UP Board Class 12 Mathematics 2026 (Set 324 CY) Question Paper with Solutions

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General Instructions

(i) This booklet contains 35 questions, each provided with a complete, step-by-step solution.

(ii) It comprises 5 single-correct multiple-choice questions.

(iii) Attempt each question on your own before reviewing the given solution.

1. If $f : R \to R$ is given by $f(x) = (5 - x^5)^{1/5}$, then $f \circ f(x)$ is equal to:

(A) $x^{1/5}$

(B) x

(C) x^5

(D) $5 - x^5$

Correct Answer: (B) x

Solution:

Step 1: Understanding the Concept:

$f \circ f(x) = f(f(x))$. We need to substitute $f(x)$ into f itself.

Step 2: Substituting:

$f(f(x)) = \left(5 - \left[(5 - x^5)^{1/5}\right]^5\right)^{1/5}$.

Step 3: Simplifying the inner power:

Since raising to the 5th power undoes the 5th root, $\left[(5 - x^5)^{1/5}\right]^5 = 5 - x^5$.

Final Answer:

So $f(f(x)) = \left(5 - (5 - x^5)\right)^{1/5} = (x^5)^{1/5} = x.$

$$\boxed{x}$$

Quick Tip: Substitute $f(x)$ into itself; the 5th power and 5th root cancel.

2. The value of $\displaystyle\int_2^{2\sqrt{3}} \frac{dx}{4+x^2}$ is:

(A) $\dfrac{\pi}{12}$

(B) $\dfrac{\pi}{18}$

(C) $\dfrac{\pi}{24}$

(D) $\dfrac{\pi}{6}$

Correct Answer: (C) $\dfrac{\pi}{24}$

Solution:

Step 1: Key Formula:

$$\int \frac{dx}{4+x^2} = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C.$$

Step 2: Applying the limits:

$$\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_2^{2\sqrt{3}} = \frac{1}{2}\left(\tan^{-1}\sqrt{3} - \tan^{-1}1\right).$$

Step 3: Evaluating the inverse tangents:

$\tan^{-1}\sqrt{3} = \dfrac{\pi}{3}$ and $\tan^{-1}1 = \dfrac{\pi}{4}$, so the bracket is $\dfrac{\pi}{3} - \dfrac{\pi}{4} = \dfrac{\pi}{12}$.

Final Answer:

$\frac{1}{2} \times \frac{\pi}{12} = \frac{\pi}{24}$.

$$\boxed{\frac{\pi}{24}}$$

Quick Tip: Use $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}(x/a)$ with $a = 2$.

3. A relation R is defined in the set N as $R = \{(x, y) : y = x + 5,\ y > 5\}$. Then which of the following is correct?

(A) $(6, 10) \in R$

(B) $(3, 6) \in R$

(C) $(4, 9) \in R$

(D) $(2, 6) \in R$

Correct Answer: (C) $(4, 9) \in R$

Solution:

Step 1: Understanding the Condition:

A pair (x, y) is in R only if $y = x + 5$ (and automatically $y > 5$ then, since $x \in N$).

Step 2: Checking each option:

(6,10): $6 + 5 = 11 \neq 10$ — fails. (3,6): $3 + 5 = 8 \neq 6$ — fails. (4,9): $4 + 5 = 9$ — holds! (2,6): $2 + 5 = 7 \neq 6$ — fails.

Final Answer:

Only $(4, 9)$ satisfies $y = x + 5$, and $9 > 5$.

$$\boxed{(4,9) \in R}$$

Quick Tip: Check y = x + 5 directly for each given pair.

4. The value of $\displaystyle\int_0^{\pi/4} \sin^3 2x \cos 2x \, dx$ is:

(A) $\dfrac{1}{2}$

(B) $\dfrac{1}{4}$

(C) $\dfrac{1}{8}$

(D) $\dfrac{1}{16}$

Correct Answer: (C) $\dfrac{1}{8}$

Solution:

Step 1: Key Substitution:

Let $u = \sin 2x$, so $du = 2\cos 2x \, dx$, i.e. $\cos 2x \, dx = \dfrac{du}{2}$.

Step 2: Rewriting the integral:

$$\int \sin^3 2x \cos 2x \, dx = \int u^3 \cdot \frac{du}{2} = \frac{u^4}{8} = \frac{\sin^4 2x}{8}.$$

Step 3: Applying the limits:

At $x = \pi/4$: $\sin(\pi/2) = 1$, giving $\dfrac{1}{8}$. At $x = 0$: $\sin 0 = 0$, giving 0.

Final Answer:

$$\boxed{\frac{1}{8}}$$

Quick Tip: Substitute u = sin2x.

5. The probability of impossible events is:

(A) 0

(B) $\frac{1}{2}$

(C) $\frac{1}{3}$

(D) $\frac{1}{4}$

Correct Answer: (A) 0

Solution:

Step 1: Understanding the Concept:

An impossible event is one that can never occur; it corresponds to the empty set $\emptyset$ in the sample space.

Step 2: Applying the probability axiom:

By the axioms of probability, $P(\emptyset) = 0$.

Final Answer:

$$\boxed{0}$$

Quick Tip: An impossible event corresponds to the empty set, which always has probability 0.

6. Find the value of $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$.

Correct Answer: —

Solution:

Step 1: Key Formula or Approach:
$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ and $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$, both standard values.

Step 2: Substituting:
$\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} + 2 \times \frac{\pi}{6} = \frac{\pi}{3} + \frac{\pi}{3}$.

Final Answer:

$$\boxed{\frac{2\pi}{3}}$$

Quick Tip: Use the standard values cos^-1(1/2)=pi/3 and sin^-1(1/2)=pi/6.

7. Write $\cot^{-1}\left(\frac{1}{\sqrt{x^2 - 1}}\right)$, $x > 1$ in the simplest form.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:
Let $\cot\theta = \dfrac{1}{\sqrt{x^2-1}}$, so $\tan\theta = \sqrt{x^2-1}$.

Step 2: Building a right triangle:
With opposite $= \sqrt{x^2-1}$ and adjacent $= 1$, the hypotenuse is $\sqrt{1+(x^2-1)} = x$.

Step 3: Reading off sec theta:
$\sec\theta = \dfrac{\text{hyp}}{\text{adj}} = \dfrac{x}{1} = x$, so $\theta = \sec^{-1} x$.

Final Answer:

$$\boxed{\sec^{-1} x}$$

Quick Tip: Set cot(theta) equal to the given ratio and build a right triangle to identify sec(theta).

8. If the direction ratios of a line are $3, -2, -6$, then find its direction cosines.

Correct Answer: —

Solution:

Step 1: Key Formula:
For direction ratios a, b, c, the direction cosines are
$$\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}.$$

Step 2: Computing the magnitude:
$\sqrt{3^2 + (-2)^2 + (-6)^2} = \sqrt{9 + 4 + 36} = \sqrt{49} = 7.$

Final Answer:
Direction cosines are $\frac{3}{7}, -\frac{2}{7}, -\frac{6}{7}$.

$$\boxed{\left(\frac{3}{7}, -\frac{2}{7}, -\frac{6}{7}\right)}$$

Quick Tip: Divide each direction ratio by the square root of the sum of their squares.

9. Find the angle between the vectors $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$.

Correct Answer: —

Solution:

Step 1: Key Formula:
$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}.$$

Step 2: Computing the dot product and magnitudes:
$\vec{a} \cdot \vec{b} = (1)(1) + (1)(-1) + (-1)(1) = 1 - 1 - 1 = -1.$ $|\vec{a}| = |\vec{b}|$
$= \sqrt{1 + 1 + 1} = \sqrt{3}.$

Final Answer:
$\cos\theta = \frac{-1}{3}$, so $\theta = \cos^{-1}\left(-\frac{1}{3}\right)$.

$$\boxed{\theta = \cos^{-1}\left(-\frac{1}{3}\right)}$$

Quick Tip: Use cos(theta) = (a.b)/(|a||b|).

10. Show that the function $f(x) = x^3 - 6x^2 + 12x,\ x \in R$, is an increasing function on R.

Correct Answer: —

Solution:

Step 1: Key Approach:

A function is increasing on R if $f'(x) \geq 0$ for all $x \in R$.

Step 2: Differentiating:

$f'(x) = 3x^2 - 12x + 12.$

Step 3: Rewriting as a perfect square:

$f'(x) = 3(x^2 - 4x + 4) = 3(x - 2)^2.$

Final Answer:

Since $(x - 2)^2 \geq 0$ for every real x, $f'(x) = 3(x - 2)^2 \geq 0$ for all $x \in R$. Hence f is increasing on R.

$$\boxed{f'(x) = 3(x-2)^2 \geq 0 \ \Rightarrow f \text{ is increasing}}$$

Quick Tip: Differentiate f(x) and show f'(x) is a non-negative perfect square.

11. If two vectors $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$, then show that $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are perpendicular.

Correct Answer: —

Solution:

Step 1: Key Approach:

Two vectors are perpendicular if their dot product is 0. We use the identity $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$.

Step 2: Computing the magnitudes:

$|\vec{a}|^2 = 5^2 + (-1)^2 + (-3)^2 = 25 + 1 + 9 = 35$. $|\vec{b}|^2 = 1^2 + 3^2 + (-5)^2 = 1 + 9 + 25 = 35$.

Final Answer:

$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 35 - 35 = 0$. Since the dot product is zero, $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are perpendicular.

$$\boxed{(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0}$$

Quick Tip: Use (a+b).(a-b) = |a|^2 - |b|^2 and check if it is zero.

12. If A and B are independent events and $P(A) = 0.3$, $P(B) = 0.4$, then find (i) $P(A \cap B)$, (ii) $P(A \cup B)$.

Correct Answer: —

Solution:

Step 1: Key Formula for independent events:

For independent events, $P(A \cap B) = P(A) \cdot P(B)$.

Step 2: Computing P(A∩B):

$P(A \cap B) = 0.3 \times 0.4 = 0.12$.

Step 3: Computing P(A∪B):

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.4 - 0.12 = 0.58$.

Final Answer:

$$\boxed{P(A \cap B) = 0.12, \quad P(A \cup B) = 0.58}$$

Quick Tip: Use P(A∩B)=P(A)P(B) for independent events, then the addition rule.

13. Prove that the relation given by $R = \{(1,1), (2,2), (3,3), (1,2), (2,3)\}$ in the set $\{1,2,3\}$ is reflexive but neither symmetric nor transitive.

Correct Answer: —

Solution:

Step 1: Checking Reflexivity:

R is reflexive if $(a,a) \in R$ for every $a \in \{1,2,3\}$. Here $(1,1), (2,2), (3,3)$ are all in R, so R is reflexive.

Step 2: Checking Symmetry:

R is symmetric if $(a,b) \in R \Rightarrow (b,a) \in R$. But $(1,2) \in R$ while $(2,1) \notin R$, so R is not symmetric.

Step 3: Checking Transitivity:

R is transitive if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$. Here $(1, 2) \in R$ and $(2, 3) \in R$, but $(1, 3) \notin R$, so R is not transitive.

Final Answer:

$$\boxed{R \text{ is reflexive, but not symmetric and not transitive}}$$

Quick Tip: Check the three properties directly by listing the pairs.

14. If $\sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$, then find the value of x.

Correct Answer: —

Solution:

Step 1: Key Approach:

Rewrite as $\sin^{-1}(1-x) = \frac{\pi}{2} + 2\sin^{-1} x$, then take sine of both sides using $\sin(\pi/2 + \theta) = \cos\theta$.

Step 2: Applying sine to both sides:

$1 - x = \cos(2\sin^{-1} x) = 1 - 2\sin^2(\sin^{-1} x) = 1 - 2x^2$.

Step 3: Solving the resulting equation:

$1 - x = 1 - 2x^2 \Rightarrow 2x^2 - x = 0 \Rightarrow x(2x-1) = 0 \Rightarrow x = 0 \text{ or } x = \frac{1}{2}$.

Step 4: Verifying which root is valid:

For $x = \frac{1}{2}$: $\sin^{-1}(1 - \frac{1}{2}) - 2\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6} - 2 \cdot \frac{\pi}{6} = -\frac{\pi}{6} \neq \frac{\pi}{2}$, so x

$= \frac{1}{2}$ is rejected. For $x = 0$: $\sin^{-1}(1) - 2\sin^{-1}(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ ✓.

Final Answer:

$$\boxed{x = 0}$$

Quick Tip: Isolate one inverse sine term, apply sine to both sides using sin(pi/2+theta)=cos(theta), then verify roots (extraneous roots are common with inverse trig equations).

15. If $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$, then find $|\vec{a}|$ and $|\vec{b}|$.

Correct Answer: —

Solution:

Step 1: Key Formula:
$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$.

Step 2: Setting up equations:
Let $|\vec{b}| = k$, so $|\vec{a}| = 8k$. Then $|\vec{a}|^2 - |\vec{b}|^2 = 64k^2 - k^2 = 63k^2 = 8$.

Step 3: Solving for k:

$$k^2 = \frac{8}{63} \Rightarrow k = \sqrt{\frac{8}{63}} = \frac{2\sqrt{2}}{3\sqrt{7}} = \frac{2\sqrt{14}}{21}.$$

Final Answer:

$$|\vec{b}| = \frac{2\sqrt{14}}{21} \text{ and } |\vec{a}| = 8|\vec{b}| = \frac{16\sqrt{14}}{21}.$$

$$\boxed{|\vec{a}| = \frac{16\sqrt{14}}{21},\ |\vec{b}| = \frac{2\sqrt{14}}{21}}$$

Quick Tip: Use (a+b).(a-b) = |a|^2-|b|^2 and substitute |a|=8|b|.

16. Simplify: $\cos\theta\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} + \sin\theta\begin{bmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}$.

Correct Answer: —

Solution:

Step 1: Multiplying each matrix by its scalar:

$$\cos\theta\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}\cos^2\theta & \sin\theta\cos\theta\\ -\sin\theta\cos\theta & \cos^2\theta\end{bmatrix}, \text{ and}$$

$$\sin\theta\begin{bmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{bmatrix} = \begin{bmatrix}\sin^2\theta & -\sin\theta\cos\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{bmatrix}.$$

Step 2: Adding entrywise:

Top-left: $\cos^2\theta + \sin^2\theta = 1$. Top-right: $\sin\theta\cos\theta - \sin\theta\cos\theta = 0$. Bottom-left: $-\sin\theta\cos\theta + \sin\theta\cos\theta = 0$. Bottom-right: $\cos^2\theta + \sin^2\theta = 1$.

Final Answer:

$$\boxed{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}$$

Quick Tip: Multiply out each matrix by its scalar and add entrywise, using sin^2+cos^2=1.

17. Simplify: $\tan^{-1}\left(\dfrac{3a^2x - x^3}{a^3 - 3ax^2}\right)$.

Correct Answer: —

Solution:

Step 1: Key Formula:

Recall the triple-angle identity $\tan 3\phi = \dfrac{3\tan\phi - \tan^3\phi}{1 - 3\tan^2\phi}$.

Step 2: Matching the given expression:

Let $\tan\phi = \dfrac{x}{a}$. Then $\dfrac{3a^2x - x^3}{a^3 - 3ax^2} = \dfrac{a^3\left(3\frac{x}{a} - \frac{x^3}{a^3}\right)}{a^3\left(1 - 3\frac{x^2}{a^2}\right)} = \dfrac{3\tan\phi - \tan^3\phi}{1 - 3\tan^2\phi}$

$= \tan 3\phi$.

Final Answer:

So the expression equals $\tan^{-1}(\tan 3\phi) = 3\phi = 3\tan^{-1}\left(\dfrac{x}{a}\right)$.

$$\boxed{3\tan^{-1}\left(\frac{x}{a}\right)}$$

Quick Tip: Divide numerator and denominator by a^3 and match against the tan(3 theta) expansion formula.

18. Show that the points $A(2, 3, -4)$, $B(1, -2, 3)$ and $C(3, 8, -11)$ are collinear.

Correct Answer: —

Solution:

Step 1: Key Approach:

Three points are collinear if the vectors joining them are parallel (scalar multiples of each other).

Step 2: Computing the vectors:

$\vec{AB} = B - A = (1-2, -2-3, 3-(-4)) = (-1, -5, 7)$. $\vec{AC} = C - A = (3 - 2, 8-3, -11-(-4)) = (1, 5, -7)$.

Step 3: Comparing the vectors:

$\vec{AC} = (1, 5, -7) = -1 \times (-1, -5, 7) = -\vec{AB}$.

Final Answer:

Since $\vec{AC}$ is a scalar multiple of $\vec{AB}$, the points A, B, C are collinear.

$$\boxed{A,\ B,\ C \text{ are collinear}}$$

Quick Tip: Compute AB and AC and check if one is a scalar multiple of the other.

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19. Find the minimum value of the objective function $Z = 3x + 5y$ under the following constraints by the graphical method: $x + 3y \geq 3,\ x + y \geq 2,\ x \geq 0,\ y \geq 0$.

Correct Answer: —

Solution:

Step 1: Understanding the Feasible Region:

Both constraints are $\geq$ type, so the feasible region lies above both lines $x + 3y = 3$ and $x + y = 2$, in the first quadrant. This region is unbounded.

Step 2: Finding the Corner Points:
$x + 3y = 3$ meets the y-axis at $(0, 1)$ — rejected since it fails $x + y \geq 2$ ($1 < 2$). $x + y = 2$ meets the x-axis at $(2, 0)$ — rejected since it fails $x + 3y \geq 3$ ($2 < 3$). Intersection of the two lines: solving $x + y = 2 \Rightarrow x = 2 - y$, substitute: $(2 - y) + 3y = 3 \Rightarrow 2y = 1 \Rightarrow y = \frac{1}{2}, x = \frac{3}{2}$.

Step 3: Listing the actual feasible corner points:
The feasible region's corners are $(0, 2)$ [where $x + y = 2$ meets the y-axis], $\left(\frac{3}{2}, \frac{1}{2}\right)$ [intersection point], and $(3, 0)$ [where $x + 3y = 3$ meets the x-axis].

Step 4: Evaluating Z at each corner:
At $(0, 2)$: $Z = 0 + 10 = 10$. At $\left(\frac{3}{2}, \frac{1}{2}\right)$: $Z = \frac{9}{2} + \frac{5}{2} = 7$. At $(3, 0)$: $Z = 9 + 0 = 9$.

Final Answer:
The smallest value is $Z = 7$ at $\left(\frac{3}{2}, \frac{1}{2}\right)$; since the region is unbounded but Z only increases moving away from this vertex (positive coefficients), this is the true minimum.

$$\boxed{Z_{min} = 7 \text{ at } \left(\frac{3}{2}, \frac{1}{2}\right)}$$

Quick Tip: Plot both boundary lines, shade the common region above both, and evaluate Z at each corner point.

20. Find the minimum distance between the lines $\vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$.

Correct Answer: —

Solution:

Step 1: Key Formula for skew lines:

For lines $\vec{r} = \vec{a}_1 + \lambda\vec{d}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{d}_2$, the shortest distance is $\dfrac{\left|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{d}_1 \times \vec{d}_2)\right|}{|\vec{d}_1 \times \vec{d}_2|}$.

Step 2: Identifying vectors:

$\vec{a}_1 = (1, 1, 0)$, $\vec{d}_1 = (2, -1, 1)$; $\vec{a}_2 = (2, 1, -1)$, $\vec{d}_2 = (3, -5, 2)$. So $\vec{a}_2 - \vec{a}_1 = (1, 0, -1)$.

Step 3: Computing the cross product:

$\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix} = \hat{i}[(-1)(2) - (1)(-5)] - \hat{j}[(2)(2) - (1)(3)] + \hat{k}[(2)(-5) - (-1)(3)] = (3, -1, -7)$.

Step 4: Computing the dot product and magnitude:

$(\vec{a}_2 - \vec{a}_1) \cdot (\vec{d}_1 \times \vec{d}_2) = (1)(3) + (0)(-1) + (-1)(-7) = 3 + 7 = 10$. $|\vec{d}_1 \times \vec{d}_2| = \sqrt{9 + 1 + 49} = \sqrt{59}$.

Final Answer:

$$\boxed{\text{Shortest distance} = \frac{10}{\sqrt{59}} = \frac{10\sqrt{59}}{59} \text{ units}}$$

Quick Tip: Use the shortest-distance-between-skew-lines formula with the cross product of the direction vectors.

21. If $y = Ae^{mx} + Be^{nx}$, then show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$.

Correct Answer: —

Solution:

Step 1: Finding the first derivative:

$$\frac{dy}{dx} = Ame^{mx} + Bne^{nx}.$$

Step 2: Finding the second derivative:

$$\frac{d^2y}{dx^2} = Am^2e^{mx} + Bn^2e^{nx}.$$

Step 3: Substituting into the left-hand side:

$$\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = \left(Am^2e^{mx} + Bn^2e^{nx}\right) - (m+n)\left(Ame^{mx} + Bne^{nx}\right) + mn\left(Ae^{mx} + Be^{nx}\right).$$

Step 4: Collecting terms in e^{mx} and e^{nx}:

Coefficient of Ae^{mx}: $m^2 - (m+n)m + mn = m^2 - m^2 - mn + mn = 0$.

Coefficient of Be^{nx}: $n^2 - (m+n)n + mn = n^2 - mn - n^2 + mn = 0$.

Final Answer:

Both coefficients vanish, so the entire expression equals 0.

$$\boxed{\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0}$$

Quick Tip: Differentiate twice and substitute; the coefficients of e^{mx} and e^{nx} each vanish identically.

22. Find the values of a and b for which the function defined by $f(x) = \begin{cases} ax+1, & x \le 3 \\ bx+3, & x > 3 \end{cases}$ is continuous at $x = 3$.

Correct Answer: —

Solution:

Step 1: Key Approach:

For continuity at $x = 3$, the left-hand limit, right-hand limit and the function's value at $x = 3$ must all be equal.

Step 2: Computing the value and left-hand limit at x=3:

$f(3) = a(3) + 1 = 3a + 1$; this is also the left-hand limit since the first branch applies for $x \le 3$.

Step 3: Computing the right-hand limit:

$\lim_{x \to 3^+} f(x) = b(3) + 3 = 3b + 3$.

Step 4: Equating and simplifying:

$3a + 1 = 3b + 3 \;\Rightarrow\; 3a - 3b = 2 \;\Rightarrow\; a - b = \frac{2}{3}$.

Final Answer:

Continuity at $x = 3$ requires exactly $a - b = \frac{2}{3}$, i.e. $a = b + \frac{2}{3}$ for any real b (one condition on two unknowns, giving a family of solutions).

$$\boxed{a - b = \frac{2}{3}}$$

Quick Tip: Match the left-hand and right-hand pieces at x=3 for continuity.

23. Find the value of the determinant $\begin{vmatrix} 18 & 22 & -13 \\ 7 & -9 & 11 \\ -11 & 7 & 17 \end{vmatrix}$.

Correct Answer: —

Solution:

Step 1: Key Formula — expansion along the first row:

$\Delta = 18\begin{vmatrix} -9 & 11 \\ 7 & 17 \end{vmatrix} - 22\begin{vmatrix} 7 & 11 \\ -11 & 17 \end{vmatrix} + (-13)\begin{vmatrix} 7 & -9 \\ -11 & 7 \end{vmatrix}.$

Step 2: Computing the 2x2 minors:

$\begin{vmatrix} -9 & 11 \\ 7 & 17 \end{vmatrix} = (-9)(17) - (11)(7) = -153 - 77 = -230.$ $\begin{vmatrix} 7 & 11 \\ -11 & 17 \end{vmatrix} = (7)(17) - (11)(-11) = 119 + 121 = 240.$ $\begin{vmatrix} 7 & -9 \\ -11 & 7 \end{vmatrix} = (7)(7) - (-9)(-11) = 49 - 99 = -50.$

Step 3: Substituting back:

$\Delta = 18(-230) - 22(240) + (-13)(-50) = -4140 - 5280 + 650.$

Final Answer:

$\Delta = -4140 - 5280 + 650 = -8770.$

$$\boxed{-8770}$$

Quick Tip: Expand along the first row using 2x2 minors.

24. Three vectors $\vec{a}, \vec{b}, \vec{c}$ satisfy the condition $\vec{a} + \vec{b} + \vec{c} = 0$. If $|\vec{a}| = 3, |\vec{b}| = 4$ and $|\vec{c}| = 2$, then find the value of $\mu = \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$.

Correct Answer: —

Solution:

Step 1: Key Approach:

Since $\vec{a} + \vec{b} + \vec{c} = 0$, we have $|\vec{a} + \vec{b} + \vec{c}|^2 = 0$.

Step 2: Expanding the square:

$$|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0.$$

Step 3: Substituting the magnitudes:

$$3^2 + 4^2 + 2^2 + 2\mu = 0 \Rightarrow 9 + 16 + 4 + 2\mu = 0 \Rightarrow 29 + 2\mu = 0.$$

Final Answer:

$$\mu = -\frac{29}{2}.$$

$$\boxed{\mu = -\frac{29}{2}}$$

Quick Tip: Square the condition a+b+c=0 and expand using the dot product.

25. Prove that the height of a cylinder of maximum volume, inscribed in a sphere of radius R, is $\frac{2R}{\sqrt{3}}$.

Correct Answer: —

Solution:

Step 1: Setting up the volume in terms of one variable:

Let the cylinder have height h and radius r. Since it is inscribed in the sphere, $r^2 + \left(\frac{h}{2}\right)^2 = R^2 \Rightarrow r^2 = R^2 - \frac{h^2}{4}$.

Step 2: Writing the volume as a function of h alone:

$V = \pi r^2 h = \pi h\left(R^2 - \frac{h^2}{4}\right) = \pi R^2 h - \frac{\pi h^3}{4}$.

Step 3: Differentiating and setting to zero:

$\frac{dV}{dh} = \pi R^2 - \frac{3\pi h^2}{4} = 0 \Rightarrow h^2 = \frac{4R^2}{3} \Rightarrow h = \frac{2R}{\sqrt{3}}$.

Step 4: Confirming it is a maximum:

$\frac{d^2V}{dh^2} = -\frac{3\pi h}{2} < 0$ for $h > 0$, confirming this critical point gives the maximum volume.

Final Answer:

$$\boxed{h = \frac{2R}{\sqrt{3}}}$$

Quick Tip: Express V as a function of h alone using the sphere constraint, then maximize with calculus.

26. Differentiate: $y = x^{\sin x} + (\cos x)^{\tan x}$.

Correct Answer: —

Solution:

Step 1: Splitting into two parts:

Let $u = x^{\sin x}$ and $v = (\cos x)^{\tan x}$, so $y = u + v$ and $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$.

Step 2: Differentiating u using logarithmic differentiation:

$\ln u = \sin x \ln x$. Differentiating: $\dfrac{1}{u}\dfrac{du}{dx} = \cos x \ln x + \dfrac{\sin x}{x}$, so $\dfrac{du}{dx}$ $= x^{\sin x}\left(\cos x \ln x + \dfrac{\sin x}{x}\right)$.

Step 3: Differentiating v using logarithmic differentiation:

$\ln v = \tan x \ln(\cos x)$. Differentiating: $\dfrac{1}{v}\dfrac{dv}{dx} = \sec^2 x \ln(\cos x) + \tan x$ $\cdot \dfrac{-\sin x}{\cos x} = \sec^2 x \ln(\cos x) - \tan^2 x$, so $\dfrac{dv}{dx}$ $= (\cos x)^{\tan x}\left[\sec^2 x \ln(\cos x) - \tan^2 x\right]$.

Final Answer:

$$\boxed{\frac{dy}{dx} = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right) + (\cos x)^{\tan x}\left[\sec^2 x \ln(\cos x) - \tan^2 x\right]}$$

Quick Tip: Use logarithmic differentiation separately on each term, since both base and exponent involve x.

27. Prove that if E and F are two independent events, then E and F' will also be independent.

Correct Answer: —

Solution:

Step 1: Key Approach:

We must show $P(E \cap F') = P(E) \cdot P(F')$, using the fact that $P(E \cap F)$ $= P(E)P(F)$ since E, F are independent.

Step 2: Splitting E using F and F':

Since F and F' partition the sample space, $E = (E \cap F) \cup (E \cap F')$, and these two pieces are disjoint. So $P(E) = P(E \cap F) + P(E \cap F')$.

Step 3: Solving for P(E∩F'):

$P(E \cap F') = P(E) - P(E \cap F) = P(E) - P(E)P(F) = P(E)\big[1 - P(F)\big]$.

Step 4: Recognizing 1-P(F) as P(F'):

By the complement rule, $1 - P(F) = P(F')$. So $P(E \cap F') = P(E) \cdot P(F')$.

Final Answer:

This is exactly the condition for independence of E and F'.

$$\boxed{P(E \cap F') = P(E)P(F') \ \Rightarrow \ E, F' \text{ independent}}$$

Quick Tip: Split E as (E∩F)∪(E∩F'), use independence of E,F, and the complement rule.

28. Find the area surrounded by the line $y = 3x + 2$, the x-axis, and the ordinates $x = -1$ and $x = 1$.

Correct Answer: —

Solution:

Step 1: Finding where the line crosses the x-axis:

Set $y = 0$: $3x + 2 = 0 \Rightarrow x = -\dfrac{2}{3}$. This lies between -1 and 1, so the line is below the x-axis on $\left[-1, -\frac{2}{3}\right]$ and above it on $\left[-\frac{2}{3}, 1\right]$.

Step 2: Setting up the two separate integrals:

$$\text{Area} = \left|\int_{-1}^{-2/3} (3x+2)\,dx\right| + \int_{-2/3}^{1} (3x+2)\,dx.$$

Step 3: Evaluating the antiderivative:

$$\int (3x+2)dx = \frac{3x^2}{2} + 2x.$$

Step 4: Computing the first piece (x=-1 to -2/3):

At $x = -\frac{2}{3}$: $\dfrac{3(4/9)}{2} + 2(-\frac{2}{3}) = \frac{2}{3} - \frac{4}{3} = -\frac{2}{3}$. At $x = -1$: $\dfrac{3}{2} - 2 = -\frac{1}{2}$.

Integral $= -\frac{2}{3} - (-\frac{1}{2}) = -\frac{1}{6}$; taking absolute value gives $\frac{1}{6}$.

Step 5: Computing the second piece (x=-2/3 to 1):

At $x = 1$: $\dfrac{3}{2} + 2 = \frac{7}{2}$. At $x = -\frac{2}{3}$: $-\frac{2}{3}$ (from above). Integral $= \frac{7}{2} - (-\frac{2}{3}) = \frac{21}{6} + \frac{4}{6} = \frac{25}{6}$.

Final Answer:

Total area $= \frac{1}{6} + \frac{25}{6} = \frac{26}{6} = \frac{13}{3}$ square units.

$$\boxed{\frac{13}{3}\text{ sq. units}}$$

Quick Tip: Split the integral at the x-intercept x=-2/3 and take absolute values on the part below the axis.

29. Solve the following system of equations by matrix method: $3x - 2y + 3z = 8,\ 2x + y - z = 1,\ 4x - 3y + 2z = 4.$

Correct Answer: –

Solution:

Step 1: Setting up matrix form AX=B:

$$A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}.$$

Step 2: Computing |A|:

$$|A| = 3\big[(1)(2) - (-1)(-3)\big] - (-2)\big[(2)(2) - (-1)(4)\big] + 3\big[(2)(-3) - (1)(4)\big] = 3(2-3) + 2(4+4) + 3(-6-4) = -3 + 16 - 30 = -17.$$

Step 3: Computing the cofactor matrix and adj(A):

Working out all nine cofactors gives $\mathrm{adj}(A) = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$ (transpose of the cofactor matrix).

Step 4: Finding A^{-1} and X = A^{-1}B:

$A^{-1} = \frac{1}{-17}\mathrm{adj}(A)$. Then $X = A^{-1}B$ gives, after multiplying out, $x = 1,\ y = 2,\ z = 3$ (verified directly by substitution back into all three equations).

Final Answer:

$$\boxed{x = 1,\ y = 2,\ z = 3}$$

Quick Tip: Write as AX=B, find |A| and adj(A), then X=A^{-1}B.

30. Express the matrix $B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ in the form of the sum of a symmetric matrix and a skew-symmetric matrix.

Correct Answer: —

Solution:

Step 1: Key Formula:

Any square matrix B can be written as $B = P + Q$ where $P = \frac{1}{2}(B + B^T)$ (symmetric) and $Q = \frac{1}{2}(B - B^T)$ (skew-symmetric).

Step 2: Computing B transpose:

$$B^T = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}.$$

Step 3: Computing the symmetric part P:

$$P = \frac{1}{2}(B + B^T) = \frac{1}{2}\begin{bmatrix} 4 & -3 & -3 \\ -3 & 6 & 2 \\ -3 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 2 & -3/2 & -3/2 \\ -3/2 & 3 & 1 \\ -3/2 & 1 & -3 \end{bmatrix}.$$

Step 4: Computing the skew-symmetric part Q:

$$Q = \frac{1}{2}(B - B^T) = \frac{1}{2}\begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1/2 & -5/2 \\ 1/2 & 0 & 3 \\ 5/2 & -3 & 0 \end{bmatrix}.$$

Final Answer:

$B = P + Q$ with $P = \begin{bmatrix} 2 & -3/2 & -3/2 \\ -3/2 & 3 & 1 \\ -3/2 & 1 & -3 \end{bmatrix}$ (symmetric) and $Q = \begin{bmatrix} 0 & -1/2 & -5/2 \\ 1/2 & 0 & 3 \\ 5/2 & -3 & 0 \end{bmatrix}$ (skew-symmetric).

$$\boxed{B = P + Q}$$

Quick Tip: Use P=(B+B^T)/2 for the symmetric part and Q=(B-B^T)/2 for the skew-symmetric part.

31. Solve the differential equation $(x\,dy - y\,dx)\,y \sin\left(\frac{y}{x}\right) = (y\,dx + x\,dy)\,x \cos\left(\frac{y}{x}\right)$.

Correct Answer: —

Solution:

Step 1: Recognising a homogeneous equation:

Every term has the same total degree, so this is homogeneous.

Substitute $y = vx$, so $dy = v\,dx + x\,dv$.

Step 2: Substituting into the equation:

$x\,dy - y\,dx = x(v\,dx + x\,dv) - vx\,dx = x^2dv$. Also $y\,dx + x\,dy = vx\,dx + x(v\,dx + x\,dv) = 2vx\,dx + x^2dv$.

Step 3: Rewriting the full equation in terms of v:

$x^2dv \cdot vx \sin v = (2vx\,dx + x^2dv) \cdot x \cos v$, i.e. $vx^3 \sin v\,dv = (2vx^2 \cos v)dx + x^3 \cos v\,dv$.

Step 4: Separating variables:

$$x^3(v\sin v - \cos v)dv = 2vx^2\cos v\, dx \Rightarrow \frac{v\sin v - \cos v}{v\cos v}dv = \frac{2\,dx}{x}$$

$$\Rightarrow \left(\tan v - \frac{1}{v}\right)dv = \frac{2\,dx}{x}.$$

Step 5: Integrating both sides:

$$\int\left(\tan v - \frac{1}{v}\right)dv = \int\frac{2\,dx}{x} \Rightarrow -\ln|\cos v| - \ln|v| = 2\ln|x| + C_1$$

$$\Rightarrow \ln|v\cos v| = -2\ln|x| - C_1.$$

Final Answer:

So $v\cos v = \frac{C}{x^2}$; substituting back $v = \frac{y}{x}$ and multiplying by x^2:

$xy\cos\left(\frac{y}{x}\right) = C.$

$$\boxed{xy\cos\left(\frac{y}{x}\right) = C}$$

Quick Tip: This is a homogeneous equation; substitute y=vx, separate variables, then integrate tan(v) - 1/v.

32. Find the value of $\int_{-1}^{2}|x^3 - x|\,dx$.

Correct Answer: —

Solution:

Step 1: Finding where x^3 - x changes sign:

$x^3 - x = x(x-1)(x+1)$, which is zero at $x = -1, 0, 1$. On $[-1, 0]$: positive test at $x = -0.5$ gives $(-0.5)^3 - (-0.5) = 0.375 > 0$. On $[0, 1]$: test $x = 0.5$ gives $-0.375 < 0$. On $[1, 2]$: test $x = 1.5$ gives $1.875 > 0$.

Step 2: Splitting the integral accordingly:

$$\int_{-1}^{2} |x^3 - x|dx = \int_{-1}^{0} (x^3 - x)dx - \int_{0}^{1} (x^3 - x)dx + \int_{1}^{2} (x^3 - x)dx.$$

Step 3: Using the antiderivative F(x) = x^4/4 - x^2/2:

$F(0) = 0,\ F(-1) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4},\ F(1) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4},\ F(2) = 4 - 2 = 2.$

Step 4: Evaluating each piece:

Piece 1: $F(0) - F(-1) = 0 - (-\frac{1}{4}) = \frac{1}{4}$. Piece 2: $-[F(1) - F(0)] = -[-\frac{1}{4} - 0] = \frac{1}{4}$. Piece 3: $F(2) - F(1) = 2 - (-\frac{1}{4}) = \frac{9}{4}$.

Final Answer:

Total $= \frac{1}{4} + \frac{1}{4} + \frac{9}{4} = \frac{11}{4}$.

$$\boxed{\frac{11}{4}}$$

Quick Tip: Find the sign of x^3-x on each subinterval and integrate piecewise.

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33. Find the value of $\int_{1}^{4} \left(|x-1| + |x-2| + |x-3|\right) dx.$

Correct Answer: —

Solution:

Step 1: Identifying sign-change points inside [1,4]:

The three break points are $x = 1, 2, 3$, all within $[1, 4]$, splitting it into

$[1,2],[2,3],[3,4]$.

Step 2: Writing the integrand explicitly on each piece:

On $[1,2]$: $|x-1| = x-1,\ |x-2| = 2-x,\ |x-3| = 3-x$, sum $= 4-x$. On $[2,3]$: $|x-1| = x-1,\ |x-2| = x-2,\ |x-3| = 3-x$, sum $= x$. On $[3,4]$: all three are $x-1, x-2, x-3$, sum $= 3x-6$.

Step 3: Integrating each piece:

$\int_1^2 (4-x)dx = \left[4x - \frac{x^2}{2}\right]_1^2 = (8-2) - (4-\frac{1}{2}) = 6 - 3.5 = 2.5$. $\int_2^3 x\,dx = \left[\frac{x^2}{2}\right]_2^3 = 4.5 - 2 = 2.5$. $\int_3^4 (3x-6)dx = \left[\frac{3x^2}{2} - 6x\right]_3^4 = (24-24) - (13.5-18) = 0 - (-4.5) = 4.5$.

Final Answer:

Total $= 2.5 + 2.5 + 4.5 = 9.5 = \dfrac{19}{2}$.

$$\boxed{\frac{19}{2}}$$

Quick Tip: Split at x=1,2,3 and simplify the sum of absolute values on each piece before integrating.

34. Find the value of the integral $\displaystyle\int \frac{x^4}{(x-1)(x^2+1)}\,dx$.

Correct Answer: —

Solution:

Step 1: Performing polynomial long division first:

Since the numerator's degree (4) exceeds the denominator's degree

(3), divide: $\dfrac{x^4}{(x-1)(x^2+1)} = \dfrac{x^4}{x^3-x^2+x-1} = x+1$ $+ \dfrac{1}{(x-1)(x^2+1)}$ (verified by multiplying back out).

Step 2: Setting up partial fractions for the remainder:

$$\frac{1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

Step 3: Solving for A, B, C:

$1 = A(x^2+1) + (Bx+C)(x-1)$. At $x = 1$: $1 = 2A \Rightarrow A = \frac{1}{2}$. Matching x^2 coefficients: $0 = A + B \Rightarrow B = -\frac{1}{2}$. Matching constants: $1 = A - C$ $\Rightarrow C = A - 1 = -\frac{1}{2}$.

Step 4: Writing the full partial fraction decomposition:

$$\frac{1}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} - \frac{x+1}{2(x^2+1)}.$$

Step 5: Integrating term by term:

$\displaystyle\int (x+1)dx = \frac{x^2}{2} + x$. $\displaystyle\int \frac{dx}{2(x-1)} = \frac{1}{2}\ln|x-1|$. $\displaystyle\int \frac{x+1}{2(x^2+1)}dx$ $= \dfrac{1}{4}\ln(x^2+1) + \dfrac{1}{2}\tan^{-1}x$ (splitting $x/(x^2+1)$ and $1/(x^2+1)$ separately).

Final Answer:

$$\boxed{\frac{x^2}{2} + x + \frac{1}{2}\ln|x-1| - \frac{1}{4}\ln(x^2+1) - \frac{1}{2}\tan^{-1}x + C}$$

Quick Tip: Divide first since the numerator's degree exceeds the denominator's, then use partial fractions on the proper-fraction remainder.

35. Show that the general solution of the differential equation $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$ is $x + y + 1 = A(1 - x - y - 2xy)$, in which A is a parameter.

Correct Answer: —

Solution:

Step 1: Separating variables:

$$\frac{dy}{y^2+y+1} = -\frac{dx}{x^2+x+1}.$$

Step 2: Completing the square in both denominators:

$y^2 + y + 1 = \left(y + \tfrac{1}{2}\right)^2 + \tfrac{3}{4}$, similarly $x^2 + x + 1 = \left(x + \tfrac{1}{2}\right)^2 + \tfrac{3}{4}$.

Step 3: Integrating both sides using the standard arctangent form:

$\int \frac{dy}{(y+1/2)^2 + 3/4} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right)$, and similarly for x. So $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) = -\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C_1$.

Step 4: Combining the two arctangents:

$\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) = C_2$. Taking tangent of both sides and using $\tan^{-1} p + \tan^{-1} q = \tan^{-1}\frac{p+q}{1-pq}$ (up to the constant), this rearranges, after simplification of the combined fraction and relabelling the arbitrary constant as A, to the required implicit form.

Step 5: Verifying by differentiating the given implicit solution:

Differentiating $x + y + 1 = A(1 - x - y - 2xy)$ implicitly with respect to x and eliminating A using the original equation reproduces exactly $\frac{dy}{dx} = -\frac{y^2+y+1}{x^2+x+1}$ (confirmed by direct symbolic computation).

Final Answer:

$$\boxed{x + y + 1 = A(1 - x - y - 2xy)}$$

Quick Tip: Separate variables, complete the square in each denominator, integrate to arctangents, and combine using the tan addition formula.