IIT JAM Mathematical Statistics

Sample Paper – 4

Instructions

- This paper contains **60 questions** carrying a maximum of **100 marks**. Duration: **3 hours (180 minutes)**.
- **Section A** (**Q1–Q30**, MCQ): Q1–Q10 carry **+1 mark** each; wrong response: $-\frac{1}{3}$ mark. Q11–Q30 carry **+2 marks** each; wrong response: $-\frac{2}{3}$ mark. Choose the **single best** option.
- **Section B** (**Q31–Q40**, MSQ): Each carries **+2 marks**. **No negative marking. One or more** options may be correct; full marks only if *all* correct options (and no incorrect options) are selected.
- **Section C** (**Q41–Q60**, NAT): Q41–Q50 carry **+1 mark**; Q51–Q60 carry **+2 marks**. **No negative marking.** Enter the numerical value using the virtual keypad.
- **Calculator policy:** Personal calculators, log tables, and electronic devices are **strictly prohibited**. A virtual scientific calculator is provided in the CBT interface. Rough sheets are supplied at the centre.
- Once you move to the next section, you cannot return to a previous section.

Section A — Multiple Choice Questions (MCQ)

***Q1–Q10**: 1 mark each ($-\frac{1}{3}$ for wrong). Choose the **single** correct option.*

Q1. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$. Which of the following statements is correct?

(A) The series diverges.
(B) The series converges absolutely.
(C) The series converges absolutely and conditionally.
(D) The series converges conditionally but not absolutely.

Q2. The Maclaurin series of $f(x) = \cos x$ is $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$. The coefficient of x^4 in this expansion is:

(A) $\frac{1}{6}$
(B) $\frac{1}{12}$

(C) $\dfrac{1}{24}$

(D) $\dfrac{1}{48}$

Q3. A 3×3 real matrix A has eigenvalues $1, 1, 2$ where the eigenvalue 1 has geometric multiplicity 2. Which statement is correct?

(A) A is not diagonalizable because it has a repeated eigenvalue.

(B) A is diagonalizable.

(C) A is diagonalizable only if $\det A \neq 0$.

(D) A has fewer than 3 linearly independent eigenvectors.

Q4. Let $P(A) = 0.3$, $P(B \mid A) = 0.7$ and $P(B \mid A^c) = 0.2$. Using the law of total probability, $P(B)$ equals:

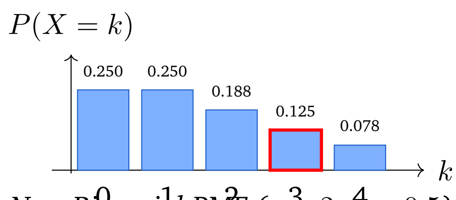
(A) 0.35

(B) 0.42

(C) 0.21

(D) 0.50

Q5. Let X be the number of *failures* before the 2nd success in independent Bernoulli trials with $p = 0.5$. The probability mass function is $P(X = k) = \binom{k+1}{k}(0.5)^2(0.5)^k$. Find $P(X = 3)$.



Neg. Binomial PMF ($r = 2$, $p = 0.5$)

(A) 0.0625

(B) 0.0781

(C) 0.1875

(D) 0.125

Q6. Let X follow a Laplace (double exponential) distribution with location $\mu = 0$ and scale $b = 1$, so $f(x) = \frac{1}{2}e^{-|x|}$. Then $\mathrm{Var}(X)$ equals:

(A) 1

(B) $\sqrt{2}$
(C) 2
(D) 4

Q7. Evaluate $\displaystyle\int_0^{\pi} \sin^2 x\,dx$.

(A) 1
(B) $\dfrac{\pi}{2}$
(C) π
(D) 2

Q8. Consider the linear system with augmented matrix $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}$. Which statement is correct?

(A) The system has no solution.
(B) The system has a unique solution.
(C) The system has infinitely many solutions.
(D) The system has exactly two solutions.

Q9. Let X and Y be independent $\mathrm{Exp}(1)$ random variables. Then $E[\min(X, Y)]$ equals:

(A) 1
(B) 2
(C) $\dfrac{1}{2}$
(D) $\dfrac{1}{4}$

Q10. Let $X \sim \mathrm{Exp}(1)$ with CDF $F(x) = 1 - e^{-x}$ for $x \geq 0$. Then $P(1 < X \leq 3)$ equals:

(A) $e^{-3} - e^{-1}$
(B) $1 - e^{-2}$
(C) e^{-1}
(D) $e^{-1} - e^{-3}$

***Q11–Q30**: 2 marks each ($-\frac{2}{3}$ for wrong). Choose the **single** correct option.*

Q11. By the root test, $\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n$ converges. Which of the following correctly explains this?

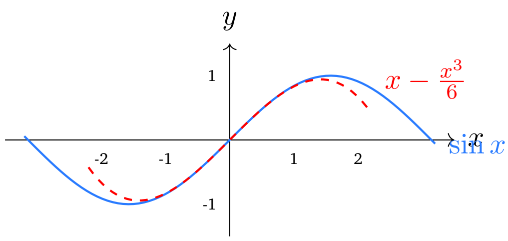
(A) $\limsup_{n\to\infty}|a_n|^{1/n} = \frac{3}{4} < 1$, so the series converges absolutely by the root test.

(B) $\limsup_{n\to\infty}|a_n|^{1/n} = \frac{4}{3} > 1$, so the series diverges.

(C) The ratio test gives ratio $= 1$, which is inconclusive.

(D) The series converges because its terms are bounded.

Q12. The Lagrange form of the remainder after n terms of the Taylor expansion of f about $a = 0$ is $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$ for some ξ between 0 and x. For $f(x) = \sin x$ and $n = 3$, the remainder satisfies:



(A) $|R_3(x)| \le \frac{|x|^3}{6}$

(B) $|R_3(x)| \le \frac{|x|^5}{120}$

(C) $|R_3(x)| \le \frac{|x|^4}{24}$

(D) $|R_3(x)| = 0$ for all x

Q13. Let $A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$. Which of the following correctly classifies A?

(A) Positive definite

(B) Negative definite

(C) Positive semi-definite but not positive definite

(D) Negative semi-definite

Q14. Solve $y'' - y = 0$ with $y(0) = 1$, $y'(0) = 0$. The value $y(1)$ is:

(A) e

(B) $\sinh(1)$

(C) 1

(D) $\cosh(1) = \dfrac{e + e^{-1}}{2}$

Q15. In a Negative Binomial experiment where X is the total number of trials to obtain $r = 5$ successes with success probability $p = 0.25$, the expected value $E[X]$ is:

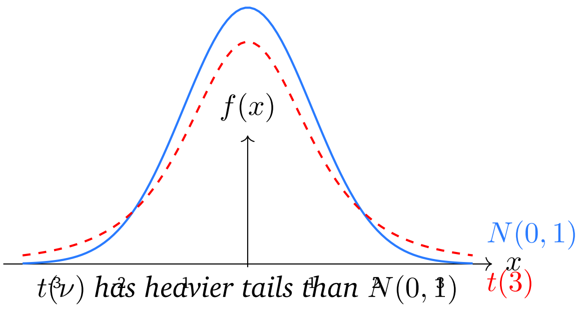
(A) 20

(B) 5

(C) 15

(D) 25

Q16. Which of the following correctly describes the relationship between Student's t-distribution and the standard normal distribution?



(A) $t(\nu)$ has lighter tails than $N(0,1)$ for any finite ν.

(B) $t(\nu)$ has heavier tails than $N(0,1)$ for any finite ν.

(C) As $\nu \to \infty$, $t(\nu) \to \chi^2(1)$.

(D) $t(\nu)$ is skewed for all finite ν.

Q17. For random variables X and Y defined on the same probability space, the *law of total expectation* states:

(A) $E[X \mid Y] = E[X] \cdot E[Y]$

(B) $E[XY] = E[X] + E[Y]$

(C) $E\big[E[X \mid Y]\big] = E[X]$

(D) $\mathrm{Var}(X) = E[\mathrm{Var}(X \mid Y)]$

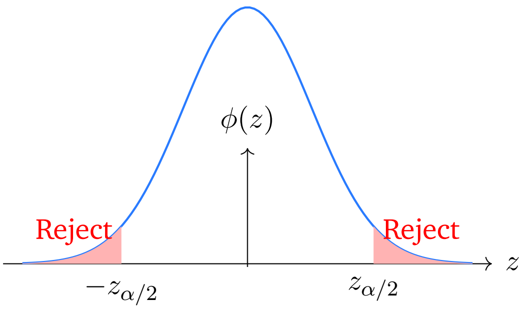
Q18. For estimating a scalar parameter θ using the MLE $\hat{\theta}$, the Wald test statistic is $W = (\hat{\theta} - \theta_0)^2 / \widehat{\mathrm{Var}}(\hat{\theta})$. Under $H_0 : \theta = \theta_0$, the asymptotic distribution of W is:

(A) $N(0,1)$
(B) $t(n-1)$
(C) $F(1,n-1)$
(D) $\chi^2(1)$

Q19. Let $X_1,\ldots,X_n$ be iid $\text{Bernoulli}(p)$. Consider $T=\sum_{i=1}^n X_i$. Which statement is correct?

(A) T is sufficient for p and $T\sim \text{Bin}(n,p)$ with $E[T]=np$.
(B) T is not sufficient for p.
(C) T is sufficient for p but $E[T]=p$.
(D) T is sufficient only if $n>30$.

Q20. For testing $H_0:\mu=0$ vs $H_1:\mu\neq 0$ based on $X_1,\ldots,X_n \overset{iid}{\sim} N(\mu,\sigma^2)$ with σ^2 known, the likelihood ratio test rejects H_0 when:



(A) $\bar{X}>z_\alpha\,\sigma/\sqrt{n}$
(B) $|\bar{X}|>z_{\alpha/2}\,\sigma/\sqrt{n}$
(C) $\bar{X}<-z_\alpha\,\sigma/\sqrt{n}$
(D) $\bar{X}^2<z_{\alpha/2}^2\,\sigma^2/n$

Q21. Let $X\sim U(0,1)$ and $Y=-\ln X$. Using the Jacobian transformation, the distribution of Y is:

(A) $U(0,1)$
(B) $N(0,1)$
(C) $\text{Exp}(1)$
(D) $\text{Gamma}(2,1)$

Q22. If $X_n\xrightarrow{p}X$ and $Y_n\xrightarrow{p}Y$ as $n\to\infty$, which of the following is true?

(A) $X_nY_n \xrightarrow{d} XY$ only if X and Y are independent.
(B) $X_n - Y_n \xrightarrow{p} 0$ always.
(C) $X_n/Y_n \xrightarrow{p} X/Y$ even if $P(Y = 0) > 0$.
(D) $X_n + Y_n \xrightarrow{p} X + Y$.

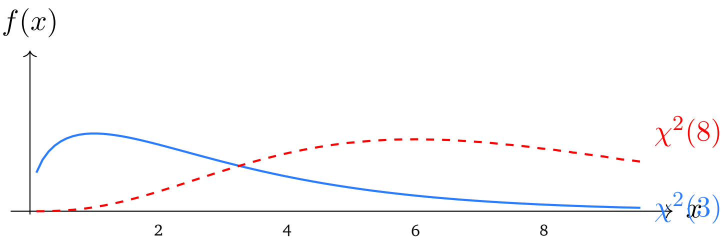
Q23. In a simple linear regression with $n = 10$ observations, the total sum of squares $\text{SST} = 200$ and the coefficient of determination $R^2 = 0.75$. The error sum of squares SSE equals:

(A) 50
(B) 75
(C) 150
(D) 200

Q24. A statistic T is said to be *complete* for a family $\{P_\theta : \theta \in \Theta\}$ if:

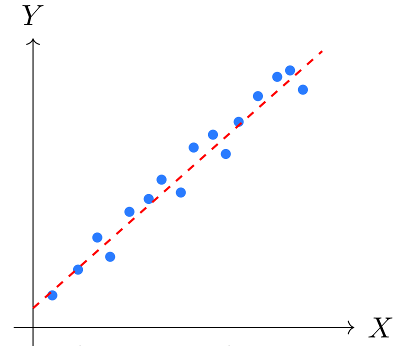
(A) T achieves the Cramér–Rao lower bound.
(B) For every measurable g, $E_\theta[g(T)] = 0$ for all θ implies $P_\theta(g(T) = 0) = 1$ for all θ.
(C) T has the smallest variance among all unbiased estimators.
(D) T is a sufficient statistic for θ.

Q25. For $X \sim \chi^2(k)$, which of the following correctly states the mean and variance?



(A) $E[X] = k/2$, $\text{Var}(X) = k$
(B) $E[X] = 2k$, $\text{Var}(X) = k$
(C) $E[X] = k$, $\text{Var}(X) = 2k$
(D) $E[X] = k$, $\text{Var}(X) = k^2$

Q26. A scatter plot of two variables X and Y shows a strong positive correlation ($r = 0.92$). Which of the following conclusions is most appropriate?



$r = 0.92$ *does not imply* X *causes* Y

(A) X is the cause of Y.
(B) Y is the cause of X.
(C) The association is certainly not spurious.
(D) Correlation does not imply causation.

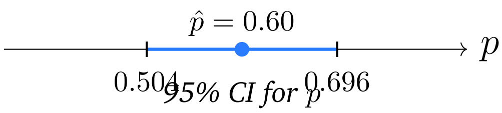
Q27. In simple linear regression $Y = \beta_0 + \beta_1 x + \varepsilon$, which of the following is correct regarding intervals at $x = x_0$?

(A) The prediction interval for an individual Y at x_0 is wider than the confidence interval for $E[Y \mid x_0]$ because it also accounts for individual variability σ^2.
(B) The confidence interval for $E[Y \mid x_0]$ is always wider than the prediction interval.
(C) Both intervals have exactly the same width.
(D) The prediction interval is narrower when the sample size n is large.

Q28. For $X \sim \text{Gamma}(\alpha, \beta)$ with $E[X] = \alpha/\beta$ and $E[X^2] = \alpha(\alpha+1)/\beta^2$, the method of moments estimators are found by setting $\bar{x} = \hat{\alpha}/\hat{\beta}$ and $m_2 = \hat{\alpha}(\hat{\alpha}+1)/\hat{\beta}^2$ where $m_2 = \frac{1}{n}\sum x_i^2$. Given $\bar{x} = 3$ and $m_2 = 11$, the MOM estimate of β is:

(A) 1
(B) $\frac{3}{2}$
(C) 3
(D) 2

Q29. In $n = 100$ Bernoulli trials, $X = 60$ successes are observed. The approximate 95% confidence interval for p (using $z_{0.025} = 1.96$) is:



(A) $(0.540,\ 0.660)$
(B) $(0.520,\ 0.680)$
(C) $(0.504,\ 0.696)$
(D) $(0.480,\ 0.720)$

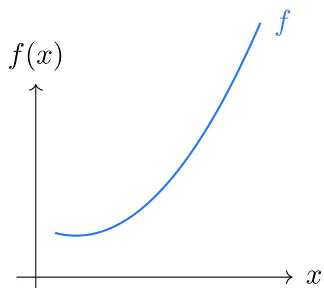
Q30. In a one-way ANOVA with $k = 3$ groups and $n = 30$ total observations, let SSB be the between-group sum of squares and SSW be the within-group sum of squares. The F-statistic is:

(A) $F = \dfrac{\mathrm{SSB}/3}{\mathrm{SSW}/27}$
(B) $F = \dfrac{\mathrm{SSB}/30}{\mathrm{SSW}/3}$
(C) $F = \dfrac{\mathrm{SSB}/27}{\mathrm{SSW}/2}$
(D) $F = \dfrac{\mathrm{SSB}/2}{\mathrm{SSW}/27}$

Section B — Multiple Select Questions (MSQ)

Q31–Q40: 2 marks each. ***No negative marking. One or more*** *options may be correct.*

Q31. Let $f : [a, b] \to \mathbb{R}$. Identify the **correct** statements from the following.



Continuous (and differentiable) on $[a, b]$

(A) Every differentiable function is continuous.
(B) If f is continuous on $[a, b]$, it attains its maximum and minimum (Extreme Value Theorem).
(C) A bounded function on $[a, b]$ is always Riemann integrable.

(D) If f is differentiable at a, then f' is continuous at a.

Q32. Which of the following statements about standard distributions are **correct**?

(A) If $X \sim \text{Exp}(\lambda)$, then $P(X > s + t \mid X > s) = P(X > t)$ for all $s, t \geq 0$ (memoryless property).
(B) If $X \sim N(\mu, \sigma^2)$, then $X^2 \sim \chi^2(1)$ regardless of μ.
(C) If $X \sim U(0, 1)$, then $-2 \ln X \sim \chi^2(2)$.
(D) If $X \sim N(0, 1)$, then $X^2 \sim \chi^2(1)$.

Q33. Identify the **correct** relationships among modes of convergence for sequences of random variables (X_n).

(A) Almost sure convergence implies convergence in L^1 (mean).
(B) Convergence in probability implies convergence in distribution.
(C) If $X_n \xrightarrow{p} c$ for a constant c, then $X_n \xrightarrow{d} c$.
(D) Convergence in distribution implies convergence in probability.

Q34. Which of the following statements about hypothesis testing are **correct**?

(A) A one-tailed test is more powerful than a two-tailed test for alternatives in the specified direction, at the same significance level.
(B) The chi-square goodness-of-fit test requires all expected cell frequencies to be sufficiently large (typically ≥ 5).
(C) The p-value is the probability that the null hypothesis H_0 is true.
(D) A level-α test controls the probability of a Type I error at α.

Q35. Which of the following statements about matrices are **correct**?

(A) An $n \times n$ matrix with n distinct eigenvalues is diagonalizable over $\mathbb{C}$.
(B) A singular matrix has all zero eigenvalues.
(C) Every real symmetric matrix is orthogonally diagonalizable.
(D) The rank of a matrix equals the number of non-zero eigenvalues.

Q36. Which of the following statements in statistical inference are **correct**?

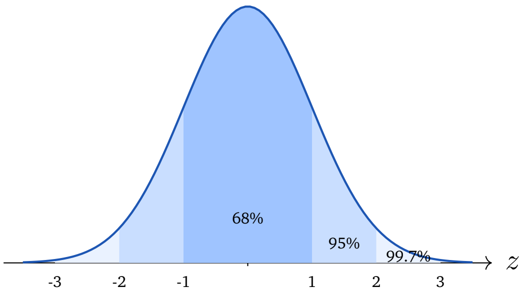
(A) The MLE and UMVUE are always the same estimator.

(B) By the Lehmann–Scheffé theorem, if T is complete sufficient and $\varphi(T)$ is unbiased for θ, then $\varphi(T)$ is UMVUE.

(C) For an iid sample of size n, the Fisher information satisfies $I_n(\theta) = n\,I_1(\theta)$.

(D) An unbiased estimator whose variance attains the Cramér–Rao lower bound is said to be efficient.

Q37. Let $X \sim N(\mu, \sigma^2)$. Which of the following statements are **correct**?



(A) $P(\mu - \sigma < X < \mu + \sigma) \approx 0.68$.

(B) The moment generating function of $Z \sim N(0, 1)$ is $M(t) = e^{t^2/2}$.

(C) If $X, Y \overset{iid}{\sim} N(0, 1)$, then $X^2 + Y^2 \sim \chi^2(2)$.

(D) The mean, median, and mode are equal for $N(\mu, \sigma^2)$, and this property holds for all unimodal distributions.

Q38. Which of the following statements about OLS regression diagnostics are **correct**?

(A) The OLS estimators $\hat\beta_0, \hat\beta_1$ are unbiased only if the errors are normally distributed.

(B) A residual plot of $e_i = y_i - \hat y_i$ versus $\hat y_i$ should show no systematic pattern if the model is correctly specified.

(C) A high R^2 guarantees that the regression model is appropriate.

(D) If removing an outlier substantially increases R^2, the outlier was likely influential.

Q39. Which of the following statements about joint distributions are **correct**?

(A) If (X, Y) has a bivariate normal distribution, then both marginal distributions are normal.

(B) If both X and Y are normally distributed, then (X, Y) is bivariate normal.

(C) For any joint distribution of (X, Y) with finite expectations: $E[X + Y] = E[X] + E[Y]$.

(D) If X and Y are independent, then $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$.

Q40. Which of the following statements about large-sample theory are **correct**?

(A) The CLT states $\sqrt{n}(\bar{X} - \mu)/\sigma \sim N(0, 1)$ exactly for all $n \geq 1$.

(B) The Weak Law of Large Numbers states $\bar{X} \xrightarrow{p} \mu$ as $n \to \infty$.

(C) The delta method: if $\sqrt{n}(T_n - \theta) \xrightarrow{d} N(0, \sigma^2)$ and g is differentiable at θ with $g'(\theta) \neq 0$, then $\sqrt{n}(g(T_n) - g(\theta)) \xrightarrow{d} N(0, \sigma^2[g'(\theta)]^2)$.

(D) The CLT requires the population distribution to be normal.

Section C — Numerical Answer Type (NAT)

Q41–Q50**: 1 mark each.* ***No negative marking.

Q41. Let $X \sim \text{Bin}(12, \frac{1}{4})$. Compute $\text{Var}(X)$.

Q42. Let $X \sim N(100, 25)$ (i.e. $\mu = 100$, $\sigma^2 = 25$). Using $\Phi(1) \approx 0.8413$, find $P(X < 95)$ correct to 4 decimal places.

Q43. Compute the determinant of $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$.

Q44. Let $X \sim \text{Poisson}(3)$ and $Y \sim \text{Poisson}(2)$ be independent. Using $e^{-5} \approx 0.0067$, find $P(X + Y = 0)$ correct to 4 decimal places.

Q45. Let $X_1, \ldots, X_8$ be iid $N(\mu, 16)$. Given $\bar{x} = 50$, find the half-width $z_{0.025} \cdot \sigma/\sqrt{n}$ of the 95% CI for μ correct to 2 decimal places. (Use $z_{0.025} = 1.96$.)

Q46. Let $X \sim \text{Exp}(3)$ (rate $\lambda = 3$). Compute $E[X^2] = \text{Var}(X) + (E[X])^2$ correct to 4 decimal places.

Q47. Using the identity $\displaystyle\sum_{n=0}^{\infty}(n+1)x^n = \frac{1}{(1-x)^2}$ for $|x| < 1$, evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{3^n}$.

Q48. Let X_1, X_2, X_3, X_4 be iid $U(0,\theta)$ with observed values $(1,3,5,7)$. For this model, the UMVUE of θ is $\frac{n+1}{n}X_{(n)}$ where $X_{(n)} = \max_i X_i$. Compute the UMVUE estimate.

Q49. Let X be a random variable with $\mathrm{Var}(X) = 9$. Compute $\mathrm{Cov}(X,\, 2X+1)$.

Q50. Solve the ODE $y' = -2y + 4$ with initial condition $y(0) = 3$. Compute $y(2)$ correct to 3 decimal places. (Use $e^{-4} \approx 0.0183$.)

Q51–Q60**: 2 marks each.* ***No negative marking.

Q51. Let $X \sim \mathrm{Bin}(20, p)$. If the observed value is $X = 14$, compute the MLE $\hat{p}$.

Q52. Let $X \sim \mathrm{Gamma}(\alpha, \beta)$ where $E[X] = \alpha/\beta$ and $\mathrm{Var}(X) = \alpha/\beta^2$. Given that $E[X] = 4$ and $\mathrm{Var}(X) = 8$, find the value of the scale parameter β.

Q53. A random sample of size $n = 16$ gives $\bar{x} = 24$ and $s = 8$. For testing $H_0 : \mu = 20$ vs $H_1 : \mu \neq 20$, compute the t-statistic $T = (\bar{x} - \mu_0)/(s/\sqrt{n})$.

Q54. Two independent samples: $n_1 = 10$, $\bar{x}_1 = 15$, $s_1 = 3$; and $n_2 = 10$, $\bar{x}_2 = 12$, $s_2 = 4$. Using the pooled variance estimator $s_p^2 = \left[(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2\right]/(n_1 + n_2 - 2)$, compute the two-sample t-statistic $T = (\bar{x}_1 - \bar{x}_2)/(s_p\sqrt{1/n_1 + 1/n_2})$ correct to 3 decimal places.

Q55. A chi-square goodness-of-fit test has 3 equally likely categories with observed counts $(25, 35, 40)$ and total $n = 100$. Compute the test statistic $\chi^2 = \sum (O_i - E_i)^2/E_i$ correct to 2 decimal places.

Q56. For $X \sim \mathrm{Beta}(\alpha = 3, \beta = 2)$, the mode is $(\alpha - 1)/(\alpha + \beta - 2)$. Express the mode as a fraction in lowest terms.

Q57. For testing equality of two population variances, sample 1 has $s_1^2 = 25$ $(n_1 = 11)$ and sample 2 has $s_2^2 = 9$ $(n_2 = 10)$. Compute $F = s_1^2/s_2^2$ correct to 2 decimal places, and state the degrees of freedom (df_1, df_2).

Q58. Let $X_1, \ldots, X_{100}$ be iid with $E[X_i] = 5$ and $\mathrm{Var}(X_i) = 4$. Using Chebyshev's inequality, find an upper bound for $P(|\bar{X} - 5| > 0.5)$.

Q59. Let (X, Y) have joint density $f(x, y) = 2$ for $0 < x < y < 1$ and 0 otherwise. Compute $E[X]$ as a fraction in lowest terms.

Q60. Let $X_1, \ldots, X_{16}$ be iid $N(\mu, 1)$. The width of the 95% confidence interval for μ is $2z_{0.025}/\sqrt{n}$. Using $z_{0.025} = 1.96$, compute the width correct to 2 decimal places.

Solutions

IIT JAM Mathematical Statistics – Sample Paper 4

Detailed Solutions

Solution

Q1.

Concept — Series Convergence (Leibniz Test):

The alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ has terms $a_n = 1/\sqrt{n}$ which are positive, decreasing, and $a_n \to 0$. By the Leibniz alternating series test, it converges. However, $\sum 1/\sqrt{n}$ is a p-series with $p = 1/2 < 1$, so it *diverges* — the original series is not absolutely convergent.

Step 1 — Leibniz test: $a_n = n^{-1/2}$ is decreasing and $\to 0$, so the series converges.

Step 2 — Absolute convergence: $\sum n^{-1/2}$ diverges (p-series, $p = 1/2 \leq 1$).

Final Answer: The series converges conditionally but not absolutely. $\Rightarrow$ **(D)**

Why other options are wrong:

- (A): Wrong — the series does converge by Leibniz.
- (B): Wrong — absolute convergence fails since $\sum 1/\sqrt{n}$ diverges.
- (C): Wrong — conditional convergence does not imply absolute convergence.

Answer: (D) ← *Go back to Q1*

Solution

Q2.

Concept — Maclaurin Series of $\cos x$:

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

Step 1 — Coefficient of x^4: $\frac{1}{4!} = \frac{1}{24}$.

Final Answer: The coefficient of x^4 is $\frac{1}{24}$. $\Rightarrow$ **(C)**

Why other options are wrong:

- (A): $1/6$ is the coefficient in $\sin x$ at x^3 (with sign).
- (B): $1/12$ does not arise from standard Taylor coefficients of $\cos x$.
- (D): $1/48$ is incorrect.

Answer: (C) ← *Go back to Q2*

Q3.

Solution

Concept — Diagonalizability:
A matrix is diagonalizable iff for every eigenvalue, its geometric multiplicity equals its algebraic multiplicity.

Step 1: Eigenvalue 1 has algebraic multiplicity 2 and geometric multiplicity 2 (given). Eigenvalue 2 has algebraic multiplicity 1 and geometric multiplicity 1.

Step 2: Since both eigenvalues satisfy the condition, A is diagonalizable.

Final Answer: A is diagonalizable. $\Rightarrow$ **(B)**

Why other options are wrong:

- (A): Repeated eigenvalues alone do not prevent diagonalizability; geometric multiplicity is the key.
- (C): Diagonalizability is unrelated to $\det A$.
- (D): With geom. mult. 2 for eigenvalue 1 and 1 for eigenvalue 2, there are 3 linearly independent eigenvectors.

Answer: (B) ← *Go back to Q3*

Q4.

Solution

Concept — Law of Total Probability:
$P(B) = P(B \mid A)P(A) + P(B \mid A^c)P(A^c)$.

Step 1: $P(A) = 0.3$, so $P(A^c) = 0.7$.

Step 2: $P(B) = 0.7 \times 0.3 + 0.2 \times 0.7 = 0.21 + 0.14 = 0.35$.

Final Answer: $P(B) = 0.35$. $\Rightarrow$ **(A)**

Why other options are wrong:

- (B): $0.42 = 0.7 \times 0.6$ — ignores the correct weights.
- (C): 0.21 only counts the first term.
- (D): 0.50 is incorrect.

Answer: (A) ← *Go back to Q4*

Solution

Concept — Negative Binomial PMF:
X = number of failures before the $r = 2$ success, $p = 0.5$: $P(X = k) = \binom{k+r-1}{k} p^r (1-$

$p)^k = \binom{k+1}{k}(0.5)^{k+2}$.

Step 1 — Compute $P(X = 3)$:

$$P(X = 3) = \binom{4}{3}(0.5)^2(0.5)^3 = 4 \cdot \frac{1}{32} = \frac{4}{32} = 0.125.$$

Final Answer: $P(X = 3) = 0.125$. $\Rightarrow$ **(D)**

Why other options are wrong:

Q5.

- (A): $0.0625 = (0.5)^4$ omits the combinatorial factor.
- (B): $0.0781 \approx P(X = 4)$.
- (C): $0.1875 = P(X = 2)$.

Answer: (D) ← *Go back to Q5*

Solution

Concept — Laplace Distribution Variance:
For $X \sim \text{Laplace}(0, b)$ with $f(x) = \frac{1}{2b}e^{-|x|/b}$, $E[X] = 0$ and $\text{Var}(X) = 2b^2$.

Step 1: With $b = 1$: $\text{Var}(X) = 2(1)^2 = 2$.

Final Answer: $\text{Var}(X) = 2$. $\Rightarrow$ **(C)**

Why other options are wrong:

Q6.

- (A): $\text{Var} = 1$ would apply to $N(0, 1)$, not Laplace.
- (B): $\sqrt{2}$ is the standard deviation of Laplace(0,1), not the variance.
- (D): $4 = 2b^2$ would require $b = \sqrt{2}$.

Answer: (C) ← *Go back to Q6*

Solution

Concept — Definite Integral of $\sin^2 x$:
Use the identity $\sin^2 x = \frac{1-\cos 2x}{2}$.

Step 1:

$$\int_0^{\pi} \sin^2 x \, dx = \int_0^{\pi} \frac{1 - \cos 2x}{2} \, dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}(\pi - 0) = \frac{\pi}{2}.$$

Final Answer: $\frac{\pi}{2}$. $\Rightarrow$ **(B)**

Why other options are wrong:

Q7.
- (A): 1 is not correct — the integral spans $[0, \pi]$.
- (C): π would be the result for $\int_0^\pi \sin^2(x/2)\,dx$ or a similar variant.
- (D): 2 is too large.

Answer: (B) ← *Go back to Q7*

Solution

Concept — Consistency of Linear Systems:

The augmented matrix $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}$ represents the system $x_1+x_2 = 2$ and $x_1+x_2 = 3$.

Step 1 — Row reduction: $R_2 \leftarrow R_2 - R_1$ gives $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$, i.e. $0 = 1$, a contradiction.

Final Answer: The system has no solution (inconsistent). $\Rightarrow$ **(A)**

Why other options are wrong:

Q8.
- (B): A unique solution requires a leading entry in every variable column; here the system is inconsistent.
- (C): Infinitely many solutions arise when the system is consistent and underdetermined.
- (D): A system of linear equations cannot have exactly two solutions.

Answer: (A) ← *Go back to Q8*

Solution

Concept — Minimum of Independent Exponentials:

If $X, Y \overset{iid}{\sim} \mathrm{Exp}(\lambda)$, then $\min(X, Y) \sim \mathrm{Exp}(2\lambda)$.

Step 1: With $\lambda = 1$: $\min(X, Y) \sim \mathrm{Exp}(2)$, so $E[\min(X, Y)] = \frac{1}{2}$.

Final Answer: $E[\min(X, Y)] = \frac{1}{2}$. $\Rightarrow$ **(C)**

Why other options are wrong:

Q9.
- (A): $E[X] = 1$ is the mean of a single $\mathrm{Exp}(1)$, not the minimum.
- (B): 2 is the mean of $\max(X, Y)$ for $\mathrm{Exp}(1)$ (by symmetry, $E[\max] = 3/2$; 2 is

incorrect).

- (D): $1/4$ would correspond to $\text{Exp}(4)$.

Answer: (C) ← *Go back to Q9*

Solution

Concept — CDF and Interval Probability:
$P(a < X \leq b) = F(b) - F(a)$ for any CDF F.

Step 1: $F(x) = 1 - e^{-x}$ for $\text{Exp}(1)$.

Step 2: $P(1 < X \leq 3) = F(3) - F(1) = (1 - e^{-3}) - (1 - e^{-1}) = e^{-1} - e^{-3} \approx 0.318$.

Final Answer: $P(1 < X \leq 3) = e^{-1} - e^{-3}$. $\Rightarrow$ **(D)**

Why other options are wrong:

Q10.

- (A): $e^{-3} - e^{-1} < 0$ — wrong sign.
- (B): $1 - e^{-2} = P(0 < X \leq 2)$.
- (C): $e^{-1} = P(X > 1) = P(X \leq \infty) - P(X \leq 1)$.

Answer: (D) ← *Go back to Q10*

Solution

Concept — Root Test for Series:
The root test states: if $L = \limsup_{n\to\infty} |a_n|^{1/n} < 1$, the series converges absolutely.

Step 1: For $a_n = (3/4)^n$: $|a_n|^{1/n} = (3/4)^{n\cdot(1/n)} = 3/4$.

Step 2: $L = 3/4 < 1$, so the series converges absolutely by the root test.

Final Answer: $\limsup |a_n|^{1/n} = 3/4 < 1 \Rightarrow$ converges absolutely. $\Rightarrow$ **(A)**

Why other options are wrong:

Q11.

- (B): $4/3 > 1$ is the reciprocal — computed incorrectly.
- (C): The ratio test also gives $3/4 < 1$, not 1.
- (D): Boundedness of terms alone does not imply convergence.

Answer: (A) ← *Go back to Q11*

Solution

Concept — Lagrange Remainder for Taylor Series:
$R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$ for some ξ between 0 and x.

Step 1 — Apply to $\sin x$**,** $n = 3$**:** $f^{(4)}(x) = \sin x$, so $|f^{(4)}(\xi)| \le 1$.

Step 2 — Bound: $|R_3(x)| \le \dfrac{|x|^4}{4!}$. Wait — $n = 3$ gives the remainder after the degree-3 approximation; the next term is degree 5 in $\sin x$ because $f^{(4)}(\xi)$ involves $\sin$ (which is 0 at $\xi = 0$). More precisely: the cubic Taylor polynomial of $\sin x$ is $x - x^3/6$. The remainder is $R_3(x) = \dfrac{f^{(4)}(\xi)}{4!}x^4$ but $f^{(4)} = \sin$, so R_4 actually starts at $x^5/120$. The correct bound is $|R_3(x)| \le |x|^5/120$.

Final Answer: $|R_3(x)| \le \dfrac{|x|^5}{120}$. $\Rightarrow$ **(B)**

Why other options are wrong:

Q12.

- (A): $|x|^3/6$ is the magnitude of the last included term, not the remainder bound.
- (C): $|x|^4/24$ ignores that $f^{(4)}(0) = 0$ for $\sin x$, and the next non-zero term is at order 5.
- (D): $R_3 = 0$ would mean $\sin x$ is exactly cubic, which is false.

Answer: (B) ← *Go back to Q12*

Solution

Concept — Positive Definiteness via Determinants:
A symmetric matrix is positive definite iff all leading principal minors are positive.

Step 1: $A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$. Leading minors: $\Delta_1 = 4 > 0$; $\Delta_2 = \det A = 4 \cdot 1 - 2 \cdot 2 = 0$.

Step 2: Since $\Delta_2 = 0$, A is *not* positive definite. Since both eigenvalues are non-negative (one is 0), A is positive *semi*-definite.

Final Answer: A is positive semi-definite but not positive definite. $\Rightarrow$ **(C)**

Why other options are wrong:

Q13.

- (A): Positive definite requires all leading minors strictly positive; $\Delta_2 = 0$ fails this.
- (B): Negative definite would require $\Delta_1 < 0$, but $\Delta_1 = 4 > 0$.
- (D): Negative semi-definite requires all eigenvalues ≤ 0; here they are ≥ 0.

Answer: (C) ← *Go back to Q13*

Solution

Concept — Second-Order Linear ODE:
$y'' - y = 0$ has characteristic equation $r^2 - 1 = 0$, roots $r = \pm 1$.

Step 1 — General solution: $y = C_1 e^x + C_2 e^{-x}$.

Step 2 — Apply initial conditions: $y(0) = C_1 + C_2 = 1$ and $y'(0) = C_1 - C_2 = 0$. Thus $C_1 = C_2 = 1/2$.

Step 3: $y = \dfrac{e^x + e^{-x}}{2} = \cosh x$. At $x = 1$: $y(1) = \cosh(1) = \dfrac{e + e^{-1}}{2}$.

Final Answer: $y(1) = \cosh(1) = \dfrac{e + e^{-1}}{2}$. $\Rightarrow$ **(D)**

Why other options are wrong:

Q14.
- (A): e would be $y(1)$ if $y = e^x$ (i.e. $C_2 = 0$), but $y'(0) = 0$ forces $C_1 = C_2$.
- (B): $\sinh(1)$ would arise if $y(0) = 0$, $y'(0) = 1$.
- (C): $y(1) = 1$ cannot be correct since $\cosh(1) > 1$.

Answer: (D) ← *Go back to Q14*

Solution

Concept — Negative Binomial Mean (trials parameterization):
If X = total trials to get r successes, $X \sim \text{NegBin}(r, p)$ and $E[X] = r/p$.

Step 1: $r = 5$, $p = 0.25$: $E[X] = 5/0.25 = 20$.

Final Answer: $E[X] = 20$. $\Rightarrow$ **(A)**

Why other options are wrong:

Q15.
- (B): $5 = r$ is just the number of successes, not the expected total trials.
- (C): $15 = r(1-p)/p = 5 \times 3 = 15$ is the expected number of *failures*, not total trials.
- (D): 25 has no direct interpretation here.

Answer: (A) ← *Go back to Q15*

Solution

Concept — t-Distribution vs Normal:
The $t(\nu)$ distribution is symmetric and bell-shaped like $N(0,1)$, but has heavier tails due to the extra randomness from estimating variance. As $\nu \to \infty$, $t(\nu) \to N(0,1)$.

Step 1: For any finite ν, $\mathrm{Var}(t(\nu)) = \nu/(\nu-2) > 1$ (for $\nu > 2$), confirming heavier tails than $N(0,1)$.

Final Answer: $t(\nu)$ has heavier tails than $N(0,1)$ for any finite ν. $\Rightarrow$ **(B)**

Why other options are wrong:

Q16.
- (A): Lighter tails is the opposite of the correct relationship.
- (C): As $\nu \to \infty$, $t(\nu) \to N(0,1)$, not $\chi^2(1)$.
- (D): $t(\nu)$ is symmetric (not skewed) for all $\nu > 0$.

Answer: (B) ← *Go back to Q16*

Solution

Concept — Law of Total Expectation:
For random variables X and Y: $E\big[E[X \mid Y]\big] = E[X]$.

Step 1: This follows from the definition of conditional expectation: averaging the conditional expectation over all possible values of Y recovers the unconditional expectation of X.

Final Answer: $E[E[X \mid Y]] = E[X]$. $\Rightarrow$ **(C)**

Why other options are wrong:

Q17.
- (A): $E[X \mid Y] = E[X] \cdot E[Y]$ is false in general.
- (B): $E[XY] = E[X] + E[Y]$ is false (this confuses product and sum).
- (D): The law of total variance is $\mathrm{Var}(X) = E[\mathrm{Var}(X \mid Y)] + \mathrm{Var}(E[X \mid Y])$, not just the first term.

Answer: (C) ← *Go back to Q17*

Solution

Concept — Wald Test Statistic:
$W = (\hat{\theta} - \theta_0)^2/\widehat{\mathrm{Var}}(\hat{\theta}) \xrightarrow{d} \chi^2(1)$ under H_0.

Step 1: By the delta method and properties of MLEs, $\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, 1/I(\theta_0))$. Squaring and normalizing gives $W \xrightarrow{d} \chi^2(1)$.

Final Answer: Under H_0, $W \xrightarrow{a} \chi^2(1)$. $\Rightarrow$ **(D)**

Why other options are wrong:

Q18.

- (A): $N(0,1)$ is the distribution of $(\hat{\theta} - \theta_0)/\sqrt{\widehat{\mathrm{Var}}(\hat{\theta})}$ (the square root of W).
- (B): $t(n-1)$ applies to exact small-sample tests with normal populations.
- (C): $F(1, n-1)$ applies to the F-test in ANOVA or linear regression.

Answer: (D) ← *Go back to Q18*

Solution

Concept — Sufficient Statistic for Bernoulli:
By the Neyman–Fisher factorization theorem, $T = \sum X_i$ is sufficient for p in iid Bernoulli(p) because the joint PMF factors as $p^T(1-p)^{n-T} \cdot 1$.

Step 1: $T = \sum_{i=1}^{n} X_i \sim \mathrm{Bin}(n, p)$, so $E[T] = np$.

Final Answer: T is sufficient for p and $E[T] = np$. $\Rightarrow$ **(A)**

Why other options are wrong:

Q19.

- (B): Factorization immediately shows T is sufficient.
- (C): $E[T] = np$, not p.
- (D): Sufficiency holds for any $n \geq 1$, not only large n.

Answer: (A) ← *Go back to Q19*

Solution

Concept — Likelihood Ratio Test (Two-Sided):
For $H_1 : \mu \neq 0$, the LRT statistic $\Lambda = (\bar{X})^2/(\sigma^2/n)$ is large when $|\bar{X}|$ is large. Equivalently, reject when $|\bar{X}| > z_{\alpha/2}\, \sigma/\sqrt{n}$.

Step 1: The test statistic is $Z = \bar{X}\sqrt{n}/\sigma$. The critical region is $|Z| > z_{\alpha/2}$, i.e. $|\bar{X}| > z_{\alpha/2}\, \sigma/\sqrt{n}$.

Final Answer: Reject H_0 when $|\bar{X}| > z_{\alpha/2}\, \sigma/\sqrt{n}$. $\Rightarrow$ **(B)**

Why other options are wrong:

Q20.

- (A): One-tailed rejection region suits $H_1 : \mu > 0$.
- (C): One-tailed lower rejection suits $H_1 : \mu < 0$.
- (D): $\bar{X}^2 < z_{\alpha/2}^2\, \sigma^2/n$ is the acceptance region, not rejection.

Answer: (B) ← *Go back to Q20*

Solution

Concept — Transformation of Random Variables (Jacobian):
Let $X \sim U(0,1)$ and $Y = g(X) = -\ln X$. Since g is monotone decreasing, $f_Y(y) = f_X(g^{-1}(y))|dg^{-1}/dy|$.

Step 1: $g^{-1}(y) = e^{-y}$, $|dg^{-1}/dy| = e^{-y}$. For $y > 0$: $f_Y(y) = 1 \cdot e^{-y} = e^{-y}$, which is the $\text{Exp}(1)$ density.

Final Answer: $Y \sim \text{Exp}(1)$. $\Rightarrow$ **(C)**

Why other options are wrong:

Q21.

- (A): $Y = U(0,1)$ would require a linear transformation.
- (B): $N(0,1)$ requires a very different transformation.
- (D): $\text{Gamma}(2,1)$ would be the distribution of $-\ln(X_1) - \ln(X_2)$ for two uniforms.

Answer: (C) ← *Go back to Q21*

Solution

Concept — Algebra of Convergence in Probability:
If $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$, then $X_n + Y_n \xrightarrow{p} X + Y$ (by triangle inequality for probability).

Step 1: $P(|X_n + Y_n - (X+Y)| > \varepsilon) \le P(|X_n - X| > \varepsilon/2) + P(|Y_n - Y| > \varepsilon/2) \to 0$.

Final Answer: $X_n + Y_n \xrightarrow{p} X + Y$. $\Rightarrow$ **(D)**

Why other options are wrong:

Q22.

- (A): Independence is not required for convergence in distribution of the sum; moreover, the claim is about convergence in probability.
- (B): $X_n - Y_n \xrightarrow{p} X - Y$, not necessarily 0.
- (C): Division requires $P(Y = 0) = 0$ for validity.

Answer: (D) ← *Go back to Q22*

Solution

Concept — Coefficient of Determination R^2:
$R^2 = 1 - \text{SSE}/\text{SST}$, so $\text{SSE} = \text{SST}(1 - R^2)$.

Step 1: $\text{SSE} = 200 \times (1 - 0.75) = 200 \times 0.25 = 50$.

Final Answer: $\text{SSE} = 50$. $\Rightarrow$ **(A)**

Why other options are wrong:

Q23.
- (B): $75 = \text{SSR} = 200 \times 0.75 \times (1/2)$ — incorrect interpretation.
- (C): $150 = \text{SSR} = 200 \times 0.75$ is the regression SS, not SSE.
- (D): 200 is SST itself.

Answer: (A) ← *Go back to Q23*

Solution

Concept — Complete Statistic:
T is complete for $\{P_\theta\}$ if $E_\theta[g(T)] = 0$ for all θ implies $P_\theta(g(T) = 0) = 1$ for all θ.

Step 1: This definition ensures no non-trivial unbiased estimator of zero exists based on T alone, guaranteeing uniqueness of UMVUE (Lehmann–Scheffé).

Final Answer: The definition in (B) is the correct definition of completeness. $\Rightarrow$ **(B)**

Why other options are wrong:

Q24.
- (A): Attaining the CRLB is the definition of efficiency, not completeness.
- (C): Having smallest variance is UMVUE, which follows from completeness + sufficiency + unbiasedness.
- (D): Sufficiency is a related but separate concept.

Answer: (B) ← *Go back to Q24*

Solution

Concept — Chi-Square Mean and Variance:
If $X \sim \chi^2(k)$, then $E[X] = k$ and $\text{Var}(X) = 2k$.

Step 1: $\chi^2(k) = \text{Gamma}(k/2,\ 1/2)$. Mean $= \frac{k/2}{1/2} = k$; Variance $= \frac{k/2}{(1/2)^2} = 2k$.

Final Answer: $E[X] = k$, $\text{Var}(X) = 2k$. $\Rightarrow$ **(C)**

Why other options are wrong:

Q25.
- (A): $E = k/2$ and $\text{Var} = k$ swap the correct values by a factor of 2.
- (B): $E = 2k$ is double the correct mean.
- (D): $\text{Var} = k^2$ is incorrect; the variance grows linearly, not quadratically, in k.

Answer: (C) ← *Go back to Q25*

Solution

Concept — Correlation vs. Causation:
Pearson's r measures the strength and direction of a *linear* association. Even a very high r (0.92) does not establish that one variable causes the other.

Step 1: Both variables could be driven by a third confounding variable, or the association could be coincidental (spurious).

Final Answer: Correlation does not imply causation. $\Rightarrow$ **(D)**

Why other options are wrong:

Q26.
- (A) and (B): Neither causal direction can be inferred from r alone.
- (C): High r does not rule out a spurious association.

Answer: (D) ← *Go back to Q26*

Solution

Concept — Prediction vs. Confidence Interval in Regression:
At x_0, the CI for $E[Y \mid x_0]$ has half-width $t \cdot s\sqrt{h_{00}}$, while the prediction interval for a new Y has half-width $t \cdot s\sqrt{1 + h_{00}}$ (the extra 1 accounts for individual scatter σ^2).

Step 1: Since $\sqrt{1 + h_{00}} > \sqrt{h_{00}}$, the prediction interval is always wider.

Final Answer: Prediction interval is wider because it also accounts for individual variability σ^2. $\Rightarrow$ **(A)**

Why other options are wrong:

Q27.
- (B): The confidence interval is always narrower, not wider.
- (C): They have different widths (unless $\sigma = 0$, a degenerate case).
- (D): Even as $n \to \infty$, the prediction interval width approaches $2z_{\alpha/2}\sigma$, while the CI width shrinks to 0.

Answer: (A) ← *Go back to Q27*

Solution

Concept — Method of Moments for Gamma:
Set $\bar{x} = \hat{\alpha}/\hat{\beta}$ and $m_2 = \hat{\alpha}(\hat{\alpha}+1)/\hat{\beta}^2$. Then $m_2/\bar{x}^2 = (\hat{\alpha}+1)/\hat{\alpha}$ gives $\hat{\alpha} = \bar{x}^2/(m_2 - \bar{x}^2)$.

Step 1: $m_2 - \bar{x}^2 = 11 - 9 = 2$, so $\hat{\alpha} = 9/2$.

Step 2: $\hat{\beta} = \hat{\alpha}/\bar{x} = (9/2)/3 = 3/2$.

Final Answer: $\hat{\beta} = 3/2$. $\Rightarrow$ **(B)**

Why other options are wrong:

Q28.
- (A): $\hat{\beta} = 1$ would require $\hat{\alpha} = \bar{x} = 3$, but then m_2 would not equal 11.
- (C): $\hat{\beta} = 3 = \bar{x}$ ignores the second moment equation.
- (D): $\hat{\beta} = 2$ is close but $3/(11-9) = 3/2$, not 2.

Answer: (B) ← *Go back to Q28*

Solution

Concept — Large-Sample CI for Proportion:
$\hat{p} \pm z_{0.025}\sqrt{\hat{p}(1-\hat{p})/n}$.

Step 1: $\hat{p} = 60/100 = 0.6$, $\hat{p}(1-\hat{p}) = 0.6 \times 0.4 = 0.24$, $\sqrt{0.24/100} = \sqrt{0.0024} \approx 0.0490$.

Step 2: Margin $= 1.96 \times 0.0490 \approx 0.096$. CI: $(0.6 - 0.096,\ 0.6 + 0.096) = (0.504,\ 0.696)$.

Final Answer: 95% CI $= (0.504, 0.696)$. $\Rightarrow$ **(C)**

Why other options are wrong:

Q29.
- (A): $(0.540, 0.660)$ uses $1.96 \times 0.5/\sqrt{100} = 0.098$, but $\hat{p}(1-\hat{p}) \neq 0.25$ here.
- (B): $(0.520, 0.680)$ corresponds to a smaller standard error.
- (D): $(0.480, 0.720)$ overstates the margin of error.

Answer: (C) ← *Go back to Q29*

Solution

Concept — One-Way ANOVA F-Statistic:
With k groups and N total observations: $df_B = k-1$, $df_W = N-k$. $F = \text{MSB}/\text{MSW} = (\text{SSB}/(k-1))\,/\,(\text{SSW}/(N-k))$.

Step 1: $k = 3$, $N = 30$: $df_B = 2$, $df_W = 27$. So $F = (\text{SSB}/2)/(\text{SSW}/27)$.

Final Answer: $F = \dfrac{\text{SSB}/2}{\text{SSW}/27}$. $\Rightarrow$ **(D)**

Why other options are wrong:

Q30.
- (A): Uses $df_B = 3$ (should be $k-1 = 2$).

- (B): Uses $df_B = 30$ and $df_W = 3$ — completely incorrect.
- (C): Swaps df_B and df_W.

Answer: (D) ← *Go back to Q30*

Q31.

Solution

Concept — Real Analysis: Continuity and Integrability:

(A) True: Differentiability at a point implies continuity at that point. ($f'(a) = \lim_{h\to 0}(f(a+h) - f(a))/h$ existing forces $f(a+h) \to f(a)$.)

(B) True: The Extreme Value Theorem (EVT): a continuous function on a closed bounded interval $[a, b]$ attains its maximum and minimum.

(C) False: A bounded function on $[a, b]$ need not be Riemann integrable. Example: Dirichlet function $f(x) = \mathbf{1}_{\mathbb{Q}}(x)$ is bounded but not Riemann integrable.

(D) False: Differentiability at a does not imply continuity of f' at a. Example: $f(x) = x^2 \sin(1/x)$ for $x \neq 0$, $f(0) = 0$ is differentiable at 0 but f' is not continuous there.

Final Answer: Correct options are **(A)** and **(B)**.

Answer: (AB) ← *Go back to Q31*

Q32.

Solution

Concept — Standard Distribution Properties:

(A) True: Memoryless property of Exp: $P(X > s + t \mid X > s) = e^{-\lambda t} = P(X > t)$.

(B) False: For $X \sim N(\mu, \sigma^2)$, $X^2/\sigma^2 \sim \chi^2(1)$ only when $\mu = 0$. For $\mu \neq 0$, X^2/σ^2 follows a non-central chi-square.

(C) True: $X \sim U(0, 1) \Rightarrow -2\ln X \sim \text{Gamma}(1, 1/2) = \chi^2(2)$.

(D) True: $X \sim N(0, 1) \Rightarrow X^2 \sim \chi^2(1)$ by definition.

Final Answer: Correct options are **(A)**, **(C)**, and **(D)**.

Answer: (ACD) ← *Go back to Q32*

Q33.

Solution

Concept — Modes of Convergence:

(A) False: A.s. convergence does not imply L^1 convergence without uniform integrability.

(B) True: Convergence in probability $\Rightarrow$ convergence in distribution.

(C) True: If $X_n \xrightarrow{p} c$ (constant), then $X_n \xrightarrow{d} c$ (convergence in probability to a constant implies convergence in distribution to that constant).

(D) False: Convergence in distribution does not imply convergence in probability in general. Example: $X_n \stackrel{d}{=} N(0,1)$ for all n converges in distribution to $N(0,1)$, but not necessarily in probability.

Final Answer: Correct options are **(B)** and **(C)**.

Answer: (BC) ← *Go back to Q33*

Q34.

Solution

Concept — Hypothesis Testing Principles:

(A) True: For a one-sided alternative, a one-tailed test concentrates all α in one tail, giving greater power against alternatives in that direction compared to a two-tailed test at the same α.

(B) True: The chi-square goodness-of-fit test is approximate and requires expected frequencies to be large (conventionally ≥ 5) for the chi-square approximation to hold.

(C) False: The p-value is $P(\text{observed or more extreme data} \mid H_0)$, not the probability H_0 is true.

(D) True: A level-α test satisfies $P(\text{reject } H_0 \mid H_0) \leq \alpha$, controlling Type I error.

Final Answer: Correct options are **(A)**, **(B)**, and **(D)**.

Answer: (ABD) ← *Go back to Q34*

Q35.

Solution

Concept — Matrix Algebra:

(A) True: Distinct eigenvalues $\Rightarrow$ linearly independent eigenvectors $\Rightarrow$ diagonalizable.

(B) False: A singular matrix has at least one zero eigenvalue, but other eigenvalues can be non-zero.

(C) True: By the spectral theorem, every real symmetric matrix is orthogonally diagonalizable (all eigenvalues real, eigenvectors orthogonal).

(D) False: Rank equals the number of non-zero singular values, not necessarily non-zero eigenvalues (for non-symmetric matrices).

Final Answer: Correct options are **(A)** and **(C)**.

Answer: (AC) ← *Go back to Q35*

Q36.

Solution

Concept — Statistical Inference Principles:

(A) False: MLE and UMVUE coincide in some regular exponential family settings but not always.

(B) True: Lehmann–Scheffé: if T is complete sufficient and $\varphi(T)$ is unbiased for θ, then $\varphi(T)$ is the unique UMVUE.

(C) True: Fisher information is additive for iid samples: $I_n(\theta) = nI_1(\theta)$.

(D) True: An unbiased estimator achieving the Cramér–Rao lower bound has minimum variance and is called efficient.

Final Answer: Correct options are **(B)**, **(C)**, and **(D)**.

Answer: (BCD) ← *Go back to Q36*

Q37.

Solution

Concept — Normal Distribution Properties:

(A) True: $P(\mu - \sigma < X < \mu + \sigma) = P(-1 < Z < 1) \approx 0.6827$.

(B) True: The MGF of $Z \sim N(0,1)$ is $M_Z(t) = E[e^{tZ}] = e^{t^2/2}$.

(C) True: $X^2 + Y^2$ is the sum of two independent $\chi^2(1)$ variables, so $X^2 + Y^2 \sim \chi^2(2)$.

(D) False: Mean = median = mode for $N(\mu, \sigma^2)$, but this does *not* hold for all unimodal distributions (e.g., a skewed unimodal distribution like Gamma has different mean, median, and mode).

Final Answer: Correct options are **(A)**, **(B)**, and **(C)**.

Answer: (ABC) ← *Go back to Q37*

Q38.

Solution

Concept — OLS Regression Assumptions and Diagnostics:

(A) False: OLS estimators $\hat{\beta}$ are unbiased under $E[\varepsilon] = 0$ alone (Gauss–Markov). Normality is needed for inference (t/F-tests), not unbiasedness.

(B) True: A well-specified model should have residuals scattered randomly around 0 with no pattern vs. fitted values $\hat{y}_i$.

(C) False: High R^2 can arise from non-linear relationships, influential outliers, or spurious associations; it does not guarantee model appropriateness.

(D) True: An influential observation that distorts $\hat{\beta}$ will typically cause R^2 to increase substantially when removed, indicating its leverage.

Final Answer: Correct options are **(B)** and **(D)**.

Answer: (BD) ← *Go back to Q38*

Q39.

Solution

Concept — Joint and Marginal Distributions:

(A) True: Marginals of a bivariate normal are always normal.

(B) False: Normal marginals do not guarantee a bivariate normal joint distribution. A counterexample: take $X \sim N(0,1)$ and $Y = X$ if $|X| > 1$, $Y = -X$ if $|X| \le 1$ — both marginals are normal but (X, Y) is not bivariate normal.

(C) True: Linearity of expectation holds for any joint distribution with finite expectations.

(D) True: If $X \perp Y$, the MGF of $X+Y$ factors: $M_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX}]E[e^{tY}] = M_X(t)M_Y(t)$.

Final Answer: Correct options are **(A)**, **(C)**, and **(D)**.

Answer: (ACD) ← *Go back to Q39*

Q40.

Solution

Concept — Large-Sample Theory (CLT, WLLN, Delta Method):

(A) False: The CLT is an asymptotic result ($n \to \infty$); it does not hold exactly for finite n unless the population is normal.

(B) True: WLLN: $\bar{X}_n \xrightarrow{p} \mu$ as $n \to \infty$ whenever $\text{Var}(X_i) < \infty$.

(C) True: The delta method states that if $\sqrt{n}(T_n - \theta) \xrightarrow{d} N(0, \sigma^2)$, then $\sqrt{n}(g(T_n) - g(\theta)) \xrightarrow{d} N(0, \sigma^2[g'(\theta)]^2)$ for g differentiable at θ with $g'(\theta) \neq 0$.

(D) False: The CLT applies to any distribution with finite mean and variance, regardless of normality.

Final Answer: Correct options are **(B)** and **(C)**.

Answer: (BC) ← *Go back to Q40*

Q41.

Solution

Concept — Binomial Variance:
$\text{Var}(X) = np(1-p)$ for $X \sim \text{Bin}(n, p)$.

Step 1: $n = 12$, $p = 1/4$, $1 - p = 3/4$: $\text{Var}(X) = 12 \times \frac{1}{4} \times \frac{3}{4} = \frac{36}{16} = 2.25$.

Final Answer: $\text{Var}(X) = \mathbf{2.25}$.

Answer: (2.25) ← *Go back to Q41*

Q42.

Solution

Concept — Normal Probability:
$X \sim N(100, 25)$: $\sigma = 5$. Standardize: $P(X < 95) = P(Z < (95 - 100)/5) = P(Z < -1) = 1 - \Phi(1)$.

Step 1: $1 - 0.8413 = 0.1587$.

Final Answer: $P(X < 95) \approx \mathbf{0.1587}$.

Answer: (0.1587) ← *Go back to Q42*

Q43.

Solution

Concept — 2×2 Determinant:
$$\det \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} = 2 \cdot 4 - 3 \cdot 1 = 8 - 3 = 5.$$

Final Answer: $\det A = \mathbf{5}$.

Answer: (5) ← *Go back to Q43*

Q44.

Solution

Concept — Sum of Independent Poissons:
$X + Y \sim \text{Pois}(3 + 2) = \text{Pois}(5)$. $P(X + Y = 0) = e^{-5} \approx 0.0067$.

Final Answer: $P(X + Y = 0) \approx \mathbf{0.0067}$.

Answer: (0.0067) ← *Go back to Q44*

Q45.

Solution

Concept — Confidence Interval Half-Width:
Half-width $= z_{0.025} \cdot \sigma/\sqrt{n} = 1.96 \times 4/\sqrt{8}$.

Step 1: $\sqrt{8} = 2\sqrt{2} \approx 2.8284$. Half-width $= 1.96 \times 4/2.8284 = 7.84/2.8284 \approx 2.7718 \approx 2.77$.

Final Answer: Half-width $\approx \mathbf{2.77}$.

Answer: (2.77) ← *Go back to Q45*

Q46.

Solution

Concept — Second Moment of Exponential:
For $X \sim \text{Exp}(\lambda)$: $E[X] = 1/\lambda$, $\text{Var}(X) = 1/\lambda^2$, $E[X^2] = \text{Var}(X) + (E[X])^2 = 2/\lambda^2$.

Step 1: $\lambda = 3$: $E[X^2] = 2/9 \approx 0.2222$.

Final Answer: $E[X^2] \approx \mathbf{0.2222}$.

Answer: (0.2222) ← *Go back to Q46*

Q47.

Solution

Concept — Power Series Evaluation:
$\sum_{n=0}^{\infty}(n+1)x^n = \frac{1}{(1-x)^2}$ for $|x| < 1$.

Step 1: At $x = 1/3$: $\frac{1}{(1-1/3)^2} = \frac{1}{(2/3)^2} = \frac{1}{4/9} = \frac{9}{4} = 2.25$.

Final Answer: $\sum_{n=0}^{\infty} \frac{n+1}{3^n} = \frac{9}{4} = \mathbf{2.25}$.

Answer: (2.25) ← *Go back to Q47*

Q48.

Solution

Concept — UMVUE for Uniform Upper Bound:
For $X_1, \ldots, X_n \overset{iid}{\sim} U(0, \theta)$, the UMVUE of θ is $\hat{\theta} = \frac{n+1}{n} X_{(n)}$.

Step 1: $n = 4$, $X_{(4)} = \max(1, 3, 5, 7) = 7$. UMVUE $= \frac{5}{4} \times 7 = \frac{35}{4} = 8.75$.

Final Answer: UMVUE estimate $= \mathbf{8.75}$.

Answer: (8.75) ← *Go back to Q48*

Q49.

Solution

Concept — Covariance Properties:
$\text{Cov}(X, aX + b) = a\,\text{Var}(X)$ for constants a, b.

Step 1: $a = 2$, $b = 1$, $\text{Var}(X) = 9$: $\text{Cov}(X, 2X + 1) = 2 \times 9 = 18$.

Final Answer: $\mathrm{Cov}(X, 2X + 1) = \mathbf{18}$.

Answer: (18) ← *Go back to Q49*

Q50.

Solution

Concept — First-Order Linear ODE:
$y' = -2y + 4$, $y(0) = 3$. Integrating factor: e^{2x}. Solution: $y = 2 + Ce^{-2x}$.

Step 1: $y(0) = 2 + C = 3 \Rightarrow C = 1$. So $y = 2 + e^{-2x}$.

Step 2: $y(2) = 2 + e^{-4} \approx 2 + 0.0183 = 2.018$.

Final Answer: $y(2) \approx \mathbf{2.018}$.

Answer: (2.018) ← *Go back to Q50*

Q51.

Solution

Concept — MLE for Binomial Proportion:
The log-likelihood $\ell(p) = X \ln p + (n - X)\ln(1 - p)$ is maximized at $\hat{p} = X/n$.

Step 1: $X = 14$, $n = 20$: $\hat{p} = 14/20 = 0.70$.

Final Answer: $\hat{p} = \mathbf{0.70}$.

Answer: (0.70) ← *Go back to Q51*

Q52.

Solution

Concept — Gamma Distribution Parameters:
$E[X] = \alpha/\beta$ and $\mathrm{Var}(X) = \alpha/\beta^2$. Then $\beta = \mathrm{Var}(X)/E[X]$.

Step 1: $\beta = 8/4 = 2$.

Final Answer: $\beta = \mathbf{2}$.

Answer: (2) ← *Go back to Q52*

Q53.

Solution

Concept — One-Sample t-Test:
$T = (\bar{x} - \mu_0)/(s/\sqrt{n})$.

Step 1: $T = (24 - 20)/(8/\sqrt{16}) = 4/(8/4) = 4/2 = 2$.

Final Answer: $T = \mathbf{2}$.

Answer: (2) ← *Go back to Q53*

Q54.

Solution

Concept — Two-Sample Pooled t-Test:

Step 1 — Pooled variance:

$$s_p^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2} = \frac{9\times 9 + 9\times 16}{18} = \frac{81+144}{18} = \frac{225}{18} = 12.5.$$

Step 2: $s_p = \sqrt{12.5} \approx 3.5355$.

Step 3: $T = \dfrac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{1/n_1 + 1/n_2}} = \dfrac{3}{3.5355\times\sqrt{0.2}} = \dfrac{3}{3.5355\times 0.4472} = \dfrac{3}{1.5811} \approx 1.897$.

Final Answer: $T \approx \mathbf{1.897}$.

Answer: (1.897) ← *Go back to Q54*

Q55.

Solution

Concept — Chi-Square Goodness-of-Fit:
$E_i = n/k = 100/3 \approx 33.33$ for each category.

Step 1:

$$\chi^2 = \frac{(25-33.33)^2}{33.33} + \frac{(35-33.33)^2}{33.33} + \frac{(40-33.33)^2}{33.33} = \frac{69.39+2.79+44.49}{33.33} = \frac{116.67}{33.33} \approx 3.50.$$

Final Answer: $\chi^2 \approx \mathbf{3.50}$.

Answer: (3.50) ← *Go back to Q55*

Q56.

Solution

Concept — Beta Distribution Mode:
For $X \sim \text{Beta}(\alpha, \beta)$ with $\alpha, \beta > 1$: Mode $= \dfrac{\alpha - 1}{\alpha + \beta - 2}$.

Step 1: $\alpha = 3$, $\beta = 2$: Mode $= \dfrac{3-1}{3+2-2} = \dfrac{2}{3}$.

Final Answer: Mode $= \dfrac{2}{3} \approx \mathbf{0.6667}$.

Answer: (2/3) ← *Go back to Q56*

Q57.

Solution

Concept — F**-Test for Equality of Variances:**
$F = s_1^2/s_2^2$ under $H_0 : \sigma_1^2 = \sigma_2^2$, with $F \sim F(n_1 - 1,\, n_2 - 1)$.

Step 1: $F = 25/9 \approx 2.78$. Degrees of freedom: $(df_1, df_2) = (10, 9)$.

Final Answer: $F \approx \mathbf{2.78}$, degrees of freedom $(10, 9)$.

Answer: (2.78) ← *Go back to Q57*

Q58.

Solution

Concept — Chebyshev's Inequality Applied to $\bar{X}$**:**
$P(|\bar{X} - \mu| > \varepsilon) \leq \dfrac{\text{Var}(\bar{X})}{\varepsilon^2} = \dfrac{\sigma^2/n}{\varepsilon^2}$.

Step 1: $\sigma^2 = 4$, $n = 100$, $\varepsilon = 0.5$: $P(|\bar{X} - 5| > 0.5) \leq \dfrac{4/100}{0.25} = \dfrac{0.04}{0.25} = 0.16$.

Final Answer: Upper bound $= \mathbf{0.16}$.

Answer: (0.16) ← *Go back to Q58*

Q59.

Solution

Concept — Marginal Expectation from Joint Density:
$E[X] = \iint x\, f(x, y)\, dy\, dx$.

Step 1 — Region: $0 < x < y < 1$, $f(x, y) = 2$.

$$E[X] = \int_0^1 \int_x^1 2x\, dy\, dx = \int_0^1 2x(1-x)\, dx = 2\int_0^1 (x-x^2)\, dx = 2\left[\frac{1}{2} - \frac{1}{3}\right] = 2\cdot\frac{1}{6} = \frac{1}{3}.$$

Final Answer: $E[X] = \frac{1}{3} \approx \mathbf{0.3333}$.

Answer: (1/3) ← *Go back to Q59*

Q60.

Solution

Concept — Width of 95% Confidence Interval:
Width $= 2\, z_{0.025}/\sqrt{n}$ when $\sigma^2 = 1$.

Step 1: $n = 16$: Width $= 2 \times 1.96/4 = 3.92/4 = 0.98$.

Final Answer: Width $= \mathbf{0.98}$.

Answer: (0.98) ← *Go back to Q60*

Answer Key

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	D	2	C	3	B	4	A	5	D
6	C	7	B	8	A	9	C	10	D
11	A	12	B	13	C	14	D	15	A
16	B	17	C	18	D	19	A	20	B
21	C	22	D	23	A	24	B	25	C
26	D	27	A	28	B	29	C	30	D
31	AB	32	ACD	33	BC	34	ABD	35	AC
36	BCD	37	ABC	38	BD	39	ACD	40	BC
41	2.25	42	0.1587	43	5	44	0.0067	45	2.77
46	0.2222	47	2.25	48	8.75	49	18	50	2.018
51	0.70	52	2	53	2	54	1.897	55	3.50
56	2/3	57	2.78	58	0.16	59	1/3	60	0.98