IIT JAM Mathematics

Sample Paper – 8

Instructions

- This paper contains **60 questions** carrying a maximum of **100 marks**. Duration: **3 hours (180 minutes)**.
- **Section A** (**Q1–Q30**, MCQ): Q1–Q10 carry **+1 mark** each; wrong response: $-\frac{1}{3}$ mark. Q11–Q30 carry **+2 marks** each; wrong response: $-\frac{2}{3}$ mark. Choose the **single best** option.
- **Section B** (**Q31–Q40**, MSQ): Each carries **+2 marks**. **No negative marking. One or more** options may be correct; full marks only if *all* correct options are selected.
- **Section C** (**Q41–Q60**, NAT): Q41–Q50 carry **+1 mark**; Q51–Q60 carry **+2 marks**. **No negative marking.** Enter the numerical value using the virtual keypad.
- **Calculator policy:** Personal calculators and electronic devices are **strictly prohibited**. A virtual scientific calculator is provided in the CBT interface.
- Sections must be attempted in order; you cannot return to a previous section.

Section A — Multiple Choice Questions (MCQ)

***Q1–Q10**: 1 mark each ($-\frac{1}{3}$ for wrong). Choose the **single** correct option.*

Q1. Let $f : [a,b] \to \mathbb{R}$ be continuously differentiable on $[a,b]$. By the Fundamental Theorem of Calculus, which of the following is TRUE?

(A) $f(b) = f(a) + \displaystyle\int_a^b f(x)\,dx$

(B) $f'(b) = f(a) - f(b)$

(C) $f(b) - f(a) = \displaystyle\int_a^b f'(x)\,dx$

(D) $\displaystyle\int_a^b f'(x)\,dx = 0$ for all f

Q2. Darboux's theorem states: if f is differentiable on (a,b) and μ is any value strictly between $f'(a)$ and $f'(b)$, then there exists $c \in (a,b)$ such that:

(A) $f(c) = \mu$

(B) $f''(c) = \mu$

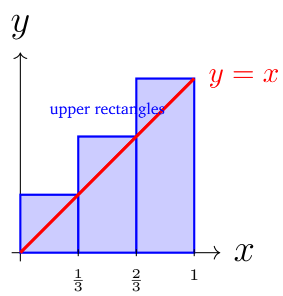
(C) $f(c) = 0$

(D) $f'(c) = \mu$

Q3. (Lebesgue's criterion) A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if:

(A) f is continuous almost everywhere (the set of discontinuities has Lebesgue measure zero)
(B) f is differentiable on (a, b)
(C) f is monotone on $[a, b]$
(D) f is continuous everywhere on $[a, b]$

Q4. Consider $f(x) = x$ on $[0, 1]$ with partition $P = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. The upper Darboux sum $U(f, P)$ equals:



(A) $\frac{1}{3}$
(B) $\frac{2}{3}$
(C) $\frac{1}{2}$
(D) $\frac{5}{9}$

Q5. The Euler-Cauchy equation $x^2y'' + axy' + by = 0$ is solved by the trial substitution:

(A) $y = e^{mx}$
(B) $y = mx^2$
(C) $y = x^m$
(D) $y = \ln x$

Q6. For the Euler-Cauchy equation $x^2y'' - 3xy' + 4y = 0$, the characteristic equation is:

(A) $m^2 - 3m + 4 = 0$
(B) $m^2 + 3m + 4 = 0$

(C) $m^2 - 2m + 4 = 0$
(D) $m(m-1) - 3m + 4 = 0$

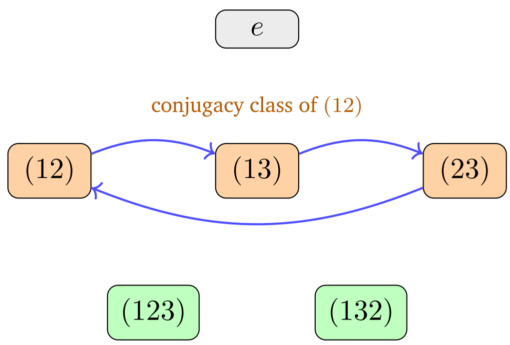
Q7. The Euler-Cauchy ODE whose characteristic equation has a **repeated** root $m = 2$ has general solution:

(A) $(C_1 + C_2 \ln x)\, x^2$
(B) $C_1 x^2 + C_2 x^{-2}$
(C) $C_1 \cos(2\ln x) + C_2 \sin(2\ln x)$
(D) $C_1 x + C_2 x^4$

Q8. Using the Wallis reduction formula $I_n = \dfrac{n-1}{n} I_{n-2}$ with $I_0 = \dfrac{\pi}{2}$, the value of $\displaystyle\int_0^{\pi/2} \sin^4 x\, dx$ is:

(A) $\dfrac{\pi}{16}$
(B) $\dfrac{3\pi}{16}$
(C) $\dfrac{\pi}{8}$
(D) $\dfrac{3\pi}{8}$

Q9. The symmetric group S_3 acts on itself by conjugation: $g \cdot x = gxg^{-1}$. The orbit of the transposition (12) under this conjugation action is:



(A) $\{(12)\}$ alone
(B) $\{e, (12)\}$
(C) $\{(12), (13), (23)\}$
(D) All of S_3

Q10. Let $\mathbb{Z}_2 = \{0, 1\}$ act on $\mathbb{R}$ by $0 \cdot x = x$ and $1 \cdot x = -x$. Which statement about this action is TRUE?

(A) The action is free (element 1 has no fixed points on $\mathbb{R}$)
(B) Element 1 fixes every point of $\mathbb{R}$
(C) Element 1 fixes every nonzero point of $\mathbb{R}$
(D) $x = 0$ is the only fixed point of element 1

***Q11–Q30**: 2 marks each ($-\frac{2}{3}$ for wrong). Choose the **single** correct option.*

Q11. The Lebesgue outer measure of any countable set $S = \{r_1, r_2, r_3, \ldots\} \subset \mathbb{R}$ is:

(A) 0
(B) 1
(C) Countably infinite
(D) Uncountably infinite

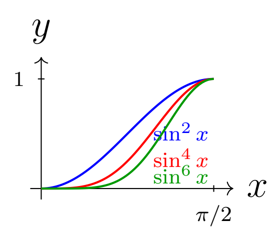
Q12. The standard Cantor set $\mathcal{C}$ (obtained by repeatedly removing the open middle third from $[0, 1]$) has:

(A) Positive Lebesgue measure
(B) Lebesgue measure 0
(C) Only finitely many elements
(D) Nonempty interior

Q13. A **Borel set** in $\mathbb{R}$ is:

(A) Any subset of $\mathbb{R}$
(B) Only an open subset of $\mathbb{R}$
(C) Any element of the σ-algebra generated by the open sets in $\mathbb{R}$
(D) Only a closed subset of $\mathbb{R}$

Q14. For $I_n = \displaystyle\int_0^{\pi/2} \sin^n x \, dx$, the reduction formula gives $I_n = \dfrac{n-1}{n} I_{n-2}$ and $I_1 = 1$. The value of I_5 is:



(A) $\frac{3}{8}$
(B) $\frac{5}{8}$
(C) $\frac{15}{8}$
(D) $\frac{8}{15}$

Q15. Using $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and the recurrence $\Gamma(z+1) = z\,\Gamma(z)$, the value of $\Gamma\left(\frac{5}{2}\right)$ is:

(A) $\frac{3\sqrt{\pi}}{4}$
(B) $\frac{\sqrt{\pi}}{4}$
(C) $\sqrt{\pi}$
(D) $\frac{3\sqrt{\pi}}{2}$

Q16. The space ℓ^2 of square-summable sequences consists of all sequences $\{a_n\}_{n=1}^{\infty}$ of real (or complex) numbers satisfying:

(A) $\sum_{n=1}^{\infty} |a_n| < \infty$
(B) $\sum_{n=1}^{\infty} |a_n|^2 < \infty$
(C) $a_n \to 0$ as $n \to \infty$
(D) $\{a_n\}$ is a bounded sequence

Q17. The space ℓ^2 equipped with the inner product $\langle a, b\rangle = \sum_{n=1}^{\infty} a_n \overline{b_n}$ is a:

(A) Banach space but not a Hilbert space
(B) Metric space but not a normed space
(C) Hilbert space (complete inner product space)
(D) Incomplete metric space

Q18. Let $e_n \in \ell^2$ denote the sequence with 1 in position n and 0 elsewhere. The set $\{e_n\}_{n=1}^{\infty}$ forms:

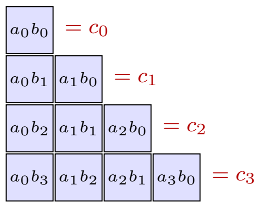
(A) An algebraic (Hamel) basis for ℓ^2
(B) A complete orthogonal but not orthonormal system

(C) A finite-dimensional basis
(D) A complete orthonormal system (Schauder basis) for ℓ^2

Q19. For $f \in L^2(\mathbb{R})$ with Fourier transform $\hat{f}$, Plancherel's theorem states:

(A) $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$ (the L^2 norm is preserved)
(B) $\|\hat{f}\|_{L^1} = \|f\|_{L^2}$
(C) $\hat{f}$ is always in $L^1(\mathbb{R})$
(D) $\hat{f}(\xi) = f(-\xi)$ for all ξ

Q20. Let $\displaystyle\sum_{n=0}^{\infty} a_n$ converge **absolutely** to A and $\displaystyle\sum_{n=0}^{\infty} b_n$ converge to B. The Cauchy product $\displaystyle\sum_{n=0}^{\infty} c_n$ (with $c_n = \sum_{k=0}^{n} a_k b_{n-k}$) converges to (Mertens' theorem):



(A) $A + B$
(B) $A \cdot B$
(C) A/B
(D) $(A + B)^2$

Q21. For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the ℓ^∞ (sup) norm $\|\mathbf{x}\|_\infty$ is defined as:

(A) $\displaystyle\left(\sum_{i=1}^{n} |x_i|^\infty\right)^{1/\infty}$
(B) $\displaystyle\sum_{i=1}^{n} |x_i|$
(C) $\max\{|x_1|, |x_2|, \ldots, |x_n|\}$
(D) $\min\{|x_1|, |x_2|, \ldots, |x_n|\}$

Q22. In a finite-dimensional normed space, any two norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are:

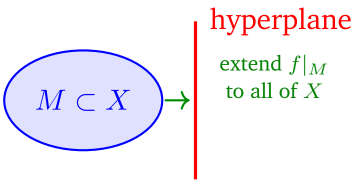
(A) Equal: $\|\mathbf{x}\|_\alpha = \|\mathbf{x}\|_\beta$ for all $\mathbf{x}$
(B) Equal on the unit ball

(C) Identical pointwise
(D) Equivalent: $\exists\, c, C > 0$ with $c\|\mathbf{x}\|_\alpha \le \|\mathbf{x}\|_\beta \le C\|\mathbf{x}\|_\alpha$ for all $\mathbf{x}$

Q23. The dual space $(\mathbb{R}^n)^*$ of all bounded linear functionals $f : \mathbb{R}^n \to \mathbb{R}$ is isomorphic to:

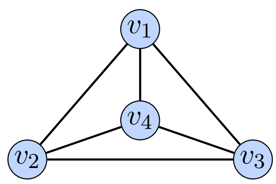
(A) $\mathbb{R}^n$ itself
(B) $\mathbb{R}^{n^2}$
(C) $\mathbb{R}^{n-1}$
(D) The zero space $\{0\}$

Q24. The Hahn-Banach extension theorem guarantees:



(A) Every linear functional can be extended without any norm constraint
(B) Every bounded linear functional f on a subspace M of a normed space X can be extended to all of X *with the same norm*
(C) All normed spaces are complete (Banach)
(D) The dual of any Banach space is finite-dimensional

Q25. For a connected planar graph G with V vertices, E edges, and F faces (including the unbounded outer face), Euler's formula states $V - E + F = ?$



$V = 4,\ E = 6,\ F = 4 \ \Rightarrow\ 4 - 6 + 4 = ?$

(A) 0
(B) $2V$
(C) 2
(D) $E - V$

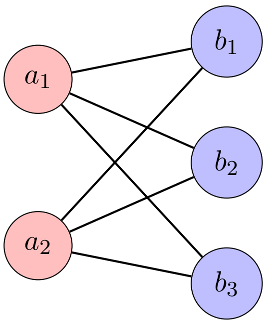
Q26. By the Four Colour Theorem, every planar graph can be properly vertex-coloured using at most:

(A) 2 colours
(B) 3 colours
(C) 6 colours
(D) 4 colours

Q27. The chromatic polynomial $P(K_n, k)$ (counting the number of proper vertex k-colourings) of the complete graph K_n is:

(A) $k(k-1)(k-2)\cdots(k-n+1)$
(B) k^n
(C) $n!$
(D) k^{n-1}

Q28. A graph G is **bipartite** if and only if:



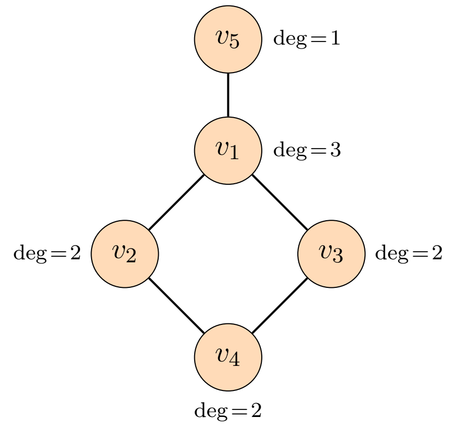
$K_{2,3}$: only even cycles

(A) G has no triangles (K_3-free)
(B) G contains no odd cycles
(C) G is a planar graph
(D) G is a connected graph

Q29. A spanning tree of a connected graph G with n vertices has exactly:

(A) n edges
(B) $n+1$ edges
(C) $n-1$ edges
(D) $2n-1$ edges

Q30. By the Handshaking Lemma, the sum of all vertex degrees in any graph $G = (V, E)$ equals:



$\Sigma\,\text{deg} = 2 + 2 + 2 + 3 + 1 = 10 = 2 \times 5 = 2|E|$

(A) The number of vertices $|V|$
(B) The number of edges $|E|$
(C) Half the number of vertices
(D) Twice the number of edges: $2|E|$

Section B — Multiple Select Questions (MSQ)

***Q31–Q40**: 2 marks each. **No negative marking. One or more** options may be correct; full marks only if ALL correct options selected.*

Q31. For the Euler-Cauchy ODE $x^2y'' - xy' + y = 0$ $(x > 0)$, which of the following are TRUE?

(A) $y = x$ is a solution
(B) $y = e^x$ is a solution
(C) The characteristic equation is $m^2 - 2m + 1 = 0$ (repeated root $m = 1$)
(D) The general solution is $C_1e^x + C_2xe^x$

Q32. For a Hilbert space H with inner product $\langle \cdot, \cdot \rangle$, which of the following are TRUE?

(A) H is a Banach space (complete normed space under $\|x\| = \sqrt{\langle x, x \rangle}$)
(B) The Cauchy-Schwarz inequality holds: $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$ for all $x, y \in H$
(C) Every Hilbert space is finite-dimensional

(D) For any closed subspace $M \subset H$, one has $H = M \oplus M^{\perp}$ (orthogonal complement decomposition)

Q33. Let $\sum a_n$ and $\sum b_n$ be convergent series. Which of the following are TRUE?

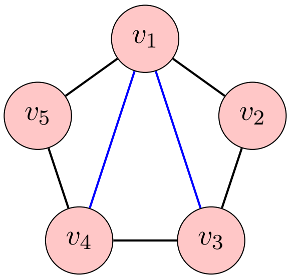
(A) The Cauchy product $\sum c_n$ always converges (for any two convergent series)

(B) If $\sum a_n$ converges *absolutely* to A and $\sum b_n$ converges to B, then the Cauchy product converges to $A \cdot B$ (Mertens' theorem)

(C) $\sum (a_n + b_n)$ converges to $\left(\sum a_n\right) + \left(\sum b_n\right)$

(D) $\sum a_n b_n$ always converges

Q34. A connected planar graph G has $V = 5$ vertices and $E = 7$ edges (shown below). By Euler's formula, which statements are TRUE?



Pentagon + 2 diagonals: $V = 5,\ E = 7$

(A) $F = 4$ (4 faces including the outer face)

(B) $V - E + F = 0$

(C) $V - E + F = 2$

(D) $F = E - V + 2 = 4$

Q35. Which of the following functions are in $L^1(\mathbb{R})$ (Lebesgue integrable over all of $\mathbb{R}$)?

(A) $f(x) = \dfrac{1}{x}$ (with $f(0) = 0$)

(B) $f(x) = e^{-x^2}$

(C) $f(x) = 1$ (constant function)

(D) $f(x) = e^{-|x|}$

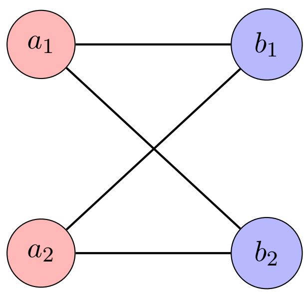
Q36. For the chromatic number $\chi(G)$ of a graph G, which of the following are TRUE?

(A) $\chi(K_n) = n$ for all $n \geq 1$
(B) $\chi(K_{2,3}) = 3$
(C) $\chi(K_{m,n}) = 2$ for all $m, n \geq 1$
(D) Every tree T (with at least 2 vertices) has $\chi(T) = 3$

Q37. Let λ denote the Lebesgue measure on $\mathbb{R}$. Which of the following are TRUE?

(A) $\lambda(\{0\}) = 0$
(B) $\lambda([0,1]) = 1$
(C) $\lambda(\mathbb{Q} \cap [0,1]) = 1$
(D) $\lambda(\mathcal{C}) = 0$, where $\mathcal{C}$ is the Cantor set

Q38. Consider the complete bipartite graph $K_{2,2}$ with parts $\{a_1, a_2\}$ and $\{b_1, b_2\}$ shown below. Which statements are TRUE?



$K_{2,2}$: 4 vertices, 4 edges

(A) $K_{2,2}$ requires 3 colours to properly colour its vertices
(B) $K_{2,2}$ is bipartite with chromatic number $\chi = 2$
(C) $K_{2,2}$ contains a Hamiltonian cycle
(D) $K_{2,2}$ has exactly 5 edges

Q39. For a bounded function $f : [0,1] \to \mathbb{R}$, which of the following conditions **ensure** Riemann integrability?

(A) f is continuous except at finitely many points
(B) f has infinitely many discontinuities
(C) f is monotone on $[0,1]$

(D) f is not bounded on $[0, 1]$

Q40. For the Euler-Cauchy equation $x^2y'' + xy' + 4y = 0$ $(x > 0)$, which of the following are TRUE?

(A) The general solution has the form $C_1x^\alpha + C_2x^\beta$ for real α, β
(B) The general solution is $C_1 \cos(2 \ln x) + C_2 \sin(2 \ln x)$
(C) The equation has only real-exponential type solutions
(D) The substitution $x = e^t$ transforms this into the constant-coefficient ODE $\ddot{y} + 4y = 0$

Section C — Numerical Answer Type (NAT)

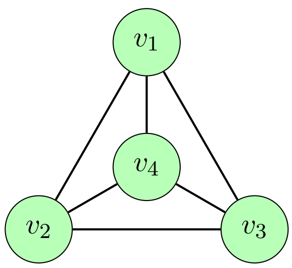
***Q41–Q50**: 1 mark each. No negative marking. Enter the exact numerical value.*

Q41. Evaluate $\displaystyle\int_0^{\pi/2} \sin^5 x \, dx$ using the Wallis reduction formula $I_n = \dfrac{n-1}{n} I_{n-2}$, $I_1 = 1$.

Q42. The Euler-Cauchy equation $x^2y'' - 2xy' + 2y = 0$ has characteristic equation $m^2 - 3m + 2 = 0$. Find the **larger** characteristic root.

Q43. A simple graph G has degree sequence $(2, 2, 3, 3, 4)$. Using the Handshaking Lemma, find the number of edges in G.

Q44. By Cayley's formula, the number of spanning trees of the complete graph K_n is n^{n-2}. Find the number of spanning trees of K_4.



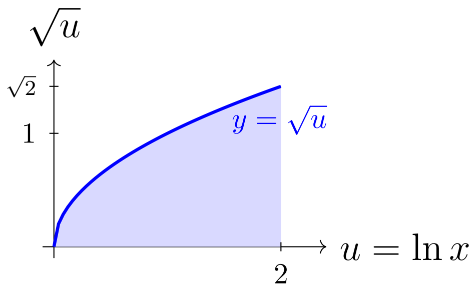
K_4: 4 vertices, 6 edges

Q45. Evaluate $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} \, dx$. (Express your answer in exact form.)

Q46. The complete graph K_4 is a connected planar graph with $V = 4$ vertices and $E = 6$ edges. Using Euler's formula $V - E + F = 2$, find the number of faces F.

Q47. How many proper 2-colourings of the complete bipartite graph $K_{3,3}$ exist? (A proper colouring assigns each vertex a colour so that no two adjacent vertices share a colour.)

Q48. Evaluate $\int_1^{e^2} \frac{\sqrt{\ln x}}{x}\, dx$ using the substitution $u = \ln x$.



Q49. The Euler-Cauchy equation $x^2y'' - 4xy' + 6y = 0$ has general solution $y = C_1x^2 + C_2x^3$. Solve the IVP with $y(1) = 2$ and $y'(1) = 5$, then find $y(1)$ for the specific solution.

Q50. Evaluate $\int_0^1 x^2 \ln x\, dx$ using integration by parts.

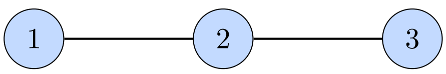
***Q51–Q60**: 2 marks each. No negative marking. Enter the exact numerical value.*

Q51. For the rotation matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$ at $\theta = \frac{\pi}{3}$, find $\operatorname{tr}(A)$.

Q52. Evaluate $\int_{-1}^{1} \left|x^2 - \frac{1}{4}\right| dx$. (Note: $x^2 - \frac{1}{4}$ changes sign at $x = \pm\frac{1}{2}$.)

Q53. Find the Lebesgue measure of the set of irrational numbers in $[0, 1]$, i.e., find $\lambda\big([0, 1] \setminus \mathbb{Q}\big)$.

Q54. For the path graph P_3 (three vertices 1–2–3 connected in a line), the chromatic polynomial is $P(P_3, k) = k(k - 1)^2$. Evaluate $P(P_3, 3)$.



P_3: path on 3 vertices

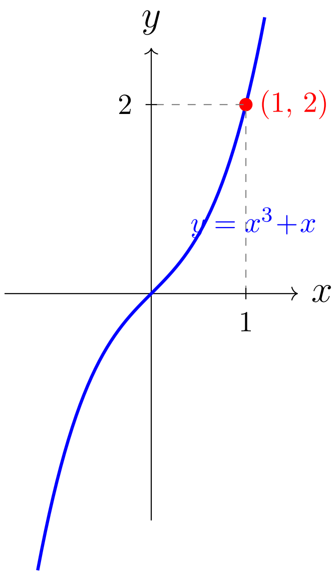
Q55. The complete graph K_5 has $\dfrac{5 \cdot 4}{2} = 10$ edges. If one edge is removed from K_5, how many edges remain?

Q56. What is the dimension of the solution space (as a vector space over $\mathbb{R}$) of the second-order linear ODE $x^2y'' + p(x)y' + q(x)y = 0$ on any interval where p, q are continuous?

Q57. Evaluate $\displaystyle\int_0^{2\pi} \cos^2 x \, dx$.

Q58. How many distinct Hamiltonian cycles does the complete graph K_4 contain? (Two Hamiltonian cycles are the same if they traverse the same set of edges, regardless of direction or starting vertex.)

Q59. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 + x$. Since f is strictly increasing, find $f^{-1}(2)$ (the unique x with $f(x) = 2$).



Q60. The Euler-Cauchy equation $x^2y'' + 5xy' + 4y = 0$ has general solution $y = (C_1 + C_2 \ln x)\, x^{-2}$. Solve the IVP $y(1) = 1$, $y'(1) = 0$, and find $y(e)$.

Solutions

IIT JAM Mathematics – Sample Paper 8

Section A Solutions — Q1–Q10 (1 mark each)

Q1.

Solution

Concept: The Fundamental Theorem of Calculus (Part 2): if f is continuously differentiable on $[a, b]$, then $\displaystyle\int_a^b f'(x)\,dx = f(b) - f(a)$.

Analysis of options:
(A) $f(b) = f(a) + \int_a^b f(x)\,dx$: FALSE. The integral involves f, not f'.
(B) $f'(b) = f(a) - f(b)$: FALSE. This has no basis in calculus.
(C) $f(b) - f(a) = \int_a^b f'(x)\,dx$: TRUE. This is exactly FTC Part 2.
(D) $\int_a^b f'(x)\,dx = 0$ for all f: FALSE. It equals $f(b) - f(a)$, which need not be 0.

Conclusion: Option (C) is the correct statement of the Fundamental Theorem of Calculus.

Answer: (C) ← *Go back to Q1*

Q2.

Solution

Concept: Darboux's theorem (Intermediate Value Property for derivatives): if f is differentiable on (a, b), then f' satisfies the IVP even if f' is not continuous.

Step 1: Suppose $f'(a) < \mu < f'(b)$ (or $f'(b) < \mu < f'(a)$). Darboux's theorem asserts there exists $c \in (a, b)$ with $f'(c) = \mu$.

Step 2: Option (D) states precisely $f'(c) = \mu$, which is the conclusion.

Why others are wrong:
(A) $f(c) = \mu$: false; the IVP applies to f', not f itself.
(B) $f''(c) = \mu$: false; Darboux says nothing about f''.
(C) $f(c) = 0$: false; this is a root of f, not a property guaranteed here.

Answer: (D) ← *Go back to Q2*

Q3.

Solution

Concept: Lebesgue's criterion for Riemann integrability: a bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if its set of discontinuities has Lebesgue measure zero (i.e., f is continuous *almost everywhere*).

Step 1: Option (A) states this criterion exactly.

Why others are wrong:
(B) Differentiability implies continuity, but far more is assumed than necessary.
(C) Monotone functions are RI (their discontinuities form a countable set, measure zero), but monotonicity is *sufficient*, not necessary.
(D) Continuity everywhere is sufficient but not necessary; f can be discontinuous on a measure-zero set and still be RI.

Answer: (A) ← *Go back to Q3*

Q4.

Solution

Concept: The upper Darboux sum $U(f,P) = \sum_{i=1}^{k} M_i\,\Delta x_i$ where $M_i = \sup_{[x_{i-1},x_i]} f$.

Step 1: For $f(x) = x$ (increasing) on $P = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, the supremum on each subinterval is the right-endpoint value:

$$M_1 = \tfrac{1}{3}, \quad M_2 = \tfrac{2}{3}, \quad M_3 = 1.$$

Step 2: Each subinterval has width $\Delta x = \frac{1}{3}$, so:

$$U(f,P) = \tfrac{1}{3}\cdot\tfrac{1}{3} + \tfrac{2}{3}\cdot\tfrac{1}{3} + 1\cdot\tfrac{1}{3} = \tfrac{1}{9} + \tfrac{2}{9} + \tfrac{3}{9} = \tfrac{6}{9} = \tfrac{2}{3}.$$

Why others are wrong:
(A) $1/3$ is the lower Darboux sum.
(C) $1/2$ is the exact integral $\int_0^1 x\,dx$; the upper sum strictly exceeds it.
(D) $5/9$ is incorrect by direct computation.

Answer: (B) ← *Go back to Q4*

Q5.

Solution

Concept: The Euler-Cauchy equation $x^2y'' + axy' + by = 0$ is an equidimensional ODE. The substitution $y = x^m$ (a power function) converts it into an algebraic equation in m.

Step 1: With $y = x^m$: $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$.

Step 2: Substitute: $x^2\cdot m(m-1)x^{m-2} + ax\cdot mx^{m-1} + bx^m = x^m\left[m(m-1)+am+b\right] = 0$.

The bracket gives the characteristic equation $m(m-1) + am + b = 0$, a quadratic in m.

Why others are wrong:
(A) $y = e^{mx}$ works for constant-coefficient ODEs $y'' - \ldots = 0$, not equidimensional ones.
(B) $y = mx^2$ is a particular form, not the general substitution.
(D) $y = \ln x$ appears in the solution of repeated-root cases, not as the initial substitution.

Answer: (C) ← *Go back to Q5*

Q6.

Solution

Concept: For $x^2y''+axy'+by = 0$, substituting $y = x^m$ yields characteristic equation $m(m-1) + am + b = 0$.

Step 1: Here $a = -3$, $b = 4$. Characteristic equation:

$$m(m-1) + (-3)m + 4 = 0 \iff m^2 - m - 3m + 4 = 0 \iff m^2 - 4m + 4 = 0.$$

Step 2: In unsimplified form this is $m(m-1) - 3m + 4 = 0$, which matches option (D).

Verification: $m^2 - 4m + 4 = (m-2)^2 = 0 \Rightarrow m = 2$ (repeated). ✓

Why others are wrong:
(A) $m^2 - 3m + 4 = 0$ corresponds to $a = -2$, not $a = -3$.
(B) $m^2 + 3m + 4 = 0$ corresponds to $a = 4$, wrong sign.
(C) $m^2 - 2m + 4 = 0$ corresponds to $a = -1$, wrong.

Answer: (D) ← *Go back to Q6*

Q7.

Solution

Concept: When the characteristic equation of an Euler-Cauchy ODE has a repeated root $m = r$, the general solution is $y = (C_1 + C_2 \ln x)\, x^r$.

Step 1: Repeated root $m = 2$ gives:

$$y = (C_1 + C_2 \ln x)\, x^2.$$

The two linearly independent solutions are x^2 and $x^2 \ln x$.

Step 2 (verification): Let $y_1 = x^2$. Then $x^2y_1'' - 3xy_1' + 4y_1 = x^2 \cdot 2 - 3x \cdot 2x + 4x^2 = 2x^2 - 6x^2 + 4x^2 = 0$. ✓

Why others are wrong:

(B) $C_1x^2 + C_2x^{-2}$: arises from distinct real roots $m = 2$ and $m = -2$, not a repeated root.
(C) Trig form arises from complex conjugate roots, not repeated real roots.
(D) $C_1x + C_2x^4$: arises from distinct roots $m = 1$ and $m = 4$.

Answer: (A) ← *Go back to Q7*

Q8.

Solution

Concept: Wallis reduction: $I_n = \dfrac{n-1}{n} I_{n-2}$ with $I_0 = \dfrac{\pi}{2}$.

Step 1: Apply the formula:

$$I_4 = \frac{3}{4} I_2 = \frac{3}{4} \cdot \frac{1}{2} I_0 = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{16}.$$

Verification: $I_2 = \dfrac{1}{2} \cdot \dfrac{\pi}{2} = \dfrac{\pi}{4}$. Then $I_4 = \dfrac{3}{4} \cdot \dfrac{\pi}{4} = \dfrac{3\pi}{16}$. ✓

Why others are wrong:
(A) $\pi/16$: incorrect reduction factor.
(C) $\pi/8$: this would be $\frac{1}{2}I_2$, mixing up the formula.
(D) $3\pi/8 = 2 \cdot I_4$: an overcount by factor 2.

Answer: (B) ← *Go back to Q8*

Q9.

Solution

Concept: The orbit of $x \in G$ under conjugation is the *conjugacy class* of x.

Step 1: In S_3, conjugating (12) by various group elements:

$$(13)(12)(13)^{-1} = (13)(12)(13) = (23), \quad (123)(12)(132) = (23) \cdots$$

More systematically, any permutation σ satisfies $\sigma(ij)\sigma^{-1} = (\sigma(i)\,\sigma(j))$.

Step 2: Applying all $g \in S_3$ to (12) yields $\{(12), (13), (23)\}$: all transpositions form a single conjugacy class.

Why others are wrong:
(A) $\{(12)\}$: this would be a normal element; (12) is not self-conjugate in S_3.
(B) $\{e, (12)\}$: e and (12) are in different conjugacy classes.
(D) All of S_3: the full group forms multiple conjugacy classes ($\{e\}$, $\{(12), (13), (23)\}$, $\{(123), (132)\}$).

Answer: (C) ← *Go back to Q9*

Q10.

Solution

Concept: A fixed point of $g \in G$ acting on X is an $x \in X$ with $g \cdot x = x$.

Step 1: The element $1 \in \mathbb{Z}_2$ acts by $1 \cdot x = -x$. For a fixed point: $-x = x \Rightarrow 2x = 0 \Rightarrow x = 0$.

Step 2: So $x = 0$ is the *only* fixed point of element 1.

Why others are wrong:
(A) The action is NOT free because $x = 0$ is a fixed point of element 1.
(B) Element 1 fixes only $x = 0$, not every point.
(C) Element 1 maps every nonzero x to $-x \neq x$; it does NOT fix any nonzero point.

Answer: (D) ← *Go back to Q10*

Section A Solutions — Q11–Q30 (2 marks each)

Q11.

Solution

Concept: Countable sets have Lebesgue outer measure zero.

Proof sketch: Given $S = \{r_1, r_2, \ldots\}$ and $\varepsilon > 0$, cover r_n by an open interval of length $\varepsilon/2^n$. The total length is $\sum_{n=1}^{\infty} \varepsilon/2^n = \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\lambda^*(S) = 0$.

Why others are wrong:
(B) Measure 1: only for "fat" sets like $[0, 1]$.
(C),(D): Measure is never ∞ for a set that can be covered by arbitrarily small intervals.

Answer: (A) ← *Go back to Q11*

Q12.

Solution

Concept: At stage n of the Cantor construction, 2^n closed intervals of total length $(2/3)^n$ remain. As $n \to \infty$, the total length $(2/3)^n \to 0$.

Step 1: $\lambda(\mathcal{C}) = \lim_{n\to\infty}(2/3)^n = 0$.

Step 2: However, $\mathcal{C}$ is uncountable (it is in bijection with $\{0, 1\}^{\mathbb{N}}$ via ternary expansions). So (C) and (D) are false.

Why others are wrong:
(A) Positive measure: false; the measure is 0.
(C) Finite: false; $\mathcal{C}$ is uncountable.

(D) Nonempty interior: false; $\mathcal{C}$ contains no open interval (it is totally disconnected).

Answer: (B) ← *Go back to Q12*

Q13.

Solution

Concept: The Borel σ-algebra $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra containing all open sets in $\mathbb{R}$. Its elements are called **Borel sets**.

Step 1: $\mathcal{B}(\mathbb{R})$ contains: all open sets, all closed sets, all F_σ sets (countable unions of closed sets), all G_δ sets (countable intersections of open sets), and so on.

Why others are wrong:
(A) Not every subset of $\mathbb{R}$ is Borel (e.g., a Vitali set is not measurable, hence not Borel).
(B) Open sets are a proper subset of $\mathcal{B}(\mathbb{R})$.
(D) Closed sets are a proper subset of $\mathcal{B}(\mathbb{R})$.

Answer: (C) ← *Go back to Q13*

Q14.

Solution

Concept: Wallis reduction: $I_n = \dfrac{n-1}{n} I_{n-2}$, $I_1 = 1$.

Step 1: Apply iteratively:

$$I_5 = \frac{4}{5} I_3 = \frac{4}{5} \cdot \frac{2}{3} I_1 = \frac{4}{5} \cdot \frac{2}{3} \cdot 1 = \frac{8}{15}.$$

Verification: $I_3 = \frac{2}{3} \cdot I_1 = \frac{2}{3}$. Then $I_5 = \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15}$. ✓

Why others are wrong:
(A) $3/8$: this is $I_4/(\pi/2)$, a mixing of the even and odd formulas.
(B) $5/8$: incorrect application of the reduction.
(C) $15/8 > 1$: impossible since $\sin^5 x \le 1$ on $[0, \pi/2]$ implies $I_5 \le \pi/2 \approx 1.57$; but $15/8 = 1.875$ is also greater than $I_1 = 1$, which is contradicted by the graph ($\sin^n x \le \sin^{n-2} x$).

Answer: (D) ← *Go back to Q14*

Q15.

Solution

Concept: Use the recurrence $\Gamma(z+1) = z\,\Gamma(z)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Step 1: $\Gamma(\frac{3}{2}) = \frac{1}{2}\,\Gamma(\frac{1}{2}) = \dfrac{\sqrt{\pi}}{2}$.

Step 2: $\Gamma(\frac{5}{2}) = \frac{3}{2}\,\Gamma(\frac{3}{2}) = \dfrac{3}{2} \cdot \dfrac{\sqrt{\pi}}{2} = \dfrac{3\sqrt{\pi}}{4}$.

Why others are wrong:
(B) $\sqrt{\pi}/4$: off by factor 3.
(C) $\sqrt{\pi} = \Gamma(1/2)$, not $\Gamma(5/2)$.
(D) $3\sqrt{\pi}/2$: off by factor 2.

Answer: (A) ← *Go back to Q15*

Q16.

Solution

Concept: By definition, $\ell^2 = \{\{a_n\} : \sum_{n=1}^{\infty} |a_n|^2 < \infty\}$, equipped with norm $\|\{a_n\}\| = \left(\sum |a_n|^2\right)^{1/2}$.

Why others are wrong:
(A) $\sum |a_n| < \infty$ defines ℓ^1, not ℓ^2.
(C) $a_n \to 0$ is necessary but not sufficient for membership in ℓ^2; for example, $a_n = 1/\sqrt{n}$ tends to 0 but $\sum 1/n$ diverges.
(D) Boundedness is necessary but not sufficient.

Answer: (B) ← *Go back to Q16*

Q17.

Solution

Concept: A Hilbert space is an inner product space that is complete (every Cauchy sequence converges in the space).

Step 1: ℓ^2 has a well-defined inner product $\langle a, b\rangle = \sum a_n \overline{b_n}$ (convergent by Cauchy-Schwarz).

Step 2: ℓ^2 is complete: if $\{a^{(k)}\}$ is a Cauchy sequence in ℓ^2, one constructs the limit coordinatewise and verifies convergence in ℓ^2.

Conclusion: ℓ^2 is a Hilbert space.

Why others are wrong:
(A) ℓ^2 is indeed a Hilbert space (hence also Banach), so "not Hilbert" is wrong.
(B) The inner product induces a norm, so ℓ^2 is definitely a normed space.
(D) ℓ^2 is complete (this is a standard theorem).

Answer: (C) ← Go back to Q17

Q18.

Solution

Concept: A Schauder basis (or complete ONS) for a Hilbert space H is a sequence $\{e_n\}$ such that every $x \in H$ can be written $x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$ (convergence in norm).

Step 1: Orthonormality: $\langle e_m, e_n \rangle = \delta_{mn}$. ✓

Step 2: Completeness: for any $\{a_n\} \in \ell^2$, $\{a_n\} = \sum_{n=1}^{\infty} a_n e_n$ (convergent in ℓ^2). ✓

Conclusion: $\{e_n\}$ is a complete orthonormal system (Schauder basis) for ℓ^2.

Why others are wrong:
(A) An algebraic (Hamel) basis for an infinite-dimensional space must be uncountable; $\{e_n\}$ is countable.
(B) The e_n are orthonormal (not just orthogonal).
(C) ℓ^2 is infinite-dimensional.

Answer: (D) ← Go back to Q18

Q19.

Solution

Concept: Plancherel's theorem: the Fourier transform $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is an isometric isomorphism, i.e., $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$.

Step 1: The Fourier transform preserves the L^2 norm (Parseval-Plancherel identity):

$$\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 \, d\xi = \int_{-\infty}^{\infty} |f(x)|^2 \, dx.$$

Why others are wrong:
(B) $\|\hat{f}\|_{L^1} = \|f\|_{L^2}$: false; $\|\hat{f}\|_{L^1}$ can be infinite even for smooth $f \in L^2$.
(C) $\hat{f} \in L^1$: not always true for $f \in L^2$ (Fourier transform maps $L^2 \to L^2$, not necessarily to L^1).
(D) $\hat{f}(\xi) = f(-\xi)$: this would mean f is its own Fourier transform; it holds for Gaussians but not in general.

Answer: (A) ← Go back to Q19

Q20.

Solution

Concept: Mertens' theorem: if $\sum a_n$ converges absolutely to A and $\sum b_n$ converges to B, then the Cauchy product $\sum c_n$ (with $c_n = \sum_{k=0}^{n} a_k b_{n-k}$) converges to $A \cdot B$.

Step 1: The triangular grid diagram illustrates how c_n is formed by summing all products $a_k b_j$ with $k + j = n$.

Step 2: Mertens proved the limit of $\sum c_n$ equals the product of the two series' sums: $A \cdot B$.

Why others are wrong:
(A) $A + B$: sums of series add, not multiply; the Cauchy product gives the product.
(C) A/B: division is not the operation of Cauchy product.
(D) $(A + B)^2$: has no algebraic basis for the Cauchy product limit.

Answer: (B) ← *Go back to Q20*

Q21.

Solution

Concept: For $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, the ℓ^∞ (supremum) norm is $\|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i|$.

Step 1: This is the limit as $p \to \infty$ of the ℓ^p norm $\|\mathbf{x}\|_p = \left(\sum |x_i|^p\right)^{1/p}$.

Verification: For $\mathbf{x} = (3, -5, 2)$: $\|\mathbf{x}\|_\infty = 5 = \max(3, 5, 2)$. ✓

Why others are wrong:
(A) The expression $(\sum |x_i|^\infty)^{1/\infty}$ is a limiting notation, not a literal formula; the correct answer is the maximum.
(B) $\sum |x_i| = \|\mathbf{x}\|_1$, the ℓ^1 norm.
(D) The minimum has no standard name as a norm.

Answer: (C) ← *Go back to Q21*

Q22.

Solution

Concept: In any finite-dimensional vector space, all norms are *equivalent*: for any two norms $\|\cdot\|_\alpha$, $\|\cdot\|_\beta$, there exist constants $0 < c \le C < \infty$ with $c\|\mathbf{x}\|_\alpha \le \|\mathbf{x}\|_\beta \le C\|\mathbf{x}\|_\alpha$ for all $\mathbf{x}$.

Proof idea: On a finite-dimensional space, the unit sphere of $\|\cdot\|_\alpha$ is compact, so $\|\cdot\|_\beta$ (continuous) attains its min and max on that sphere, giving c and C.

Why others are wrong:
(A),(B),(C): Two different norms need not be equal. For example, on $\mathbb{R}^2$, $\|(1,1)\|_1 = 2$ while $\|(1,1)\|_\infty = 1$.

Answer: (D) ← *Go back to Q22*

Q23.

Solution

Concept: The Riesz representation theorem for $\mathbb{R}^n$: every bounded linear functional $f : \mathbb{R}^n \to \mathbb{R}$ is of the form $f(\mathbf{x}) = \langle \mathbf{v}, \mathbf{x} \rangle = \sum v_i x_i$ for a unique $\mathbf{v} \in \mathbb{R}^n$.

Step 1: The map $\mathbf{v} \mapsto f_{\mathbf{v}}$ (where $f_{\mathbf{v}}(\mathbf{x}) = \langle \mathbf{v}, \mathbf{x} \rangle$) is a linear isomorphism from $\mathbb{R}^n$ to $(\mathbb{R}^n)^*$.

Conclusion: $(\mathbb{R}^n)^* \cong \mathbb{R}^n$.

Why others are wrong:
(B) $\mathbb{R}^{n^2}$: this is the dimension of $n \times n$ matrices, not functionals on $\mathbb{R}^n$.
(C) $\mathbb{R}^{n-1}$: functionals on $\mathbb{R}^n$ have n degrees of freedom (the components of $\mathbf{v}$).
(D) The zero space: trivially wrong; there are infinitely many nontrivial functionals.

Answer: (A) ← *Go back to Q23*

Q24.

Solution

Concept: The Hahn-Banach theorem (extension form): if M is a subspace of a normed space X and $f \in M^*$ is a bounded linear functional on M, then there exists an extension $F \in X^*$ with $F|_M = f$ and $\|F\|_{X^*} = \|f\|_{M^*}$.

Key point: The extension preserves the *norm exactly*.

Why others are wrong:
(A) The theorem requires the extended functional to have the same norm; there IS a norm constraint.
(C) The Hahn-Banach theorem says nothing about completeness of normed spaces.
(D) Dual spaces of Banach spaces can be infinite-dimensional (e.g., the dual of L^1 is L^∞).

Answer: (B) ← *Go back to Q24*

Q25.

Solution

Concept: Euler's formula for connected planar graphs: $V - E + F = 2$.

Verification with K_4: $V = 4$, $E = 6$, $F = 4$ (three inner triangular faces + one outer face). Then $4 - 6 + 4 = 2$. ✓

Another example: A triangle: $V = 3$, $E = 3$, $F = 2$ (inner face + outer face). $3 - 3 + 2 = 2$. ✓

Why others are wrong:
(A) $V - E + F = 0$: false; Euler's formula gives 2, not 0.
(B) $2V$: depends on V, not a constant.

(D) $E - V$: equals $F - 2$ by Euler, not a constant.

Answer: (C) ← *Go back to Q25*

Q26.

Solution

Concept: The Four Colour Theorem (Appel-Haken, 1976): every planar graph is properly vertex-colourable using at most 4 colours.

Step 1: The theorem gives the sharp bound of 4: the icosahedron graph requires exactly 4 colours.

Step 2: Planar graphs need at most 4 colours; some require exactly 4 (e.g., K_4).

Why others are wrong:
(A) 2 colours: only bipartite planar graphs need just 2.
(B) 3 colours: not always sufficient (e.g., K_4 needs 4).
(C) 6 colours: the Heawood conjecture applies to higher-genus surfaces; for planar graphs, 4 suffices.

Answer: (D) ← *Go back to Q26*

Q27.

Solution

Concept: In K_n, every pair of vertices is adjacent, so all n vertices must receive distinct colours.

Step 1: Assign colours one vertex at a time: vertex 1 can receive any of k colours, vertex 2 any of $k - 1, \ldots,$ vertex n any of $k - (n - 1)$.

Step 2:

$$P(K_n, k) = k(k - 1)(k - 2) \cdots (k - n + 1).$$

This is the falling factorial $k^{\underline{n}}$.

Why others are wrong:
(B) k^n: applies if all vertices could receive any colour independently, ignoring the graph structure.
(C) $n!$: this would fix $k = n$ and count permutations, not k-colourings in general.
(D) k^{n-1}: incorrect for K_n (a cycle C_n has chromatic polynomial $(k-1)^n+(-1)^n(k-1)$, not k^{n-1}).

Answer: (A) ← *Go back to Q27*

Q28.

Solution

Concept: König's theorem: G is bipartite if and only if G contains no odd cycle.

Step 1: ($\Rightarrow$) In a bipartite graph with parts A, B, every edge crosses between A and B; any cycle alternates A-B-A-$\ldots$, so must have even length.

Step 2: ($\Leftarrow$) If G has no odd cycles, 2-colour it: fix a root r, set colour by parity of distance from r. No conflict arises (odd cycle would create one).

Verification: $K_{2,3}$: every cycle has length 4 or 6, no odd cycles. $K_{2,3}$ is bipartite. ✓

Why others are wrong:
(A) K_3-free is NOT equivalent to bipartite. Example: C_5 is triangle-free but not bipartite (it has the odd cycle of length 5).
(C) Bipartite graphs need not be planar ($K_{3,3}$ is non-planar).
(D) Bipartite graphs need not be connected.

Answer: (B) ← *Go back to Q28*

Q29.

Solution

Concept: A spanning tree of a connected graph G with n vertices is a connected acyclic spanning subgraph.

Step 1: A tree on n vertices always has exactly $n-1$ edges. (Proof: a connected graph with k edges on n vertices satisfies $k \geq n-1$; equality holds iff the graph is acyclic, i.e., a tree.)

Step 2: This follows from Euler's formula applied to trees ($F = 1$ for a planar tree): $V - E + F = n - (n-1) + 1 = 2$. Alternatively, induct: remove a leaf to get a tree on $n-1$ vertices with $n-2$ edges.

Why others are wrong:
(A) n edges: would create exactly one cycle (a unicyclic graph, not a tree).
(B) $n+1$ edges: impossible in a simple connected graph with n vertices without multiple cycles.
(D) $2n-1$: not a tree for any $n > 1$.

Answer: (C) ← *Go back to Q29*

Q30.

Solution

Concept: Handshaking Lemma: $\sum_{v \in V} \deg(v) = 2|E|$.

Proof: Each edge $\{u, v\}$ contributes 1 to $\deg(u)$ and 1 to $\deg(v)$, so it contributes exactly 2 to $\sum \deg(v)$.

Verification from diagram: Degrees $3+2+2+2+1=10=2\times 5=2|E|$. ✓

Corollary: The number of vertices of odd degree is always even.

Why others are wrong:
(A) $|V|$: independent of the edge structure; the sum of degrees encodes edge information.
(B) $|E|$: off by factor 2; each edge contributes to two vertex-degrees.
(C) Half the number of vertices: unrelated to edges.

Answer: (D) ← *Go back to Q30*

Section B Solutions — Q31–Q40 (MSQ, 2 marks each)

Q31.

Solution

Characteristic equation of $x^2y'' - xy' + y = 0$ (with $a=-1$, $b=1$):

$$m(m-1)+(-1)m+1 = m^2-2m+1 = (m-1)^2 = 0 \;\Rightarrow\; m=1 \text{ (repeated)}.$$

General solution: $y = (C_1 + C_2 \ln x)\, x$.

Checking options:
(A) $y=x$: $y'=1$, $y''=0$. Check: $x^2(0)-x(1)+x=0$. **TRUE.** ✓
(B) $y=e^x$: $y'=e^x$, $y''=e^x$. Check: $x^2e^x - xe^x + e^x = e^x(x^2-x+1)\neq 0$. **FALSE.**
(C) Characteristic equation is $m^2-2m+1=0$: derived above. **TRUE.** ✓
(D) General solution $C_1e^x + C_2xe^x$: this is for the ODE $y''-2y'+y=0$ (constant coefficients). **FALSE.**

Correct options: A and C.

Answer: (AC) ← *Go back to Q31*

Q32.

Solution

Checking each option for a Hilbert space H:

(A) H is a Banach space: TRUE. A Hilbert space is complete by definition; with norm $\|x\| = \sqrt{\langle x, x\rangle}$, H is a complete normed space, i.e., a Banach space. ✓

(B) Cauchy-Schwarz: $|\langle x, y\rangle| \le \|x\|\cdot\|y\|$: TRUE. This is a fundamental inequality for any inner product space. ✓

(C) Every Hilbert space is finite-dimensional: FALSE. ℓ^2, $L^2(\mathbb{R})$, etc. are infinite-dimensional Hilbert spaces.

(D) $H = M \oplus M^{\perp}$ for any closed subspace M: TRUE. The orthogonal complement theorem guarantees this decomposition in any Hilbert space. ✓

Correct options: A, B, D.

Answer: (ABD) ← *Go back to Q32*

Q33.

Solution

Checking each option:

(A) Cauchy product of any two convergent series converges: FALSE. Example: $\sum(-1)^n/\sqrt{n+1}$ converges (alternating), but its Cauchy product with itself diverges.

(B) If $\sum a_n$ converges absolutely and $\sum b_n$ converges, then Cauchy product $\to A \cdot B$: TRUE (Mertens' theorem). ✓

(C) $\sum(a_n + b_n) = (\sum a_n) + (\sum b_n)$: TRUE by linearity of series (if both converge). ✓

(D) $\sum a_n b_n$ always converges: FALSE. Take $a_n = b_n = 1/\sqrt{n}$; each $\sum a_n$ diverges, but even if they converged, $\sum a_n b_n = \sum 1/n$ diverges.

Correct options: B and C.

Answer: (BC) ← *Go back to Q33*

Q34.

Solution

Step 1: By Euler's formula $V - E + F = 2$ with $V = 5$, $E = 7$:

$$F = 2 - V + E = 2 - 5 + 7 = 4.$$

The 4 faces are: 3 inner triangular regions (created by the 2 diagonals) + 1 outer (unbounded) face.

Checking options:
(A) $F = 4$: TRUE. ✓
(B) $V - E + F = 0$: FALSE. Euler gives $5 - 7 + 4 = 2 \neq 0$.
(C) $V - E + F = 2$: TRUE. This is Euler's formula. ✓
(D) $F = E - V + 2 = 7 - 5 + 2 = 4$: TRUE (rearrangement of Euler). ✓

Correct options: A, C, D.

Answer: (ACD) ← *Go back to Q34*

Q35.

Solution

Checking each option for membership in $L^1(\mathbb{R})$:

(A) $f(x) = 1/x$: $\int_{-\infty}^{\infty} |1/x|\,dx$ diverges (at both 0 and ∞). NOT in L^1. FALSE.

(B) $f(x) = e^{-x^2}$: $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} < \infty$. **TRUE.** ✓

(C) $f(x) = 1$: $\int_{-\infty}^{\infty} 1\,dx = \infty$. NOT in L^1. FALSE.

(D) $f(x) = e^{-|x|}$: $\int_{-\infty}^{\infty} e^{-|x|}\,dx = 2\int_0^{\infty} e^{-x}\,dx = 2 < \infty$. **TRUE.** ✓

Correct options: B and D.

Answer: (BD) ← *Go back to Q35*

Q36.

Solution

Checking each option:

(A) $\chi(K_n) = n$: TRUE. In K_n every vertex is adjacent to all others, requiring all n colours. ✓

(B) $\chi(K_{2,3}) = 3$: FALSE. $K_{2,3}$ is bipartite, so $\chi(K_{2,3}) = 2$ (2-colour it by the two parts).

(C) $\chi(K_{m,n}) = 2$ for $m, n \geq 1$: TRUE. Every complete bipartite graph $K_{m,n}$ is bipartite, hence 2-colourable. ✓

(D) Every tree has $\chi(T) = 3$: FALSE. Every tree (with ≥ 2 vertices) is bipartite (no odd cycles), so $\chi(T) = 2$.

Correct options: A and C.

Answer: (AC) ← *Go back to Q36*

Q37.

Solution

Checking each option:

(A) $\lambda(\{0\}) = 0$: TRUE. Singletons are countable (measure-zero) sets. ✓

(B) $\lambda([0,1]) = 1$: TRUE. The Lebesgue measure of a closed interval is its length. ✓

(C) $\lambda(\mathbb{Q} \cap [0,1]) = 1$: FALSE. $\mathbb{Q}$ is countable, so $\lambda(\mathbb{Q} \cap [0,1]) = 0$, not 1.

(D) $\lambda(\mathcal{C}) = 0$ for the Cantor set $\mathcal{C}$: TRUE. At stage n, total measure remaining is $(2/3)^n \to 0$. ✓

Correct options: A, B, D.

Answer: (ABD) ← *Go back to Q37*

Q38.

Solution

Analysis of $K_{2,2}$ with parts $\{a_1, a_2\}$ and $\{b_1, b_2\}$, edges $\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}$:

(A) Requires 3 colours: FALSE. $K_{2,2}$ is bipartite; colour part $\{a_1, a_2\}$ red and part $\{b_1, b_2\}$ blue. $\chi(K_{2,2}) = 2$.

(B) $K_{2,2}$ is bipartite with $\chi = 2$: TRUE. ✓

(C) $K_{2,2}$ has a Hamiltonian cycle: TRUE. The cycle $a_1 \to b_1 \to a_2 \to b_2 \to a_1$ visits all 4 vertices. ✓

(D) $K_{2,2}$ has 5 edges: FALSE. $K_{2,2}$ has $2 \times 2 = 4$ edges, not 5.

Correct options: B and C.

Answer: (BC) ← *Go back to Q38*

Q39.

Solution

By Lebesgue's criterion: a bounded $f : [0, 1] \to \mathbb{R}$ is RI iff its discontinuity set has measure zero.

(A) Finitely many discontinuities: TRUE. A finite set has measure zero $\Rightarrow$ RI. ✓

(B) Infinitely many discontinuities: this is NOT a condition that ensures RI by itself. A function with infinitely many discontinuities (not forming a measure-zero set, e.g., the Dirichlet function) fails to be RI. The condition does not ENSURE RI. FALSE.

(C) Monotone on $[0, 1]$: TRUE. A bounded monotone function has at most countably many discontinuities (measure zero), so it is RI. ✓

(D) f is not bounded: FALSE. Riemann integration requires boundedness by definition; an unbounded function is not Riemann integrable.

Correct options: A and C.

Answer: (AC) ← *Go back to Q39*

Q40.

Solution

Characteristic equation of $x^2y'' + xy' + 4y = 0$ (with $a = 1$, $b = 4$):

$$m(m-1) + 1 \cdot m + 4 = m^2 + 4 = 0 \Rightarrow m = \pm 2i \quad (\alpha = 0, \ \beta = 2).$$

General solution for complex roots $m = \alpha \pm \beta i$: $y = x^\alpha[C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)]$. With $\alpha = 0$, $\beta = 2$: $y = C_1 \cos(2 \ln x) + C_2 \sin(2 \ln x)$.

Checking options:

(A) $C_1 x^\alpha + C_2 x^\beta$ (real root form): FALSE; applies only to distinct real roots.

(B) $C_1 \cos(2 \ln x) + C_2 \sin(2 \ln x)$: TRUE. ✓

(C) Only real-exponential solutions: FALSE; complex characteristic roots give trigonometric (real-valued) solutions.

(D) $x = e^t$ transforms to $\ddot{y} + 4y = 0$: TRUE. Under $x = e^t$, the Euler-Cauchy ODE $x^2y'' + xy' + 4y = 0$ becomes $\ddot{y} + 4y = 0$ (constant coefficients). ✓

Correct options: B and D.

Answer: (BD) ← *Go back to Q40*

Section C Solutions — Q41–Q50 (NAT, 1 mark each)

Q41.

Solution

Apply the Wallis reduction formula with $I_1 = 1$:

$$I_3 = \frac{2}{3} I_1 = \frac{2}{3}, \qquad I_5 = \frac{4}{5} I_3 = \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15}.$$

Answer: ($8/15$) ← *Go back to Q41*

Q42.

Solution

Step 1: For $x^2y'' - 2xy' + 2y = 0$ ($a = -2$, $b = 2$), characteristic equation:

$$m(m-1) + (-2)m + 2 = m^2 - 3m + 2 = (m-1)(m-2) = 0.$$

Step 2: Roots $m = 1$ and $m = 2$. The larger root is **2**.

Answer: (2) ← *Go back to Q42*

Q43.

Solution

By the Handshaking Lemma: $\sum_v \deg(v) = 2|E|$.

Sum of degrees: $2+2+3+3+4=14$. Therefore $|E| = 14/2 = \mathbf{7}$.

Answer: (7) ← *Go back to Q43*

Q44.

Solution

Cayley's formula: The number of spanning trees of K_n is n^{n-2}.

For K_4: $4^{4-2} = 4^2 = \mathbf{16}$.

Direct verification: One can enumerate all spanning trees of K_4; there are exactly 16 distinct labelled trees.

Answer: (16) ← *Go back to Q44*

Q45.

Solution

Step 1: Let $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$. Then

$$I^2 = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta = 2\pi \cdot \tfrac{1}{2} = \pi.$$

Step 2: Since $I > 0$, $I = \sqrt{\pi}$.

Answer: ($\sqrt{\pi}$) ← *Go back to Q45*

Q46.

Solution

By Euler's formula $V - E + F = 2$ with $V = 4$, $E = 6$:

$$F = 2 - V + E = 2 - 4 + 6 = 4.$$

The 4 faces are: the outer (unbounded) face, plus 3 inner triangular faces.

Answer: (4) ← *Go back to Q46*

Solution

$K_{3,3}$ is bipartite with parts $A = \{v_1, v_2, v_3\}$ and $B = \{v_4, v_5, v_6\}$. In a proper 2-colouring using colours {red, blue}:

Q47.

- Colouring 1: $A \mapsto$red, $B \mapsto$blue.
- Colouring 2: $A \mapsto$blue, $B \mapsto$red.

No other valid 2-colourings exist (all vertices in the same part must share a colour). Answer: **2**.

Answer: (2) ← Go back to Q47

Q48.

Solution

Substitution: $u = \ln x$, $du = dx/x$. When $x = 1$, $u = 0$; when $x = e^2$, $u = 2$.

$$\int_1^{e^2} \frac{\sqrt{\ln x}}{x}\,dx = \int_0^2 \sqrt{u}\,du = \left[\frac{2u^{3/2}}{3}\right]_0^2 = \frac{2 \cdot 2^{3/2}}{3} = \frac{2 \cdot 2\sqrt{2}}{3} = \frac{4\sqrt{2}}{3}.$$

Answer: ($4\sqrt{2}/3$) ← Go back to Q48

Q49.

Solution

General solution: $y = C_1x^2 + C_2x^3$.

IVP: $y(1) = C_1 + C_2 = 2$ and $y' = 2C_1x + 3C_2x^2 \Rightarrow y'(1) = 2C_1 + 3C_2 = 5$.

Solving: Subtract the first equation from the second: $C_2 = 1$, then $C_1 = 1$.

Specific solution: $y = x^2 + x^3$. $\quad y(1) = 1 + 1 = \mathbf{2}$.

Answer: (2) ← Go back to Q49

Q50.

Solution

Integration by parts: $u = \ln x$, $dv = x^2\,dx$; so $du = dx/x$, $v = x^3/3$.

$$\int_0^1 x^2 \ln x\,dx = \left[\frac{x^3 \ln x}{3}\right]_0^1 - \int_0^1 \frac{x^2}{3}\,dx.$$

Evaluate: $[x^3 \ln x/3]_0^1 = 0 - 0 = 0$ (since $x^3 \ln x \to 0$ as $x \to 0^+$).

$$-\int_0^1 \frac{x^2}{3}\,dx = -\frac{1}{3} \cdot \frac{1}{3} = -\frac{1}{9}.$$

Answer: ($-1/9$) ← Go back to Q50

Section C Solutions — Q51–Q60 (NAT, 2 marks each)

Q51.

Solution

The trace of $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$ is $\mathrm{tr}(A) = 1 + \cos\theta + \cos\theta = 1 + 2\cos\theta$.

At $\theta = \pi/3$: $\cos(\pi/3) = 1/2$, so $\mathrm{tr}(A) = 1 + 2 \cdot (1/2) = 1 + 1 = \mathbf{2}$.

Answer: (2) ← *Go back to Q51*

Q52.

Solution

$x^2 - \frac{1}{4} = 0$ at $x = \pm\frac{1}{2}$; $f < 0$ for $|x| < \frac{1}{2}$ and $f > 0$ for $|x| > \frac{1}{2}$ on $[-1, 1]$.

By symmetry:

$$\int_{-1}^{1} \left|x^2 - \tfrac{1}{4}\right| dx = 2\left[\int_0^{1/2} \left(\tfrac{1}{4} - x^2\right) dx + \int_{1/2}^{1} \left(x^2 - \tfrac{1}{4}\right) dx\right].$$

First integral: $\left[\frac{x}{4} - \frac{x^3}{3}\right]_0^{1/2} = \frac{1}{8} - \frac{1}{24} = \frac{3-1}{24} = \frac{1}{12}$.

Second integral: $\left[\frac{x^3}{3} - \frac{x}{4}\right]_{1/2}^{1} = \left(\frac{1}{3} - \frac{1}{4}\right) - \left(\frac{1}{24} - \frac{1}{8}\right) = \frac{1}{12} - \left(-\frac{1}{12}\right) = \frac{2}{12} = \frac{1}{6}$.

Total: $2\left(\frac{1}{12} + \frac{1}{6}\right) = 2 \cdot \frac{1}{4} = \dfrac{1}{2}$.

Answer: (1/2) ← *Go back to Q52*

Q53.

Solution

$\mathbb{Q}$ is countable, hence $\lambda(\mathbb{Q}) = 0$. By countable additivity:

$$\lambda([0,1]) = \lambda(\mathbb{Q} \cap [0,1]) + \lambda([0,1] \setminus \mathbb{Q}) \;\Rightarrow\; 1 = 0 + \lambda([0,1] \setminus \mathbb{Q}).$$

Therefore $\lambda\big([0,1] \setminus \mathbb{Q}\big) = \mathbf{1}$.

Answer: (1) ← *Go back to Q53*

Q54.

Solution

The chromatic polynomial of the path P_3 (vertices 1-2-3):

$$P(P_3, k) = k(k-1)^2.$$

Derivation: Vertex 1 gets any of k colours; vertex 2 gets any colour $\neq$ colour of 1 ($k-1$ choices); vertex 3 gets any colour $\neq$ colour of 2 ($k-1$ choices).

At $k = 3$: $P(P_3, 3) = 3 \cdot (3-1)^2 = 3 \cdot 4 = \mathbf{12}$.

Answer: (12) ← Go back to Q54

Q55.

Solution

K_5 has $\dfrac{5 \cdot 4}{2} = 10$ edges. Removing one edge leaves $10 - 1 = \mathbf{9}$ edges.

Answer: (9) ← Go back to Q55

Q56.

Solution

A second-order linear homogeneous ODE on an interval where p, q are continuous has a two-dimensional solution space (the existence-uniqueness theorem guarantees a unique solution for each initial condition $(y(x_0), y'(x_0))$, and the solution space is isomorphic to $\mathbb{R}^2$). Dimension $= \mathbf{2}$.

Answer: (2) ← Go back to Q56

Q57.

Solution

Using the identity $\cos^2 x = \dfrac{1+\cos 2x}{2}$:

$$\int_0^{2\pi} \cos^2 x \, dx = \int_0^{2\pi} \frac{1+\cos 2x}{2} \, dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{2\pi} = \frac{1}{2}(2\pi + 0) = \pi.$$

Answer: (π) ← Go back to Q57

Q58.

Solution

The number of distinct Hamiltonian cycles in K_n is $\frac{(n-1)!}{2}$.

For K_4: $\frac{3!}{2} = \frac{6}{2} = \mathbf{3}$.

Listing: With vertices $\{1,2,3,4\}$: fix vertex 1, permute the rest in $(3!) = 6$ orders and divide by 2 for directionality: cycles $\{1-2-3-4,\ 1-2-4-3,\ 1-3-2-4\}$.

Answer: (3) ← *Go back to Q58*

Q59.

Solution

$f(x) = x^3 + x$ is strictly increasing (since $f'(x) = 3x^2 + 1 > 0$ for all x), so it is invertible.

To find $f^{-1}(2)$: solve $x^3 + x = 2$. By inspection, $x = 1$: $1^3 + 1 = 2$. ✓

Therefore $f^{-1}(2) = \mathbf{1}$.

Answer: (1) ← *Go back to Q59*

Q60.

Solution

General solution of $x^2y'' + 5xy' + 4y = 0$ (characteristic: $m^2 + 4m + 4 = (m+2)^2 = 0 \Rightarrow m = -2$ repeated):

$$y = (C_1 + C_2 \ln x)\, x^{-2}.$$

IVP: $y(1) = C_1 \cdot 1 = C_1 = 1 \Rightarrow C_1 = 1$.

$y' = -2C_1x^{-3} + C_2\left(x^{-2} \cdot \frac{1}{x} + \ln x \cdot (-2x^{-3})\right) = -2C_1x^{-3} + C_2x^{-3}(1 - 2\ln x)$.

$y'(1) = -2C_1 + C_2(1 - 0) = -2 + C_2 = 0 \Rightarrow C_2 = 2$.

Evaluate at $x = e$**:** $\ln e = 1$.

$$y(e) = (1 + 2 \cdot 1)\, e^{-2} = 3e^{-2} = \frac{3}{e^2}.$$

Answer: ($3/e^2$) ← *Go back to Q60*

Answer Key — IIT JAM Mathematics Sample Paper 8

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	C	2	D	3	A	4	B	5	C
6	D	7	A	8	B	9	C	10	D
11	A	12	B	13	C	14	D	15	A
16	B	17	C	18	D	19	A	20	B
21	C	22	D	23	A	24	B	25	C
26	D	27	A	28	B	29	C	30	D
31	AC	32	ABD	33	BC	34	ACD	35	BD
36	AC	37	ABD	38	BC	39	AC	40	BD
41	$8/15$	42	2	43	7	44	16	45	$\sqrt{\pi}$
46	4	47	2	48	$4\sqrt{2}/3$	49	2	50	$-1/9$
51	2	52	$1/2$	53	1	54	12	55	9
56	2	57	π	58	3	59	1	60	$3/e^2$