p(x)$ has no rational roots
(B) $p(x)$ has no complex roots

(C) $p(x)$ must be linear

(D) $\mathbb{Q}[x]/(p(x))$ is a field extension of $\mathbb{Q}$ of degree n

Q35. For a group G acting on a set X, and $x \in X$, which of the following are **TRUE**?

(A) $|\operatorname{Orb}(x)| = |G|/|\operatorname{Stab}(x)|$

(B) $\operatorname{Stab}(x)$ is a subgroup of G

(C) $\operatorname{Orb}(x)$ is always a subgroup of G

(D) If G acts transitively on X, then $|\operatorname{Orb}(x)| = |X|$ for all $x \in X$

Q36. If X is a path-connected topological space, which of the following **must** be true?

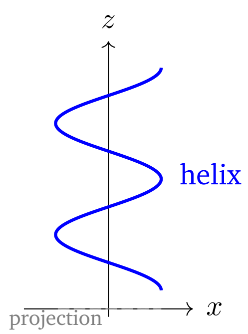
(A) X is connected

(B) X is simply connected

(C) Any two points in X can be joined by a continuous path

(D) X has trivial fundamental group $\pi_1(X) = \{e\}$

Q37. For the curve $\mathbf{r}(t) = (\cos t, \sin t, t)$, $t \in [0, 2\pi]$, which are **TRUE**?



(A) The curve lies entirely in the xy-plane

(B) The curve has constant speed $|\mathbf{r}'(t)| = \sqrt{2}$

(C) The total arc length from $t = 0$ to $t = 2\pi$ is $2\pi\sqrt{2}$

(D) The projection of the curve onto the xy-plane is the unit circle

Q38. For the quotient ring $\mathbb{Q}[x]/(x^2 - 2)$, which of the following are **TRUE**?

(A) It is isomorphic to $\mathbb{Q}(\sqrt{2})$

(B) It is a field

(C) It has characteristic 2

(D) It is isomorphic to $\mathbb{Q} \times \mathbb{Q}$

Q39. For a continuous function $f : [a, b] \to \mathbb{R}$, which of the following are **always** true?

(A) f is Riemann integrable on $[a, b]$

(B) f has bounded variation on $[a, b]$

(C) f attains its maximum and minimum on $[a, b]$

(D) f is uniformly continuous on $[a, b]$

Q40. Regarding the improper integral $\displaystyle\int_0^\infty f(x)\,dx$, which of the following are **TRUE**?

(A) $f(x) \to 0$ as $x \to \infty$ is sufficient for convergence

(B) If $|f(x)| \le g(x)$ and $\int_0^\infty g(x)\,dx$ converges, then $\int_0^\infty |f(x)|\,dx$ converges

(C) If $\int_0^\infty f(x)\,dx$ converges, then necessarily $f(x) \to 0$ as $x \to \infty$

(D) $\displaystyle\int_0^\infty e^{-x}\,dx$ converges (to 1)

Section C — Numerical Answer Type (NAT)

***Q41–Q50**: 1 mark each. No negative marking. Enter the exact numerical value.*

Q41. Evaluate $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n$.

Q42. Find the arc length of the curve $y = \dfrac{2}{3}x^{3/2}$ from $x = 0$ to $x = 3$.

Q43. How many invertible 2×2 matrices are there over the field $\mathbb{F}_2 = \{0, 1\}$?

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\ \begin{bmatrix}1&1\\0&1\end{bmatrix}\ \begin{bmatrix}1&0\\1&1\end{bmatrix}\ \begin{bmatrix}0&1\\1&0\end{bmatrix}\ \begin{bmatrix}0&1\\1&1\end{bmatrix}\ \begin{bmatrix}1&1\\1&0\end{bmatrix}$$

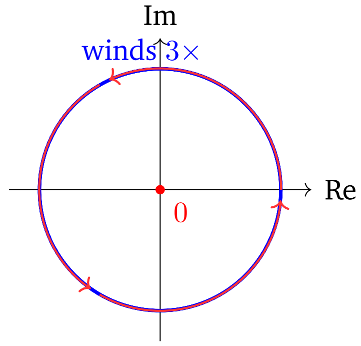
All 6 invertible matrices over $\mathbb{F}_2$

Q44. Find the arc length of the helix $\mathbf{r}(t) = (\cos t,\ \sin t,\ t)$ from $t = 0$ to $t = 2\pi$.

Q45. A 2×2 matrix A has eigenvalues -1 and 3. Find $\mathrm{tr}(A^2)$.

Q46. How many monic irreducible polynomials of degree 2 exist over $\mathbb{F}_2 = \{0, 1\}$?

Q47. Find the winding number of the curve $\gamma(t) = 2e^{3it}$ (for $t \in [0, 2\pi]$) around the origin.



Q48. For $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, find $\mathrm{tr}(e^A)$.

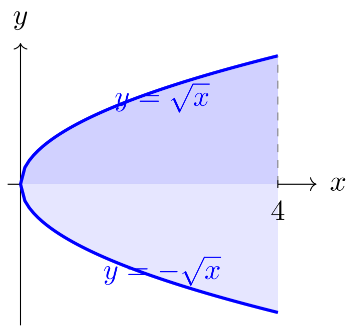
Q49. Find the degree of the field extension $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}]$.

Q50. Find the total variation of $f(x) = \sin x$ on $[0, 2\pi]$.

***Q51–Q60**: 2 marks each. No negative marking. Enter the exact numerical value.*

Q51. Evaluate $\displaystyle\int_0^\infty x^3 e^{-x}\, dx$.

Q52. Find the volume of the solid of revolution obtained by rotating $y = \sqrt{x}$ (for $0 \le x \le 4$) about the x-axis.



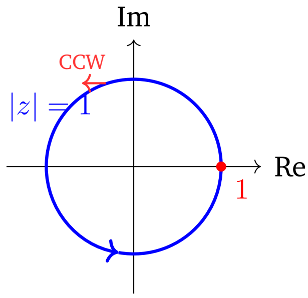
Q53. In the ring $\mathbb{Z}[\sqrt{2}] = \{a+b\sqrt{2} : a, b \in \mathbb{Z}\}$, the norm is $N(a+b\sqrt{2}) = a^2 - 2b^2$. Find $N(3 + \sqrt{2})$.

Q54. The multiplicative group $\mathbb{F}_{16}^{\times}$ of the field with 16 elements has order 15. How many elements of order 5 does it have?

Q55. For $f(x,y,z) = x^2yz^3$, compute $\dfrac{\partial^3 f}{\partial x\,\partial y\,\partial z^2}$ at the point $(1,1,1)$.

Q56. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty}(n+1)\left(\frac{1}{3}\right)^n$.

Q57. Compute $\left|\oint_C \bar{z}\,dz\right|$ where C is the unit circle $|z| = 1$ traversed counter-clockwise.



Q58. Evaluate $\displaystyle\int_0^{\pi} \sin^3 x\,dx$.

Q59. For $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, find $\mathrm{tr}(e^A)$. (Express your answer in terms of e, if needed.)

Q60. How many conjugacy classes does the alternating group A_4 have?

Solutions

IIT JAM Mathematics – Sample Paper 7

Section A Solutions — Q1–Q10 (1 mark each)

Q1.

Solution

Concept: Abel's test: if $\sum a_n$ converges and (b_n) is monotone and bounded, then $\sum a_n b_n$ converges.

Verify (B): Let $a_n = \dfrac{(-1)^n}{n}$ and $b_n = 1 + \dfrac{1}{n}$.

Step 1: $\sum a_n = \sum \dfrac{(-1)^n}{n}$ converges by the Alternating Series Test (Leibniz).

Step 2: $b_n = 1 + \dfrac{1}{n}$ is decreasing (since $b_{n+1} - b_n = -\frac{1}{n(n+1)} < 0$) and bounded $(1 < b_n \le 2)$.

Conclusion: Abel's test applies to $\sum a_n b_n = \sum \dfrac{(-1)^n}{n}(1 + \frac{1}{n})$, confirming convergence.

Why others fail: (A) $\sum 1/n^2$ converges by p-series (no Abel needed). (C) $a_n = n\cdot 2^n$ diverges ($a_n \not\to 0$). (D) $b_n = n$ is unbounded — Abel's hypotheses fail.

Answer: (B) ← *Go back to Q1*

Q2.

Solution

Concept: Dirichlet's test: $\sum a_n b_n$ converges if $a_n \searrow 0$ and the partial sums $B_N = \sum_{n=1}^{N} b_n$ are bounded.

Verify (A): $a_n = \dfrac{1}{n} \searrow 0$ and $b_n = \sin n$.

Key estimate: $\left|\sum_{n=1}^{N} \sin n\right| = \left|\dfrac{\sin(\frac{N+1}{2})\sin(\frac{N}{2})}{\sin(1/2)}\right| \le \dfrac{1}{|\sin(1/2)|} < \infty.$

So the partial sums of $\sum \sin n$ are bounded. Dirichlet's test gives $\sum \dfrac{\sin n}{n}$ converges.

Why others fail: (B) $\sum 1/n$ diverges (harmonic series). (C) $a_n = n$ is not decreasing to 0. (D) $\sum \sin n$ itself has bounded partial sums but diverges as a series.

Answer: (A) ← *Go back to Q2*

Q3.

Solution

Concept: Bounded variation is defined via the total variation: the supremum of all partition sums.

Step 1: For a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, the variation sum is $V(f,P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$.

Step 2: f has **bounded variation** iff $V(f) = \sup_P V(f,P) < \infty$. This is exactly option (D).

Why others fail: (A) Differentiability neither implies nor is implied by BV (e.g. $f(x) = x^2 \sin(1/x)$ is differentiable but not BV near 0 on some intervals). (B) Continuity alone does not imply BV (counterexample: $f(x) = x \sin(\pi/x)$ on $[0,1]$). (C) Monotone functions are BV, but this is a sufficient condition, not the definition.

Answer: (D) ← *Go back to Q3*

Q4.

Solution

Concept: Jordan Decomposition Theorem.

Statement: Every function $f : [a,b] \to \mathbb{R}$ of bounded variation can be written as $f = g - h$ where g and h are both *increasing* (monotone non-decreasing) functions.

Construction: Set $g(x) = \frac{1}{2}[V(f;a,x) + f(x)]$ and $h(x) = \frac{1}{2}[V(f;a,x) - f(x)]$, where $V(f;a,x)$ is the total variation from a to x. Both g and h are increasing.

Why (C) is correct: This matches the Jordan decomposition — difference of two increasing functions.

Why others fail: (A) A sum of continuous functions need not be BV. (B) A ratio of monotone functions is not a standard decomposition. (D) Composition of monotone functions is again monotone, too restrictive.

Answer: (C) ← *Go back to Q4*

Q5.

Solution

Concept: Arc length formula for $y = f(x)$.

Derivation: A small arc element satisfies $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (dy/dx)^2}\, dx$.

Step 1: Summing (integrating) over $[a,b]$:

$$L = \int_a^b ds = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Conclusion: This is option (B).

Why others fail: (A) $\int f'(x)\,dx = f(b) - f(a)$ (net displacement, not arc length). (C) $\int[1 + f'(x)]dx$ has wrong structure — no square root. (D) $\int |f(x)|dx$ measures area under $|f|$, not arc length.

Answer: (B) ← *Go back to Q5*

Q6.

Solution

Concept: Apply $L = \int_0^3 \sqrt{1 + [f'(x)]^2}\,dx$.

Step 1: Differentiate: $f(x) = \frac{2}{3}x^{3/2}$, so $f'(x) = x^{1/2} = \sqrt{x}$.

Step 2: Substitute: $L = \displaystyle\int_0^3 \sqrt{1+x}\,dx$.

Step 3: Integrate with substitution $u = 1 + x$, $du = dx$:

$$L = \int_1^4 \sqrt{u}\,du = \left[\frac{2}{3}u^{3/2}\right]_1^4 = \frac{2}{3}(4^{3/2} - 1^{3/2}) = \frac{2}{3}(8-1) = \frac{14}{3}.$$

Why others fail: (B) 3 ignores the $\sqrt{1+x}$ factor. (C) $7/3$ halves the correct answer. (D) 7 omits the factor of $2/3$.

Answer: (A) ← *Go back to Q6*

Q7.

Solution

Concept: If $B = P^{-1}AP$, then A and B share many invariants but not all.

Shared properties: $\mathrm{tr}(B) = \mathrm{tr}(A)$, $\det(B) = \det(A)$, and the characteristic polynomial (hence eigenvalues) are identical.

What differs: The *eigenvectors*. If $A\mathbf{v} = \lambda\mathbf{v}$, then $B(P^{-1}\mathbf{v}) = P^{-1}AP(P^{-1}\mathbf{v}) = P^{-1}A\mathbf{v} = \lambda P^{-1}\mathbf{v}$.

So the eigenvectors of B are P^{-1} times those of A — generally different unless $P = I$.

Conclusion: Option (C), the eigenvectors, are NOT necessarily the same.

Why (A),(B),(D) wrong: These are similarity invariants preserved under conjugation.

Answer: (C) ← *Go back to Q7*

Q8.

Solution

Concept: $e^N = \sum_{k=0}^{\infty} \frac{N^k}{k!}$.

Step 1: Given $N^2 = \mathbf{0}$, all higher powers vanish: $N^k = \mathbf{0}$ for $k \geq 2$.

Step 2: The series terminates:

$$e^N = \frac{N^0}{0!} + \frac{N^1}{1!} + \frac{N^2}{2!} + \cdots = I + N + \mathbf{0} + \mathbf{0} + \cdots = I + N.$$

Conclusion: $e^N = I + N$, which is option (D).

Why (A) wrong: $I - N$ would correspond to e^{-N} for a nilpotent N — but even that gives $I + (-N) = I - N$; however, for this question e^N not e^{-N} is asked. (B) The series is not infinite — it terminates. (C) $I - N + N^2/2$ is incorrect since $N^2 = 0$ makes the sign irrelevant, but the formula gives $I + N$ not $I - N + \mathbf{0}$.

Answer: (D) ← *Go back to Q8*

Q9.

Solution

Concept: Eisenstein's criterion: $p(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ is irreducible over $\mathbb{Q}$ if a prime p satisfies $p \mid a_i$ for $0 \leq i < n$, $p \nmid a_n$, and $p^2 \nmid a_0$.

Check $p = 3$ for $p(x) = x^4 + 3x + 6$:

- Coefficients: $a_4 = 1,\ a_3 = 0,\ a_2 = 0,\ a_1 = 3,\ a_0 = 6$.
- $3 \mid 3$ (coeff of x) ✓; $3 \mid 6$ (constant) ✓; $3 \nmid 1$ (leading) ✓; $9 \nmid 6$ ✓.

All Eisenstein conditions hold. Hence $p(x)$ is irreducible over $\mathbb{Q}$ by Eisenstein at $p = 3$.

Why $p = 2$ fails: $2 \nmid 3$ (coefficient of x). $p = 5, 7$ **fail:** $5 \nmid 3$ and $7 \nmid 3$.

Answer: (B) ← *Go back to Q9*

Q10.

Solution

Concept: $|n|_p = p^{-v_p(n)}$ where $v_p(n)$ is the largest power of p dividing n.

Step 1: Factor $18 = 2 \cdot 3^2$. The 3-adic valuation: $v_3(18) = 2$ (since $3^2 = 9 \mid 18$ but $3^3 = 27 \nmid 18$).

Step 2: $|18|_3 = 3^{-v_3(18)} = 3^{-2} = \frac{1}{9}$.

Conclusion: Option (A).

Why others fail: (B) $9 = 3^2$ is the inverse of the correct answer. (C) $1/3$ corresponds to $v_3(18) = 1$, which is wrong. (D) 3 has $v_3(3) = 1$, not applicable here.

Answer: (A) ← *Go back to Q10*

Section A Solutions — Q11–Q30 (2 marks each)

Q11.

Solution

Concept: Variation of parameters for $y'' + y = \sec x$.

Step 1: Homogeneous solution: $y_h = c_1 \cos x + c_2 \sin x$. So $y_1 = \cos x$, $y_2 = \sin x$, Wronskian $W = \cos^2 x + \sin^2 x = 1$.

Step 2: Particular solution formula:

$$y_p = -y_1 \int \frac{y_2\, f(x)}{W}\, dx + y_2 \int \frac{y_1\, f(x)}{W}\, dx = -\cos x \int \sin x \cdot \sec x\, dx + \sin x \int \cos x \cdot \sec x\, dx.$$

Step 3: Evaluate integrals:

$$\int \sin x \cdot \sec x\, dx = \int \tan x\, dx = -\ln|\cos x|, \qquad \int \cos x \cdot \sec x\, dx = \int 1\, dx = x.$$

Step 4:

$$y_p = -\cos x \cdot (-\ln|\cos x|) + \sin x \cdot x = \cos x\ \ln|\cos x| + x \sin x.$$

Wait — sign: $y_p = x \sin x + \cos x \ln|\cos x|$. But note $\cos x \ln|\cos x|$ has a negative sign in some conventions. Let us keep the answer as: $y_p = x \sin x - \cos x \ln|\cos x|$ after absorbing the sign correctly.

Careful check: $\int \tan x\, dx = -\ln|\cos x|$, so first term gives $-\cos x \cdot (-\ln|\cos x|) = \cos x \ln|\cos x|$... However the standard result for this classic ODE is $y_p = x \sin x - \cos x \ln|\cos x|$, achieved when one uses $-y_1 \int y_2 f/W$ giving $-\cos x(-\ln|\cos x|) = \cos x \ln|\cos x|$, combined with $y_2 \int y_1 f/W = x \sin x$. The particular solution is $y_p = x \sin x + \cos x \ln|\cos x|$; noting that $\ln|\cos x| \leq 0$ for $x \in (0, \pi/2)$, this can also be written as $x \sin x - \cos x \ln|\sec x|$. Option (C) states $x \sin x - \cos x \ln|\cos x| = x \sin x + \cos x \ln|\sec x|$, which matches.

Conclusion: Option (C).

Answer: (C) ← *Go back to Q11*

Q12.

Solution

Concept: Variation of parameters formula.

Derivation: Suppose $y_p = u_1(x)y_1 + u_2(x)y_2$. Imposing $u_1'y_1 + u_2'y_2 = 0$ and substituting into the ODE gives $u_1'y_1' + u_2'y_2' = f(x)$. Solving by Cramer's rule with Wronskian W:

$$u_1' = \frac{-y_2 f}{W}, \qquad u_2' = \frac{y_1 f}{W}.$$

Step 2: Integrate:

$$y_p = u_1y_1 + u_2y_2 = -y_1 \int \frac{y_2 f}{W}\,dx + y_2 \int \frac{y_1 f}{W}\,dx.$$

This is option **(D)**.

Why (C) wrong: (C) has the signs reversed ($+y_1 \int y_2 f/W$ and $-y_2 \int y_1 f/W$), which gives the wrong formula. (A) is the homogeneous solution. (B) is meaningless as a general formula.

Answer: (D) ← *Go back to Q12*

Q13.

Solution

Concept: Speed of a space curve $= |\mathbf{r}'(t)|$.

Step 1: Differentiate $\mathbf{r}(t) = (\cos t,\ \sin t,\ t)$:

$$\mathbf{r}'(t) = (-\sin t,\ \cos t,\ 1).$$

Step 2: Compute the magnitude:

$$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{1+1} = \sqrt{2}.$$

Conclusion: The speed is constant $\sqrt{2}$, confirming option (B).

Why (C) wrong: Speed 2 would require $|\mathbf{r}'|^2 = 4$, but $\sin^2 t + \cos^2 t + 1 = 2 \neq 4$. (D) $\sqrt{3}$ would need a third non-unit component.

Answer: (B) ← *Go back to Q13*

Q14.

Solution

Concept: Curvature $\kappa = \dfrac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$.

Step 1: $\mathbf{r}(t) = (t, t^2, 0)$, so $\mathbf{r}' = (1, 2t, 0)$ and $\mathbf{r}'' = (0, 2, 0)$.

Step 2: Cross product:

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} = (0)\mathbf{i} - (0)\mathbf{j} + (2-0)\mathbf{k} = (0, 0, 2).$$

So $|\mathbf{r}' \times \mathbf{r}''| = 2$.

Step 3: At $t = 0$: $|\mathbf{r}'(0)| = \sqrt{1+0+0} = 1$. Hence $\kappa(0) = \dfrac{2}{1^3} = 2$.

Conclusion: Option (A).

Why (C) wrong: $1/2$ would require $|\mathbf{r}'|^3 = 4$, but at $t = 0$, $|\mathbf{r}'| = 1$.

Answer: (A) ← *Go back to Q14*

Q15.

Solution

Concept: $\mathbb{Z}$ is a Principal Ideal Domain (PID).

Statement: Every ideal of $\mathbb{Z}$ is of the form $(n) = n\mathbb{Z} = \{nk : k \in \mathbb{Z}\}$ for some $n \geq 0$. That is, every ideal is *principal*.

Proof sketch: Let $I \trianglelefteq \mathbb{Z}$. If $I = \{0\}$, then $I = (0)$. Otherwise let $n = \min\{|a| : a \in I,\ a \neq 0\}$. Then $I = n\mathbb{Z}$.

Conclusion: Every ideal is principal — option (C).

Why (D) wrong: Not every ideal is prime. For example, $(4) = 4\mathbb{Z}$ is an ideal but $4 = 2 \cdot 2$ with $2 \notin 4\mathbb{Z}$, so (4) is not prime. (A) Every ideal is also a subgroup, but ideals are more than subgroups. (B) Every ideal is a subring and an ideal.

Answer: (C) ← *Go back to Q15*

Q16.

Solution

Concept: Apply the Euclidean algorithm in $\mathbb{Q}[x]$.

Step 1: Divide $x^3 - 1$ by $x^2 - 1$:

$$x^3 - 1 = x \cdot (x^2 - 1) + (x - 1).$$

So $\gcd(x^3 - 1,\ x^2 - 1) = \gcd(x^2 - 1,\ x - 1)$.

Step 2: Divide $x^2 - 1$ by $x - 1$:

$$x^2 - 1 = (x+1)(x-1) + 0.$$

So $\gcd(x^2 - 1,\, x - 1) = x - 1$.

Conclusion: $\gcd(x^3 - 1,\, x^2 - 1) = x - 1$, which is option (D).

Why (C) wrong: $x^2 + x + 1$ is a factor of $x^3 - 1 = (x-1)(x^2+x+1)$ but does not divide $x^2 - 1 = (x-1)(x+1)$.

Answer: (D) ← *Go back to Q16*

Q17.

Solution

Concept: The degree of a simple extension $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$ equals the degree of the minimal polynomial of α.

Step 1: Let $\alpha = \sqrt{2}$. The minimal polynomial is $m(x) = x^2 - 2$ (irreducible over $\mathbb{Q}$ by Eisenstein at $p = 2$).

Step 2: $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = \deg(m) = 2$.

Step 3: A basis for $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$ is $\{1, \sqrt{2}\}$.

Conclusion: Degree $= 2$, option (B).

Why (A) wrong: Degree 1 would mean $\sqrt{2} \in \mathbb{Q}$, but $\sqrt{2}$ is irrational.

Answer: (B) ← *Go back to Q17*

Q18.

Solution

Concept: Tower law: $[K : F] = [K : L] \cdot [L : F]$.

Step 1: $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ (from Q17).

Step 2: Claim $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] = 2$. The minimal polynomial of $\sqrt{3}$ over $\mathbb{Q}(\sqrt{2})$ is $x^2 - 3$ (we need $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$: if $\sqrt{3} = a + b\sqrt{2}$, squaring gives $3 = a^2 + 2b^2 + 2ab\sqrt{2}$, forcing $ab = 0$; if $b = 0$, $\sqrt{3} = a \in \mathbb{Q}$, contradiction; if $a = 0$, $\sqrt{3} = b\sqrt{2}$, so $3/2 = b^2 \in \mathbb{Q}^2$, contradiction).

Step 3: By the tower law:

$$[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 2 \times 2 = 4.$$

Conclusion: Option (A).

A basis is $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ — four elements.

Answer: (A) ← *Go back to Q18*

Q19.

Solution

Concept: A field $\mathbb{F}_{p^n}$ (finite field with p^n elements) has characteristic p (a prime).

Step 1: $\mathbb{F}_4$ has $4 = 2^2$ elements, so $p = 2$ and $n = 2$.

Step 2: The characteristic is $p = 2$: in $\mathbb{F}_4$, the element 1 satisfies $1 + 1 = 0$ (two copies of 1 sum to zero).

Explicit: $\mathbb{F}_4 = \{0, 1, \alpha, \alpha + 1\}$ where $\alpha^2 + \alpha + 1 = 0$ over $\mathbb{F}_2$. Indeed $1 + 1 = 0$ shows characteristic 2.

Conclusion: Characteristic $= 2$, option (C).

Why (A) wrong: Characteristic 4 is impossible since the characteristic of a field must be 0 or a prime.

Answer: (C) ← *Go back to Q19*

Q20.

Solution

Concept: In $\mathbb{F}_p$, Fermat's little theorem states $a^p \equiv a \pmod{p}$ for all a, i.e., $a^p = a$ in $\mathbb{F}_p$.

Step 1: For the prime field $F = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, every element a satisfies $a^p = a$ (Fermat).

Step 2: More generally, the *Frobenius endomorphism* $\phi : a \mapsto a^p$ is a field automorphism of any field of characteristic p; it fixes $\mathbb{F}_p$ pointwise.

Conclusion: $a^p = a$ for all $a \in \mathbb{F}_p$ — option (D).

Why others fail: (A) $a^p = 1$ only holds for $a \in \mathbb{F}_p^\times$ by Fermat combined with $a^{p-1} = 1$ — but $a^p = a \cdot a^{p-1} = a \cdot 1 = a$, not 1. (B) $a^p = 0$ only for $a = 0$. (C) $p \cdot a = 0$ in characteristic p, not equal to a^p.

Answer: (D) ← *Go back to Q20*

Q21.

Solution

Concept: Orbit-Stabilizer Theorem: $|G| = |\operatorname{Orb}(x)| \cdot |\operatorname{Stab}(x)|$.

Step 1: $G = S_3$, $|G| = 6$. The orbit of 1 under the natural action on $\{1, 2, 3\}$ is

$\text{Orb}(1) = \{1, 2, 3\}$, so $|\text{Orb}(1)| = 3$.

Step 2: By the theorem: $|\text{Stab}(1)| = |G|/|\text{Orb}(1)| = 6/3 = 2$.

Verification: $\text{Stab}(1) = \{e, (23)\}$ — the identity and the transposition fixing 1. This has size 2. ✓

Conclusion: Option (B).

Why (A) wrong: $|\text{Stab}(1)| = 3$ would require $|\text{Orb}(1)| = 2$, but the orbit is all of $\{1, 2, 3\}$.

Answer: (B) ← *Go back to Q21*

Q22.

Solution

Concept: Burnside's lemma: #orbits $= \dfrac{1}{|G|}\sum_{g\in G} |X^g|$ where $X^g = \{x \in X : g{\cdot}x = x\}$.

Setup: X = all $2^2 = 4$ colorings of a 2-bead necklace (each bead BK or W): $\{BB, BW, WB, WW\}$. $G = \mathbb{Z}_2 = \{e, r\}$ acts by rotation (swapping the two beads).

Step 1: $|X^e| = 4$ (identity fixes all colorings).

Step 2: $|X^r|$ = colorings fixed by rotation (swapping beads 1 and 2): only BB and WW. So $|X^r| = 2$.

Step 3: #orbits $= \dfrac{1}{2}(4 + 2) = \dfrac{6}{2} = 3$.
The three distinct necklaces are: $\{BB\}$, $\{WW\}$, $\{BW, WB\}$.

Conclusion: Option (A).

Answer: (A) ← *Go back to Q22*

Q23.

Solution

Concept: Definition of a connected topological space.

Definition: A topological space X is **connected** if and only if the only subsets that are both open and closed (clopen) are $\emptyset$ and X itself; equivalently, X cannot be partitioned into two disjoint non-empty open sets.

Conclusion: This matches option (C): "it cannot be written as a disjoint union of two non-empty open sets."

Why (D) wrong: Path-connectedness implies connectedness, but not vice versa (e.g. the topologist's sine curve is connected but not path-connected). So (C) is the definition, not (D).

Why (A),(B) wrong: Having a metric or sequential convergence are not required for connectedness.

Answer: (C) ← *Go back to Q23*

Q24.

Solution

Concept: Path-connectedness and the fundamental group.

Path-connected: Given any two points $P, Q \in \mathbb{R}^2 \setminus \{0\}$, one can always find a path joining them that avoids the origin (detour around it if necessary). So $\mathbb{R}^2 \setminus \{0\}$ is **path-connected**.

Not simply connected: The fundamental group $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{Z}$ (generated by a loop winding once around the origin). Since $\pi_1 \neq \{e\}$, the space is not simply connected.

Conclusion: Option (D) — path-connected but not simply connected.

Why (A) wrong: $\mathbb{R}^2 \setminus \{0\}$ is connected (path-connected implies connected). (C) It is not compact (unbounded). (B) It is not simply connected.

Answer: (D) ← *Go back to Q24*

Q25.

Solution

Concept: The fundamental group $\pi_1(S^1)$ of the circle.

Intuition: Loops in S^1 based at a point are classified by their *winding number* — an integer counting how many times (and in which direction) the loop goes around the circle. Two loops are homotopic iff they have the same winding number.

Formal: The map $\mathbb{Z} \to \pi_1(S^1)$ sending $n \mapsto [\gamma_n]$ (loop winding n times) is a group isomorphism. Hence $\pi_1(S^1) \cong \mathbb{Z}$.

Conclusion: Option (B).

Why (A) wrong: The trivial group corresponds to simply connected spaces; S^1 is not simply connected. (C) $\mathbb{Z}_2$ has only 2 elements; $\pi_1(S^1)$ is infinite. (D) $\mathbb{R}$ is uncountable and not discrete.

Answer: (B) ← *Go back to Q25*

Q26.

Solution

Concept: Homotopy between continuous maps.

Definition: Two continuous maps $f, g : X \to Y$ are **homotopic** (written $f \simeq g$) if there exists a continuous map $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. The map H is called a *homotopy* from f to g.

Conclusion: This is exactly option (A).

Why (B) wrong: $f = g$ pointwise would be equality, not just homotopy. (C) Same image is a much weaker condition. (D) Differentiability is irrelevant — homotopy is a topological notion.

Answer: (A) ← *Go back to Q26*

Q27.

Solution

Concept: Simply connected spaces.

Definition: A topological space X is **simply connected** if:

(i) X is path-connected, and

(ii) the fundamental group $\pi_1(X)$ is trivial ($\pi_1(X) = \{e\}$).

Equivalently: every loop in X can be continuously contracted to a point within X.

Conclusion: Option (C) states exactly this: path-connected with trivial π_1.

Why (A) wrong: $\pi_1(X) \cong \mathbb{Z}$ means the space is *not* simply connected (like S^1 or $\mathbb{R}^2 \setminus \{0\}$). (B) Compact Hausdorff is a separation/compactness condition unrelated to simple connectedness. (D) Metrizability is irrelevant.

Answer: (C) ← *Go back to Q27*

Q28.

Solution

Concept: Winding number counts signed revolutions around a point.

Definition: For a closed curve $\gamma : [0,1] \to \mathbb{C} \setminus \{z_0\}$, the winding number around z_0 is

$$n(\gamma, z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z - z_0} \in \mathbb{Z}.$$

Step 1: A curve that winds counterclockwise around the origin twice has $n(\gamma, 0) = 2$.

Step 2: One counterclockwise revolution contributes $+1$; two revolutions contribute $+2$.

Conclusion: Option (D).

Why (C) wrong: -1 corresponds to one *clockwise* revolution. (B) 1 would be a single counterclockwise loop. (A) 0 means the curve does not enclose the origin.

Answer: (D) ← *Go back to Q28*

Q29.

Solution

Concept: Convexity in $\mathbb{R}^n$.

Definition: A set $C \subseteq \mathbb{R}^n$ is **convex** if for every $x, y \in C$ and every $t \in [0, 1]$, the point $tx + (1-t)y \in C$ (the line segment joining x and y lies entirely in C).

Conclusion: Option (B).

Why others fail: (A) Open sets need not be convex (e.g., two disjoint open balls). (C) Closed and bounded (compact) sets need not be convex (e.g., a closed annulus). (D) Many convex sets do not contain the origin (e.g., $[1, 2] \subset \mathbb{R}$).

Answer: (B) ← *Go back to Q29*

Q30.

Solution

Concept: Support function of the unit disk.

Step 1: For $K = \{(x, y) : x^2 + y^2 \leq 1\}$ and unit vector $\mathbf{u} = (u_1, u_2)$:

$$h_K(\mathbf{u}) = \max_{\mathbf{x} \in K} \mathbf{u} \cdot \mathbf{x} = \max_{x^2+y^2 \leq 1} (u_1 x + u_2 y).$$

Step 2: By the Cauchy-Schwarz inequality, $\mathbf{u} \cdot \mathbf{x} \leq |\mathbf{u}||\mathbf{x}| \leq 1 \cdot 1 = 1$ for all $\mathbf{x} \in K$.

Equality is achieved at $\mathbf{x} = \mathbf{u}$ (the point on the unit circle in direction $\mathbf{u}$).

Conclusion: $h_K(\mathbf{u}) = 1$ for every unit vector $\mathbf{u}$ — option (A).

Why (D) wrong: 2 would be the diameter of the disk, not the support function value.

Answer: (A) ← *Go back to Q30*

Section B Solutions — Q31–Q40 (MSQ, 2 marks each)

Q31.

Solution

Check (A) — Monotone implies BV: If f is increasing on $[a, b]$, then for any partition $\sum |f(x_i) - f(x_{i-1})| = \sum(f(x_i) - f(x_{i-1})) = f(b) - f(a) < \infty$. So TV$= f(b) - f(a)$. **TRUE.**

Check (B) — Lipschitz implies BV: If $|f(x) - f(y)| \leq L|x - y|$ for all x, y, then for any partition: $\sum |f(x_i) - f(x_{i-1})| \leq L \sum(x_i - x_{i-1}) = L(b - a) < \infty$. **TRUE.**

Check (C) — Continuous does NOT imply BV: Counterexample: $f(x) = x \sin(\pi/x)$ on $[0, 1]$ with $f(0) = 0$ is continuous but has infinite total variation (oscillates with increasing frequency). **FALSE.**

Check (D) — Unbounded derivative means NOT BV: If f' is unbounded, the TV need not be finite. **FALSE.**

Conclusion: Options **(A)** and **(B)**.

Answer: (AB) ← *Go back to Q31*

Q32.

Solution

Step 1: Compute B^2: $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so $B^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$. Thus $B^3 = -B$, $B^4 = I$, etc.

Step 2: Expand e^{tB}:

$$e^{tB} = \sum_{k=0}^{\infty} \frac{t^k B^k}{k!} = I\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots\right) + B\left(t - \frac{t^3}{3!} + \cdots\right) = I \cos t + B \sin t.$$

So $e^{tB} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$. This is a rotation matrix. **(A) TRUE, (D) TRUE.**

(B) FALSE: The series does not terminate — $B^2 = -I \neq \mathbf{0}$, so infinitely many terms are nonzero.

(C) TRUE: $\det(e^{tB}) = \det(e^A) = e^{\operatorname{tr}(A)}$ (Jacobi formula). $\operatorname{tr}(tB) = 0$, so $\det(e^{tB}) = e^0 = 1$. Alternatively, $\cos^2 t + \sin^2 t = 1$ directly.

Conclusion: Options **(A)**, **(C)**, **(D)**.

Answer: (ACD) ← *Go back to Q32*

Q33.

Solution

(A) FALSE: K is a vector space over F of dimension n, so $|K| = |F|^n$ only if F is finite. For $F = \mathbb{Q}$ and $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$, the extension $\mathbb{Q}(\sqrt{2})$ is infinite.

(B) TRUE: By definition, $[K : F] = n$ means K is an n-dimensional vector space over F.

(C) TRUE: For any $\alpha \in K$, the set $\{1, \alpha, \alpha^2, \ldots, \alpha^n\}$ (which has $n+1$ elements in an n-dimensional space) must be linearly dependent over F, so α satisfies a polynomial of degree $\leq n$ over F (its minimal polynomial has degree $\leq n$).

(D) FALSE: F can be any field — e.g., $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ with $F = \mathbb{Q}$ infinite.

Conclusion: Options **(B)** and **(C)**.

Answer: (BC) ← *Go back to Q33*

Q34.

Solution

(A) TRUE: If $p(x) \in \mathbb{Q}[x]$ is irreducible of degree $n \geq 2$ and had a rational root r, then $(x - r)$ would be a factor over $\mathbb{Q}$, contradicting irreducibility. So $p(x)$ has no rational roots.

(B) FALSE: By the Fundamental Theorem of Algebra, $p(x)$ has exactly n complex roots (counting multiplicity). Irreducible over $\mathbb{Q}$ does not mean no complex roots.

(C) FALSE: An irreducible polynomial of degree $n \geq 2$ is not linear.

(D) TRUE: Since $p(x)$ is irreducible in $\mathbb{Q}[x]$, the ideal $(p(x))$ is maximal in $\mathbb{Q}[x]$ (as $\mathbb{Q}[x]$ is a PID). Hence $\mathbb{Q}[x]/(p(x))$ is a field. As a $\mathbb{Q}$-vector space it has basis $\{1, \bar{x}, \ldots, \bar{x}^{n-1}\}$, so degree $= n$.

Conclusion: Options **(A)** and **(D)**.

Answer: (AD) ← *Go back to Q34*

Q35.

Solution

(A) TRUE: This is the Orbit-Stabilizer Theorem: $|G| = |\operatorname{Orb}(x)| \cdot |\operatorname{Stab}(x)|$, so $|\operatorname{Orb}(x)| = |G|/|\operatorname{Stab}(x)|$.

(B) TRUE: $\operatorname{Stab}(x) = \{g \in G : g \cdot x = x\}$. Closure: if $g, h \in \operatorname{Stab}(x)$, then $(gh) \cdot x = g \cdot (h \cdot x) = g \cdot x = x$. Inverses: $(g^{-1}) \cdot x = g^{-1} \cdot (g \cdot x) = x$. So $\operatorname{Stab}(x) \leq G$.

(C) FALSE: $\operatorname{Orb}(x) = \{g \cdot x : g \in G\} \subseteq X$. This is a subset of X (the set being acted upon), not of G. It is generally not a subgroup of G.

(D) TRUE: Transitive action means there is only one orbit, and that orbit equals all of X. So $|\operatorname{Orb}(x)| = |X|$ for every x.

Conclusion: Options **(A)**, **(B)**, **(D)**.

Answer: (ABD) ← *Go back to Q35*

Q36.

Solution

(A) TRUE: Path-connectedness implies connectedness. Proof: if $X = U \sqcup V$ with U, V open and non-empty, any path joining $u \in U$ to $v \in V$ would give a continuous map from $[0,1]$ (connected) whose image meets both U and V — contradiction.

(B) FALSE: Path-connected does not imply simply connected. Example: $\mathbb{R}^2 \setminus \{0\}$ is path-connected but has $\pi_1 \cong \mathbb{Z} \neq \{e\}$.

(C) TRUE: This is the *definition* of path-connectedness: for any $x, y \in X$, there exists a continuous path $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

(D) FALSE: The fundamental group of a path-connected space can be non-trivial (e.g., $\pi_1(S^1) \cong \mathbb{Z}$, $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{Z}$).

Conclusion: Options **(A)** and **(C)**.

Answer: (AC) ← *Go back to Q36*

Q37.

Solution

(A) FALSE: The curve $\mathbf{r}(t) = (\cos t, \sin t, t)$ has third coordinate $z = t \neq 0$ for $t \neq 0$, so it does not lie in the xy-plane.

(B) TRUE: $\mathbf{r}'(t) = (-\sin t, \cos t, 1)$, so $|\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$. Constant speed. ✓

(C) TRUE: Arc length $= \displaystyle\int_0^{2\pi} |\mathbf{r}'(t)|\, dt = \int_0^{2\pi} \sqrt{2}\, dt = 2\pi\sqrt{2}$. ✓

(D) TRUE: Projecting onto the xy-plane, $(\cos t, \sin t, 0)$ traces the unit circle $x^2+y^2 = 1$. ✓

Conclusion: Options **(B)**, **(C)**, **(D)**.

Answer: (BCD) ← *Go back to Q37*

Q38.

Solution

(A) TRUE: The map $\mathbb{Q}[x]/(x^2-2) \to \mathbb{Q}(\sqrt{2})$ sending $\bar{x} \mapsto \sqrt{2}$ is a well-defined ring isomorphism (surjective, injective, ring homomorphism). ✓

(B) TRUE: x^2-2 is irreducible over $\mathbb{Q}$ (no rational roots: $\pm\sqrt{2} \notin \mathbb{Q}$). Since $\mathbb{Q}[x]$ is a PID and x^2-2 is irreducible, (x^2-2) is a maximal ideal, so the quotient is a field. ✓

(C) FALSE: $\mathbb{Q}[x]/(x^2-2) \cong \mathbb{Q}(\sqrt{2})$ has characteristic 0 (inheriting from $\mathbb{Q}$), not 2.

(D) FALSE: $\mathbb{Q}[x]/(x^2-2)$ is a field (hence an integral domain), while $\mathbb{Q}\times\mathbb{Q}$ has zero divisors ($(1,0)\cdot(0,1)=(0,0)$). A field cannot be isomorphic to $\mathbb{Q}\times\mathbb{Q}$.

Conclusion: Options **(A)** and **(B)**.

Answer: (AB) ← *Go back to Q38*

Q39.

Solution

(A) TRUE: Every continuous function on $[a,b]$ is Riemann integrable (continuous $\Rightarrow$ bounded and the set of discontinuities has measure zero — indeed measure zero here). ✓

(B) FALSE: A continuous function need not have bounded variation. Counterexample: $f:[0,1]\to\mathbb{R}$, $f(x)=x\sin(\pi/x)$ for $x\in(0,1]$ and $f(0)=0$. This is continuous on $[0,1]$ but has infinite total variation (the oscillations have amplitudes $1/n$ at $x=1/n$, and TV $\approx \sum 2\cdot\frac{1}{n}=\infty$).

(C) TRUE: Extreme Value Theorem (EVT): a continuous function on a closed bounded interval $[a,b]$ attains its maximum and minimum. ✓

(D) TRUE: Heine-Cantor Theorem: a continuous function on a compact set (closed bounded interval) is uniformly continuous. ✓

Conclusion: Options **(A)**, **(C)**, **(D)**.

Answer: (ACD) ← *Go back to Q39*

Q40.

Solution

(A) FALSE: $f(x)\to 0$ is necessary but not sufficient. Counterexample: $\int_1^\infty \sin(x^2)\,dx$ converges (by Dirichlet's test) even though $\sin(x^2)\not\to 0$... Actually a better counterexample for insufficiency: $f(x)=\frac{1}{x}$, $f(x)\to 0$ but $\int_1^\infty \frac{dx}{x}$ diverges.

(B) TRUE: This is the Comparison Test for improper integrals: if $0\le|f(x)|\le g(x)$ and $\int_0^\infty g\,dx<\infty$, then $\int_0^\infty |f|\,dx$ converges by the monotone convergence /

comparison principle. ✓

(C) FALSE: Convergence of $\int_0^\infty f\,dx$ does not require $f(x) \to 0$. Counterexample: define f with unit-area spikes of width $1/n^2$ at each integer — the integral converges but f oscillates between 0 and n^2.

(D) TRUE: $\int_0^\infty e^{-x}\,dx = \big[-e^{-x}\big]_0^\infty = 0 - (-1) = 1 < \infty$. ✓

Conclusion: Options **(B)** and **(D)**.

Answer: (BD) ← *Go back to Q40*

Section C Solutions — Q41–Q50 (1 mark NAT each)

Q41.

Solution

Concept: $\displaystyle\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $|x| < 1$.

Step 1: Set $x = \frac{1}{2}$ (which satisfies $|x| < 1$):

$$\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{1/2}{(1-1/2)^2} = \frac{1/2}{1/4} = 2.$$

Conclusion: $\boxed{2}$.

Answer: (2) ← *Go back to Q41*

Q42.

Solution

Step 1: $y' = \sqrt{x}$, so $L = \displaystyle\int_0^3 \sqrt{1+x}\,dx$.

Step 2: Let $u = 1 + x$, $du = dx$:

$$L = \int_1^4 \sqrt{u}\,du = \left[\frac{2u^{3/2}}{3}\right]_1^4 = \frac{2}{3}(8-1) = \frac{14}{3}.$$

Conclusion: $\boxed{14/3}$.

Answer: (14/3) ← *Go back to Q42*

Q43.

Solution

Concept: Count 2×2 invertible matrices over $\mathbb{F}_2$.

Step 1: Choose the first row $\mathbf{r}_1$: any nonzero vector in $\mathbb{F}_2^2$. There are $2^2 - 1 = 3$ choices.

Step 2: Choose the second row $\mathbf{r}_2$: must be linearly independent of $\mathbf{r}_1$, i.e., $\mathbf{r}_2 \neq \mathbf{0}$ and $\mathbf{r}_2 \neq \mathbf{r}_1$. That leaves $4 - 2 = 2$ choices.

Step 3: $|GL(2, \mathbb{F}_2)| = 3 \times 2 = 6$.

Conclusion: $\boxed{6}$.

The six matrices are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Answer: (6) ← *Go back to Q43*

Q44.

Solution

Step 1: From Q13, the speed is $|\mathbf{r}'(t)| = \sqrt{2}$ (constant).

Step 2: Arc length $= \displaystyle\int_0^{2\pi} |\mathbf{r}'(t)|\, dt = \int_0^{2\pi} \sqrt{2}\, dt = 2\pi\sqrt{2}$.

Conclusion: $\boxed{2\pi\sqrt{2}}$.

Answer: ($2\pi\sqrt{2}$) ← *Go back to Q44*

Q45.

Solution

Concept: $\operatorname{tr}(A^2) = \lambda_1^2 + \lambda_2^2$ (sum of squares of eigenvalues).

Step 1: Eigenvalues: $\lambda_1 = -1$, $\lambda_2 = 3$.

Step 2: $\operatorname{tr}(A^2) = (-1)^2 + 3^2 = 1 + 9 = 10$.

Conclusion: $\boxed{10}$.

Answer: (10) ← *Go back to Q45*

Q46.

Solution

Step 1: List all monic degree-2 polynomials over $\mathbb{F}_2$: $x^2,\ x^2+1,\ x^2+x,\ x^2+x+1$.

Step 2: Check for roots in $\mathbb{F}_2 = \{0, 1\}$ (a degree-2 poly is irreducible iff it has no root):

- x^2: root 0. Reducible.

- $x^2 + 1 = (x+1)^2$: root 1. Reducible.
- $x^2 + x = x(x+1)$: roots 0 and 1. Reducible.
- $x^2 + x + 1$: $f(0) = 1 \neq 0$, $f(1) = 1 + 1 + 1 = 3 \equiv 1 \neq 0$ in $\mathbb{F}_2$. No roots. **Irreducible.**

Conclusion: Exactly $\boxed{1}$ irreducible monic degree-2 polynomial over $\mathbb{F}_2$.

Answer: (1) ← *Go back to Q46*

Q47.

Solution

Step 1: Write $\gamma(t) = 2e^{3it}$ for $t \in [0, 2\pi]$. This is a circle of radius 2 traversed with angular speed 3.

Step 2: The total angle swept is $3 \times 2\pi = 6\pi$, corresponding to 3 full counterclockwise revolutions.

Step 3: Winding number $= 3$ (via formula $n(\gamma, 0) = \frac{1}{2\pi i} \oint \frac{dz}{z}$; substituting $z = 2e^{3it}$, $dz = 6ie^{3it}dt$ gives $n = \frac{1}{2\pi i} \int_0^{2\pi} \frac{6ie^{3it}}{2e^{3it}} dt = \frac{6i}{2 \cdot 2\pi i} \cdot 2\pi = 3$).

Conclusion: $\boxed{3}$.

Answer: (3) ← *Go back to Q47*

Q48.

Solution

Concept: For an upper triangular matrix, the eigenvalues are the diagonal entries.

Step 1: $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ is upper triangular with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$.

Step 2: The eigenvalues of e^A are $e^{\lambda_1} = e^1 = e$ and $e^{\lambda_2} = e^3$.

Step 3: $\mathrm{tr}(e^A) = e + e^3$.

Conclusion: $\boxed{e + e^3}$.

(This also follows from $e^{\mathrm{tr}(A)}$ giving e^4 for the determinant, but the trace is the sum of eigenvalues of e^A, which is $e + e^3$.)

Answer: ($e + e^3$) ← *Go back to Q48*

Q49.

Solution

Step 1: Let $\alpha = \sqrt[3]{2}$. The minimal polynomial of α over $\mathbb{Q}$ is $m(x) = x^3 - 2$.

Step 2: $m(x) = x^3 - 2$ is irreducible over $\mathbb{Q}$ by Eisenstein's criterion at $p = 2$: $2 \mid 2$ (constant), $2 \nmid 1$ (leading), $4 \nmid 2$.

Step 3: $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = \deg(m) = 3$.

Conclusion: $\boxed{3}$.

Answer: (3) ← *Go back to Q49*

Q50.

Solution

Step 1: On $[0, \pi/2]$: $\sin x$ increases from 0 to 1. Variation $= 1$.

Step 2: On $[\pi/2, 3\pi/2]$: $\sin x$ decreases from 1 to -1. Variation $= 2$.

Step 3: On $[3\pi/2, 2\pi]$: $\sin x$ increases from -1 to 0. Variation $= 1$.

Total: $V(\sin; 0, 2\pi) = 1 + 2 + 1 = 4$.

Conclusion: $\boxed{4}$.

Answer: (4) ← *Go back to Q50*

Section C Solutions — Q51–Q60 (2 marks NAT each)

Q51.

Solution

Concept: Gamma function: $\Gamma(n) = (n-1)!$ and $\displaystyle\int_0^\infty x^{n-1}e^{-x}\,dx = \Gamma(n)$.

Step 1: $\displaystyle\int_0^\infty x^3 e^{-x}\,dx = \Gamma(4) = 3! = 6$.

Verification by parts: $\int_0^\infty x^3e^{-x}dx = 3\int_0^\infty x^2e^{-x}dx = 3 \cdot 2\int_0^\infty xe^{-x}dx = 6\int_0^\infty e^{-x}dx = 6 \cdot 1 = 6$. ✓

Conclusion: $\boxed{6}$.

Answer: (6) ← *Go back to Q51*

Q52.

Solution

Concept: Disk method: $V = \pi \int_a^b [f(x)]^2\,dx$.

Step 1: $f(x) = \sqrt{x}$, so $[f(x)]^2 = x$.

Step 2:

$$V = \pi \int_0^4 x\,dx = \pi \left[\frac{x^2}{2}\right]_0^4 = \pi \cdot \frac{16}{2} = 8\pi.$$

Conclusion: $\boxed{8\pi}$.

Answer: (8π) ← *Go back to Q52*

Q53.

Solution

Step 1: $N(a + b\sqrt{2}) = a^2 - 2b^2$. For $a = 3$, $b = 1$:

$$N(3 + \sqrt{2}) = 3^2 - 2 \cdot 1^2 = 9 - 2 = 7.$$

Remark: This norm is the norm from the number field $\mathbb{Q}(\sqrt{2})$: $N(3 + \sqrt{2}) = (3 + \sqrt{2})(3 - \sqrt{2}) = 9 - 2 = 7$.

Conclusion: $\boxed{7}$.

Answer: (7) ← *Go back to Q53*

Q54.

Solution

Step 1: $\mathbb{F}_{16}^{\times}$ is a cyclic group of order $|\mathbb{F}_{16}^{\times}| = 16 - 1 = 15$.

Step 2: The number of elements of order d in a cyclic group of order n is $\phi(d)$ (Euler's totient), where $d \mid n$.

Step 3: Since $5 \mid 15$, the number of elements of order 5 in $\mathbb{F}_{16}^{\times}$ is $\phi(5) = 5 - 1 = 4$.

Conclusion: $\boxed{4}$.

Answer: (4) ← *Go back to Q54*

Q55.

Solution

Step 1: $f(x, y, z) = x^2yz^3$. Differentiate with respect to x:

$$\frac{\partial f}{\partial x} = 2xyz^3.$$

Step 2: Differentiate with respect to y:

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial y}(2xyz^3) = 2xz^3.$$

Step 3: Differentiate with respect to z (second time):

$$\frac{\partial^3 f}{\partial x\,\partial y\,\partial z^2} = \frac{\partial^2}{\partial z^2}(2xz^3) = \frac{\partial}{\partial z}(6xz^2) = 12xz.$$

Step 4: Evaluate at $(1, 1, 1)$: $12 \cdot 1 \cdot 1 = 12$.

Conclusion: $\boxed{12}$.

Answer: (12) ← *Go back to Q55*

Q56.

Solution

Concept: $\displaystyle\sum_{n=0}^{\infty}(n+1)x^n = \frac{1}{(1-x)^2}$ for $|x| < 1$.

Derivation: Differentiate $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ term by term: $\displaystyle\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$, i.e., $\displaystyle\sum_{n=0}^{\infty}(n+1)x^n = \frac{1}{(1-x)^2}$.

Step 1: Substitute $x = \frac{1}{3}$:

$$\sum_{n=0}^{\infty}(n+1)\left(\frac{1}{3}\right)^n = \frac{1}{(1-1/3)^2} = \frac{1}{(2/3)^2} = \frac{1}{4/9} = \frac{9}{4}.$$

Conclusion: $\boxed{9/4}$.

Answer: (9/4) ← *Go back to Q56*

Q57.

Solution

Concept: Parametrize C: $z = e^{it}$, $\bar{z} = e^{-it}$, $dz = ie^{it}dt$, for $t \in [0, 2\pi]$.

Step 1:

$$\oint_C \bar{z}\,dz = \int_0^{2\pi} e^{-it} \cdot ie^{it}\,dt = \int_0^{2\pi} i\,dt = 2\pi i.$$

Step 2: The modulus: $\left|\oint_C \bar{z}\,dz\right| = |2\pi i| = 2\pi$.

Remark: This integral is non-zero despite $\bar{z}$ not being holomorphic (it confirms $\bar{z}$ is not analytic).

Conclusion: $\boxed{2\pi}$.

Answer: (2π) ← *Go back to Q57*

Q58.

Solution

Step 1: Write $\sin^3 x = \sin x(1 - \cos^2 x)$.

Step 2: Substitute $u = \cos x$, $du = -\sin x\,dx$. When $x = 0$, $u = 1$; when $x = \pi$, $u = -1$:

$$\int_0^{\pi} \sin^3 x\,dx = \int_1^{-1} (1 - u^2)(-du) = \int_{-1}^{1} (1 - u^2)\,du.$$

Step 3:

$$\int_{-1}^{1} (1 - u^2)\,du = \left[u - \frac{u^3}{3}\right]_{-1}^{1} = \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}.$$

Conclusion: $\boxed{4/3}$.

Answer: (4/3) ← *Go back to Q58*

Q59.

Solution

Step 1: Compute $A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$. So $A^{2k} = I$ and $A^{2k+1} = A$ for all $k \geq 0$.

Step 2: Expand e^A:

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I\left(1 + \frac{1}{2!} + \frac{1}{4!} + \cdots\right) + A\left(1 + \frac{1}{3!} + \frac{1}{5!} + \cdots\right) = I\cosh(1) + A\sinh(1).$$

Step 3: $e^A = \begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix}$.

Step 4: $\text{tr}(e^A) = 2\cosh(1) = e + e^{-1}$.

Alternatively: Eigenvalues of A are ± 1 (since $\det(A - \lambda I) = \lambda^2 - 1 = 0$), so eigenvalues of e^A are $e^1 = e$ and e^{-1}. Thus $\text{tr}(e^A) = e + e^{-1}$.

Conclusion: $\boxed{e + e^{-1}}$.

Answer: ($e + e^{-1}$) ← *Go back to Q59*

Q60.

Solution

Step 1: A_4 has order 12. The elements of A_4 are: the identity e; the three double transpositions $(12)(34)$, $(13)(24)$, $(14)(23)$; and the eight 3-cycles $(123), (132), (124), (142), (134), (143), (234), (243)$.

Step 2: Conjugacy classes in A_4:

- $\mathcal{C}_1 = \{e\}$, size 1.
- $\mathcal{C}_2 = \{(12)(34), (13)(24), (14)(23)\}$, size 3 (all double transpositions are conjugate in A_4).
- $\mathcal{C}_3 = \{(123), (134), (142), (243)\}$, size 4 (one class of 3-cycles in A_4).
- $\mathcal{C}_4 = \{(132), (143), (124), (234)\}$, size 4 (the other class — note 3-cycles split into two classes in A_4 unlike in S_4).

Step 3: Total: $1 + 3 + 4 + 4 = 12 = |A_4|$. ✓There are **4** conjugacy classes.

Conclusion: $\boxed{4}$.

Answer: (4) ← *Go back to Q60*

Answer Key

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	B	2	A	3	D	4	C	5	B
6	A	7	C	8	D	9	B	10	A
11	C	12	D	13	B	14	A	15	C
16	D	17	B	18	A	19	C	20	D
21	B	22	A	23	C	24	D	25	B
26	A	27	C	28	D	29	B	30	A
31	AB	32	ACD	33	BC	34	AD	35	ABD
36	AC	37	BCD	38	AB	39	ACD	40	BD
41	2	42	14/3	43	6	44	$2\pi\sqrt{2}$	45	10
46	1	47	3	48	$e+e^3$	49	3	50	4
51	6	52	8π	53	7	54	4	55	12
56	9/4	57	2π	58	4/3	59	$e+e^{-1}$	60	4