IIT JAM Mathematical Statistics

Sample Paper – 2

Instructions

- This paper contains **60 questions** carrying a maximum of **100 marks**. Duration: **3 hours (180 minutes)**.
- **Section A** (**Q1–Q30**, MCQ): Q1–Q10 carry **+1 mark** each; wrong response: $-\frac{1}{3}$ mark. Q11–Q30 carry **+2 marks** each; wrong response: $-\frac{2}{3}$ mark. Choose the **single best** option.
- **Section B** (**Q31–Q40**, MSQ): Each carries **+2 marks**. **No negative marking. One or more** options may be correct; full marks only if *all* correct options (and no incorrect options) are selected.
- **Section C** (**Q41–Q60**, NAT): Q41–Q50 carry **+1 mark**; Q51–Q60 carry **+2 marks**. **No negative marking.** Enter the numerical value using the virtual keypad.
- **Calculator policy:** Personal calculators, log tables, and electronic devices are **strictly prohibited**. A virtual scientific calculator is provided in the CBT interface. Rough sheets are supplied at the centre.
- Once you move to the next section, you cannot return to a previous section.

Section A — Multiple Choice Questions (MCQ)

***Q1–Q10**: 1 mark each ($-\frac{1}{3}$ for wrong). Choose the **single** correct option.*

Q1. Which of the following statements is **correct** about the convergence of infinite series?

(A) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

(B) Both $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ diverge

(C) Both $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge

(D) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ converges and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ diverges

Q2. Evaluate $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{\sin 2x}$.

(A) 1

(B) $\dfrac{3}{2}$

(C) $\dfrac{2}{3}$
(D) 2

Q3. The value of $\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$ is:

(A) 8
(B) 12
(C) 18
(D) 24

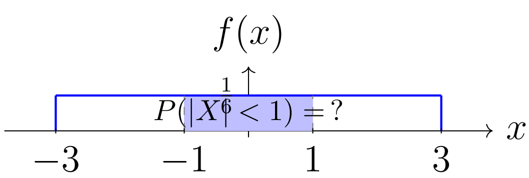
Q4. Events A and B satisfy $P(A) = 0.6,\ \ P(B) = 0.5,\ \ P(A \cup B) = 0.8$. Find $P(A \cap B)$.

(A) 0.1
(B) 0.2
(C) 0.3
(D) 0.4

Q5. Let X follow a Geometric distribution with success probability $p = \frac{1}{3}$, where X counts the trial number of the *first* success ($X = 1, 2, 3, \ldots$). Find $E[X]$.

(A) $\dfrac{1}{3}$
(B) 3
(C) $\dfrac{2}{3}$
(D) 9

Q6. Let $X \sim U(-3, 3)$. The figure shows the PDF of X with the shaded region indicating $P(|X| < 1)$. Find $P(|X| < 1)$.



(A) $\dfrac{1}{3}$

(B) $\frac{1}{6}$
(C) $\frac{1}{2}$
(D) $\frac{2}{3}$

Q7. Suppose f is continuous on $[a, b]$ with $f(a) < 0$ and $f(b) > 0$. Which theorem guarantees the existence of $c \in (a, b)$ such that $f(c) = 0$?

(A) Mean Value Theorem
(B) Rolle's Theorem
(C) Extreme Value Theorem
(D) Intermediate Value Theorem

Q8. Evaluate the improper integral $\int_0^\infty e^{-3x}\, dx$.

(A) $\frac{1}{5}$
(B) $\frac{1}{4}$
(C) $\frac{1}{3}$
(D) 3

Q9. Given $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cap B) = 0.2$. Which statement **best** describes A and B?

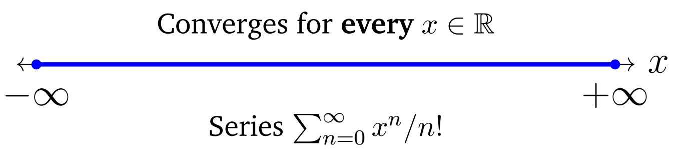
(A) A and B are mutually exclusive
(B) A and B are independent
(C) A and B are positively correlated but not independent
(D) $A \subset B$

Q10. Let $X \sim \text{Poisson}(2)$. Find $E[X(X-1)]$.

(A) 4
(B) 2
(C) 6
(D) 8

***Q11–Q30**: 2 marks each ($-\frac{2}{3}$ for wrong). Choose the **single** correct option.*

Q11. The number line below illustrates the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. What is its radius of convergence R?



(A) $R = 0$
(B) $R = 1$
(C) $R = 2$
(D) $R = \infty$

Q12. Let $f(x, y) = x^2 y + \sin(xy)$. Find $\frac{\partial f}{\partial x}$.

(A) $2x + \cos(xy)$
(B) $x^2 + x\cos(xy)$
(C) $2xy + y\cos(xy)$
(D) $2y + \cos(x)$

Q13. Matrix A is 4×6 with $\text{rank}(A) = 3$. By the Rank-Nullity theorem, the *nullity* of A equals:

(A) 2
(B) 3
(C) 4
(D) 6

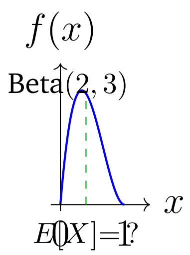
Q14. Solve $\frac{dy}{dx} + y = e^x$ with $y(0) = 0$. The value of $y(1)$ is:

(A) $\frac{e - e^{-1}}{2}$
(B) $\frac{e}{2}$
(C) $\frac{e + e^{-1}}{2}$
(D) $\frac{e^{-1}}{2}$

Q15. An urn contains 5 red and 3 blue balls. Four balls are drawn *without* replacement. The probability that **exactly 2** of the selected balls are red equals:

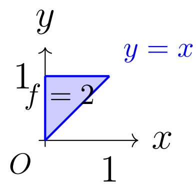
(A) $\frac{1}{7}$
(B) $\frac{2}{7}$
(C) $\frac{1}{4}$
(D) $\frac{3}{7}$

Q16. The curve below shows the PDF of $X \sim \text{Beta}(2,3)$, where $f(x) = 12x(1-x)^2$ for $0 < x < 1$. What is $E[X]$?



(A) $\frac{1}{3}$
(B) $\frac{1}{2}$
(C) $\frac{2}{5}$
(D) $\frac{3}{5}$

Q17. The joint PDF of (X, Y) is $f(x, y) = 2$ on the triangular region $0 < x < y < 1$ (shown below), zero elsewhere. Find the conditional PDF $f_{X|Y}(x \mid y)$ for $0 < x < y$.



(A) $\frac{1}{2y}$
(B) $\frac{1}{y}$

(C) $2y$

(D) $\dfrac{2x}{y^2}$

Q18. Let $T \sim t(\nu)$ denote the Student t-distribution with ν degrees of freedom. As $\nu \to \infty$, which statement is **correct**?

(A) $t(\nu)$ converges in distribution to $N(0, 1)$

(B) $t(\nu)$ converges in distribution to the Cauchy distribution

(C) $t(\nu)$ converges in distribution to $\chi^2(1)$

(D) $t(\nu)$ converges in distribution to $U(-1, 1)$

Q19. Let $X_1, \ldots, X_n$ be i.i.d. $U(0, \theta)$. The method-of-moments estimator is $\hat{\theta} = 2\bar{X}$. For the sample $\{1, 3, 5, 7, 9\}$, the estimate $\hat{\theta}$ equals:

(A) 5

(B) 8

(C) 9

(D) 10

Q20. In $n = 100$ Bernoulli trials, $X = 45$ successes are observed. For the test $H_0 : p = 0.5$ vs. $H_1 : p \neq 0.5$, the test statistic $Z = \dfrac{X - np_0}{\sqrt{np_0(1 - p_0)}}$ equals:

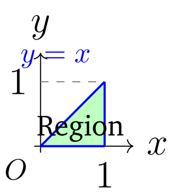
(A) $Z = 1$

(B) $Z = -2$

(C) $Z = -1$

(D) $Z = 0.5$

Q21. The region of integration in $\int_0^1 \int_y^1 f(x, y)\, dx\, dy$ is the shaded triangle below. After changing the order of integration, the equivalent double integral is:



(A) $\int_0^1 \int_0^y f(x, y)\, dx\, dy$

(B) $\int_0^1 \int_0^x f(x,y)\,dy\,dx$

(C) $\int_0^1 \int_x^1 f(x,y)\,dy\,dx$

(D) $\int_0^1 \int_0^1 f(x,y)\,dy\,dx$

Q22. The sequence $a_n = \left(1 + \frac{1}{n}\right)^n$ converges as $n \to \infty$ to:

(A) e

(B) 1

(C) ∞

(D) 0

Q23. The matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ has eigenvalues 0 and 2. Which of the following **correctly** classifies A?

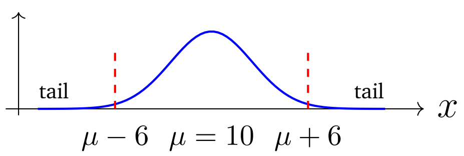
(A) Positive definite

(B) Negative definite

(C) Negative semi-definite

(D) Positive semi-definite

Q24. A random variable X has $E[X] = 10$ and $\mathrm{Var}(X) = 9$. The figure below sketches the distribution with dashed lines at $\mu \pm 6$. Using Chebyshev's inequality, an upper bound on $P(|X - 10| \geq 6)$ is:



(A) 0.10

(B) 0.15

(C) 0.25

(D) 0.50

Q25. Let $X_1, \ldots, X_n$ be i.i.d. $N(\mu, \sigma^2)$ with sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Which of the following is **correct**?

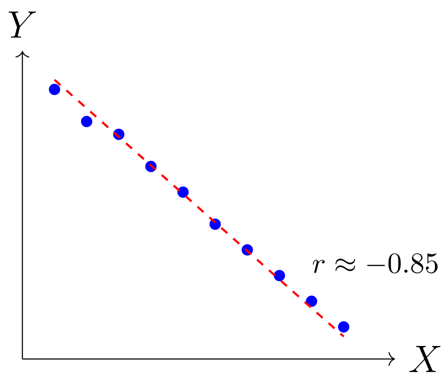
(A) $\frac{\sum X_i}{\sigma} \sim \chi^2(n)$

(B) $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

(C) $\frac{nS^2}{\sigma^2} \sim \chi^2(n-1)$

(D) $\frac{(n-1)S^2}{\sigma} \sim \chi^2(n)$

Q26. The scatter plot below displays bivariate data with $r \approx -0.85$. Which statement **correctly** describes the relationship between X and Y?



(A) As X increases, Y tends to **decrease**; the association is **negative**

(B) As X increases, Y tends to increase; the association is positive

(C) X and Y are uncorrelated ($r \approx 0$)

(D) X and Y have *perfect* negative correlation ($r = -1$)

Q27. In a simple linear regression of Y on X, the coefficient of determination is $R^2 = 0.81$. Which statement is **correct**?

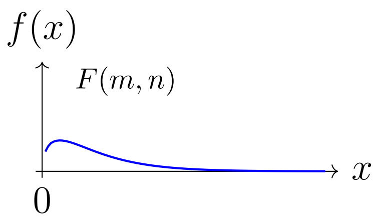
(A) The estimated slope $\hat{\beta}_1 = 0.81$

(B) The regression is significant at the 5% level of significance

(C) 81% of the variation in X is explained by Y

(D) 81% of the variation in Y is explained by X

Q28. The curve below shows the PDF of an $F(m, n)$ distribution. If $T \sim t(n)$, which of the following is **correct**?

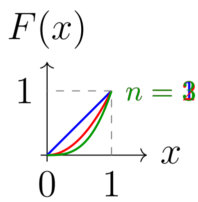


(A) $T^2 \sim \chi^2(1)$
(B) $T^2 \sim F(n, 1)$
(C) $T^2 \sim F(1, n)$
(D) $T \sim F(1, n)$

Q29. Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli(p). A *sufficient statistic* for p is:

(A) The sample variance S^2
(B) $T = \sum_{i=1}^{n} X_i$
(C) The sample median
(D) X_1 alone

Q30. Let $X_1, \ldots, X_n$ be i.i.d. $U(0, 1)$. The graph shows the CDF of $X_{(n)} = \max(X_1, \ldots, X_n)$ for $n = 1, 2, 3$. What is $F_{X_{(n)}}(x)$ for $0 < x < 1$?



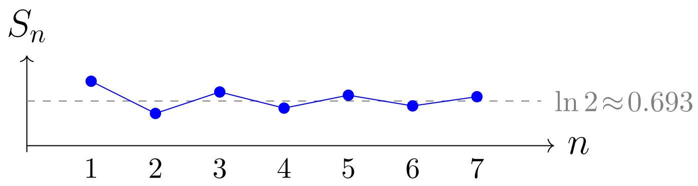
(A) $F_{X_{(n)}}(x) = x^n \quad$ for $0 < x < 1$
(B) $F_{X_{(n)}}(x) = nx^{n-1} \quad$ for $0 < x < 1$
(C) $F_{X_{(n)}}(x) = 1 - (1 - x)^n \quad$ for $0 < x < 1$
(D) $F_{X_{(n)}}(x) = x/n \quad$ for $0 < x < 1$

Section B — Multiple Select Questions (MSQ)

***Q31–Q40**: 2 marks each. **No negative marking. One or more** options may be correct.*

Q31. The graph shows partial sums of the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

Select **all correct** statements about series convergence.



(A) The ratio test is conclusive whenever the limit $\lim_{n\to\infty} |a_{n+1}/a_n|$ equals 1

(B) If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges

(C) The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p \geq 1$

(D) If $b_n > 0$, b_n is non-increasing, and $b_n \to 0$, then $\sum_{n=1}^{\infty} (-1)^n b_n$ converges

Q32. Select **all correct** statements about statistical estimators.

(A) An unbiased estimator T_n is consistent if $\mathrm{Var}(T_n) \to 0$ as $n \to \infty$

(B) Every consistent estimator must be unbiased

(C) The sample mean $\bar{X}$ is an unbiased estimator of the population mean μ

(D) The estimator $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$ is an unbiased estimator of σ^2

Q33. Select **all correct** statements about linear transformations of random variables.

(A) If $X \sim N(0,1)$, then $aX + b \sim N(b,\, a^2)$ for any constants $a \neq 0,\ b$

(B) $E[aX + b] = a\,E[X] + b$ for any random variable X and constants a, b

(C) $\mathrm{Var}(aX + b) = a \cdot \mathrm{Var}(X)$ for constant a

(D) If X and Y are independent, then $\mathrm{Var}(X - Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

Q34. Select **all correct** statements about hypothesis testing.

(A) For a fixed sample size, both the Type I (α) and Type II (β) error probabilities can be simultaneously reduced to zero

(B) For a fixed sample size, decreasing α (significance level) generally *increases* β (Type II error probability)

(C) The p-value is the probability of observing a test statistic at least as extreme as the observed value, *assuming* H_0 is true
(D) Rejecting H_0 when it is actually true constitutes a Type II error

Q35. Let A be a real $n \times n$ matrix. Select **all correct** statements.

(A) If A is symmetric, all its eigenvalues are real
(B) If λ is an eigenvalue of A, then 2λ is an eigenvalue of $2A + I$
(C) $\det(A)$ equals the product of all eigenvalues of A (counting multiplicity)
(D) $\mathrm{trace}(A)$ equals the sum of all eigenvalues of A (counting multiplicity)

Q36. Let A and B be events with $P(B) > 0$. Select **all correct** statements.

(A) $P(A \mid B) = P(B \mid A)$ for any events A, B
(B) $P(A \mid B) + P(A^c \mid B) = 1$
(C) $P(A \mid A \cap B) = P(A)$ whenever $P(A \cap B) > 0$
(D) If A and B are independent, then $P(A \mid B) = P(A)$

Q37. Let $X \sim \mathrm{Bin}(n, p)$. Select **all correct** statements.

(A) $E[X] = np$
(B) $\mathrm{Var}(X) = np(1 - p)$
(C) For large n and small p with $np = \lambda$ fixed, $\mathrm{Bin}(n, p)$ converges in distribution to $\mathrm{Poisson}(\lambda)$
(D) $\mathrm{Bin}(n, p)$ is symmetric for *all* values of p

Q38. In the model $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$ (fixed X_i, OLS estimation). Select **all correct** statements.

(A) The OLS residuals satisfy $\sum_{i=1}^{n} \hat{\varepsilon}_i = 0$ when an intercept is included
(B) OLS *estimation* of β_0 and β_1 requires the errors ε_i to be normally distributed
(C) The OLS slope estimator is $\hat{\beta}_1 = S_{XY}/S_{YY}$
(D) Under $E[\varepsilon_i] = 0$, the OLS estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ are unbiased

Q39. Select **all correct** statements about integration.

(A) The improper integral $\displaystyle\int_{-\infty}^{\infty} e^x\,dx$ converges

(B) Every Riemann integrable function on a closed interval $[a,b]$ is bounded on $[a,b]$

(C) If $F'(x) = f(x)$ on $[a,b]$, then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$ (Fundamental Theorem of Calculus)

(D) A monotone function on a closed bounded interval need not be Riemann integrable

Q40. Let $X_1, \ldots, X_n$ be i.i.d. from a continuous distribution with order statistics $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$. Select **all correct** statements.

(A) $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ are the sorted (ordered) sample values

(B) Order statistics from any continuous distribution are always mutually independent

(C) If the population distribution is symmetric about 0, then $E[X_{(k)}] = -E[X_{(n+1-k)}]$ for all k

(D) The sample range $R = X_{(n)} - X_{(1)}$ is a measure of statistical dispersion

Section C — Numerical Answer Type (NAT)

Q41–Q50**: 1 mark each. **No negative marking.

Q41. Let X follow a Geometric distribution with success probability $p = 0.25$, where X counts the number of trials up to and including the first success. Find $E[X]$.

Q42. Evaluate $\displaystyle\int_0^2 x\,e^{-x}\,dx$. Give your answer correct to **3 decimal places**.

Q43. Let $A = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$. Find $\mathrm{trace}(A^2)$.

Q44. Events A and B are *independent* with $P(A) = 0.3$ and $P(B) = 0.4$. Find $P(A \cup B)$.

Q45. Observations from a Poisson(λ) distribution are: $3,\ 1,\ 2,\ 4$. Using the method of moments, find $\hat{\lambda}$.

Q46. If $X \sim N(0,1)$, find $P(-1.96 < X < 1.96)$. Give your answer correct to **2 decimal places**.

Q47. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n$.

Q48. Let $X \sim \text{Exp}(\lambda)$ with rate parameter $\lambda = \frac{1}{2}$ (so $E[X] = 2$). Find $P(X > 4)$, correct to **4 decimal places**.

Q49. Let X and Y be *independent* random variables with $E[X] = 3$, $E[Y] = 5$, $\text{Var}(X) = 1$, $\text{Var}(Y) = 4$. Find $E[(X - Y)^2]$.

Q50. Solve $x\dfrac{dy}{dx} - y = x^2$ with $y(1) = 0$. Find $y(2)$.

Q51–Q60**: 2 marks each.* ***No negative marking.

Q51. A random sample of size $n = 9$ is drawn from $N(\mu, \sigma^2)$ with known $\sigma^2 = 16$. The sample mean is $\bar{x} = 14$. Construct a 90% confidence interval for μ using the z-distribution ($z_{0.05} = 1.645$). Report the **upper bound** of the interval, correct to **2 decimal places**.

Q52. Let $X \sim N(0,1)$ and define $Y = 2X + 3$. Find $\text{Cov}(X, Y)$.

Q53. Let $X_1, \dots, X_{10}$ be i.i.d. Poisson(λ). To test $H_0 : \lambda = 1$ vs. $H_1 : \lambda > 1$ at level $\alpha = 0.05$, the UMP test rejects H_0 when $T = \sum_{i=1}^{10} X_i > c$. Under H_0, $T \sim \text{Poisson}(10)$. Given $P(T > 15 \mid \lambda_0 = 1) \approx 0.0487$, find the critical value c.

Q54. Let X_1, X_2 be i.i.d. Exp(1) (rate $= 1$). Find $P(X_1 + X_2 < 2)$, correct to **3 decimal places**.

Q55. Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Find the $(1,1)$ entry (top-left entry) of A^{-1}, correct to **4 decimal places**.

Q56. Let $X_1, \ldots, X_5$ be i.i.d. $U(0, \theta)$. The UMVUE of θ is $\hat{\theta} = \dfrac{n+1}{n} X_{(n)}$. For an observed sample maximum $X_{(5)} = 8$, compute the UMVUE estimate of θ.

Q57. Test $H_0 \colon \mu = 20$ vs. $H_1 \colon \mu > 20$ with known $\sigma = 5$, $n = 25$, and $\alpha = 0.01$ ($z_{0.01} = 2.326$). If the true mean is $\mu = 23$, find the power of the test, correct to **3 decimal places**. [Use $\Phi(0.674) \approx 0.750$].

Q58. Let $X \sim \mathrm{Bin}(64, 0.5)$. Using the normal approximation *with* continuity correction, find $P(X \geq 36)$, correct to **3 decimal places**. [Use $\Phi(0.875) \approx 0.809$].

Q59. In a Bayesian model, the prior is $p \sim \mathrm{Beta}(2, 2)$ and the likelihood is $X \mid p \sim \mathrm{Bin}(10, p)$. Given $X = 6$, the posterior is $p \mid X = 6 \sim \mathrm{Beta}(8, 6)$. Find the posterior variance $\mathrm{Var}(p \mid X = 6)$, correct to **4 decimal places**.

Q60. The joint PDF of (X, Y) is $f(x, y) = cx^2y$ for $0 < y < x < 1$, and 0 otherwise. Find the constant c.

Solutions

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Detailed Solutions

Solution

Concept — p-Series and Harmonic Series:
The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Step 1 — Classify each series:

Q1.
- $\sum \frac{1}{n}$: this is the harmonic series, a p-series with $p = 1$. Since $p \leq 1$, it **diverges**.
- $\sum \frac{1}{n^2}$: p-series with $p = 2 > 1$. It **converges** (to $\pi^2/6$).

Final Answer: $\sum 1/n$ diverges and $\sum 1/n^2$ converges $\Rightarrow$ **(A)**

Why other options are wrong:

- Option B: $\sum 1/n^2$ converges, so both do NOT diverge.
- Option C: $\sum 1/n$ diverges, so both do NOT converge.
- Option D: Swaps the two conclusions; incorrect.

Answer: (A) ← *Go back to Q1*

Solution

Concept — L'Hôpital's Rule / Small-Angle Approximation:
For $x \to 0$: $\sin(kx) \approx kx$, so $\frac{\sin 3x}{\sin 2x} \approx \frac{3x}{2x} = \frac{3}{2}$.

Step 1 — Formal L'Hôpital: The limit is $0/0$ at $x = 0$.

$$\lim_{x\to 0} \frac{\sin 3x}{\sin 2x} = \lim_{x\to 0} \frac{3\cos 3x}{2\cos 2x} = \frac{3\cdot 1}{2\cdot 1} = \frac{3}{2}.$$

Final Answer: $\frac{3}{2} \Rightarrow$ **(B)**

Why other options are wrong:

Q2.
- Option A (1): Would be the limit of $\frac{\sin x}{\sin x}$, not $\frac{\sin 3x}{\sin 2x}$.
- Option C (2/3): This is the reciprocal; a common sign-flip error.
- Option D (2): Incorrect arithmetic.

Answer: (B) ← *Go back to Q2*

Solution

Concept — Determinant of an Upper-Triangular Matrix:
For a triangular matrix, $\det =$ product of diagonal entries.

Step 1 — Read off the diagonal:

$$\det\begin{pmatrix}1 & 2 & 3\\ 0 & 4 & 5\\ 0 & 0 & 6\end{pmatrix} = 1\times 4\times 6 = 24.$$

Final Answer: $24 \Rightarrow$ **(D)**

Why other options are wrong:

Q3.
- Option A (8): Would require multiplying different entries.
- Option B (12): $1\times 4\times 3 = 12$ — wrong diagonal selection.
- Option C (18): Incorrect computation.

Answer: (D) ← *Go back to Q3*

Solution

Concept — Inclusion-Exclusion Principle:
$P(A\cup B) = P(A) + P(B) - P(A\cap B)$.

Step 1 — Solve for $P(A\cap B)$:

$$P(A\cap B) = P(A) + P(B) - P(A\cup B) = 0.6 + 0.5 - 0.8 = 0.3.$$

Final Answer: $P(A\cap B) = 0.3 \Rightarrow$ **(C)**

Why other options are wrong:

Q4.
- Option A (0.1): Underestimates; arises from $0.6 - 0.5 = 0.1$.
- Option B (0.2): Arithmetic error in inclusion-exclusion.
- Option D (0.4): $0.8 - 0.6 + 0.2 = 0.4$ — sign error.

Answer: (C) ← *Go back to Q4*

Solution

Concept — Expected Value of the Geometric Distribution:
If $X \sim \mathrm{Geometric}(p)$ (number of trials to first success), then $E[X] = \dfrac{1}{p}$.

Step 1 — Compute:

$$E[X] = \frac{1}{p} = \frac{1}{1/3} = 3.$$

Final Answer: $E[X] = 3 \Rightarrow$ **(B)**

Why other options are wrong:

Q5.

- Option A $(1/3)$: This is p, not $E[X]$.
- Option C $(2/3)$: This is $1-p$ (failure probability), not $E[X]$.
- Option D (9): This equals $1/p^2 = E[X]^2$; not the expectation.

Answer: (B) ← *Go back to Q5*

Solution

Concept — Uniform Distribution and Probability via Area:
For $X \sim U(a,b)$, the PDF is $f(x) = \frac{1}{b-a}$ on (a,b) and $P(c < X < d) = \frac{d-c}{b-a}$.

Step 1 — Identify parameters: $X \sim U(-3,3)$, so $f(x) = \frac{1}{6}$ for $-3 < x < 3$.

Step 2 — Compute probability:

$$P(|X| < 1) = P(-1 < X < 1) = \frac{1-(-1)}{3-(-3)} = \frac{2}{6} = \frac{1}{3}.$$

The shaded rectangle in the figure has width 2 and height $1/6$; its area $= 1/3$ confirms this.

Final Answer: $P(|X| < 1) = \frac{1}{3} \Rightarrow$ **(A)**

Why other options are wrong:

Q6.

- Option B $(1/6)$: The height of the PDF, not the probability.
- Option C $(1/2)$: Would correspond to an interval of length 3.
- Option D $(2/3)$: Twice the correct answer; error by factor 2.

Answer: (A) ← *Go back to Q6*

Solution

Concept — Intermediate Value Theorem (IVT):
If f is continuous on $[a,b]$ and $f(a)$ and $f(b)$ have opposite signs, then $\exists\, c \in (a,b)$ with $f(c) = 0$.

Step 1 — Match to the theorem: The hypothesis "f continuous on $[a,b]$, $f(a) <$

$0,\ f(b) > 0$" exactly matches the IVT (also known as Bolzano's theorem). The conclusion $f(c) = 0$ follows.

Final Answer: Intermediate Value Theorem $\Rightarrow$ **(D)**

Why other options are wrong:

Q7.

- Option A (Mean Value Theorem): MVT gives $f'(c) = (f(b) - f(a))/(b - a)$; requires differentiability.
- Option B (Rolle's Theorem): Requires $f(a) = f(b)$; guarantees $f'(c) = 0$.
- Option C (Extreme Value Theorem): Guarantees f attains its maximum and minimum; no zero-crossing claim.

Answer: (D) ← *Go back to Q7*

Solution

Concept — Improper Integral of Exponential:
$\displaystyle\int_0^{\infty} e^{-kx}\,dx = \frac{1}{k}$ for $k > 0$.

Step 1 — Evaluate:

$$\int_0^{\infty} e^{-3x}\,dx = \left[-\frac{e^{-3x}}{3}\right]_0^{\infty} = 0 - \left(-\frac{1}{3}\right) = \frac{1}{3}.$$

Final Answer: $\frac{1}{3} \Rightarrow$ **(C)**

Why other options are wrong:

Q8.

- Option A $(1/5)$: Would arise from $k = 5$.
- Option B $(1/4)$: Would arise from $k = 4$.
- Option D (3): This is k itself, not $1/k$.

Answer: (C) ← *Go back to Q8*

Solution

Concept — Statistical Independence of Events:
A and B are independent $\iff P(A \cap B) = P(A) \cdot P(B)$.

Step 1 — Check independence:

$$P(A) \cdot P(B) = 0.4 \times 0.5 = 0.20 = P(A \cap B).$$

The equality holds, so A and B are **independent**.

Final Answer: A and B are independent $\Rightarrow$ **(B)**

Why other options are wrong:

Q9.

- Option A: Mutually exclusive requires $P(A \cap B) = 0$, but here $P(A \cap B) = 0.2 \neq 0$.
- Option C: No residual dependence once independence is confirmed.
- Option D: $A \subset B$ would force $P(A \cap B) = P(A) = 0.4 \neq 0.2$.

Answer: (B) ← *Go back to Q9*

Solution

Concept — Factorial Moments of the Poisson Distribution:
For $X \sim \text{Poisson}(\lambda)$: $E[X(X-1)] = \lambda^2$ (second factorial moment).

Step 1 — Apply the formula with $\lambda = 2$:

$$E[X(X-1)] = \lambda^2 = 2^2 = 4.$$

Verification: $E[X^2] - E[X] = (\text{Var}(X) + E[X]^2) - E[X] = (2+4) - 2 = 4.$

Final Answer: $E[X(X-1)] = 4 \Rightarrow$ **(A)**

Why other options are wrong:

Q10.

- Option B (2): Equals $E[X] = \lambda$, not $E[X(X-1)]$.
- Option C (6): Equals $E[X^2] = \text{Var}(X) + E[X]^2 = 2 + 4 = 6$; not the required quantity.
- Option D (8): Incorrect; no standard formula gives 8 here.

Answer: (A) ← *Go back to Q10*

Solution

Concept — Radius of Convergence via the Ratio Test:
For a power series $\sum a_n x^n$, the radius of convergence is $R = \lim \frac{1}{|a_{n+1}/a_n|}$ (when the limit exists), or equivalently $1/L$ where $L = \lim |a_{n+1}/a_n|$ with x factored out.

Step 1 — Identify $a_n = 1/n!$ and apply the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| \cdot |x| = \frac{1}{(n+1)!} \cdot n! \cdot |x| = \frac{|x|}{n+1} \xrightarrow{n\to\infty} 0,$$

for *every* $x \in \mathbb{R}$. The ratio $\to 0 < 1$ regardless of x, so the series converges everywhere.

Final Answer: $R = \infty \Rightarrow$ **(D)**

Why other options are wrong:

Q11.

- Option A ($R = 0$): Series $\sum n!x^n$ has $R = 0$, not $\sum x^n/n!$.
- Option B ($R = 1$): Would mean convergence only on $(-1, 1)$; incorrect here.
- Option C ($R = 2$): No basis; the series converges everywhere.

Answer: (D) ← *Go back to Q11*

Solution

Concept — Partial Differentiation (Chain Rule for $\sin$):
Differentiate each term of $f(x, y) = x^2y + \sin(xy)$ with respect to x, treating y as a constant.

Step 1 — Differentiate term by term:

$$\frac{\partial}{\partial x}(x^2y) = 2xy, \qquad \frac{\partial}{\partial x}\sin(xy) = \cos(xy) \cdot y = y\cos(xy).$$

Therefore $\dfrac{\partial f}{\partial x} = 2xy + y\cos(xy)$.

Final Answer: $2xy + y\cos(xy) \Rightarrow$ **(C)**

Why other options are wrong:

Q12.

- Option A: Misses the factor y in the product x^2y and misses y in the chain-rule factor.
- Option B: Differentiates x^2y as x^2 (dropping factor y) and gets the chain-rule factor of x instead of y.
- Option D: Differentiates with respect to y, not x.

Answer: (C) ← *Go back to Q12*

Solution

Concept — Rank-Nullity Theorem:
For an $m \times n$ matrix A: $\text{rank}(A) + \text{nullity}(A) = n$ (number of columns).

Step 1 — Apply the theorem:

$$\text{nullity}(A) = n - \text{rank}(A) = 6 - 3 = 3.$$

Final Answer: Nullity $= 3 \Rightarrow$ **(B)**

Why other options are wrong:

Q13.
- Option A (2): $6 - 4 = 2$ — uses wrong rank.
- Option C (4): $4 - \text{rank}$ uses number of rows instead of columns.
- Option D (6): This is n; forgetting to subtract the rank.

Answer: (B) ← *Go back to Q13*

Solution

Concept — First-Order Linear ODE via Integrating Factor:
$\dfrac{dy}{dx} + P(x)y = Q(x)$. Integrating factor $\mu = e^{\int P\,dx}$.

Step 1 — Compute the integrating factor: $P(x) = 1$, so $\mu = e^x$.

Step 2 — Multiply through and integrate:

$$\frac{d}{dx}(ye^x) = e^x \cdot e^x = e^{2x} \implies ye^x = \int e^{2x}\,dx = \frac{e^{2x}}{2} + C.$$

Step 3 — Apply initial condition $y(0) = 0$:

$$0 \cdot e^0 = \frac{1}{2} + C \implies C = -\frac{1}{2}.$$

Step 4 — Solve for y and evaluate at $x = 1$:

$$y = \frac{e^x}{2} - \frac{e^{-x}}{2} = \sinh(x) \implies y(1) = \frac{e - e^{-1}}{2}.$$

Final Answer: $y(1) = \dfrac{e - e^{-1}}{2} \Rightarrow$ **(A)**

Why other options are wrong:

Q14.
- Option B ($e/2$): Ignores the $e^{-x}/2$ term; omits the homogeneous solution contribution.
- Option C ($(e + e^{-1})/2 = \cosh 1$): This is $\cosh(1)$, not $\sinh(1)$.
- Option D ($e^{-1}/2$): Evaluates the particular but not the full solution.

Answer: (A) ← *Go back to Q14*

Solution

Concept — Hypergeometric Distribution:
Drawing $n = 4$ balls without replacement from $N = 8$ (5 red, 3 blue). The probability of exactly $k = 2$ red balls is:

$$P(X = 2) = \frac{\binom{5}{2}\binom{3}{2}}{\binom{8}{4}}.$$

Step 1 — Compute combinations:

$$\binom{5}{2} = 10, \quad \binom{3}{2} = 3, \quad \binom{8}{4} = 70.$$

Step 2 — Evaluate:

$$P(X = 2) = \frac{10 \times 3}{70} = \frac{30}{70} = \frac{3}{7}.$$

Final Answer: $P = \frac{3}{7} \Rightarrow$ **(D)**

Why other options are wrong:

Q15.

- Option A (1/7): Would be $\binom{5}{1}\binom{3}{3}/70 = 5/70$.
- Option B (2/7): Numerator error in combination calculation.
- Option C (1/4): Uses incorrect denominator (not the hypergeometric formula).

Answer: (D) ← *Go back to Q15*

Solution

Concept — Mean of the Beta Distribution:
For $X \sim \text{Beta}(\alpha, \beta)$, $E[X] = \frac{\alpha}{\alpha + \beta}$.

Step 1 — Substitute $\alpha = 2$, $\beta = 3$:

$$E[X] = \frac{2}{2 + 3} = \frac{2}{5} = 0.4.$$

The green dashed line at $x = 0.4$ in the figure marks this expected value.

Final Answer: $E[X] = \dfrac{2}{5} \Rightarrow$ **(C)**

Why other options are wrong:

Q16.

- Option A $(1/3)$: This is the *mode* of Beta$(2, 3)$, not the mean.
- Option B $(1/2)$: This would be $E[X]$ for Beta$(1, 1) = U(0, 1)$.
- Option D $(3/5)$: This is $\beta/(\alpha + \beta) = 3/5$, not $\alpha/(\alpha + \beta)$.

Answer: (C) ← *Go back to Q16*

Solution

Concept — Conditional PDF from Joint PDF:
$f_{X|Y}(x \mid y) = \dfrac{f(x, y)}{f_Y(y)}$.

Step 1 — Find the marginal $f_Y(y)$: The joint is $f(x, y) = 2$ for $0 < x < y$ (at fixed y, x ranges from 0 to y):

$$f_Y(y) = \int_0^y 2\,dx = 2y, \quad 0 < y < 1.$$

Step 2 — Compute the conditional:

$$f_{X|Y}(x \mid y) = \frac{2}{2y} = \frac{1}{y}, \quad 0 < x < y.$$

This is the PDF of $U(0, y)$, which is $1/y$ on $(0, y)$ — consistent.

Final Answer: $f_{X|Y}(x \mid y) = \dfrac{1}{y} \Rightarrow$ **(B)**

Why other options are wrong:

Q17.

- Option A $(1/(2y))$: Divides by $2y$ but forgets the factor of 2 in the joint PDF.
- Option C $(2y)$: This is the marginal $f_Y(y)$, not the conditional.
- Option D $(2x/y^2)$: No such formula arises from this joint; confuses x and y roles.

Answer: (B) ← *Go back to Q17*

Solution

Concept — Limiting Distribution of the t-Distribution:
The $t(\nu)$ distribution has heavier tails than the standard normal, but as $\nu \to \infty$, these tails approach those of $N(0, 1)$.

Step 1 — Formal argument: The PDF of $t(\nu)$ is proportional to $(1 + x^2/\nu)^{-(\nu+1)/2}$. As $\nu \to \infty$:

$$\left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2} \to e^{-x^2/2},$$

which is the kernel of $N(0, 1)$.

Final Answer: $t(\nu) \xrightarrow{d} N(0, 1) \Rightarrow$ **(A)**

Why other options are wrong:

Q18.

- Option B (Cauchy): $t(1)$ is Cauchy; but $t(\nu)$ with $\nu \to \infty$ moves *away* from Cauchy.
- Option C ($\chi^2(1)$): $\chi^2(1)$ is supported on $(0, \infty)$; $t(\nu)$ is symmetric about 0.
- Option D ($U(-1, 1)$): No limiting relationship to the uniform distribution.

Answer: (A) ← *Go back to Q18*

Solution

Concept — Method of Moments for $U(0, \theta)$:
$E[X] = \theta/2$ for $X \sim U(0, \theta)$. Setting $\theta/2 = \bar{X}$ gives $\hat{\theta} = 2\bar{X}$.

Step 1 — Compute $\bar{X}$:

$$\bar{X} = \frac{1 + 3 + 5 + 7 + 9}{5} = \frac{25}{5} = 5.$$

Step 2 — Estimate θ:

$$\hat{\theta} = 2\bar{X} = 2 \times 5 = 10.$$

Final Answer: $\hat{\theta} = 10 \Rightarrow$ **(D)**

Why other options are wrong:

Q19.

- Option A (5): This is $\bar{X}$, not $\hat{\theta} = 2\bar{X}$.
- Option B (8): Confuses with the sample maximum; MLE would give $\hat{\theta}_{\text{MLE}} = 9$ here.
- Option C (9): This is the sample maximum (MLE), not the MOM estimator.

Answer: (D) ← *Go back to Q19*

Solution

Concept — One-Sample z-Test for a Proportion:
$Z = \dfrac{X - np_0}{\sqrt{np_0(1-p_0)}}$.

Step 1 — Substitute values:

$$Z = \frac{45 - 100 \times 0.5}{\sqrt{100 \times 0.5 \times 0.5}} = \frac{45 - 50}{\sqrt{25}} = \frac{-5}{5} = -1.$$

Final Answer: $Z = -1 \Rightarrow$ **(C)**

Why other options are wrong:

Q20.

- Option A $(+1)$: Drops the negative sign (computes $50 - 45$ in numerator).
- Option B (-2): Would require $X = 40$ or a different n.
- Option D $(+0.5)$: Incorrect denominator computation.

Answer: (C) ← *Go back to Q20*

Solution

Concept — Changing the Order of Double Integration:
For the integral $\int_0^1 \int_y^1 f(x,y)\,dx\,dy$, the region of integration is $\{(x,y) : 0 \le y \le 1,\ y \le x \le 1\}$, equivalently $0 \le y \le x \le 1$.

Step 1 — Describe the region via the x-first perspective: For fixed $x \in [0,1]$, y ranges from 0 to x. Therefore:

$$\int_0^1 \int_y^1 f(x,y)\,dx\,dy = \int_0^1 \int_0^x f(x,y)\,dy\,dx.$$

Final Answer: $\int_0^1 \int_0^x f(x,y)\,dy\,dx \Rightarrow$ **(B)**

Why other options are wrong:

Q21.

- Option A: Keeps x as the inner variable; only changes upper limit.
- Option C: Integrates y from x to 1 — this is the complementary triangle.
- Option D: Full unit square, not just the triangular region.

Answer: (B) ← *Go back to Q21*

Solution

Concept — Definition of Euler's Number e:
One of the fundamental definitions is $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

Step 1 — Identify the limit: The sequence $a_n = (1+1/n)^n$ is monotone increasing and bounded above. Its limit is exactly Euler's constant $e \approx 2.71828$.

Final Answer: limit $= e \Rightarrow$ **(A)**

Why other options are wrong:

Q22.
- Option B (1): $\lim(1+1/n)^1 = 1$, but the exponent grows with n.
- Option C (∞): The sequence is bounded.
- Option D (0): The terms are > 1 for all finite n.

Answer: (A) ← *Go back to Q22*

Solution

Concept — Positive Semi-Definiteness via Eigenvalues:
A symmetric matrix A is positive semi-definite (PSD) iff all eigenvalues ≥ 0, and positive definite iff all eigenvalues > 0.

Step 1 — Analyse the eigenvalues of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$: Eigenvalues are 0 and 2 (given). Since both are ≥ 0 but one equals 0, A is PSD but **not** positive definite.

Step 2 — Verify: For $\mathbf{v} = (1, -1)^T$, $\mathbf{v}^T A \mathbf{v} = 0$; so A is not positive definite.

Final Answer: Positive semi-definite $\Rightarrow$ **(D)**

Why other options are wrong:

Q23.
- Option A (Positive definite): Requires *all* eigenvalues > 0; here $\lambda = 0$.
- Option B (Negative definite): Requires all eigenvalues < 0; $\lambda = 2 > 0$.
- Option C (Negative semi-definite): Requires all eigenvalues ≤ 0; $\lambda = 2 > 0$.

Answer: (D) ← *Go back to Q23*

Solution

Concept — Chebyshev's Inequality:
$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$.

Step 1 — Identify parameters: $\mu = 10$, $\sigma^2 = 9$, $k = 6$.

Step 2 — Apply the bound:

$$P(|X - 10| \geq 6) \leq \frac{9}{6^2} = \frac{9}{36} = \frac{1}{4} = 0.25.$$

Final Answer: Upper bound $= 0.25 \Rightarrow$ **(C)**

Why other options are wrong:

Q24.

- Option A (0.10): Too tight; Chebyshev only guarantees 0.25.
- Option B (0.15): Also too tight for the given parameters.
- Option D (0.50): Over-estimates; comes from using $k = \sigma = 3$ instead of $k = 6$.

Answer: (C) ← *Go back to Q24*

Solution

Concept — Chi-Square Distribution of the Sample Variance:
For i.i.d. $N(\mu, \sigma^2)$: $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$, independent of $\bar{X}$.

Step 1 — Identify the correct statement: $(n-1)S^2/\sigma^2$ is a sum of $n-1$ squared standard normals in the eigen-decomposition of the sample; its distribution is $\chi^2(n-1)$.

Final Answer: $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1) \Rightarrow$ **(B)**

Why other options are wrong:

Q25.

- Option A: $\sum X_i/\sigma$ is not a chi-square; $\sum (X_i - \mu)^2/\sigma^2 \sim \chi^2(n)$.
- Option C: nS^2/σ^2 uses n not $(n-1)$ in the scaling; incorrect degrees of freedom.
- Option D: Divides by σ not σ^2; changes the distribution completely.

Answer: (B) ← *Go back to Q25*

Solution

Concept — Pearson Correlation Coefficient and Scatter Plots:
A negative r (here $r \approx -0.85$) indicates a strong negative linear association: as one variable increases, the other tends to decrease.

Step 1 — Interpret the scatter plot: The points trend downward from left to right (high $X \to$ low Y), and the regression line has a negative slope. $|r| = 0.85$ indicates a strong (but not perfect) relationship.

Final Answer: As X increases, Y tends to decrease; negative association $\Rightarrow$ **(A)**

Why other options are wrong:

Q26.

- Option B (positive): The slope of the cloud is clearly downward.
- Option C (uncorrelated): $|r| = 0.85$ is far from zero.
- Option D (perfect $r = -1$): Perfect correlation means all points lie exactly on a line; here there is scatter.

$\boxed{\textbf{Answer: (A)}}$ ← *Go back to Q26*

Solution

Concept — Coefficient of Determination R^2:
R^2 measures the proportion of the total variation in the *response* variable Y that is explained by the regression on X.

Step 1 — Interpret $R^2 = 0.81$: 81% of the variability in Y is accounted for by the linear regression on X. The remaining 19% is unexplained (residual variance).

Final Answer: 81% of the variation in Y explained by $X \Rightarrow$ **(D)**

Why other options are wrong:

Q27.

- Option A: $\hat{\beta}_1 = \sqrt{R^2} \cdot (s_Y/s_X)$ in sign-adjusted form; not R^2 itself.
- Option B: Significance requires comparing F-statistic to a critical value; R^2 alone does not determine it.
- Option C: R^2 measures variation explained *in* Y, not in X.

$\boxed{\textbf{Answer: (D)}}$ ← *Go back to Q27*

Solution

Concept — Relationship between the t- and F-Distributions:
If $T \sim t(n)$, then $T^2 \sim F(1, n)$. This follows because $T = Z/\sqrt{\chi^2(n)/n}$, so $T^2 = Z^2/(\chi^2(n)/n) \sim F(1, n)$.

Step 1 — Apply the result: $T \sim t(n) \implies T^2 \sim F(1, n)$ (numerator d.f. $= 1$, denominator d.f. $= n$).

Final Answer: $T^2 \sim F(1, n) \Rightarrow$ **(C)**

Why other options are wrong:

Q28.

- Option A: $\chi^2(1) = F(1, \infty)$; squaring a $t(n)$ gives $F(1, n)$, not $\chi^2(1)$ unless $n \to \infty$.
- Option B: $F(n, 1)$ reverses the numerator and denominator degrees of freedom.
- Option D: T itself is t-distributed, not F-distributed.

Answer: (C) ← *Go back to Q28*

Solution

Concept — Sufficient Statistic for Bernoulli(p):
By the Factorization (Neyman–Fisher) Theorem, T is sufficient for θ iff the likelihood factors as $g(T(\mathbf{x}), \theta) \cdot h(\mathbf{x})$.

Step 1 — Likelihood for Bernoulli:

$$L(p; \mathbf{x}) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{\sum x_i}(1-p)^{n-\sum x_i}.$$

This depends on $\mathbf{x}$ only through $T = \sum_{i=1}^{n} x_i$, so T is sufficient for p.

Final Answer: $T = \sum X_i$ is sufficient $\Rightarrow$ **(B)**

Why other options are wrong:

Q29.

- Option A (Sample variance): Sufficient for σ^2 in a normal model, not for p in Bernoulli.
- Option C (Sample median): Not sufficient; loses information compared to $\sum X_i$.
- Option D (X_1 alone): Contains less information than the entire sample; not sufficient for $n > 1$.

Answer: (B) ← *Go back to Q29*

Solution

Concept — CDF of the Maximum Order Statistic from $U(0, 1)$:
For $X_1, \ldots, X_n$ i.i.d. with CDF F, the CDF of $X_{(n)} = \max$ is $[F(x)]^n$.

Step 1 — For $U(0, 1)$, $F(x) = x$:

$$F_{X_{(n)}}(x) = P(X_{(n)} \le x) = P(\text{all } X_i \le x) = [F(x)]^n = x^n, \quad 0 < x < 1.$$

The graph confirms that for $n = 1$: linear, $n = 2$: concave parabola, $n = 3$: more concave cubic.

Final Answer: $F_{X_{(n)}}(x) = x^n$ for $0 < x < 1 \Rightarrow$ **(A)**

Why other options are wrong:

Q30.
- Option B (nx^{n-1}): This is the *PDF* (density), not the CDF.
- Option C ($1 - (1-x)^n$): This is the CDF of the *minimum* $X_{(1)}$, not the maximum.
- Option D (x/n): No standard result; the maximum CDF has no $1/n$ factor.

Answer: (A) ← *Go back to Q30*

Q31.

Solution

Concept — Convergence Tests for Series:

Option A — INCORRECT: The ratio test is *inconclusive* when $\lim |a_{n+1}/a_n| = 1$ (e.g., the harmonic series and $\sum 1/n^2$ both have ratio limit equal to 1, yet one diverges and one converges).

Option B — CORRECT: Absolute convergence implies convergence: if $\sum |a_n| < \infty$, then $\sum a_n$ converges. This is a standard theorem.

Option C — INCORRECT: The p-series $\sum 1/n^p$ converges iff $p > 1$ (strict inequality). At $p = 1$ (the harmonic series), it diverges.

Option D — CORRECT: The Alternating Series Test (Leibniz): if $b_n > 0$, b_n is non-increasing, and $b_n \to 0$, then $\sum(-1)^n b_n$ converges. The graph shows the partial sums oscillating toward $\ln 2$.

Final Answer: Options **(B)** and **(D)** are correct $\Rightarrow$ **(BD)**

Answer: (BD) ← *Go back to Q31*

Q32.

Solution

Concept — Properties of Statistical Estimators:

Option A — CORRECT: If T_n is unbiased ($E[T_n] = \theta$) and $\text{Var}(T_n) \to 0$, then by Chebyshev's inequality $T_n \xrightarrow{P} \theta$: T_n is consistent.

Option B — INCORRECT: Consistency only requires convergence in probability; an estimator can be biased yet consistent (bias $\to 0$ as $n \to \infty$ even if non-zero for finite n).

Option C — CORRECT: $E[\bar{X}] = \mu$ by linearity of expectation; the sample mean is always unbiased for the population mean.

Option D — INCORRECT: $\hat{\sigma}^2 = \frac{1}{n}\sum(X_i - \bar{X})^2$ has $E[\hat{\sigma}^2] = \frac{n-1}{n}\sigma^2 \neq \sigma^2$; it is biased. The unbiased estimator uses $(n-1)$ in the denominator.

Final Answer: Options **(A)** and **(C)** are correct $\Rightarrow$ **(AC)**

Answer: (AC) ← *Go back to Q32*

Q33.

Solution

Concept — Linear Transformations of Random Variables:

Option A — CORRECT: If $X \sim N(0,1)$ and $a \neq 0$, then $aX + b \sim N(b, a^2)$. The mean shifts by b and variance scales by a^2.

Option B — CORRECT: By linearity of expectation: $E[aX + b] = aE[X] + b$. This holds for any random variable and any constants.

Option C — INCORRECT: $\text{Var}(aX + b) = a^2\,\text{Var}(X)$, NOT $a \cdot \text{Var}(X)$. The constant b shifts the mean and does not affect variance; a squares in the variance.

Option D — CORRECT: For independent X, Y: $\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(-Y) = \text{Var}(X) + \text{Var}(Y)$.

Final Answer: Options **(A)**, **(B)**, and **(D)** are correct $\Rightarrow$ **(ABD)**

Answer: (ABD) ← *Go back to Q33*

Q34.

Solution

Concept — Type I and Type II Errors in Hypothesis Testing:

Option A — INCORRECT: For a fixed sample size, α and β are linked: making the rejection region smaller to reduce α enlarges β. Only increasing n can reduce both simultaneously.

Option B — CORRECT: With a fixed n, the rejection region is set by α. A smaller α shrinks the rejection region, so more truly false H_0 scenarios lead to failure-to-reject, increasing β.

Option C — CORRECT: By definition: p-value $= P(\text{test statistic} \geq \text{observed} \mid H_0 \text{ true})$. We reject H_0 when p-value $< \alpha$.

Option D — INCORRECT: Rejecting H_0 when it is true is a **Type I** error (false positive). A Type II error is *failing* to reject H_0 when it is false.

Final Answer: Options **(B)** and **(C)** are correct $\Rightarrow$ **(BC)**

Answer: (BC) ← *Go back to Q34*

Q35.

Solution

Concept — Properties of Eigenvalues of Real Matrices:

Option A — CORRECT: Every real symmetric matrix is diagonalisable over $\mathbb{R}$ and all its eigenvalues are real (Spectral Theorem).

Option B — INCORRECT: If $A\mathbf{v} = \lambda\mathbf{v}$, then $(2A + I)\mathbf{v} = 2\lambda\mathbf{v} + \mathbf{v} = (2\lambda + 1)\mathbf{v}$. So the eigenvalue of $2A + I$ is $\mathbf{2\lambda + 1}$, not 2λ.

Option C — CORRECT: $\det(A) = \prod_{i=1}^{n} \lambda_i$ (product of all eigenvalues, counting multiplicity); this follows from the characteristic polynomial.

Option D — CORRECT: $\operatorname{trace}(A) = \sum_{i=1}^{n} \lambda_i$ (sum of eigenvalues, counting multiplicity).

Final Answer: Options **(A)**, **(C)**, and **(D)** are correct $\Rightarrow$ **(ACD)**

Answer: (ACD) ← *Go back to Q35*

Q36.

Solution

Concept — Conditional Probability Properties:

Option A — INCORRECT: $P(A \mid B) = P(A \cap B)/P(B)$ and $P(B \mid A) = P(A \cap B)/P(A)$. These are equal only if $P(A) = P(B)$.

Option B — CORRECT: Conditioning on B creates a sub-sample space in which A and A^c partition the space: $P(A \mid B) + P(A^c \mid B) = 1$.

Option C — INCORRECT: $P(A \mid A \cap B) = P(A \cap (A \cap B))/P(A \cap B) = P(A \cap B)/P(A \cap B) = 1 \neq P(A)$ in general.

Option D — CORRECT: Independence of A and B is defined by $P(A \cap B) =$

$P(A)P(B)$, which directly gives $P(A \mid B) = P(A \cap B)/P(B) = P(A)$.

Final Answer: Options **(B)** and **(D)** are correct $\Rightarrow$ **(BD)**

Answer: (BD) ← *Go back to Q36*

Q37.

Solution

Concept — Properties of the Binomial Distribution:

Option A — CORRECT: $E[X] = np$. Direct from $E[X] = \sum_{k=0}^{n} k\binom{n}{k}p^k(1-p)^{n-k} = np$.

Option B — CORRECT: $\text{Var}(X) = np(1-p)$. Standard result from second moment calculation.

Option C — CORRECT: Poisson limit theorem: for $n \to \infty$, $p \to 0$ with $np = \lambda$ fixed, $\text{Bin}(n, p) \xrightarrow{d} \text{Poisson}(\lambda)$.

Option D — INCORRECT: $\text{Bin}(n, p)$ is symmetric only when $p = 1/2$. For $p \neq 1/2$, the distribution is skewed (positively skewed for $p < 1/2$, negatively for $p > 1/2$).

Final Answer: Options **(A)**, **(B)**, and **(C)** are correct $\Rightarrow$ **(ABC)**

Answer: (ABC) ← *Go back to Q37*

Q38.

Solution

Concept — Simple Linear Regression (OLS):

Option A — CORRECT: The OLS normal equations require $\sum \hat{\varepsilon}_i = 0$ whenever an intercept is included. This is a direct consequence of the first normal equation.

Option B — INCORRECT: OLS estimation (minimising sum of squared residuals) does not require normality. Normality is needed only for inference (confidence intervals, t-tests on coefficients).

Option C — INCORRECT: The OLS slope is $\hat{\beta}_1 = S_{XY}/S_{XX}$, where $S_{XX} = \sum(X_i - \bar{X})^2$ (not S_{YY}). Using S_{YY} in the denominator is the formula for r^2 decomposition, not $\hat{\beta}_1$.

Option D — CORRECT: Under $E[\varepsilon_i] = 0$ and fixed X_i: $E[\hat{\beta}_0] = \beta_0$ and $E[\hat{\beta}_1] = \beta_1$ (unbiasedness of OLS), proved via the normal equations.

Final Answer: Options **(A)** and **(D)** are correct $\Rightarrow$ **(AD)**

Answer: (AD) ← *Go back to Q38*

Q39.

Solution

Concept — Real Analysis: Integrability and the Fundamental Theorem:

Option A — INCORRECT: $\int_0^\infty e^x\,dx = \lim_{b\to\infty}[e^x]_0^b = \infty$. The integral diverges (to $+\infty$). Similarly, $\int_{-\infty}^0 e^x\,dx$ converges to 1, but the two-sided integral diverges.

Option B — CORRECT: A Riemann integrable function on $[a,b]$ must be bounded there. Unbounded functions on $[a,b]$ are not Riemann integrable (only improperly integrable).

Option C — CORRECT: Fundamental Theorem of Calculus (Part 2): if $F' = f$ on $[a,b]$ and f is integrable, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Option D — INCORRECT: Every monotone function on a closed bounded interval is Riemann integrable (standard result; its jump discontinuities are countable, hence a set of measure zero).

Final Answer: Options **(B)** and **(C)** are correct $\Rightarrow$ **(BC)**

Answer: (BC) ← *Go back to Q39*

Q40.

Solution

Concept — Order Statistics:

Option A — CORRECT: By definition, $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ are the original sample values arranged in non-decreasing order.

Option B — INCORRECT: Order statistics are *dependent*: knowing $X_{(k)}$ restricts the possible values of $X_{(k+1)}$. They are not independent (except in degenerate cases).

Option C — CORRECT: If the parent distribution is symmetric about 0, then $X_{(k)} \overset{d}{=} -X_{(n+1-k)}$, so $E[X_{(k)}] = -E[X_{(n+1-k)}]$.

Option D — CORRECT: The range $R = X_{(n)} - X_{(1)}$ measures the spread of the data; it is a commonly used non-parametric measure of dispersion.

Final Answer: Options **(A)**, **(C)**, and **(D)** are correct $\Rightarrow$ **(ACD)**

Answer: (ACD) ← Go back to Q40

Q41.

Solution

Concept — Expected Value of the Geometric Distribution:
For $X \sim \text{Geometric}(p)$ (first-success model), $E[X] = 1/p$.

Step 1 — Apply the formula:

$$E[X] = \frac{1}{p} = \frac{1}{0.25} = \frac{1}{1/4} = 4.$$

Final Answer: $E[X] = 4$

Answer: (4) ← Go back to Q41

Q42.

Solution

Concept — Integration by Parts:
$\int u\, dv = uv - \int v\, du$. Choose $u = x$, $dv = e^{-x}dx$, so $du = dx$, $v = -e^{-x}$.

Step 1 — Apply integration by parts:

$$\int_0^2 xe^{-x}\, dx = \left[-xe^{-x}\right]_0^2 + \int_0^2 e^{-x}\, dx = -2e^{-2} + \left[-e^{-x}\right]_0^2 = -2e^{-2} + (-e^{-2} + 1).$$

Step 2 — Simplify:

$$= 1 - 3e^{-2} \approx 1 - 3(0.13534) \approx 1 - 0.40601 = 0.594.$$

Final Answer: 0.594 (to 3 d.p.)

Answer: (0.594) ← Go back to Q42

Q43.

Solution

Concept — Powers of Diagonal Matrices and Trace:
For a diagonal matrix, A^2 is also diagonal with entries squared.

Step 1 — Compute A^2:

$$A^2 = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}^2 = \begin{pmatrix} 9 & 0 \\ 0 & 25 \end{pmatrix}.$$

Step 2 — Trace:

$$\text{trace}(A^2) = 9 + 25 = 34.$$

Final Answer: $\text{trace}(A^2) = \mathbf{34}$

Answer: (34) ← *Go back to Q43*

Q44.

Solution

Concept — Probability of Union for Independent Events:
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Independence gives $P(A \cap B) = P(A)P(B)$.

Step 1 — Compute:

$$P(A \cap B) = 0.3 \times 0.4 = 0.12.$$

$$P(A \cup B) = 0.3 + 0.4 - 0.12 = 0.58.$$

Final Answer: $P(A \cup B) = \mathbf{0.58}$

Answer: (0.58) ← *Go back to Q44*

Q45.

Solution

Concept — Method of Moments Estimator for Poisson(λ):
$E[X] = \lambda$. Setting $\bar{X} = \lambda$ gives the MOM estimator $\hat{\lambda} = \bar{X}$.

Step 1 — Compute $\bar{X}$:

$$\bar{X} = \frac{3 + 1 + 2 + 4}{4} = \frac{10}{4} = 2.5.$$

Final Answer: $\hat{\lambda} = \mathbf{2.5}$

Answer: (2.5) ← *Go back to Q45*

Q46.

Solution

Concept — Standard Normal Probability:
$P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha$ for a standard normal Z.

Step 1 — Apply: $z = 1.96$ is the 97.5th percentile of $N(0,1)$. By symmetry and standard tables:

$$P(-1.96 < Z < 1.96) = 2\Phi(1.96) - 1 = 2(0.975) - 1 = 0.95.$$

Final Answer: $P(-1.96 < X < 1.96) = \mathbf{0.95}$

Answer: (0.95) ← *Go back to Q46*

Q47.

Solution

Concept — Sum of the Series $\sum nx^n$:
For $|x| < 1$: $\displaystyle\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$.

Step 1 — Substitute $x = 1/2$:

$$\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{1/2}{(1-1/2)^2} = \frac{1/2}{1/4} = 2.$$

Final Answer: Sum $= \mathbf{2}$

Answer: (2) ← *Go back to Q47*

Q48.

Solution

Concept — Tail Probability of the Exponential Distribution:
For $X \sim \text{Exp}(\lambda)$: $P(X > t) = e^{-\lambda t}$.

Step 1 — Compute:

$$P(X > 4) = e^{-\lambda \cdot 4} = e^{-(1/2)(4)} = e^{-2} \approx 0.1353.$$

Final Answer: $P(X > 4) = e^{-2} \approx \mathbf{0.1353}$

Answer: (0.1353) ← *Go back to Q48*

Q49.

Solution

Concept — Second Moment Decomposition:
$E[(X-Y)^2] = \text{Var}(X-Y) + [E(X-Y)]^2$.

Step 1 — Compute each part:

$$E[X-Y] = E[X] - E[Y] = 3 - 5 = -2.$$

Since X, Y are independent:

$$\text{Var}(X-Y) = \text{Var}(X) + \text{Var}(Y) = 1 + 4 = 5.$$

Step 2 — Combine:

$$E[(X-Y)^2] = 5 + (-2)^2 = 5 + 4 = 9.$$

Final Answer: $E[(X-Y)^2] = \mathbf{9}$

Answer: (9) ← *Go back to Q49*

Q50.

Solution

Concept — First-Order Linear ODE via Integrating Factor:
Rewrite $x\,dy/dx - y = x^2$ as $dy/dx - (1/x)y = x$. Integrating factor: $\mu = e^{\int(-1/x)dx} = e^{-\ln x} = 1/x$.

Step 1 — Multiply by $\mu = 1/x$:

$$\frac{d}{dx}\left(\frac{y}{x}\right) = 1 \implies \frac{y}{x} = x + C.$$

Step 2 — Apply $y(1) = 0$:

$$0 = 1 + C \implies C = -1 \implies y = x^2 - x.$$

Step 3 — Evaluate at $x = 2$:

$$y(2) = 4 - 2 = 2.$$

Final Answer: $y(2) = \mathbf{2}$

Answer: (2) ← *Go back to Q50*

Q51.

Solution

Concept — z-Based Confidence Interval for a Normal Mean (Known σ):
$\bar{X} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$.

Step 1 — Identify parameters: $\bar{x} = 14$, $\sigma = 4$, $n = 9$, $z_{0.05} = 1.645$.

Step 2 — Compute margin of error:

$$z_{0.05} \cdot \frac{\sigma}{\sqrt{n}} = 1.645 \times \frac{4}{3} = 1.645 \times 1.\overline{3} \approx 2.1933.$$

Step 3 — Upper bound:

$$14 + 2.1933 = 16.1933 \approx 16.19.$$

Final Answer: Upper bound $\approx$ **16.19**

Answer: (16.19) ← *Go back to Q51*

Q52.

Solution

Concept — Covariance with a Linear Function:
$\mathrm{Cov}(X, aX + b) = a\,\mathrm{Var}(X)$.

Step 1 — Apply with $a = 2$, $b = 3$:

$$\mathrm{Cov}(X, 2X + 3) = 2\,\mathrm{Var}(X) = 2 \times 1 = 2.$$

($X \sim N(0, 1)$ has $\mathrm{Var}(X) = 1$.)

Final Answer: $\mathrm{Cov}(X, Y) =$ **2**

Answer: (2) ← *Go back to Q52*

Q53.

Solution

Concept — UMP Test for Poisson (One-Sided, NP Lemma):
For $H_0 : \lambda = 1$ vs. $H_1 : \lambda > 1$, the UMP test based on $T = \sum X_i$ rejects H_0 when

$T > c$, where c is chosen so that $P(T > c \mid \lambda = 1) = \alpha$.

Step 1 — Identify the null distribution: Under H_0, $T \sim \text{Poisson}(n \cdot \lambda_0) = \text{Poisson}(10)$.

Step 2 — Use the given tail probability: $P(T > 15 \mid \text{Pois}(10)) \approx 0.0487 < 0.05$. The next lower critical value ($T > 14$) gives $P(T > 14 \mid \text{Pois}(10)) > 0.05$. To maintain $\alpha \leq 0.05$, we set $c = 15$.

Final Answer: Critical value $c = \mathbf{15}$

Answer: (15) ← *Go back to Q53*

Q54.

Solution

Concept — Sum of i.i.d. Exponential Variables (Gamma Distribution):
If $X_1, X_2 \overset{\text{iid}}{\sim} \text{Exp}(1)$, then $X_1 + X_2 \sim \text{Gamma}(2, 1)$ with PDF $f(t) = te^{-t}$ for $t > 0$.

Step 1 — Use the Gamma CDF:

$$P(X_1 + X_2 < 2) = \int_0^2 te^{-t}\,dt = 1 - (1 + 2)e^{-2} = 1 - 3e^{-2}.$$

Step 2 — Evaluate:

$$1 - 3e^{-2} \approx 1 - 3(0.13534) \approx 0.594.$$

Final Answer: $P(X_1 + X_2 < 2) \approx \mathbf{0.594}$

Answer: (0.594) ← *Go back to Q54*

Q55.

Solution

Concept — Inverse of a 2×2 Matrix:
For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $A^{-1} = \dfrac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Step 1 — Compute $\det(A)$:

$$\det(A) = 2 \times 2 - (-1)(-1) = 4 - 1 = 3.$$

Step 2 — Find A^{-1}:

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Step 3 — Read off $(1,1)$ entry:

$$(A^{-1})_{11} = \frac{2}{3} \approx 0.6667.$$

Final Answer: $(A^{-1})_{11} = \frac{2}{3} \approx \mathbf{0.6667}$

Answer: (0.6667) ← Go back to Q55

Q56.

Solution

Concept — UMVUE for the Uniform Maximum θ:
For i.i.d. $U(0,\theta)$, the UMVUE is $\hat{\theta}^* = \dfrac{n+1}{n}X_{(n)}$.

Step 1 — Substitute $n = 5$, $X_{(5)} = 8$:

$$\hat{\theta}^* = \frac{5+1}{5} \times 8 = \frac{6}{5} \times 8 = \frac{48}{5} = 9.6.$$

Verification: $E\left[\frac{n+1}{n}X_{(n)}\right] = \frac{n+1}{n} \cdot \frac{n}{n+1}\theta = \theta$. Unbiased. ✓

Final Answer: $\hat{\theta}^* = \mathbf{9.6}$

Answer: (9.6) ← Go back to Q56

Q57.

Solution

Concept — Power of a One-Sided z-Test:
Reject H_0 when $\bar{X} > 20 + z_{0.01} \cdot (\sigma/\sqrt{n})$.

Step 1 — Compute the rejection threshold:

$$c^* = 20 + 2.326 \times \frac{5}{\sqrt{25}} = 20 + 2.326 \times 1 = 22.326.$$

Step 2 — Compute power at $\mu = 23$:

$$\pi = P(\bar{X} > 22.326 \mid \mu = 23) = P\left(Z > \frac{22.326 - 23}{1}\right) = P(Z > -0.674).$$

$$P(Z > -0.674) = P(Z < 0.674) = \Phi(0.674) \approx 0.750.$$

Final Answer: Power $\approx$ **0.750**

Answer: (0.750) ← *Go back to Q57*

Q58.

Solution

Concept — Normal Approximation to Binomial with Continuity Correction:
$X \sim \text{Bin}(64, 0.5)$, so $\mu = 32$, $\sigma^2 = 16$, $\sigma = 4$.

Step 1 — Apply continuity correction:

$$P(X \geq 36) = P(X \geq 35.5) \approx P\left(Z \geq \frac{35.5 - 32}{4}\right) = P(Z \geq 0.875).$$

Step 2 — Compute:

$$P(Z \geq 0.875) = 1 - \Phi(0.875) \approx 1 - 0.809 = 0.191.$$

Final Answer: $P(X \geq 36) \approx$ **0.191**

Answer: (0.191) ← *Go back to Q58*

Q59.

Solution

Concept — Variance of the Beta Distribution (Posterior):
For $p \mid X \sim \text{Beta}(\alpha, \beta)$:

$$\text{Var}(p \mid X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}.$$

Step 1 — Substitute posterior parameters $\alpha = 8$, $\beta = 6$:

$$\alpha + \beta = 14, \quad \alpha + \beta + 1 = 15.$$

$$\text{Var}(p \mid X = 6) = \frac{8 \times 6}{14^2 \times 15} = \frac{48}{196 \times 15} = \frac{48}{2940} = \frac{4}{245}.$$

Step 2 — Evaluate:

$$\frac{4}{245} \approx 0.016327 \approx 0.0163.$$

Final Answer: $\text{Var}(p \mid X = 6) \approx \mathbf{0.0163}$

Answer: (0.0163) ← *Go back to Q59*

Q60.

Solution

Concept — Normalisation Constant of a Joint PDF:
Integrate $f(x, y) = cx^2y$ over the support $\{0 < y < x < 1\}$ and set equal to 1.

Step 1 — Set up the integral (integrate y first):

$$\int_0^1 \int_0^x cx^2y \, dy \, dx = c \int_0^1 x^2 \left[\frac{y^2}{2}\right]_0^x dx = c \int_0^1 \frac{x^4}{2} \, dx = \frac{c}{2} \cdot \frac{x^5}{5}\Bigg|_0^1 = \frac{c}{10}.$$

Step 2 — Solve for c:

$$\frac{c}{10} = 1 \implies c = 10.$$

Final Answer: $c = 10$

Answer: (10) ← *Go back to Q60*

Answer Key

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	A	2	B	3	D	4	C	5	B
6	A	7	D	8	C	9	B	10	A
11	D	12	C	13	B	14	A	15	D
16	C	17	B	18	A	19	D	20	C
21	B	22	A	23	D	24	C	25	B
26	A	27	D	28	C	29	B	30	A
31	BD	32	AC	33	ABD	34	BC	35	ACD
36	BD	37	ABC	38	AD	39	BC	40	ACD
41	4	42	0.594	43	34	44	0.58	45	2.5
46	0.95	47	2	48	0.1353	49	9	50	2
51	16.19	52	2	53	15	54	0.594	55	0.6667
56	9.6	57	0.750	58	0.191	59	0.0163	60	10