IIT JAM Mathematical Statistics

Sample Paper – 1

Instructions

- This paper contains **60 questions** carrying a maximum of **100 marks**. Duration: **3 hours (180 minutes)**.
- **Section A** (**Q1–Q30**, MCQ): Q1–Q10 carry **+1 mark** each; wrong response: $-\frac{1}{3}$ mark. Q11–Q30 carry **+2 marks** each; wrong response: $-\frac{2}{3}$ mark. Choose the **single best** option.
- **Section B** (**Q31–Q40**, MSQ): Each carries **+2 marks**. **No negative marking. One or more** options may be correct; full marks only if *all* correct options (and no incorrect options) are selected.
- **Section C** (**Q41–Q60**, NAT): Q41–Q50 carry **+1 mark**; Q51–Q60 carry **+2 marks**. **No negative marking.** Enter the numerical value using the virtual keypad.
- **Calculator policy:** Personal calculators, log tables, and electronic devices are **strictly prohibited**. A virtual scientific calculator is provided in the CBT interface. Rough sheets are supplied at the centre.
- Once you move to the next section, you cannot return to a previous section.

Section A — Multiple Choice Questions (MCQ)

***Q1–Q10**: 1 mark each ($-\frac{1}{3}$ for wrong). Choose the **single** correct option.*

Q1. Which of the following statements about sequences in $\mathbb{R}$ is **true**?

(A) Every Cauchy sequence in $\mathbb{R}$ is monotone.
(B) Every bounded monotone sequence in $\mathbb{R}$ converges.
(C) Every convergent sequence in $\mathbb{R}$ is monotone.
(D) Every bounded sequence in $\mathbb{R}$ converges.

Q2. Let $f(x) = x^3 - 3x$ on the closed interval $[-\sqrt{3}, \sqrt{3}]$. By Rolle's theorem, the value(s) of $c \in (-\sqrt{3}, \sqrt{3})$ satisfying $f'(c) = 0$ are:

(A) $c = \pm 1$
(B) $c = \pm\sqrt{3}$
(C) $c = \pm 2$
(D) $c = 0$

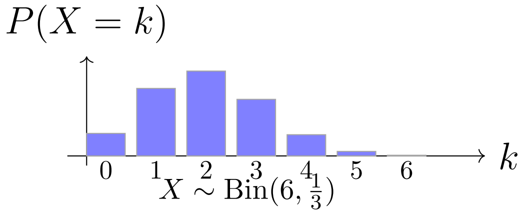
Q3. The determinant of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{pmatrix}$ equals:

(A) 18
(B) 20
(C) 22
(D) 24

Q4. A diagnostic test for a disease has sensitivity $P(+ \mid D) = 0.95$ and false positive rate $P(+ \mid D^c) = 0.05$. The prevalence of the disease is $P(D) = 0.01$. The probability that a person has the disease given a positive test result, $P(D \mid +)$, is closest to:

(A) 0.950
(B) 0.050
(C) 0.099
(D) 0.161

Q5. Let $X \sim \mathrm{Bin}(6, \frac{1}{3})$. The bar chart below shows the probability mass function of X.



The mean $E[X]$ and variance $\mathrm{Var}(X)$ of X are, respectively:

(A) $E[X] = 2, \quad \mathrm{Var}(X) = \dfrac{4}{3}$
(B) $E[X] = 2, \quad \mathrm{Var}(X) = 2$
(C) $E[X] = 3, \quad \mathrm{Var}(X) = 2$
(D) $E[X] = \dfrac{1}{3}, \quad \mathrm{Var}(X) = \dfrac{4}{3}$

Q6. Let $X \sim N(10, 4)$ (mean 10, variance 4). The probability $P(8 < X < 12)$ is closest to:

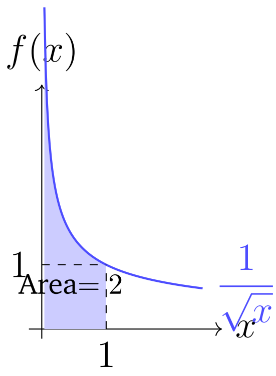
(A) 0.9545

(B) 0.6827
(C) 0.9973
(D) 0.5000

Q7. Let $X \sim U(1,7)$ (uniform on the interval $(1,7)$). The mean and variance of X are, respectively:

(A) $E[X] = 4, \quad \text{Var}(X) = 4$
(B) $E[X] = 3, \quad \text{Var}(X) = 3$
(C) $E[X] = 4, \quad \text{Var}(X) = 3$
(D) $E[X] = 3, \quad \text{Var}(X) = 4$

Q8. Consider the two improper integrals $I_1 = \int_0^1 x^{-1/2}\,dx$ and $I_2 = \int_1^\infty x^{-1/2}\,dx$. The shaded region below corresponds to I_1.



Which of the following correctly describes the convergence of I_1 and I_2?

(A) Both I_1 and I_2 converge.
(B) Both I_1 and I_2 diverge.
(C) I_1 diverges; I_2 converges.
(D) I_1 converges to 2; I_2 diverges.

Q9. Events A and B satisfy $P(A) = 0.5$, $P(B) = 0.4$, and $P(A \cap B) = 0.2$. The conditional probability $P(A \mid B)$ equals:

(A) 0.5
(B) 0.4
(C) 0.2
(D) 0.8

Q10. A random variable X has probability mass function:

$$P(X = 0) = 0.3, \quad P(X = 1) = 0.4, \quad P(X = 2) = 0.2, \quad P(X = 3) = 0.1.$$

The expected value $E[X]$ equals:

(A) 1.0
(B) 1.1
(C) 1.2
(D) 1.3

***Q11–Q30**: 2 marks each ($-\frac{2}{3}$ for wrong). Choose the **single** correct option.*

Q11. Consider the series $\displaystyle\sum_{n=1}^{\infty} n\, r^n$ where r is a real number. Which test **correctly** proves that this series converges for all $|r| < 1$?

(A) Integral test, since $\int_1^{\infty} x\, r^x\, dx$ converges for $|r| < 1$.
(B) Comparison test with $\sum r^n$.
(C) Ratio test: $\displaystyle\lim_{n\to\infty} \left| \frac{(n+1)r^{n+1}}{n\, r^n} \right| = |r| < 1$, so the series converges.
(D) Root test: $\displaystyle\lim_{n\to\infty} (n\, |r|^n)^{1/n} = |r| < 1$.

Q12. The Maclaurin series of $f(x) = (1 + x)^{1/2}$ begins $1 + \frac{x}{2} + c_2 x^2 + \cdots$. The coefficient c_2 equals:

(A) $\dfrac{1}{2}$
(B) $\dfrac{1}{8}$
(C) $-\dfrac{1}{4}$
(D) $-\dfrac{1}{8}$

Q13. The eigenvalues of the matrix $A = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}$ are:

(A) $\{3, 7\}$
(B) $\{5, 2\}$
(C) $\{7, 5\}$
(D) $\{3, 5\}$

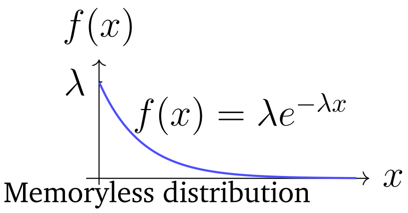
Q14. The initial value problem $\dfrac{dy}{dx} = 2xy,\ \ y(0) = 1$ has the solution $y = y(x)$. The value $y(1)$ equals:

(A) 1
(B) e
(C) e^2
(D) $2e$

Q15. Let $X \sim \text{Poisson}(4)$. The probability $P(X \leq 1)$ equals:

(A) $4e^{-4}$
(B) e^{-4}
(C) $5e^{-4}$
(D) $6e^{-4}$

Q16. Let $X \sim \text{Exp}(\lambda)$ with PDF $f(x) = \lambda e^{-\lambda x}$ for $x > 0$, shown below.



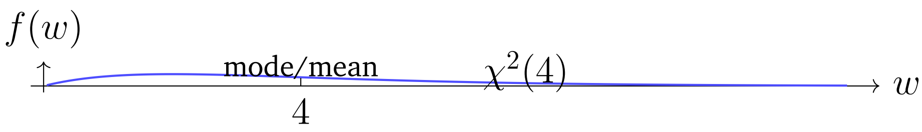
Which statement **correctly** expresses the memoryless property of X?

(A) $P(X > s + t \mid X > s) = P(X > s + t)$
(B) $E[X^2] = \dfrac{1}{\lambda^2}$
(C) $P(X > s + t \mid X > s) = P(X > s)$
(D) $P(X > s + t \mid X > s) = P(X > t)$ for all $s, t > 0$

Q17. The joint PDF of (X, Y) is $f(x, y) = 4xy$ for $0 < x < 1$, $0 < y < 1$, and 0 otherwise. The marginal PDF of X is:

(A) $f_X(x) = 2x, \quad 0 < x < 1$
(B) $f_X(x) = 4x, \quad 0 < x < 1$
(C) $f_X(x) = 2x^2, \quad 0 < x < 1$
(D) $f_X(x) = x, \quad 0 < x < 1$

Q18. Let Z_1, Z_2, Z_3, Z_4 be i.i.d. $N(0,1)$ random variables and let $W = Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2$. The PDF of the $\chi^2(k)$ distribution (for $k = 4$) is sketched below.



Which statement about $E[W]$ is **correct**?

(A) $E[\chi^2(k)] = k/2$
(B) $E[\chi^2(k)] = k$
(C) $E[\chi^2(k)] = 2k$
(D) $E[\chi^2(k)] = k^2$

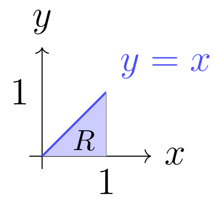
Q19. Let $X_1, \ldots, X_n$ be i.i.d. from the exponential distribution with rate parameter $\theta > 0$ and PDF $f(x;\theta) = \theta e^{-\theta x}$ for $x > 0$. The maximum likelihood estimator $\hat{\theta}$ of θ is:

(A) $\hat{\theta} = \bar{X}$
(B) $\hat{\theta} = \dfrac{n}{\sum X_i^2}$
(C) $\hat{\theta} = \dfrac{1}{\bar{X}}$
(D) $\hat{\theta} = \dfrac{\sum X_i}{n^2}$

Q20. A sample of $n = 16$ observations from a normal population gives $\bar{X} = 52$ and sample standard deviation $S = 8$. The one-sample t-statistic for testing $H_0 : \mu = 50$ vs. $H_1 : \mu \neq 50$ is $T = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}}$. The value of T is:

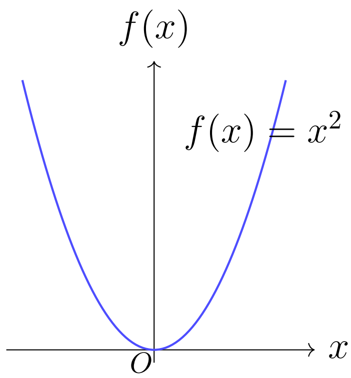
(A) 0.5
(B) 2.0
(C) 0.25
(D) 1.0

Q21. Evaluate the double integral $\displaystyle\int_0^1 \int_0^x (x+y)\, dy\, dx$ over the triangular region R shaded below.



(A) $\frac{1}{2}$
(B) $\frac{1}{3}$
(C) $\frac{2}{3}$
(D) $\frac{1}{4}$

Q22. The graph of $f(x) = x^2$ is shown below.



Which statement about $f(x) = x^2$ and uniform continuity is **correct**?

(A) $f(x) = x^2$ is uniformly continuous on all of $\mathbb{R}$.
(B) $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$, but is uniformly continuous on every bounded closed interval $[0, M]$.
(C) $f(x) = x^2$ is nowhere uniformly continuous.
(D) $f(x) = x^2$ is uniformly continuous on $(0, \infty)$ but not on any closed bounded interval.

Q23. The quadratic form $Q(x_1, x_2) = 2x_1^2 + 3x_1x_2 + 2x_2^2$ has associated symmetric matrix $A = \begin{pmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & 2 \end{pmatrix}$. The nature of Q is:

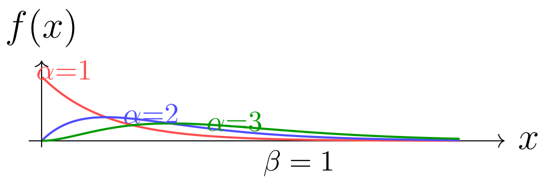
(A) Positive semi-definite but not positive definite.
(B) Indefinite.
(C) Positive definite.

(D) Negative definite.

Q24. Let $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ be independent random variables. The distribution of $X + Y$ is:

(A) $N\big(\mu_1 + \mu_2,\ (\sigma_1 + \sigma_2)^2\big)$
(B) $N\big(\mu_1\mu_2,\ \sigma_1^2\sigma_2^2\big)$
(C) $N(\mu_1 + \mu_2,\ \sigma_1\sigma_2)$
(D) $N\big(\mu_1 + \mu_2,\ \sigma_1^2 + \sigma_2^2\big)$

Q25. Let $X \sim \text{Gamma}(\alpha, \beta)$ with shape α and rate β. PDFs for $\beta = 1$ and $\alpha = 1, 2, 3$ are shown below.



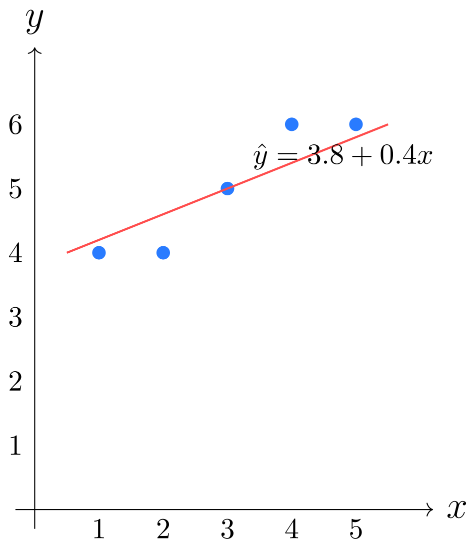
For $X \sim \text{Gamma}(\alpha, \beta)$, the mean $E[X]$ and variance $\text{Var}(X)$ are:

(A) $E[X] = \dfrac{\alpha}{\beta}, \quad \text{Var}(X) = \dfrac{\alpha}{\beta^2}$
(B) $E[X] = \dfrac{\beta}{\alpha}, \quad \text{Var}(X) = \dfrac{\beta}{\alpha^2}$
(C) $E[X] = \alpha, \quad \text{Var}(X) = \beta$
(D) $E[X] = \alpha\beta, \quad \text{Var}(X) = \alpha\beta^2$

Q26. For a bivariate data set with $n = 5$, the summary statistics are:

$$\bar{x} = 3, \quad \bar{y} = 5, \quad S_{xx} = \sum x_i^2 - n\bar{x}^2 = 10, \quad S_{yy} = 10, \quad S_{xy} = \sum x_i y_i - n\bar{x}\bar{y} = 4.$$

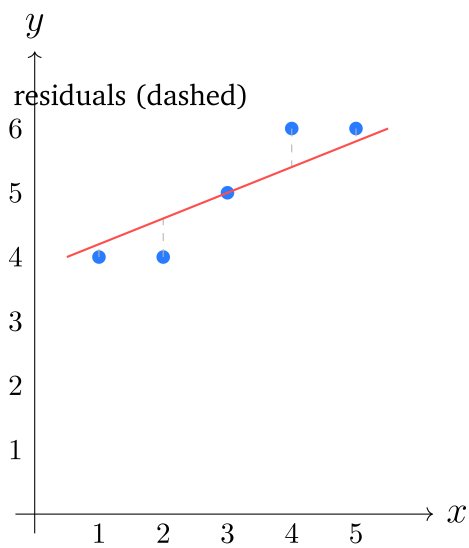
A scatter plot of the data is shown below.



The sample correlation coefficient $r = S_{xy}/\sqrt{S_{xx} \cdot S_{yy}}$ equals:

(A) 0.2
(B) 0.4
(C) 0.6
(D) 0.8

Q27. Using the same bivariate data set as in Q26 ($\bar{x} = 3$, $\bar{y} = 5$, $S_{xx} = 10$, $S_{xy} = 4$), the OLS regression line $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$ is fitted as shown below.



The OLS estimates $\hat{\beta}_1$ and $\hat{\beta}_0$ are, respectively:

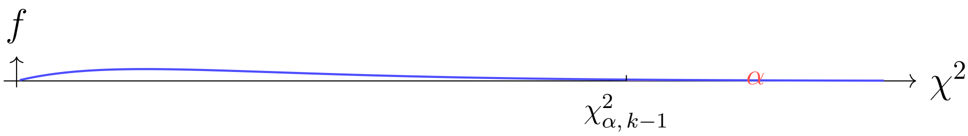
(A) $\hat{\beta}_1 = 0.4, \quad \hat{\beta}_0 = 5.0$
(B) $\hat{\beta}_1 = 0.2, \quad \hat{\beta}_0 = 3.8$

(C) $\hat{\beta}_1 = 0.4, \quad \hat{\beta}_0 = 3.8$
(D) $\hat{\beta}_1 = 0.8, \quad \hat{\beta}_0 = 2.6$

Q28. Which of the following statements about uniformly minimum variance unbiased estimators (UMVUE) is **correct**?

(A) Every sufficient statistic is also complete.
(B) The sample mean $\bar{X}$ is always the UMVUE of the population mean.
(C) The Cramér–Rao lower bound is always attained by the MLE.
(D) By the Rao–Blackwell theorem, if T is unbiased for θ and S is a complete sufficient statistic, then $E[T \mid S]$ is the UMVUE of θ.

Q29. In a chi-square goodness-of-fit test with k categories, the test statistic is $\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$. The figure below shows the χ^2 distribution with critical region (shaded) in the right tail.



Under H_0, the test statistic χ^2 follows a chi-square distribution with how many degrees of freedom?

(A) $k-1$, where k is the number of categories.
(B) k, where k is the number of categories.
(C) $k-2$, where k is the number of categories.
(D) $n-1$, where n is the total sample size.

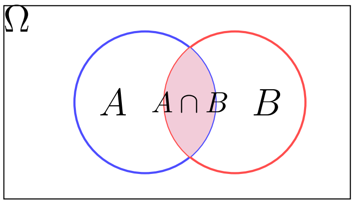
Q30. Let $X_1, \ldots, X_n$ be i.i.d. $U(0,1)$ random variables with order statistics $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$. The expected value of the k-th order statistic $X_{(k)}$ is:

(A) $\dfrac{k}{n}$
(B) $\dfrac{k}{n+1}$
(C) $\dfrac{k-1}{n-1}$
(D) $\dfrac{k}{2n}$

Section B — Multiple Select Questions (MSQ)

***Q31–Q40**: 2 marks each. **No negative marking. One or more** options may be correct; full marks only when* all *correct options (and no incorrect options) are selected.*

Q31. A probability space $(\Omega, \mathcal{F}, P)$ is given. The Venn diagram below shows two events A and B.



Which of the following statements are **correct**?

(A) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
(B) If $P(A) > 0$ and $P(B) > 0$, then $P(A \cap B) > 0$.
(C) $P(A^c) = 1 - P(A)$
(D) $P(A \mid B) = P(B \mid A)$ for all events A and B.

Q32. Let $X \sim N(\mu, \sigma^2)$. Which of the following statements are **correct**?

(A) For constants $a \neq 0$ and b, $aX + b \sim N(a\mu + b,\ a\sigma^2)$.
(B) For constants $a \neq 0$ and b, $aX + b \sim N(a\mu + b,\ a^2\sigma^2)$.
(C) The standard normal distribution $N(0, 1)$ has positive skewness.
(D) If $X_1, X_2, \ldots, X_n$ are independent normal random variables, then $\sum_{i=1}^{n} X_i$ is also normally distributed.

Q33. Let A be an $m \times n$ real matrix and B be an $n \times p$ real matrix. Which of the following statements are **correct**?

(A) $\text{rank}(A) = \text{rank}(A^\top)$
(B) $\text{rank}(A) + \text{nullity}(A) = n$ (the number of columns of A)
(C) $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$
(D) $\text{rank}(A + B) = \text{rank}(A) + \text{rank}(B)$ for any two $m \times n$ matrices A and B.

Q34. Which of the following probability inequalities or identities are **correct**?

(A) **Markov's inequality**: For a non-negative random variable X and $a > 0$, $P(X \geq a) \leq \frac{E[X]}{a}$.

(B) **Chebyshev's inequality**: $P(|X - \mu| \geq k\sigma) \leq k^2$ for all $k > 0$.

(C) **Jensen's inequality**: If f is convex, then $f(E[X]) \leq E[f(X)]$.

(D) **Linearity of expectation**: $E[aX + b] = aE[X] + b$ for constants a, b.

Q35. Which of the following statements about the method of maximum likelihood estimation are **correct**?

(A) The MLE is always an unbiased estimator of the true parameter.

(B) The MLE is invariant under one-to-one (bijective) transformations of the parameter.

(C) The MLE always achieves the Cramér–Rao lower bound for all sample sizes.

(D) Under standard regularity conditions, the MLE is asymptotically normal and asymptotically efficient.

Q36. Which of the following statements about linear ordinary differential equations (ODEs) are **correct**?

(A) The general solution of an nth-order linear ODE is $y = y_h + y_p$, where y_h is the general solution of the associated homogeneous equation and y_p is any particular solution.

(B) If the Wronskian $W[y_1, y_2](x_0) = 0$ at some point x_0, then y_1 and y_2 are linearly independent solutions.

(C) If y_1 and y_2 are both solutions of the homogeneous linear ODE $Ly = 0$, then $c_1y_1 + c_2y_2$ is also a solution for any constants c_1, c_2.

(D) For the ODE $ay'' + by' + cy = f(x)$ with $f \not\equiv 0$, the particular solution y_p is always a polynomial.

Q37. Let $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\lambda_2)$, independent. Which of the following statements are **correct**?

(A) $E[X] = \text{Var}(X) = \lambda$

(B) $X + Y \sim \text{Poisson}(\lambda + \lambda_2)$

(C) The Poisson distribution is symmetric about its mean for all $\lambda > 0$.

(D) The Poisson distribution is the limiting distribution of $\mathrm{Bin}(n, \lambda/n)$ as $n \to \infty$.

Q38. In hypothesis testing, let α denote the significance level (Type I error probability) and β the Type II error probability. Which of the following statements are **correct**?

(A) Increasing the significance level α always decreases the power of the test.
(B) Type II error probability: $\beta = P(\text{fail to reject } H_0 \mid H_1 \text{ is true})$.
(C) The power of a test equals $1 - \beta$.
(D) The critical (rejection) region is always located in the right tail of the test statistic distribution.

Q39. Let r denote the sample Pearson correlation coefficient between X and Y. Which of the following statements are **correct**?

(A) $-1 \leq r \leq 1$
(B) $r = 0$ implies that X and Y are independent.
(C) $r = \pm 1$ if and only if there is a perfect linear relationship between X and Y.
(D) The correlation coefficient r is dimensionless and invariant under positive linear rescaling of X and Y.

Q40. In simple linear regression $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$ with $\varepsilon_i \overset{\text{iid}}{\sim} N(0, \sigma^2)$, which of the following statements are **correct**?

(A) The OLS estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ are unbiased even without the assumption $E[\varepsilon_i] = 0$.
(B) The OLS method minimises $\sum_{i=1}^{n} (y_i - \hat{y}_i)^2$.
(C) The coefficient of determination R^2 can be negative.
(D) Under the Gauss–Markov conditions, OLS estimators are BLUE (Best Linear Unbiased Estimators).

Section C — Numerical Answer Type (NAT)

***Q41–Q50**: 1 mark each. **No negative marking.** Enter the exact numerical value.*

Q41. Let $X \sim \mathrm{Bin}(10, 0.3)$. The expected value $E[X]$ equals _____.

Q42. If $Z \sim N(0, 1)$, the value z such that $P(Z > z) = 0.025$ is _____.

Q43. The **larger** eigenvalue of the matrix $A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$ equals _____.

Q44. For a data set with $n = 5$, $\bar{x} = 3$, $\bar{y} = 5$, $S_{xx} = 10$, $S_{yy} = 10$, and $S_{xy} = 4$, the sample correlation coefficient r (to 2 decimal places) equals _____.

Q45. Let $X_1, \ldots, X_5$ be i.i.d. $\mathrm{Poisson}(\lambda)$ with observed values $2, 3, 1, 4, 0$. The maximum likelihood estimate of λ equals _____.

Q46. A 2×2 contingency table has observed frequencies $O = \begin{pmatrix} 10 & 20 \\ 20 & 10 \end{pmatrix}$ with $n = 60$. Under independence, all expected frequencies are 15. The chi-square statistic $\chi^2 = \sum (O - E)^2/E$ (to 2 decimal places) equals _____.

Q47. The sum of the infinite geometric series $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ equals _____.

Q48. Let $X \sim \mathrm{Exp}(2)$ (rate parameter 2). The probability $P(X > 1)$ to 4 significant figures equals _____.

Q49. Let X and Y be independent random variables with $\mathrm{Var}(X) = 4$ and $\mathrm{Var}(Y) = 9$. The value of $\mathrm{Var}(2X + 3Y)$ equals _____.

Q50. The solution of the initial value problem $\frac{dy}{dx} + 2y = 6$, $y(0) = 0$ satisfies $y(1) =$ _____ (to 3 decimal places). Use $e^{-2} \approx 0.1353$.

Q51–Q60**: 2 marks each.* ***No negative marking.

Q51. A random sample of size $n = 16$ from $N(\mu, 25)$ gives $\bar{x} = 52$. The lower bound of the 95% confidence interval for μ (using $z_{0.025} = 1.96$) equals _____.

Q52. Given the fitted regression line $\hat{y} = 2 + 3x$, the predicted value of y when $x = 5$ is _____.

Q53. For testing $H_0 : \mu = 10$ vs. $H_1 : \mu = 12$ with $\sigma = 5$, $n = 25$, and $\alpha = 0.05$ (one-sided), the rejection region is $\bar{X} > 11.645$. The power of the test when $\mu = 12$ (to 3 decimal places) equals _____. Use $\Phi(0.355) \approx 0.639$.

Q54. Let X and Y be independent $N(0, 1)$ random variables. The probability $P(X + Y < 2)$ (to 3 decimal places) equals _____. Use $\Phi(1.414) \approx 0.921$.

Q55. The determinant of the matrix $B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ equals _____.

Q56. Let $X_1, \ldots, X_9$ be i.i.d. $\text{Bernoulli}(p)$ with $p = 0.5$. The Cramér–Rao lower bound for the variance of any unbiased estimator of p (to 4 decimal places) equals _____.

Q57. A random sample of size $n = 9$ from $N(\mu, 4)$ gives $\bar{x} = 11.5$. The Z-statistic for testing $H_0 : \mu = 10$ vs. $H_1 : \mu \neq 10$ (to 2 decimal places) equals _____.

Q58. Let $X \sim \text{Bin}(100, 0.4)$. Using the normal approximation with continuity correction, $P(X \leq 45) \approx \Phi(z)$ where the z-value (to 2 decimal places) equals _____. (Use $\sqrt{24} \approx 4.899$).

Q59. A prior $p \sim \text{Beta}(1, 1)$ (uniform) is updated with likelihood $X \mid p \sim \text{Bin}(10, p)$ and observed $X = 7$. The posterior mean $E[p \mid X = 7]$ (to 3 decimal places) equals _____.

Q60. Let $f(x) = k \cdot x^2 e^{-x}$ for $x > 0$ and $f(x) = 0$ otherwise. The value of k that makes f a valid probability density function equals _____.

Solutions

IIT JAM Mathematical Statistics – Sample Paper 1

Detailed Solutions

Q1.

Solution

Concept — Monotone Convergence Theorem:
A fundamental result in real analysis states that if a sequence $\{a_n\}$ in $\mathbb{R}$ is both bounded and monotone (either non-decreasing or non-increasing), then it must converge. This is the Monotone Convergence Theorem.

Step 1 — Evaluating each option:
(A) FALSE: A Cauchy sequence need not be monotone. For example, $a_n = (-1)^n/n$ is Cauchy and convergent but not monotone.
(B) TRUE: By the Monotone Convergence Theorem, every bounded monotone sequence in $\mathbb{R}$ converges. This follows from the completeness of $\mathbb{R}$.
(C) FALSE: A convergent sequence need not be monotone. Example: $a_n = (-1)^n/n \to 0$ but is not monotone.
(D) FALSE: A bounded sequence need not converge; it may oscillate. Example: $a_n = (-1)^n$ is bounded in $[-1, 1]$ but diverges.

Final Answer: Every bounded monotone sequence in $\mathbb{R}$ converges $\Rightarrow$ **(B)**

Answer: (B) ← *Go back to Q1*

Solution

Concept — Rolle's Theorem:
If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then there exists $c \in (a, b)$ with $f'(c) = 0$.

Step 1 — Verify conditions:
$f(x) = x^3 - 3x$ is a polynomial (continuous and differentiable everywhere).
$f(-\sqrt{3}) = (-\sqrt{3})^3 - 3(-\sqrt{3}) = -3\sqrt{3} + 3\sqrt{3} = 0$ and $f(\sqrt{3}) = 3\sqrt{3} - 3\sqrt{3} = 0$.
So $f(-\sqrt{3}) = f(\sqrt{3}) = 0$. All conditions of Rolle's theorem are satisfied.

Step 2 — Find c:
$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 0 \;\Rightarrow\; x^2 = 1 \;\Rightarrow\; x = \pm 1$.
Both $c = 1$ and $c = -1$ lie in $(-\sqrt{3}, \sqrt{3})$. Hence $c = \pm 1$.

Why other options are wrong:

Q2.
- (B) $c = \pm\sqrt{3}$: These are the endpoints, not interior points.
- (C) $c = \pm 2$: $f'(\pm 2) = 3(4) - 3 = 9 \neq 0$.
- (D) $c = 0$: $f'(0) = -3 \neq 0$.

Final Answer: $c = \pm 1 \Rightarrow$ **(A)**

Answer: (A) ← *Go back to Q2*

Solution

Concept — Determinant by cofactor expansion:
Expand along the first row: $\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$.

Step 1 — Cofactor expansion along row 1:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{pmatrix}$$

$$\det(A) = 1 \cdot \det\begin{pmatrix} 4 & 5 \\ 0 & 6 \end{pmatrix} - 2 \cdot \det\begin{pmatrix} 0 & 5 \\ 1 & 6 \end{pmatrix} + 3 \cdot \det\begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$$

Step 2 — Evaluate 2×2 determinants:
$\det\begin{pmatrix} 4 & 5 \\ 0 & 6 \end{pmatrix} = 24 - 0 = 24;\quad \det\begin{pmatrix} 0 & 5 \\ 1 & 6 \end{pmatrix} = 0 - 5 = -5;\quad \det\begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} = 0 - 4 = -4.$
$\det(A) = 1(24) - 2(-5) + 3(-4) = 24 + 10 - 12 = \mathbf{22}.$

Why other options are wrong:

Q3.
- (A) 18: Arises from an arithmetic error in the cofactor expansion.
- (B) 20: Incorrect; misses the -12 term.
- (D) 24: This is only the first minor; the full expansion gives 22.

Final Answer: $\det(A) = 22 \Rightarrow$ **(C)**

Answer: (C) ← *Go back to Q3*

Solution

Concept — Bayes' Theorem:

$$P(D \mid +) = \frac{P(+ \mid D)\, P(D)}{P(+ \mid D)\, P(D) + P(+ \mid D^c)\, P(D^c)}$$

Step 1 — Identify values:
$P(D) = 0.01,\ P(+ \mid D) = 0.95,\ P(+ \mid D^c) = 0.05,\ P(D^c) = 0.99.$

Step 2 — Apply Bayes' theorem:

$$P(D \mid +) = \frac{0.95 \times 0.01}{0.95 \times 0.01 + 0.05 \times 0.99} = \frac{0.0095}{0.0095 + 0.0495} = \frac{0.0095}{0.0590} \approx 0.161.$$

Why other options are wrong:

Q4.

- (A) 0.950: This is the sensitivity $P(+ \mid D)$, not the posterior.
- (B) 0.050: This is the false positive rate $P(+ \mid D^c)$.
- (C) 0.099: Incorrect computation; forgetting to divide by $P(+)$.

Final Answer: $P(D \mid +) \approx 0.161 \Rightarrow$ **(D)**

Answer: (D) ← *Go back to Q4*

Solution

Concept — Binomial Distribution:
For $X \sim \text{Bin}(n, p)$: $E[X] = np$ and $\text{Var}(X) = np(1-p)$.

Step 1 — Compute mean and variance:
With $n = 6$ and $p = \frac{1}{3}$:

$$E[X] = 6 \times \tfrac{1}{3} = 2, \qquad \text{Var}(X) = 6 \times \tfrac{1}{3} \times \tfrac{2}{3} = \frac{12}{9} = \frac{4}{3}.$$

Why other options are wrong:

Q5.

- (B) $\text{Var}(X) = 2$: Incorrect; $np(1-p) = 4/3 \neq 2$.
- (C) $E[X] = 3$: Incorrect; $E[X] = 2$, not 3.
- (D) $E[X] = 1/3$: This is p, not $E[X] = np$.

Final Answer: $E[X] = 2$, $\text{Var}(X) = \frac{4}{3} \Rightarrow$ **(A)**

Answer: (A) ← *Go back to Q5*

Solution

Concept — Normal Distribution Standardisation:
If $X \sim N(\mu, \sigma^2)$, then $Z = (X - \mu)/\sigma \sim N(0, 1)$, and probabilities are read from the standard normal table.

Step 1 — Standardise:

$X \sim N(10, 4)$ means $\mu = 10$, $\sigma^2 = 4$, $\sigma = 2$.

$$P(8 < X < 12) = P\left(\frac{8-10}{2} < Z < \frac{12-10}{2}\right) = P(-1 < Z < 1).$$

Step 2 — Read from the standard normal table:
$P(-1 < Z < 1) = 2\Phi(1) - 1 = 2(0.8413) - 1 = 0.6826 \approx 0.6827$.

Why other options are wrong:

Q6.

- (A) 0.9545: This is $P(-2 < Z < 2)$, i.e., within 2σ of mean.
- (C) 0.9973: This is $P(-3 < Z < 3)$, i.e., within 3σ of mean.
- (D) 0.5000: Corresponds to $P(-0 < Z < 0) = 0$, or only one tail.

Final Answer: $P(8 < X < 12) \approx 0.6827 \Rightarrow$ **(B)**

Answer: (B) ← *Go back to Q6*

Solution

Concept — Uniform Distribution:
For $X \sim U(a, b)$: $E[X] = \dfrac{a+b}{2}$ and $\mathrm{Var}(X) = \dfrac{(b-a)^2}{12}$.

Step 1 — Apply formulas:
With $a = 1$, $b = 7$:

$$E[X] = \frac{1+7}{2} = 4, \qquad \mathrm{Var}(X) = \frac{(7-1)^2}{12} = \frac{36}{12} = 3.$$

Why other options are wrong:

Q7.

- (A) $\mathrm{Var}(X) = 4$: Incorrect; $(b-a)^2/12 = 36/12 = 3$.
- (B) $E[X] = 3$: Incorrect; $(1+7)/2 = 4$.
- (D) $E[X] = 3$, $\mathrm{Var}(X) = 4$: Both values are wrong.

Final Answer: $E[X] = 4$, $\mathrm{Var}(X) = 3 \Rightarrow$ **(C)**

Answer: (C) ← *Go back to Q7*

Solution

Concept — Convergence of Improper Integrals:
$\int_0^1 x^{-p}\,dx$ converges iff $p < 1$. $\int_1^\infty x^{-p}\,dx$ converges iff $p > 1$.

Step 1 — Evaluate I_1:

$I_1 = \int_0^1 x^{-1/2}\,dx = \lim_{\epsilon\to 0^+} \int_\epsilon^1 x^{-1/2}\,dx = \lim_{\epsilon\to 0^+} \left[2\sqrt{x}\right]_\epsilon^1 = \lim_{\epsilon\to 0^+} (2 - 2\sqrt{\epsilon}) = 2.$ **Converges.**

Step 2 — Evaluate I_2:
$I_2 = \int_1^\infty x^{-1/2}\,dx = \lim_{R\to\infty} \left[2\sqrt{x}\right]_1^R = \lim_{R\to\infty} (2\sqrt{R} - 2) = +\infty.$ **Diverges.**
Here $p = 1/2 < 1$, so I_1 converges, and $p = 1/2 < 1$, so I_2 diverges.

Why other options are wrong:

Q8.
- (A) "Both converge": I_2 diverges.
- (B) "Both diverge": $I_1 = 2$ converges.
- (C) "I_1 diverges": Incorrect; $I_1 = 2$.

Final Answer: $I_1 = 2$ (converges), I_2 diverges $\Rightarrow$ **(D)**

Answer: (D) ← *Go back to Q8*

Solution

Concept — Conditional Probability:
$P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ provided $P(B) > 0$.

Step 1 — Apply the formula:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.4} = 0.5.$$

Why other options are wrong:

Q9.
- (B) 0.4: This is $P(B)$, not $P(A \mid B)$.
- (C) 0.2: This is $P(A \cap B)$, not the conditional probability.
- (D) 0.8: Incorrect; $P(B \mid A) = P(A \cap B)/P(A) = 0.2/0.5 = 0.4$, not 0.8.

Final Answer: $P(A \mid B) = 0.5 \Rightarrow$ **(A)**

Answer: (A) ← *Go back to Q9*

Solution

Concept — Expected Value of a Discrete RV:
$E[X] = \sum_x x\,P(X = x)$.

Step 1 — Compute the expectation:

$$E[X] = 0 \times 0.3 + 1 \times 0.4 + 2 \times 0.2 + 3 \times 0.1 = 0 + 0.4 + 0.4 + 0.3 = 1.1.$$

Why other options are wrong:

Q10.

- (A) 1.0: Underestimates; ignores the contribution of $X = 3$.
- (C) 1.2: Overestimates; arithmetic error.
- (D) 1.3: This would require a higher weight on larger values.

Final Answer: $E[X] = 1.1 \Rightarrow$ **(B)**

Answer: (B) ← *Go back to Q10*

Solution

Concept — Ratio Test for Series:
For a series $\sum a_n$, if $\lim_{n\to\infty} |a_{n+1}/a_n| = L < 1$, the series converges absolutely.

Step 1 — Apply the ratio test to $\sum nr^n$:

$$\lim_{n\to\infty} \left| \frac{(n+1)r^{n+1}}{nr^n} \right| = \lim_{n\to\infty} \frac{n+1}{n} |r| = 1 \times |r| = |r|.$$

Step 2 — Conclude:
Since $|r| < 1$, the ratio test gives $L = |r| < 1$, so the series $\sum_{n=1}^{\infty} nr^n$ converges absolutely for all $|r| < 1$.

Why other options are wrong:

Q11.

- (A) Integral test: Requires a_n to be a continuous, positive, decreasing function; the integral test applies but is less direct here.
- (B) Comparison with $\sum r^n$: Since $nr^n \geq r^n$, comparison is in the wrong direction for convergence.
- (D) Root test: $\lim (n|r|^n)^{1/n} = |r| \cdot \lim n^{1/n} = |r|$. The root test also works, but option (C) is the most direct and correctly stated.

Final Answer: Ratio test proves convergence for $|r| < 1 \Rightarrow$ **(C)**

Answer: (C) ← *Go back to Q11*

Solution

Concept — Binomial (Generalised) Maclaurin Series:
$(1+x)^\alpha = \displaystyle\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$ where $\binom{\alpha}{k} = \dfrac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$.

Step 1 — Find coefficient of x^2 for $\alpha = \frac{1}{2}$:

$$c_2 = \binom{1/2}{2} = \frac{(1/2)(1/2-1)}{2!} = \frac{(1/2)(-1/2)}{2} = \frac{-1/4}{2} = -\frac{1}{8}.$$

Step 2 — Verify the series:
$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots$

Why other options are wrong:

Q12.

- (A) $1/2$: This is c_1 (coefficient of x), not c_2.
- (B) $1/8$: Correct magnitude but wrong sign; $c_2 = -1/8$.
- (C) $-1/4$: Off by a factor of 2; $\binom{1/2}{2} = -1/8$.

Final Answer: $c_2 = -\frac{1}{8} \Rightarrow$ **(D)**

Answer: (D) ← *Go back to Q12*

Solution

Concept — Eigenvalues of a Symmetric Matrix:
Eigenvalues λ satisfy $\det(A - \lambda I) = 0$.

Step 1 — Set up the characteristic equation:

$$\det\begin{pmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{pmatrix} = (5-\lambda)^2 - 4 = 0.$$

Step 2 — Solve:
$(5-\lambda)^2 = 4 \Rightarrow 5-\lambda = \pm 2 \Rightarrow \lambda = 5 \mp 2$, so $\lambda_1 = 7$ and $\lambda_2 = 3$.

Why other options are wrong:

Q13.

- (B) $\{5, 2\}$: These are the entries of the matrix, not the eigenvalues.
- (C) $\{7, 5\}$: $\lambda = 5$ does not satisfy the characteristic equation.
- (D) $\{3, 5\}$: $\lambda = 5$ is not an eigenvalue.

Final Answer: Eigenvalues are $\{3, 7\} \Rightarrow$ **(A)**

Answer: (A) ← *Go back to Q13*

Solution

Concept — Separable ODE:
Separate variables, integrate both sides, apply the initial condition.

Step 1 — Separate and integrate:
$\frac{dy}{y} = 2x\,dx \;\Rightarrow\; \ln|y| = x^2 + C \;\Rightarrow\; y = Ae^{x^2}$.

Step 2 — Apply initial condition and evaluate:
$y(0) = Ae^0 = A = 1$, so $y = e^{x^2}$.
$y(1) = e^{1^2} = e \approx 2.718$.

Why other options are wrong:

Q14.

- (A) $y(1) = 1$: Would require $y = 1$ (constant), but $y' = 2xy = 2x \neq 0$.
- (C) $y(1) = e^2$: This would come from $y = e^{2x}$, which solves $y' = 2y$, not $y' = 2xy$.
- (D) $y(1) = 2e$: No basis for this value.

Final Answer: $y(1) = e \Rightarrow$ **(B)**

Answer: (B) ← *Go back to Q14*

Solution

Concept — Poisson Probability:
For $X \sim \text{Poisson}(\lambda)$: $P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$.

Step 1 — Compute $P(X \leq 1)$:

$$P(X \leq 1) = P(X = 0) + P(X = 1) = e^{-4}\frac{4^0}{0!} + e^{-4}\frac{4^1}{1!} = e^{-4}(1 + 4) = 5e^{-4}.$$

Step 2 — Numerical value:
$5e^{-4} \approx 5 \times 0.01832 \approx 0.0916$.

Why other options are wrong:

Q15.

- (A) $4e^{-4}$: This is only $P(X = 1)$.
- (B) e^{-4}: This is only $P(X = 0)$.
- (D) $6e^{-4}$: Includes $P(X = 2) = 8e^{-4}/2 = 4e^{-4}$, overcounting.

Final Answer: $P(X \leq 1) = 5e^{-4} \Rightarrow$ **(C)**

Answer: (C) ← *Go back to Q15*

Solution

Concept — Memoryless Property of Exponential Distribution:
The exponential distribution is the only continuous distribution with the memory-

less property. For $X \sim \text{Exp}(\lambda)$:

$$P(X > s+t \mid X > s) = P(X > t) \quad \text{for all } s, t > 0.$$

Step 1 — Verify the property:

$$P(X > s+t \mid X > s) = \frac{P(X > s+t)}{P(X > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t).$$

Why other options are wrong:

Q16.

- (A) States $P(X > s+t \mid X > s) = P(X > s+t)$: This is $P(X > s+t) = e^{-\lambda(s+t)} \neq P(X > t)$ in general.
- (B) $E[X^2] = 1/\lambda^2$: False; $E[X^2] = \text{Var}(X) + (E[X])^2 = 1/\lambda^2 + 1/\lambda^2 = 2/\lambda^2$.
- (C) States $P(X > s+t \mid X > s) = P(X > s)$: Incorrect formula.

Final Answer: $P(X > s+t \mid X > s) = P(X > t) \Rightarrow$ **(D)**

Answer: (D) ← *Go back to Q16*

Solution

Concept — Marginal PDF from Joint PDF:
$f_X(x) = \displaystyle\int_{-\infty}^{\infty} f(x, y)\, dy$.

Step 1 — Integrate out y:
For $0 < x < 1$:

$$f_X(x) = \int_0^1 4xy\, dy = 4x\left[\frac{y^2}{2}\right]_0^1 = 4x \cdot \frac{1}{2} = 2x.$$

Why other options are wrong:

Q17.

- (B) $f_X(x) = 4x$: Forgets the factor of $1/2$ from integrating y.
- (C) $f_X(x) = 2x^2$: Incorrect integration of y; $\int_0^1 y\, dy = 1/2$, not $x/2$.
- (D) $f_X(x) = x$: Misses the factor of 2 from $4 \times (1/2)$.

Final Answer: $f_X(x) = 2x$ for $0 < x < 1 \Rightarrow$ **(A)**

Answer: (A) ← *Go back to Q17*

Solution

Concept — Chi-square Distribution:
If $Z_1, \ldots, Z_k$ are i.i.d. $N(0, 1)$, then $W = \sum_{i=1}^{k} Z_i^2 \sim \chi^2(k)$.

Key moments: $E[\chi^2(k)] = k$ and $\text{Var}(\chi^2(k)) = 2k$.

Step 1 — Identify the distribution of W:
$W = Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 \sim \chi^2(4)$, so $k = 4$.

Step 2 — Apply the formula:
$E[W] = E[\chi^2(4)] = 4 = k$.

In general, $E[\chi^2(k)] = k$ for any $k \geq 1$.

Why other options are wrong:

Q18.

- (A) $k/2$: This is the mode of $\chi^2(k)$ for $k > 2$, not the mean.
- (C) $2k$: This is $\text{Var}(\chi^2(k))$, not the mean.
- (D) k^2: No statistical basis.

Final Answer: $E[\chi^2(k)] = k \Rightarrow$ **(B)**

Answer: (B) ← *Go back to Q18*

Solution

Concept — Maximum Likelihood Estimation for Exponential:
For $X \sim \text{Exp}(\theta)$ with PDF $\theta e^{-\theta x}$, maximise the log-likelihood.

Step 1 — Write the log-likelihood:

$$\ell(\theta) = n \ln \theta - \theta \sum_{i=1}^{n} x_i.$$

Step 2 — Differentiate and set to zero:

$$\frac{d\ell}{d\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i = 0 \Rightarrow \hat{\theta} = \frac{n}{\sum x_i} = \frac{1}{\bar{X}}.$$

Why other options are wrong:

Q19.

- (A) $\hat{\theta} = \bar{X}$: This is the MLE of the mean $1/\theta$, not of θ.
- (B) $n/\sum X_i^2$: Incorrect; setting $d\ell/d\theta = 0$ does not involve $\sum X_i^2$.
- (D) $\sum X_i/n^2$: Algebraic error.

Final Answer: $\hat{\theta} = 1/\bar{X} \Rightarrow$ **(C)**

Answer: (C) ← *Go back to Q19*

Solution

Concept — One-sample t-statistic:
$T = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t(n-1)$ under H_0.

Step 1 — Substitute values:

$$T = \frac{52 - 50}{8/\sqrt{16}} = \frac{2}{8/4} = \frac{2}{2} = 1.0.$$

Why other options are wrong:

Q20.

- (A) 0.5: Divides by S instead of $S/\sqrt{n}$.
- (B) 2.0: Uses $\sqrt{n} = 4$ correctly but forgets to divide: $2/(8/4) = 1$, not 2.
- (C) 0.25: Uses wrong denominator.

Final Answer: $T = 1.0 \Rightarrow$ **(D)**

Answer: (D) ← *Go back to Q20*

Solution

Concept — Iterated Integration over a Triangular Region:
Integrate the inner integral first (with respect to y), then the outer (with respect to x).

Step 1 — Inner integral (fix x, integrate over y):

$$\int_0^x (x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_0^x = x^2 + \frac{x^2}{2} = \frac{3x^2}{2}.$$

Step 2 — Outer integral:

$$\int_0^1 \frac{3x^2}{2}\,dx = \frac{3}{2}\left[\frac{x^3}{3}\right]_0^1 = \frac{3}{2}\cdot\frac{1}{3} = \frac{1}{2}.$$

Why other options are wrong:

Q21.

- (B) $1/3$: Arises if the integrand is taken as x alone (missing y).
- (C) $2/3$: Double the correct answer; possibly using range $[0, 2]$.
- (D) $1/4$: Results from integrating x^2 instead of $3x^2/2$.

Final Answer: The double integral equals $\dfrac{1}{2} \Rightarrow$ **(A)**

Answer: (A) ← *Go back to Q21*

Solution

Concept — Uniform Continuity:
A function f is uniformly continuous on S if for every $\varepsilon > 0$ there exists $\delta > 0$ (depending only on ε, not on points in S) such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ for all $x, y \in S$.

Step 1 — Check on $\mathbb{R}$:
$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y|$. For any fixed δ, if x is large, $|x + y|$ can be arbitrarily large, so no single δ works for all $x, y \in \mathbb{R}$. Thus $f(x) = x^2$ is **not** uniformly continuous on $\mathbb{R}$.

Step 2 — Check on $[0, M]$:
On a closed bounded interval $[0, M]$, any continuous function is uniformly continuous (Heine-Cantor theorem). So $f(x) = x^2$ **is** uniformly continuous on $[0, M]$ for any $M > 0$.

Why other options are wrong:

Q22.

- (A) Uniformly continuous on all of $\mathbb{R}$: FALSE, as shown in Step 1.
- (C) Nowhere uniformly continuous: FALSE; it is uniformly continuous on bounded closed intervals.
- (D) Uniformly continuous on $(0, \infty)$ but not on $[0, M]$: Backwards; it is not UC on $(0, \infty)$.

Final Answer: Not uniformly continuous on $\mathbb{R}$, but uniformly continuous on each $[0, M] \Rightarrow$ **(B)**

$\boxed{\textbf{Answer: (B)}}$ ← *Go back to Q22*

Solution

Concept — Nature of a Quadratic Form:
A quadratic form $Q(\mathbf{x}) = \mathbf{x}^\top A\mathbf{x}$ is positive definite iff all eigenvalues of the symmetric matrix A are strictly positive.

Step 1 — Associated matrix and its eigenvalues:
$A = \begin{pmatrix} 2 & 3/2 \\ 3/2 & 2 \end{pmatrix}$. Characteristic equation: $(2-\lambda)^2 - \frac{9}{4} = 0 \;\Rightarrow\; 2 - \lambda = \pm\frac{3}{2}$.
$\lambda_1 = 2 + \frac{3}{2} = \frac{7}{2} > 0$ and $\lambda_2 = 2 - \frac{3}{2} = \frac{1}{2} > 0$.

Step 2 — Conclude:
Both eigenvalues are positive, so Q is **positive definite**. (Alternatively, $\Delta_1 = 2 > 0$ and $\det(A) = 4 - 9/4 = 7/4 > 0$ by Sylvester's criterion.)

Why other options are wrong:

Q23.

- (A) Positive semi-definite but not PD: Requires at least one zero eigenvalue; here all are positive.
- (B) Indefinite: Requires eigenvalues of mixed sign; here both are positive.
- (D) Negative definite: Requires all eigenvalues negative; here both are positive.

Final Answer: Q is positive definite $\Rightarrow$ **(C)**

Answer: (C) ← *Go back to Q23*

Solution

Concept — Sum of Independent Normal Random Variables:
If $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ independently, then $X + Y \sim N(\mu_1 + \mu_2,\, \sigma_1^2 + \sigma_2^2)$.

Step 1 — Use the MGF approach:
$M_{X+Y}(t) = M_X(t) \cdot M_Y(t) = e^{\mu_1 t + \sigma_1^2 t^2/2} \cdot e^{\mu_2 t + \sigma_2^2 t^2/2} = e^{(\mu_1+\mu_2)t + (\sigma_1^2+\sigma_2^2)t^2/2}$.
This is the MGF of $N(\mu_1 + \mu_2,\, \sigma_1^2 + \sigma_2^2)$.

Why other options are wrong:

Q24.

- (A) Variance $(\sigma_1 + \sigma_2)^2$: Variances add, not standard deviations.
- (B) Mean $\mu_1\mu_2$, variance $\sigma_1^2\sigma_2^2$: These are for products, not sums.
- (C) Variance $\sigma_1\sigma_2$: Standard deviations do not multiply for sums.

Final Answer: $X + Y \sim N(\mu_1 + \mu_2,\, \sigma_1^2 + \sigma_2^2) \Rightarrow$ **(D)**

Answer: (D) ← *Go back to Q24*

Solution

Concept — Gamma Distribution Moments:
For $X \sim \text{Gamma}(\alpha, \beta)$ with shape α and rate β: $E[X] = \alpha/\beta$ and $\text{Var}(X) = \alpha/\beta^2$.

Step 1 — Derive from the PDF:
The PDF is $f(x) = \dfrac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$ for $x > 0$.

$$E[X] = \int_0^\infty x \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\, dx = \frac{\Gamma(\alpha+1)}{\beta\,\Gamma(\alpha)} = \frac{\alpha}{\beta}.$$

Step 2 — Variance:
$E[X^2] = \dfrac{\alpha(\alpha+1)}{\beta^2}$, so $\text{Var}(X) = E[X^2] - (E[X])^2 = \dfrac{\alpha(\alpha+1)}{\beta^2} - \dfrac{\alpha^2}{\beta^2} = \dfrac{\alpha}{\beta^2}$.

Why other options are wrong:

Q25.

- (B) β/α: Swaps the roles of α and β.
- (C) $E[X] = \alpha$, $\text{Var} = \beta$: True only for $\beta = 1$ (special case).
- (D) $E[X] = \alpha\beta$: This is the scale parameterisation (shape α, scale $\beta = 1/\text{rate}$); not this parameterisation.

Final Answer: $E[X] = \alpha/\beta$, $\text{Var}(X) = \alpha/\beta^2 \Rightarrow$ **(A)**

Answer: (A) ← *Go back to Q25*

Solution

Concept — Sample Pearson Correlation Coefficient:
$r = \dfrac{S_{xy}}{\sqrt{S_{xx} \cdot S_{yy}}}$ where $S_{xy} = \sum x_i y_i - n\bar{x}\bar{y}$, $S_{xx} = \sum x_i^2 - n\bar{x}^2$, $S_{yy} = \sum y_i^2 - n\bar{y}^2$.

Step 1 — Substitute given values:
$S_{xx} = 10$, $S_{yy} = 10$, $S_{xy} = 4$.

Step 2 — Compute r:

$$r = \frac{4}{\sqrt{10 \times 10}} = \frac{4}{\sqrt{100}} = \frac{4}{10} = 0.4.$$

Why other options are wrong:

Q26.

- (A) 0.2: Halves the correct answer; possibly divides by 20 instead of 10.
- (C) 0.6: Incorrect arithmetic.
- (D) 0.8: Doubles the S_{xy} value erroneously.

Final Answer: $r = 0.4 \Rightarrow$ **(B)**

Answer: (B) ← *Go back to Q26*

Solution

Concept — OLS Regression Estimators:
$\hat{\beta}_1 = \dfrac{S_{xy}}{S_{xx}}$ and $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$.

Step 1 — Compute $\hat{\beta}_1$:

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = \frac{4}{10} = 0.4.$$

Step 2 — Compute $\hat{\beta}_0$:

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x} = 5 - 0.4 \times 3 = 5 - 1.2 = 3.8.$$

Why other options are wrong:

Q27.
- (A) $\hat{\beta}_0 = 5.0$: Uses $\bar{y}$ directly without subtracting $\hat{\beta}_1\bar{x}$.
- (B) $\hat{\beta}_1 = 0.2$: Halves the slope by error.
- (D) $\hat{\beta}_1 = 0.8$, $\hat{\beta}_0 = 2.6$: Uses S_{xy}/S_{xx} with an error.

Final Answer: $\hat{\beta}_1 = 0.4$, $\hat{\beta}_0 = 3.8 \Rightarrow$ **(C)**

Answer: (C) ← *Go back to Q27*

Solution

Concept — Rao-Blackwell Theorem and UMVUE:
If T is an unbiased estimator of θ and S is a complete sufficient statistic, then $\delta(S) = E[T \mid S]$ is the unique UMVUE (uniformly minimum variance unbiased estimator) of θ.

Step 1 — Verify option (D):
The Rao-Blackwell theorem states: $\mathrm{Var}(E[T \mid S]) \leq \mathrm{Var}(T)$. Combined with completeness of S, $E[T \mid S]$ is the unique unbiased estimator that is a function of the complete sufficient statistic, hence the UMVUE.

Why other options are wrong:

Q28.
- (A) Every sufficient statistic is complete: FALSE. A statistic can be sufficient without being complete (e.g., the sufficient statistic for a two-parameter family may not be complete in either parameter separately).
- (B) $\bar{X}$ is always the UMVUE of μ: FALSE. For non-regular families this may not hold.
- (C) The Cramér-Rao lower bound is always attained by MLE: FALSE. The CRLB is not always attained; attainment requires the score function to be linear in the sufficient statistic.

Final Answer: Rao-Blackwell + completeness gives UMVUE $\Rightarrow$ **(D)**

Answer: (D) ← *Go back to Q28*

Solution

Concept — Chi-square Goodness-of-Fit Test:
For testing whether observed data follows a specified distribution with k categories and no estimated parameters, the test statistic $\chi^2 = \sum_{i=1}^{k}(O_i - E_i)^2/E_i$ follows a

$\chi^2(k-1)$ distribution under H_0.

Step 1 — Degrees of freedom:
One degree of freedom is lost because $\sum O_i = \sum E_i = n$ (the total is fixed). Thus $\text{df} = k - 1$.

Step 2 — Interpretation:
With $k = 5$ categories, $\text{df} = 4$, and the critical value at $\alpha = 0.05$ is $\chi^2_{0.05,4} \approx 9.488$ (as shown in the figure).

Why other options are wrong:

Q29.

- (B) k: Does not account for the constraint $\sum E_i = n$.
- (C) $k-2$: Would apply only if one parameter were estimated from the data.
- (D) $n-1$: This is the df for a t-test or variance test, not chi-square goodness-of-fit.

Final Answer: $\text{df} = k - 1 \Rightarrow$ **(A)**

Answer: (A) ← *Go back to Q29*

Solution

Concept — Order Statistics of Uniform Distribution:
For $X_1, \ldots, X_n$ i.i.d. $U(0,1)$, the k-th order statistic $X_{(k)}$ has a $\text{Beta}(k, n-k+1)$ distribution.

Step 1 — Mean of the Beta distribution:
For $X_{(k)} \sim \text{Beta}(k, n-k+1)$:

$$E[X_{(k)}] = \frac{k}{k + (n-k+1)} = \frac{k}{n+1}.$$

Why other options are wrong:

Q30.

- (A) k/n: Off by one in the denominator; this would be the empirical quantile, not the expected order statistic.
- (C) $(k-1)/(n-1)$: Incorrect; does not match the Beta distribution mean.
- (D) $k/(2n)$: No valid basis for this formula.

Final Answer: $E[X_{(k)}] = k/(n+1) \Rightarrow$ **(B)**

Answer: (B) ← *Go back to Q30*

Q31.

Solution

Concept — Axioms and Rules of Probability:
Evaluate each option against the Kolmogorov axioms and standard probability rules.

Option-by-option analysis:

- (A) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$: **TRUE**. This is the inclusion-exclusion principle.
- (B) "If $P(A) > 0$ and $P(B) > 0$ then $P(A \cap B) > 0$": **FALSE**. A and B can be mutually exclusive with $P(A \cap B) = 0$ even if both have positive probability.
- (C) $P(A^c) = 1 - P(A)$: **TRUE**. Since $P(A) + P(A^c) = P(\Omega) = 1$.
- (D) $P(A \mid B) = P(B \mid A)$ always: **FALSE**. $P(A \mid B) = P(A \cap B)/P(B)$ and $P(B \mid A) = P(A \cap B)/P(A)$; these are equal only if $P(A) = P(B)$.

Final Answer: Options (A) and (C) are correct $\Rightarrow$ **(AC)**

Answer: (AC) ← *Go back to Q31*

Q32.

Solution

Concept — Properties of Normal Distribution:
Analyse each statement about the normal distribution.

Option-by-option analysis:

- (A) "$aX + b \sim N(a\mu + b,\ a\sigma^2)$": **FALSE**. The variance transforms as $a^2\sigma^2$, not $a\sigma^2$; since $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X) = a^2\sigma^2$.
- (B) "$aX+b \sim N(a\mu+b,\ a^2\sigma^2)$": **TRUE**. Correct linear transformation formula.
- (C) "$N(0, 1)$ has positive skewness": **FALSE**. $N(0, 1)$ is symmetric about 0, so skewness $= 0$.
- (D) "Sum of independent normal RVs is normal": **TRUE**. By the moment generating function: if $X_i \sim N(\mu_i, \sigma_i^2)$ independently, then $\sum X_i \sim N(\sum \mu_i, \sum \sigma_i^2)$.

Final Answer: Options (B) and (D) are correct $\Rightarrow$ **(BD)**

Answer: (BD) ← *Go back to Q32*

Solution

Concept — Properties of Matrix Rank:
Rank-nullity theorem and sub-multiplicativity of rank.

Option-by-option analysis:

Q33.

- (A) $\text{rank}(A) = \text{rank}(A^\top)$: **TRUE**. Row rank equals column rank.
- (B) $\text{rank}(A) + \text{nullity}(A) = n$: **TRUE**. This is the Rank-Nullity Theorem (n = number of columns).
- (C) $\text{rank}(AB) \le \min\{\text{rank}(A), \text{rank}(B)\}$: **TRUE**. The image of AB is a subspace of the image of A, and the rank cannot exceed that of B either (Sylvester's inequality and sub-multiplicativity).
- (D) $\text{rank}(A + B) = \text{rank}(A) + \text{rank}(B)$: **FALSE**. In general $\text{rank}(A + B) \le \text{rank}(A) + \text{rank}(B)$; equality need not hold (e.g., $B = -A$ gives $\text{rank}(A + B) = 0$).

Final Answer: Options (A), (B), and (C) are correct ⇒ **(ABC)**

Answer: (ABC) ← *Go back to Q33*

Solution

Concept — Probability Inequalities:
Key inequalities: Markov, Chebyshev, Jensen, and linearity of expectation.

Option-by-option analysis:

Q34.

- (A) Markov: $P(X \ge a) \le E[X]/a$ for $X \ge 0$, $a > 0$: **TRUE**. Follows directly from $E[X] \ge a \cdot P(X \ge a)$.
- (B) "$P(|X - \mu| \ge k\sigma) \le k^2$": **FALSE**. The correct Chebyshev bound is $P(|X - \mu| \ge k\sigma) \le \mathbf{1/k^2}$, which goes to 0 as $k \to \infty$; k^2 increases with k, making (B) trivially useless.
- (C) Jensen: f convex ⇒ $f(E[X]) \le E[f(X)]$: **TRUE**. Standard Jensen's inequality.
- (D) Linearity: $E[aX + b] = aE[X] + b$: **TRUE**. Follows from the definition of expectation.

Final Answer: Options (A), (C), and (D) are correct ⇒ **(ACD)**

Answer: (ACD) ← *Go back to Q34*

Solution

Concept — Properties of Maximum Likelihood Estimation:
MLE has several well-known properties but also limitations.

Option-by-option analysis:

Q35.

- (A) MLE is always unbiased: **FALSE**. A classic counterexample is the MLE $\hat{\sigma}^2 = \frac{1}{n}\sum(X_i - \bar{X})^2$ for the variance, which is biased (the unbiased estimator divides by $n-1$).

- (B) MLE is invariant under one-to-one transformations: **TRUE**. By the invariance property of MLE: if $\hat{\theta}$ is the MLE of θ, then $g(\hat{\theta})$ is the MLE of $g(\theta)$ for any function g.
- (C) MLE always achieves minimum variance: **FALSE**. The MLE achieves the CRLB only asymptotically (for large n) and not for every distribution in finite samples.
- (D) MLE is asymptotically normal and efficient: **TRUE**. Under regularity conditions, $\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N(0, 1/I(\theta))$, where $I(\theta)$ is the Fisher information.

Final Answer: Options (B) and (D) are correct $\Rightarrow$ **(BD)**

Answer: (BD) ← *Go back to Q35*

Solution

Concept — Theory of Linear ODEs:
Key results: superposition, general solution structure, and the Wronskian.

Option-by-option analysis:

Q36.

- (A) General solution = homogeneous + particular: **TRUE**. Standard structure theorem for linear non-homogeneous ODEs.
- (B) "If $W[y_1, y_2](x_0) = 0$ at one point, solutions are linearly independent": **FALSE**. If the Wronskian is zero at one point (and they are solutions of a linear ODE with continuous coefficients), then it is zero everywhere, meaning the solutions are linearly *dependent*, not independent.
- (C) If y_1, y_2 solve $Ly = 0$, then $c_1y_1 + c_2y_2$ also solves $Ly = 0$: **TRUE**. By linearity of the differential operator L: $L(c_1y_1 + c_2y_2) = c_1Ly_1 + c_2Ly_2 = 0$.
- (D) Particular solution is always a polynomial: **FALSE**. The particular solution depends on $f(x)$; if $f(x) = e^x$, the particular solution is of exponential form.

Final Answer: Options (A) and (C) are correct $\Rightarrow$ **(AC)**

Answer: (AC) ← *Go back to Q36*

Solution

Concept — Properties of Poisson Distribution:
Key facts: mean equals variance, reproductive property, and Poisson limit theorem.

Option-by-option analysis:

Q37.

- (A) $E[X] = \mathrm{Var}(X) = \lambda$: **TRUE**. Fundamental property of the Poisson distribution; both the mean and variance equal the rate parameter λ.

- (B) $X + Y \sim \text{Poisson}(\lambda + \lambda_2)$ for X, Y independent: **TRUE**. Reproductive property of the Poisson; MGF of $X + Y$ equals $e^{(\lambda+\lambda_2)(e^t-1)}$.
- (C) Poisson is symmetric about its mean: **FALSE**. The Poisson distribution is right-skewed (skewness $= 1/\sqrt{\lambda} > 0$) for all finite $\lambda > 0$. It approaches symmetry only as $\lambda \to \infty$.
- (D) Poisson is the limit of $\text{Bin}(n, \lambda/n)$ as $n \to \infty$: **TRUE**. This is the classical Poisson limit theorem (law of rare events).

Final Answer: Options (A), (B), and (D) are correct $\Rightarrow$ **(ABD)**

Answer: (ABD) ← *Go back to Q37*

Solution

Concept — Errors in Hypothesis Testing:
Type I error (α): reject H_0 when H_0 is true. Type II error (β): fail to reject H_0 when H_1 is true. Power $= 1 - \beta$.

Option-by-option analysis:

Q38.

- (A) "Increasing α decreases power": **FALSE**. Increasing α widens the rejection region, which *increases* power $(1 - \beta)$.
- (B) $\beta = P(\text{fail to reject } H_0 \mid H_1 \text{ true})$: **TRUE**. This is the definition of Type II error.
- (C) Power $= 1 - \beta$: **TRUE**. Power is the probability of correctly rejecting H_0 when H_1 is true.
- (D) "Critical region is always in the right tail": **FALSE**. For a two-sided test, the critical region is split between both tails; for a left-sided test it is in the left tail.

Final Answer: Options (B) and (C) are correct $\Rightarrow$ **(BC)**

Answer: (BC) ← *Go back to Q38*

Solution

Concept — Sample Correlation Coefficient:
Properties of the Pearson correlation coefficient r.

Option-by-option analysis:

Q39.

- (A) $-1 \le r \le 1$: **TRUE**. By the Cauchy-Schwarz inequality, $|S_{xy}| \le \sqrt{S_{xx}S_{yy}}$, so $|r| \le 1$.

- (B) "$r = 0$ implies X and Y are independent": **FALSE**. Zero correlation implies independence only for bivariate normal distributions. In general, $r = 0$ means no *linear* relationship; non-linear dependence can still exist (e.g., $Y = X^2$ when X is symmetric).
- (C) "$r = \pm 1$ iff perfect linear relationship": **TRUE**. $r = \pm 1$ iff all (x_i, y_i) lie exactly on a straight line.
- (D) "r is dimensionless and invariant under positive rescaling": **TRUE**. r is standardised by the product of standard deviations and is unaffected by unit changes in X or Y.

Final Answer: Options (A), (C), and (D) are correct $\Rightarrow$ **(ACD)**

Answer: (ACD) ← *Go back to Q39*

Q40.

Solution

Concept — OLS Regression Properties:
Properties of ordinary least squares estimators.

Option-by-option analysis:

- (A) "OLS estimates are unbiased even without $E[\varepsilon_i] = 0$": **FALSE**. Unbiasedness of OLS requires $E[\varepsilon_i] = 0$; if the errors have non-zero mean, the intercept estimator $\hat{\beta}_0$ is biased.
- (B) OLS minimises $\sum(y_i - \hat{y}_i)^2$: **TRUE**. By definition, OLS finds $\hat{\beta}_0, \hat{\beta}_1$ minimising the sum of squared residuals.
- (C) R^2 can be negative: **FALSE**. For regression with an intercept, $R^2 \in [0, 1]$ always. (R^2 can be negative only for regression *through the origin*, which is not standard.)
- (D) Under Gauss-Markov, OLS is BLUE: **TRUE**. The Gauss-Markov theorem states that among all linear unbiased estimators, OLS has the smallest variance.

Final Answer: Options (B) and (D) are correct $\Rightarrow$ **(BD)**

Answer: (BD) ← *Go back to Q40*

Q41.

Solution

Step 1 — Apply Binomial mean formula:
For $X \sim \text{Bin}(n, p)$: $E[X] = np$.
With $n = 10$, $p = 0.3$: $E[X] = 10 \times 0.3 = \mathbf{3}$.

Answer: (3) ← *Go back to Q41*

Q42.

Solution

Step 1 — Standard normal quantile:

$P(Z > z) = 0.025 \;\Leftrightarrow\; \Phi(z) = 0.975.$

From standard normal tables: $z = \mathbf{1.96}$.

This is the 97.5th percentile of $N(0, 1)$, widely used in 95% confidence intervals.

Answer: (1.96) ← *Go back to Q42*

Q43.

Solution

Step 1 — Characteristic equation:

$$\det(A - \lambda I) = \det\begin{pmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix} = (4-\lambda)(3-\lambda) - 2 = 0.$$

Step 2 — Expand and solve:

$\lambda^2 - 7\lambda + 12 - 2 = 0 \;\Rightarrow\; \lambda^2 - 7\lambda + 10 = 0 \;\Rightarrow\; (\lambda - 5)(\lambda - 2) = 0.$

Eigenvalues are $\lambda = 5$ and $\lambda = 2$. The **larger** eigenvalue is **5**.

Answer: (5) ← *Go back to Q43*

Q44.

Solution

Step 1 — Apply correlation formula:

$$r = \frac{S_{xy}}{\sqrt{S_{xx} \cdot S_{yy}}} = \frac{4}{\sqrt{10 \times 10}} = \frac{4}{10} = \mathbf{0.40}.$$

Answer: (0.40) ← *Go back to Q44*

Q45.

Solution

Step 1 — MLE for Poisson is the sample mean:

For $X_1, \ldots, X_n$ i.i.d. $\text{Poisson}(\lambda)$, the MLE is $\hat{\lambda} = \bar{X}$.

Step 2 — Compute:

Observed values: $2, 3, 1, 4, 0$. Sum $= 10$, $n = 5$.

$\hat{\lambda} = \bar{X} = 10/5 = \mathbf{2}$.

Answer: (2) ← *Go back to Q45*

Q46.

Solution

Step 1 — Expected frequencies under independence:
With $n = 60$ and equal marginals, $E_{ij} = n/4 = 15$ for all cells.

Step 2 — Compute chi-square statistic:

$$\chi^2 = \frac{(10-15)^2}{15} + \frac{(20-15)^2}{15} + \frac{(20-15)^2}{15} + \frac{(10-15)^2}{15} = \frac{25+25+25+25}{15} = \frac{100}{15} \approx \mathbf{6.67}.$$

Answer: (6.67) ← *Go back to Q46*

Q47.

Solution

Step 1 — Geometric series formula:
For $|r| < 1$: $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$.

Step 2 — Substitute $r = 2/3$:
$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \frac{2/3}{1-2/3} = \frac{2/3}{1/3} = \mathbf{2}.$$

Answer: (2) ← *Go back to Q47*

Q48.

Solution

Step 1 — Survival function of $\mathrm{Exp}(\lambda)$:
For $X \sim \mathrm{Exp}(\lambda)$: $P(X > t) = e^{-\lambda t}$.

Step 2 — Evaluate with $\lambda = 2$, $t = 1$:
$P(X > 1) = e^{-2\times 1} = e^{-2} \approx \mathbf{0.1353}$.

Answer: (0.1353) ← *Go back to Q48*

Q49.

Solution

Step 1 — Variance of a linear combination:
For independent X, Y: $\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$.

Step 2 — Compute:
$\mathrm{Var}(2X + 3Y) = 4 \times \mathrm{Var}(X) + 9 \times \mathrm{Var}(Y) = 4 \times 4 + 9 \times 9 = 16 + 81 = \mathbf{97}$.

Answer: (97) ← *Go back to Q49*

Q50.

Solution

Step 1 — Solve the first-order linear ODE:

$y' + 2y = 6$. Integrating factor $\mu = e^{2x}$.

$\frac{d}{dx}(e^{2x}y) = 6e^{2x} \;\Rightarrow\; e^{2x}y = 3e^{2x} + C \;\Rightarrow\; y = 3 + Ce^{-2x}$.

Step 2 — Apply initial condition $y(0) = 0$:

$0 = 3 + C \;\Rightarrow\; C = -3$, so $y = 3(1 - e^{-2x})$.

Step 3 — Evaluate at $x = 1$:

$y(1) = 3(1 - e^{-2}) = 3(1 - 0.1353) = 3 \times 0.8647 = \mathbf{2.594}$.

Answer: (2.594) ← *Go back to Q50*

Q51.

Solution

Step 1 — Confidence interval for μ with known variance:

$X_1, \ldots, X_{16}$ i.i.d. $N(\mu, 25)$, so $\sigma = 5$, $n = 16$, $\sigma/\sqrt{n} = 5/4 = 1.25$.

Step 2 — Compute the interval:

95% CI: $\bar{x} \pm z_{0.025} \cdot \frac{\sigma}{\sqrt{n}} = 52 \pm 1.96 \times 1.25 = 52 \pm 2.45$.

Lower bound $= 52 - 2.45 = \mathbf{49.55}$.

Answer: (49.55) ← *Go back to Q51*

Q52.

Solution

Step 1 — Substitute into the regression equation:

$\hat{y} = 2 + 3x$. At $x = 5$:

$$\hat{y} = 2 + 3 \times 5 = 2 + 15 = \mathbf{17}.$$

Answer: (17) ← *Go back to Q52*

Q53.

Solution

Step 1 — Set up the power calculation:

$\sigma = 5$, $n = 25$, so $\sigma/\sqrt{n} = 5/5 = 1$. The rejection region at $\alpha = 0.05$ (one-sided) is $\bar{X} > 10 + 1.645 = 11.645$.

Step 2 — Compute power under $\mu = 12$:
Under $H_1 : \mu = 12$, $\bar{X} \sim N(12, 1)$.

$$\text{Power} = P(\bar{X} > 11.645 \mid \mu = 12) = P\left(Z > \frac{11.645 - 12}{1}\right) = P(Z > -0.355) = \Phi(0.355) \approx \mathbf{0.639}.$$

Answer: (0.639) ← *Go back to Q53*

Q54.

Solution

Step 1 — Distribution of $X + Y$:
X, Y i.i.d. $N(0, 1)$, so $X + Y \sim N(0, 1 + 1) = N(0, 2)$. Thus $\sigma_{X+Y} = \sqrt{2}$.

Step 2 — Standardise and compute:

$$P(X + Y < 2) = P\left(Z < \frac{2}{\sqrt{2}}\right) = P(Z < \sqrt{2}) = P(Z < 1.414) \approx \mathbf{0.921}.$$

Answer: (0.921) ← *Go back to Q54*

Q55.

Solution

Step 1 — Cofactor expansion along row 1:

$$\det(B) = 2 \det \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} - 1 \det \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} + 0 \det \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}.$$

Step 2 — Compute 2×2 determinants:
$\det \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = 6 - 1 = 5$; $\quad \det \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = 2 - 0 = 2$.
$\det(B) = 2(5) - 1(2) + 0 = 10 - 2 = \mathbf{8}$.

Answer: (8) ← *Go back to Q55*

Q56.

Solution

Step 1 — Fisher information for Bernoulli:
For $X \sim \text{Bernoulli}(p)$: $I_1(p) = \dfrac{1}{p(1-p)}$. For $n = 9$ i.i.d. observations: $I_n(p) = \dfrac{n}{p(1-p)} = \dfrac{9}{0.5 \times 0.5} = \dfrac{9}{0.25} = 36$.

Step 2 — Cramér-Rao lower bound:
$\text{CRLB} = \frac{1}{I_n(p)} = \frac{p(1-p)}{n} = \frac{0.25}{9} \approx \mathbf{0.0278}.$

Answer: (0.0278) ← *Go back to Q56*

Q57.

Solution

Step 1 — Z-statistic formula:
$\sigma^2 = 4 \ \Rightarrow \ \sigma = 2$. Standard error: $\sigma/\sqrt{n} = 2/\sqrt{9} = 2/3$.

Step 2 — Compute:

$$Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{11.5 - 10}{2/3} = \frac{1.5 \times 3}{2} = \frac{4.5}{2} = \mathbf{2.25}.$$

Answer: (2.25) ← *Go back to Q57*

Q58.

Solution

Step 1 — Normal approximation parameters:
$X \sim \text{Bin}(100, 0.4)$: $\mu = 100 \times 0.4 = 40$, $\sigma^2 = 100 \times 0.4 \times 0.6 = 24$, $\sigma = \sqrt{24} \approx 4.899$.

Step 2 — Apply continuity correction:

$$P(X \le 45) \approx P\left(Z \le \frac{45.5 - 40}{\sqrt{24}}\right) = P\left(Z \le \frac{5.5}{4.899}\right) = P(Z \le \mathbf{1.12}).$$

Answer: (1.12) ← *Go back to Q58*

Q59.

Solution

Step 1 — Bayesian updating (Beta-Binomial conjugacy):
Prior: $p \sim \text{Beta}(1,1)$. Likelihood: $X \mid p \sim \text{Bin}(10, p)$ with $X = 7$.
Posterior: $p \mid X \sim \text{Beta}(1+7,\ 1+10-7) = \text{Beta}(8,4)$.

Step 2 — Posterior mean:
$E[p \mid X = 7] = \frac{8}{8+4} = \frac{8}{12} = \frac{2}{3} \approx \mathbf{0.667}.$

Answer: (0.667) ← *Go back to Q59*

Q60.

Solution

Step 1 — Normalisation condition:

$$\int_0^\infty f(x)\,dx = 1 \;\Rightarrow\; k\int_0^\infty x^2 e^{-x}\,dx = 1.$$

Step 2 — Evaluate the integral using the Gamma function:

$$\int_0^\infty x^2 e^{-x}\,dx = \Gamma(3) = 2! = 2.$$

Step 3 — Solve for k:

$$k \times 2 = 1 \;\Rightarrow\; k = \frac{1}{2} = \mathbf{0.5}.$$

Note: $f(x) = \frac{1}{2}x^2 e^{-x}$ is the $\mathrm{Gamma}(3,1)$ PDF (shape 3, rate 1).

Answer: (0.5) ← *Go back to Q60*

Answer Key

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	B	2	A	3	C	4	D	5	A
6	B	7	C	8	D	9	A	10	B
11	C	12	D	13	A	14	B	15	C
16	D	17	A	18	B	19	C	20	D
21	A	22	B	23	C	24	D	25	A
26	B	27	C	28	D	29	A	30	B
31	AC	32	BD	33	ABC	34	ACD	35	BD
36	AC	37	ABD	38	BC	39	ACD	40	BD
41	3	42	1.96	43	5	44	0.40	45	2
46	6.67	47	2	48	0.1353	49	97	50	2.594
51	49.55	52	17	53	0.639	54	0.921	55	8
56	0.0278	57	2.25	58	1.12	59	0.667	60	0.5