UP Board Class 12 Mathematics 2026 (Set 324 CX) Question Paper with Solutions

General Instructions

(i) This booklet contains 9 questions, each provided with a complete, step-by-step solution.

(ii) It comprises 5 single-correct multiple-choice questions.

(iii) Attempt each question on your own before reviewing the given solution.

1(a). If $\dfrac{dy}{dx} = e^{x+y}$, then its solution is:

(A) $e^{-y} = e^{x} + c$

(B) $e^{x} + e^{-y} = c$

(C) $e^{-x} - e^{-y} = c$

(D) $e^{-x} + e^{-y} = c$

Correct Answer: (B) $e^{x} + e^{-y} = c$

Solution:

Step 1: Understanding the Concept:

The equation $\dfrac{dy}{dx} = e^{x+y}$ can be split using the law of exponents $e^{x+y} = e^{x} \cdot e^{y}$.

Once split this way, all the x terms can be moved to one side and all the y terms to the other, so the equation is variable separable.

Step 2: Separating the variables:

Write $\dfrac{dy}{dx} = e^{x} \cdot e^{y}$.

Divide both sides by e^y so that y terms stay on the left and x terms move to the right.

$$e^{-y}\,dy = e^{x}\,dx$$

Step 3: Integrating both sides:

Integrate the left side with respect to y and the right side with respect to x.

$$\int e^{-y}\,dy = \int e^{x}\,dx$$

$$-e^{-y} = e^{x} + c_1$$

Multiply through by -1 and rename the constant to get the final relation.

$$e^{-y} = -e^{x} - c_1 \quad \Rightarrow \quad e^{x} + e^{-y} = c$$

Step 4: Why option A is wrong:

Option A, $e^{-y} = e^{x} + c$, drops the sign that comes from integrating e^{-y}; it does not satisfy the original equation on differentiation.

Step 5: Why option C is wrong:

Option C, $e^{-x} - e^{-y} = c$, differentiates to $-e^{-x} + e^{-y}\dfrac{dy}{dx} = 0$, giving $\dfrac{dy}{dx} = e^{y-x}$, not e^{x+y}.

Step 6: Why option D is wrong:

Option D, $e^{-x} + e^{-y} = c$, differentiates to $-e^{-x} - e^{-y}\dfrac{dy}{dx} = 0$, giving $\dfrac{dy}{dx} = -e^{y+x}$, which has the wrong sign.

Final Answer:

Only $e^x + e^{-y} = c$ reproduces the given equation on differentiation.

$$\boxed{e^x + e^{-y} = c}$$

Quick Tip: Write $e^{x+y} = e^x e^y$, separate variables as $e^{-y}dy = e^x dx$, then integrate.

1(b). $\sin(\tan^{-1} x)$, $|x| < 1$ is equal to:

(A) $\dfrac{x}{\sqrt{1+x^2}}$

(B) $\dfrac{x}{\sqrt{1-x^2}}$

(C) $\dfrac{1}{\sqrt{1+x^2}}$

(D) $\dfrac{1}{\sqrt{1-x^2}}$

Correct Answer: (A) $\dfrac{x}{\sqrt{1+x^2}}$

Solution:

Step 1: Understanding the Concept:

Let $\theta = \tan^{-1} x$, so $\tan\theta = x$ and θ lies in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

The task is to find $\sin\theta$ in terms of x using a right triangle picture.

Step 2: Building the right triangle:

Since $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{x}{1}$, take the opposite side as x and the adjacent side as 1.

By the Pythagorean theorem, the hypotenuse is $\sqrt{x^2+1}$.

Step 3: Computing sine:

Sine is opposite over hypotenuse, so substitute the side lengths found above.

$$\sin\theta = \frac{x}{\sqrt{1+x^2}}$$

This matches option A exactly.

Step 4: Why option B is wrong:

Option B, $\dfrac{x}{\sqrt{1-x^2}}$, is actually the formula for $\tan(\sin^{-1}x)$, not $\sin(\tan^{-1}x)$; the sign under the root is wrong here.

Step 5: Why option C is wrong:

Option C, $\dfrac{1}{\sqrt{1+x^2}}$, equals $\cos(\tan^{-1}x)$, the adjacent over hypotenuse ratio, not the sine ratio.

Step 6: Why option D is wrong:

Option D, $\dfrac{1}{\sqrt{1-x^2}}$, matches neither the sine nor the cosine of $\tan^{-1}x$; it resembles a secant-type expression from a different substitution.

Final Answer:

Using the right triangle for $\tan\theta = x$ gives sine directly.

$$\boxed{\sin(\tan^{-1} x) = \frac{x}{\sqrt{1+x^2}}}$$

Quick Tip: Draw a right triangle with opposite x, adjacent 1, hypotenuse $\sqrt{1+x^2}$.

1(c). The interval in which the function $f(x) = x^2 - 4x + 6$ is increasing, is:

(A) $(2, 10)$

(B) $(2, \infty)$

(C) $(-2, \infty)$

(D) $(0, \infty)$

Correct Answer: (B) $(2, \infty)$

Solution:

Step 1: Understanding the Concept:

A function is increasing on an interval where its first derivative is positive throughout that interval.

So the task is to find $f'(x)$ and determine where it is greater than zero.

Step 2: Finding the derivative:

Differentiate $f(x) = x^2 - 4x + 6$ with respect to x.

$$f'(x) = 2x - 4$$

Step 3: Solving the inequality:

Set $f'(x) > 0$ and solve for x.

$$2x - 4 > 0 \quad \Rightarrow \quad x > 2$$

Since x can be any real number greater than 2 with no upper bound, the increasing interval is $(2, \infty)$.

Step 4: Why option A is wrong:

Option A, $(2, 10)$, stops the interval at 10 for no reason; $f'(x)$ stays positive for all x beyond 10 as well, so the function keeps increasing past 10.

Step 5: Why option C is wrong:

Option C, $(-2, \infty)$, includes values like $x = 0$, where $f'(0) = -4 < 0$, so f is actually decreasing there, not increasing.

Step 6: Why option D is wrong:

Option D, $(0, \infty)$, includes values like $x = 1$, where $f'(1) = -2 < 0$, so f is decreasing on part of this interval too.

Final Answer:

The derivative is positive only for x greater than 2, so that is the increasing interval.

$$\boxed{(2, \infty)}$$

Quick Tip: Find $f'(x) = 2x - 4$ and solve $f'(x) > 0$.

1(d). The function $f(x) = 2x, x \in R$ is:

(A) one-one but not onto

(B) one-one and onto

(C) many-one and onto

(D) many-one but not onto

Correct Answer: (B) one-one and onto

Solution:

Step 1: Understanding the Concept:

Here $f : R \to R$ is defined by $f(x) = 2x$. To classify it, check whether it is one-one (injective) and whether it is onto (surjective).

Step 2: Checking one-one:

Assume $f(x_1) = f(x_2)$ for some $x_1, x_2 \in R$.

$$2x_1 = 2x_2 \quad \Rightarrow \quad x_1 = x_2$$

Since equal outputs force equal inputs, f is one-one.

Step 3: Checking onto:

Take any real number y in the codomain and try to find x in the domain with $f(x) = y$.

$$2x = y \quad \Rightarrow \quad x = \frac{y}{2}$$

Since $\frac{y}{2}$ is a real number for every real y, every y has a pre-image, so f is onto.

Step 4: Why option A is wrong:
Option A says one-one but not onto; f was shown to be onto in Step 3, so this option is incorrect.

Step 5: Why option C is wrong:
Option C says many-one and onto; f was shown to be one-one in Step 2, so calling it many-one is incorrect.

Step 6: Why option D is wrong:
Option D says many-one but not onto; f is neither many-one nor fails to be onto, so this option is incorrect on both counts.

Final Answer:
A linear function like $2x$ with nonzero slope from R to R is always a bijection.

$$\boxed{\text{one-one and onto}}$$

Quick Tip: Check if $2x_1 = 2x_2$ forces $x_1 = x_2$, and if every real y has a pre-image $y/2$.

1(e). If $3P(A) = P(B) = \frac{5}{13}$ and $P(A/B) = \frac{2}{5}$, then $P(A \cup B)$ will be:

(A) $\frac{20}{39}$

(B) $\frac{16}{39}$

(C) $\frac{11}{39}$

(D) $\frac{14}{39}$

Correct Answer: (D) $\frac{14}{39}$

Solution:

Step 1: Understanding the Concept:

Given $3P(A) = P(B) = \frac{5}{13}$, this means $P(B) = \frac{5}{13}$ and $P(A) = \frac{1}{3}P(B)$.

Also given is the conditional probability $P(A/B) = \frac{2}{5}$, which needs the formula $P(A/B) = \frac{P(A \cap B)}{P(B)}$.

Step 2: Finding P(A):

Divide $P(B) = \frac{5}{13}$ by 3.

$$P(A) = \frac{1}{3} \times \frac{5}{13} = \frac{5}{39}$$

Step 3: Finding P(A intersection B):

Use the conditional probability formula and substitute the known values.

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \quad \Rightarrow \quad \frac{2}{5} = \frac{P(A \cap B)}{5/13}$$

$$P(A \cap B) = \frac{2}{5} \times \frac{5}{13} = \frac{2}{13}$$

Step 4: Applying the union formula:

Use $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, converting all fractions to a common denominator of 39.

$$P(A \cup B) = \frac{5}{39} + \frac{15}{39} - \frac{6}{39} = \frac{14}{39}$$

Step 5: Why option A is wrong:

Option A, $\frac{20}{39}$, results from adding P(A) and P(B) without subtracting $P(A \cap B)$ at all.

Step 6: Why option B is wrong:

Option B, $\frac{16}{39}$, would come from subtracting only $\frac{4}{39}$ for the intersection, an incorrect value of $P(A \cap B)$.

Step 7: Why option C is wrong:

Option C, $\frac{11}{39}$, would come from wrongly subtracting P(A) instead of $P(A \cap B)$ or a similar arithmetic slip.

Final Answer:

Substituting the correct values of P(A), P(B) and $P(A \cap B)$ into the union formula gives 14/39.

$$\boxed{P(A \cup B) = \frac{14}{39}}$$

Quick Tip: Find $P(A)$ from $3P(A) = P(B)$, then $P(A \cap B) = P(A/B) \cdot P(B)$, then use the union formula.

2(a). If $f : R \to R$ and $g : R \to R$ are defined by $f(x) = \cos x$ and $g(x) = 3x^2$ respectively, find gof.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

The composition gof means apply f first, then apply g to the result.

So $(gof)(x) = g(f(x))$ for every real x.

Step 2: Substitute f(x) into g:

We are given $f(x) = \cos x$ and $g(x) = 3x^2$, so g squares its input and multiplies by 3.

Replacing the input of g with $f(x) = \cos x$ gives:

$$(gof)(x) = g(f(x)) = g(\cos x) = 3(\cos x)^2$$

Step 3: Simplify:

Writing $(\cos x)^2$ as $\cos^2 x$, we get the final composed function.

Final Answer:

The composition gof is 3 cos squared x.

$$\boxed{(gof)(x) = 3\cos^2 x}$$

Quick Tip: Composition gof means g(f(x)); replace x in g(x) = 3x^2 by cos x.

2(b). If a line makes $90°$, $60°$ and $30°$ with the x, y and z-axes respectively in the positive direction, then find its direction cosines.

Correct Answer: —

Solution:

Step 1: Key Formula:

If a line makes angles α, β, γ with the positive x, y, z axes, its direction cosines are $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

Here $\alpha = 90^\circ, \beta = 60^\circ, \gamma = 30^\circ$.

Step 2: Compute each cosine:

$\cos 90^\circ = 0$, $\cos 60^\circ = \frac{1}{2}$, $\cos 30^\circ = \frac{\sqrt{3}}{2}$.

So the direction cosines are:

$$l = 0, \quad m = \frac{1}{2}, \quad n = \frac{\sqrt{3}}{2}$$

Step 3: Verify using l^2+m^2+n^2=1:

Check the standard identity that direction cosines must satisfy:

$$0^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 0 + \frac{1}{4} + \frac{3}{4} = 1$$

The identity holds, so the values are correct.

Final Answer:

The direction cosines of the line are 0, one half, and root 3 by 2.

$$\boxed{l = 0,\ m = \frac{1}{2},\ n = \frac{\sqrt{3}}{2}}$$

Quick Tip: Direction cosines are l = cos(angle with x-axis), m = cos(angle with y-axis), n = cos(angle with z-axis).

2(c). Find the value of $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$.

Correct Answer: —

Solution:

Step 1: Evaluate tan inverse root 3:

The principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Since $\tan\frac{\pi}{3} = \sqrt{3}$, we get $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$.

Step 2: Evaluate sec inverse of -2:

The principal value branch of $\sec^{-1}$ is $[0, \pi]$, excluding $\frac{\pi}{2}$.

For a negative input the angle lies in $\left(\frac{\pi}{2}, \pi\right)$. Since $\sec\frac{2\pi}{3}$ $= \frac{1}{\cos(2\pi/3)} = \frac{1}{-1/2} = -2$, we get:

$$\sec^{-1}(-2) = \frac{2\pi}{3}$$

Step 3: Subtract the two values:

Substitute both principal values into the expression:

$$\tan^{-1}\sqrt{3} - \sec^{-1}(-2) = \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$$

Final Answer:

The value of the expression is negative pi by 3.

$$\boxed{-\frac{\pi}{3}}$$

Quick Tip: Use tan^-1(root3) = pi/3, and sec^-1(-2) = pi - sec^-1(2) = 2pi/3.

2(d). Find the value of the integral $\int x^2 \tan(x^3 + 2)\, dx$.

Correct Answer: —

Solution:

Step 1: Choose a substitution:

The argument of tan is $x^3 + 2$, and its derivative $3x^2$ is present (up to a constant) in the integrand as x^2.

So let $u = x^3 + 2$, which gives $du = 3x^2\, dx$, or $x^2\, dx = \frac{du}{3}$.

Step 2: Rewrite the integral in terms of u:

Substituting into the original integral:

$$\int x^2 \tan(x^3 + 2)\, dx = \int \tan u \cdot \frac{du}{3} = \frac{1}{3} \int \tan u\, du$$

Step 3: Integrate tan u:

Use the standard result $\int \tan u\, du = \ln|\sec u| + C$.

$$\frac{1}{3}\int \tan u\, du = \frac{1}{3}\ln|\sec u| + C$$

Step 4: Substitute back u = x^3+2:

Replace u by its original expression to get the final antiderivative.

Final Answer:

The integral evaluates to one third times log of sec of x cubed plus 2.

$$\boxed{\int x^2\tan(x^3+2)\,dx = \frac{1}{3}\ln\left|\sec(x^3+2)\right| + C}$$

Quick Tip: Put u = x^3 + 2 so du = 3x^2 dx, then integrate tan u.

2(e). If for a unit vector $\vec{a}$, $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a}) = 12$, then find $|\vec{x}|$.

Correct Answer: —

Solution:

Step 1: Expand the dot product:

Use the distributive property of the dot product, similar to the algebraic identity $(A-B)\cdot(A+B) = A\cdot A - B\cdot B$ for vectors.

$$(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a}) = \vec{x}\cdot\vec{x} + \vec{x}\cdot\vec{a} - \vec{a}\cdot\vec{x} - \vec{a}\cdot\vec{a}$$

Since the dot product is commutative, $\vec{x}\cdot\vec{a} = \vec{a}\cdot\vec{x}$, so these two middle terms cancel.

Step 2: Simplify using magnitudes:
What remains is:

$$(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = \vec{x} \cdot \vec{x} - \vec{a} \cdot \vec{a} = |\vec{x}|^2 - |\vec{a}|^2$$

Since $\vec{a}$ is a unit vector, $|\vec{a}| = 1$, so $|\vec{a}|^2 = 1$.

Step 3: Solve for the magnitude of x:
Substitute the given value 12 and $|\vec{a}|^2 = 1$ into the simplified equation:

$$|\vec{x}|^2 - 1 = 12 \quad \Rightarrow \quad |\vec{x}|^2 = 13 \quad \Rightarrow \quad |\vec{x}| = \sqrt{13}$$

Final Answer:
The magnitude of vector x is root 13.

$$\boxed{|\vec{x}| = \sqrt{13}}$$

Quick Tip: Expand as (x-a).(x+a) = |x|^2 - |a|^2, and use |a| = 1.

3(a). If $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 5\hat{j} - 2\hat{k}$, find $|\vec{a} \times \vec{b}|$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:
The cross product of two vectors gives a new vector that is perpendicular to both of them.
Its magnitude can be found by first computing the vector $\vec{a} \times \vec{b}$ using

the determinant method, then finding the length of that resulting vector.

Step 2: Key Formula:

For $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Step 3: Detailed Explanation:

Here $a_1 = 2, a_2 = 1, a_3 = 3$ and $b_1 = 3, b_2 = 5, b_3 = -2$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix}$$

Expanding along the first row gives the vector $-17\hat{i} + 13\hat{j} + 7\hat{k}$.

The magnitude is found using $|\vec{v}| = \sqrt{x^2 + y^2 + z^2}$.

$$|\vec{a} \times \vec{b}| = \sqrt{(-17)^2 + 13^2 + 7^2} = \sqrt{289 + 169 + 49} = \sqrt{507} = 13\sqrt{3}$$

Final Answer:

The magnitude of the cross product is 13 root 3.

$$\boxed{|\vec{a} \times \vec{b}| = 13\sqrt{3}}$$

Quick Tip: Use the determinant method to find a cross b, then take its magnitude with the square root of the sum of squares.

3(b). Prove that the function $f : R \to \{x \in R : -1 < x < 1\}$, where $f(x) = \frac{2x}{1+|x|}$, $x \in R$, is one-one.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

A function is one-one (injective) if different inputs always give different outputs.

This means we must show that whenever $f(x_1) = f(x_2)$, it forces $x_1 = x_2$.

Since the formula involves $|x|$, we split the proof into cases based on the sign of x.

Step 2: Case When Both Inputs Are Non-negative:

Let $x_1, x_2 \geq 0$. Then $|x_1| = x_1$ and $|x_2| = x_2$, so $f(x) = \frac{2x}{1+x}$.

Assume $f(x_1) = f(x_2)$.

$$\frac{2x_1}{1+x_1} = \frac{2x_2}{1+x_2}$$

Cross multiplying gives $x_1(1 + x_2) = x_2(1 + x_1)$, which simplifies to $x_1 + x_1x_2 = x_2 + x_1x_2$.

This reduces to $x_1 = x_2$.

Step 3: Case When Both Inputs Are Negative:

Let $x_1, x_2 < 0$. Then $|x_1| = -x_1$ and $|x_2| = -x_2$, so $f(x) = \dfrac{2x}{1-x}$.
Assume $f(x_1) = f(x_2)$.

$$\frac{2x_1}{1-x_1} = \frac{2x_2}{1-x_2}$$

Cross multiplying gives $x_1(1-x_2) = x_2(1-x_1)$, which simplifies to $x_1 - x_1x_2 = x_2 - x_1x_2$.
This again reduces to $x_1 = x_2$.

Step 4: Case When Inputs Have Opposite Signs:
Let $x_1 > 0$ and $x_2 < 0$ (the other order is similar). Then $f(x_1) = \dfrac{2x_1}{1+x_1} > 0$ while $f(x_2) = \dfrac{2x_2}{1-x_2} < 0$.
Since a positive number can never equal a negative number, $f(x_1) \neq f(x_2)$ in this case.
So equal outputs are impossible unless x_1 and x_2 lie in the same case, which already forces $x_1 = x_2$.

Final Answer:
In every case $f(x_1) = f(x_2)$ implies $x_1 = x_2$, so f is one-one, hence proved.

$$\boxed{f \text{ is one-one}}$$

Quick Tip: Split into cases based on the sign of x and show f(x1)=f(x2) forces x1=x2 in every case.

3(c). If $f(x) = \sin x$, $x \in [0, \pi/2]$ and $g(x) = \cos x$, $x \in [0, \pi/2]$, prove that $(f+g)$ is not one-one.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

A function h is not one-one if we can find two different points in its domain that give the same output value.

Here $h(x) = (f + g)(x) = \sin x + \cos x$, defined on $[0, \pi/2]$.

Step 2: Evaluate h at the Two Endpoints of the Domain:

Compute $h(0)$ and $h(\pi/2)$, both of which lie in the domain $[0, \pi/2]$.

$$h(0) = \sin 0 + \cos 0 = 0 + 1 = 1$$

$$h(\pi/2) = \sin(\pi/2) + \cos(\pi/2) = 1 + 0 = 1$$

Step 3: Compare the Two Points:

The two inputs $x_1 = 0$ and $x_2 = \pi/2$ are clearly different, since 0 $\neq \pi/2$.

But both give the same output, $h(x_1) = h(x_2) = 1$.

This is exactly the situation that breaks the one-one property.

Final Answer:

Since two distinct points of the domain give equal values, (f+g) is not one-one, hence proved.

$$\boxed{(f+g)(0) = (f+g)(\pi/2) = 1, \text{ so } f+g \text{ is not one-one}}$$

Quick Tip: Check the values of sin x plus cos x at the two endpoints of the interval.

3(d). If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are mutually perpendicular, find the value of k.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

Two lines in space are perpendicular when their direction vectors have a dot product equal to zero.

We first read off the direction ratios of both lines from their symmetric (Cartesian) form.

Step 2: Identify the Direction Ratios:

Line 1 has direction ratios $(-3, 2k, 2)$.

Line 2 has direction ratios $(3k, 1, -5)$.

Step 3: Apply the Perpendicularity Condition:

For perpendicular lines, $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

$$(-3)(3k) + (2k)(1) + (2)(-5) = 0$$

Simplify each term and collect like terms.

$$-9k + 2k - 10 = 0$$

$$-7k = 10$$

Step 4: Solve for k:

Divide both sides by -7 to isolate k.

$$k = -\frac{10}{7}$$

Final Answer:

The value of k that makes the lines mutually perpendicular is negative ten sevenths.

$$\boxed{k = -\frac{10}{7}}$$

Quick Tip: Set the dot product of the direction ratios of both lines equal to zero and solve for k.

4(a). Find the Cartesian equation of the line parallel to the vector $3\hat{i} + 2\hat{j} - 8\hat{k}$ and passing through the point $(5, 2, -4)$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

A line through a fixed point, parallel to a given vector, has a standard Cartesian (symmetric) equation.

If the line passes through (x_1, y_1, z_1) and is parallel to a vector with direction ratios (a, b, c), its equation follows a known formula.

Step 2: Key Formula:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Step 3: Detailed Explanation:

Here the point is $(5, 2, -4)$, so $x_1 = 5, y_1 = 2, z_1 = -4$.

The direction ratios come from the parallel vector $3\hat{i} + 2\hat{j} - 8\hat{k}$, so $a = 3, b = 2, c = -8$.

Substitute these values directly into the standard formula.

$$\frac{x - 5}{3} = \frac{y - 2}{2} = \frac{z - (-4)}{-8}$$

Final Answer:

This simplifies to the required Cartesian equation of the line.

$$\boxed{\frac{x - 5}{3} = \frac{y - 2}{2} = \frac{z + 4}{-8}}$$

Quick Tip: Use the point and the parallel vector's components as direction ratios in the standard line formula.

4(b). If $x \in \left(0, \frac{\pi}{4}\right)$, then prove that $\cot^{-1}\left(\frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}\right) = \frac{x}{2}$.

Correct Answer: —

Solution:

Step 1: Write what has to be shown:

We must prove that the given inverse cotangent expression equals $x/2$ for $x \in (0, \pi/4)$.

The key idea is to write $1 + \sin x$ and $1 - \sin x$ as perfect squares using half angle values.

Step 2: Convert to half angle form:

Using $\sin x = 2\sin\dfrac{x}{2}\cos\dfrac{x}{2}$ and $1 = \sin^2\dfrac{x}{2} + \cos^2\dfrac{x}{2}$, we get two perfect squares.

$$1 + \sin x = \left(\cos\tfrac{x}{2} + \sin\tfrac{x}{2}\right)^2, \qquad 1 - \sin x = \left(\cos\tfrac{x}{2} - \sin\tfrac{x}{2}\right)^2$$

Since $x \in (0, \pi/4)$, we have $x/2 \in (0, \pi/8)$, so $\cos\frac{x}{2} > \sin\frac{x}{2} > 0$.

Both $\cos\frac{x}{2} + \sin\frac{x}{2}$ and $\cos\frac{x}{2} - \sin\frac{x}{2}$ are positive, so square roots give these values directly, no sign issue.

Step 3: Take square roots and simplify the ratio:

$$\sqrt{1 + \sin x} = \cos\tfrac{x}{2} + \sin\tfrac{x}{2}, \qquad \sqrt{1 - \sin x} = \cos\tfrac{x}{2} - \sin\tfrac{x}{2}$$

Adding and subtracting these two results:

$$\sqrt{1 + \sin x} + \sqrt{1 - \sin x} = 2\cos\tfrac{x}{2}, \qquad \sqrt{1 + \sin x} - \sqrt{1 - \sin x} = 2\sin\tfrac{x}{2}$$

So the ratio inside the inverse cotangent becomes:

$$\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}} = \frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}} = \cot\frac{x}{2}$$

Step 4: Apply the inverse function:

Since $x/2 \in (0, \pi/8) \subset (0, \pi)$, which is exactly the principal range of $\cot^{-1}$, we can invert directly.

$$\cot^{-1}\left(\cot\tfrac{x}{2}\right) = \frac{x}{2}$$

This matches the right hand side of the identity, so the statement is true for every x in the given interval.

Final Answer:

Hence the identity $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}$ is proved.

$$\boxed{\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}}$$

Quick Tip: Write $1 \pm \sin x$ as $(\cos\frac{x}{2} \pm \sin\frac{x}{2})^2$ using half angle formulas.

4(c). Find the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

We need to expand a 3x3 determinant with rows that are cyclic shifts of a, b, c.

We use direct cofactor expansion along the first row.

Step 2: Key Formula or Approach:

For a determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, expansion along row 1 gives $a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$.

Step 3: Detailed Explanation:

Here the matrix is $\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$, so expanding along row 1:

$$\Delta = a(c \cdot b - a \cdot a) - b(b \cdot b - a \cdot c) + c(b \cdot a - c \cdot c)$$

$$= a(bc - a^2) - b(b^2 - ac) + c(ab - c^2)$$

$$= abc - a^3 - b^3 + abc + abc - c^3$$

$$= 3abc - a^3 - b^3 - c^3$$

This matches the standard algebraic identity $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$, so we factor with a negative sign in front.

Step 4: Factor the result:

$$\Delta = -(a^3 + b^3 + c^3 - 3abc) = -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

Final Answer:
The determinant equals the negative of the standard cubic sum minus three times the product identity.

$$\boxed{\Delta = 3abc - a^3 - b^3 - c^3 = -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)}$$

Quick Tip: Expand along the first row and match terms to the identity $a^3 + b^3 + c^3 - 3abc$.

4(d). If $P(A) = \frac{7}{13}$, $P(B) = \frac{9}{13}$ and $P(A \cap B) = \frac{4}{13}$, then find $P(A \cup B)$ and $P(A/B)$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:
We are given probabilities of two events A and B and their intersection.
We need the union probability and the conditional probability of A given B.

Step 2: Key Formula or Approach:
Use the addition rule $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and the

conditional probability formula $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$.

Step 3: Detailed Explanation:

First find the union probability by substituting the given values:

$$P(A \cup B) = \frac{7}{13} + \frac{9}{13} - \frac{4}{13} = \frac{7+9-4}{13} = \frac{12}{13}$$

Next find the conditional probability of A given B, which is the intersection divided by P(B):

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{4/13}{9/13} = \frac{4}{9}$$

Final Answer:

Both results follow directly from the standard formulas of probability.

$$\boxed{P(A \cup B) = \frac{12}{13}, \quad P(A/B) = \frac{4}{9}}$$

Quick Tip: Use $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and $P(A/B) = P(A \cap B)/P(B)$.

5(a). Suppose that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors such that $|\vec{a}| = 3$, $|\vec{b}| = 4$, $|\vec{c}| = 5$, and each of them is perpendicular to the sum of the other two vectors. Find $|\vec{a} + \vec{b} + \vec{c}|$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

Each vector being perpendicular to the sum of the other two means their dot products with that sum are zero.

This gives us three separate equations which we can add together to find the sum of all pairwise dot products.

Step 2: Key Formula or Approach:

The condition translates to $\vec{a} \cdot (\vec{b} + \vec{c}) = 0$, $\vec{b} \cdot (\vec{c} + \vec{a}) = 0$, $\vec{c} \cdot (\vec{a} + \vec{b}) = 0$.

We use $|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a})$.

Step 3: Detailed Explanation:

Write the three perpendicularity conditions as equations:

$$\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} = 0, \quad \vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{a} = 0, \quad \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} = 0$$

Adding all three equations, each dot product term appears exactly twice:

$$2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0 \implies \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = 0$$

Now expand the square of the magnitude of the sum vector using this result:

$$|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a})$$

$$= 3^2 + 4^2 + 5^2 + 2(0) = 9 + 16 + 25 = 50$$

Step 4: Take the square root:

$$|\vec{a}+\vec{b}+\vec{c}| = \sqrt{50} = 5\sqrt{2}$$

Final Answer:

The magnitude of the resultant vector sum is found using the vanishing pairwise dot product sum.

$$\boxed{|\vec{a}+\vec{b}+\vec{c}| = 5\sqrt{2}}$$

Quick Tip: Add the three perpendicularity conditions to get $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=0$, then expand the square.

5(b). Find the maximum value of the objective function $Z = 5x + 10y$ by graphical method under the following constraints: $x + 2y \leq 120$, $x + y \geq 60$, $x - 2y \geq 0$, $x \geq 0$, $y \geq 0$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

We must find the feasible region formed by all constraints, locate its corner points, and evaluate Z at each corner point.

The graphical method says the maximum of a linear objective function over a bounded feasible region occurs at a corner point.

Step 2: Key Formula or Approach:

Plot the boundary lines $x + 2y = 120$, $x + y = 60$, $x - 2y = 0$, find their pairwise intersections that lie inside all constraints, then evaluate $Z = 5x + 10y$ at every corner.

Step 3: Detailed Explanation, find the corner points:

Solve $x + 2y = 120$ and $x - 2y = 0$ together: adding gives $2x = 120$, so $x = 60$, then $y = 30$. Point $(60, 30)$.
Solve $x + y = 60$ and $x - 2y = 0$ together: substituting $x = 2y$ gives $2y + y = 60$, so $y = 20$, $x = 40$. Point $(40, 20)$.
Solve $x + y = 60$ with $y = 0$ (the x axis): gives $x = 60$. Point $(60, 0)$.
Solve $x + 2y = 120$ with $y = 0$: gives $x = 120$. Point $(120, 0)$.
Check that $x + 2y = 120$ and $x + y = 60$ meet at $(0, 60)$, but this fails $x - 2y \geq 0$ since $0 - 120 < 0$, so it is rejected as not feasible.
So the feasible region is a quadrilateral with corners $(60, 0)$, $(120, 0)$, $(60, 30)$, $(40, 20)$.

Step 4: Evaluate Z at each corner point:

Corner Point	Z = 5x + 10y
(60, 0)	300
(120, 0)	600
(60, 30)	600
(40, 20)	400

The largest value, 600, is achieved at two corner points, $(120, 0)$ and $(60, 30)$.
Since both lie on the line $x + 2y = 120$, and $Z = 5(x + 2y)$ on this whole line, Z equals 600 at every point of the segment joining them, not just at the two corners.

Final Answer:

The maximum value of Z is 600, attained along the whole edge from (60,30) to (120,0), giving infinitely many optimal solutions.

$$\boxed{Z_{max} = 600 \text{ at all points on the segment joining } (60, 30) \text{ and } (120, 0)}$$

Quick Tip: Find corner points by solving the boundary lines in pairs, then evaluate Z at each one.

5(c). For what value of λ, is the function

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & x \leq 0 \\ 4x + 1, & x > 0 \end{cases}$$

continuous at $x = 0$?

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

A function is continuous at a point when the left hand limit, the right hand limit and the value of the function at that point are all equal. Here f is defined by two separate rules on either side of x = 0, so each of these three quantities must be found using the correct rule.

Step 2: Find the Left Hand Limit:

For x less than or equal to 0, the rule is $f(x) = \lambda(x^2 - 2x)$.

As x approaches 0 from the left, both x^2 and x tend to 0, so the bracket tends to 0 for any value of λ.

$$\lim_{x \to 0^-} f(x) = \lambda(0^2 - 2 \cdot 0) = 0$$

Step 3: Find the Right Hand Limit and f(0):

For x greater than 0, the rule is $f(x) = 4x + 1$, so the right hand limit is obtained by putting x = 0 in this rule.

Since 0 satisfies $x \leq 0$, the value $f(0)$ is taken from the first rule and equals 0 as well.

$$\lim_{x \to 0^+} f(x) = 4(0) + 1 = 1, \qquad f(0) = \lambda(0) = 0$$

Step 4: Compare the Three Values:

The left hand limit is 0 and the right hand limit is 1, and neither of these values contains λ, so they are fixed numbers.

For continuity we need $0 = 1$, which is false for every real number, so no choice of λ can satisfy the condition.

Final Answer:

The one sided limits 0 and 1 can never be made equal, so continuity at x = 0 is impossible for any lambda.

$$\boxed{\text{No real value of } \lambda \text{ makes } f(x) \text{ continuous at } x = 0}$$

Quick Tip: Find the left hand limit and right hand limit of f(x) at x = 0 and check whether they can be made equal.

5(d). If the position vectors of points A, B, C and D are respectively $\hat{i} + \hat{j} + \hat{k}$, $2\hat{i} + 5\hat{j}$, $3\hat{i} + 2\hat{j} - 3\hat{k}$ and $\hat{i} - 6\hat{j} - \hat{k}$, then find the angle between the lines AB and CD. Prove that AB and CD are collinear.

Correct Answer: –

Solution:

Step 1: Understanding the Concept:

The vector along the line joining two points is found by subtracting the position vector of the starting point from that of the end point. The angle between two lines is the angle between their direction vectors, found using the dot product formula $\cos\theta = \dfrac{\vec{AB} \cdot \vec{CD}}{|\vec{AB}||\vec{CD}|}$.

Step 2: Find the Vectors AB and CD:

Position vectors are $A = \hat{i} + \hat{j} + \hat{k}$, $B = 2\hat{i} + 5\hat{j}$, $C = 3\hat{i} + 2\hat{j} - 3\hat{k}$, $D = \hat{i} - 6\hat{j} - \hat{k}$.

Subtract to get the direction of each line.

$$\vec{AB} = B - A = \hat{i} + 4\hat{j} - \hat{k}, \qquad \vec{CD} = D - C = -2\hat{i} - 8\hat{j} + 2\hat{k}$$

Step 3: Compute the Angle:

Find the magnitudes and the dot product of the two vectors.

$$|\vec{AB}| = \sqrt{1 + 16 + 1} = 3\sqrt{2}, \qquad |\vec{CD}| = \sqrt{4 + 64 + 4} = 6\sqrt{2}$$

$$\vec{AB} \cdot \vec{CD} = (1)(-2) + (4)(-8) + (-1)(2) = -36$$

$$\cos\theta = \frac{-36}{3\sqrt{2}\times 6\sqrt{2}} = \frac{-36}{36} = -1 \implies \theta = 180^\circ$$

Step 4: Prove AB and CD are Collinear:

Compare the two vectors found in Step 2.

$$\vec{CD} = -2\hat{i} - 8\hat{j} + 2\hat{k} = -2(\hat{i} + 4\hat{j} - \hat{k}) = -2\vec{AB}$$

Since $\vec{CD}$ is a scalar multiple of $\vec{AB}$, the two vectors are parallel, which means lines AB and CD are collinear.

The scalar -2 is negative, so the vectors point in opposite senses, which is exactly why the angle between them came out as 180° in Step 3.

Final Answer:

The angle between AB and CD is 180 degrees, and CD = -2 AB proves the lines are collinear.

$$\boxed{\theta = 180^\circ, \quad \vec{CD} = -2\vec{AB}}$$

Quick Tip: Find vectors AB and CD by subtraction, then check if one is a scalar multiple of the other.

5(e). Find the interval in which the function $f(x) = \dfrac{4\sin x - 2x - x\cos x}{2 + \cos x}$ is strictly increasing and strictly decreasing.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

A function is strictly increasing where its derivative is positive and strictly decreasing where its derivative is negative.

Since f is built from sin x and cos x, which repeat every 2π, it is enough to study one full period, so we take $x \in [0, 2\pi]$.

Step 2: Differentiate Using the Quotient Rule:

Let $u = 4\sin x - 2x - x\cos x$ and $v = 2 + \cos x$, so $f = u/v$.

Differentiate u using the product rule on the term $x\cos x$, and differentiate v directly.

$$u' = 4\cos x - 2 - (\cos x - x\sin x) = 3\cos x - 2 + x\sin x, \qquad v' = -\sin x$$

Step 3: Simplify f'(x):

Apply the quotient rule $f' = \dfrac{u'v - uv'}{v^2}$ and expand the numerator.

The numerator works out to $4\sin^2 x + 3\cos^2 x + 4\cos x - 4$, which simplifies using $\sin^2 x = 1 - \cos^2 x$.

$$4(1 - \cos^2 x) + 3\cos^2 x + 4\cos x - 4 = -\cos^2 x + 4\cos x = \cos x(4 - \cos x)$$

$$f'(x) = \frac{\cos x(4 - \cos x)}{(2 + \cos x)^2}$$

Step 4: Study the Sign of f'(x):

Since $\cos x$ always lies between -1 and 1, the factor $4 - \cos x$ is always positive, and $(2 + \cos x)^2$ is always positive.

So the sign of $f'(x)$ is exactly the sign of $\cos x$, nothing else affects it. On $[0, 2\pi]$, $\cos x > 0$ on $(0, \pi/2) \cup (3\pi/2, 2\pi)$ and $\cos x < 0$ on $(\pi/2, 3\pi/2)$.

Final Answer:

f is strictly increasing where cos x is positive and strictly decreasing where cos x is negative.

$$\boxed{\text{Increasing on } \left(0, \tfrac{\pi}{2}\right) \cup \left(\tfrac{3\pi}{2}, 2\pi\right), \quad \text{Decreasing on } \left(\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right)}$$

Quick Tip: Differentiate f(x) with the quotient rule, simplify using sin squared x plus cos squared x equals 1, then check the sign of cos x.

6(a). If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, then show that $A^3 - 23A - 40I = 0$.

Correct Answer: —

Solution:

Step 1: Understanding What Is to Be Shown:

We must show the matrix expression $A^3 - 23A - 40I$ equals the zero matrix, where I is the 3 by 3 identity matrix.

The direct way is to compute A^2, then A^3, and finally combine the terms and check every entry becomes 0.

Step 2: Compute A Squared:

Multiply A by itself row by column.

$$A^2 = AA = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$$

Step 3: Compute A Cubed:

Multiply the result by A once more.

$$A^3 = A^2 A = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix}$$

Step 4: Form the Full Expression:

Compute $23A$ and $40I$ separately, then subtract both from A^3, entry by entry.

$$23A = \begin{bmatrix} 23 & 46 & 69 \\ 69 & -46 & 23 \\ 92 & 46 & 23 \end{bmatrix}, \qquad 40I = \begin{bmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{bmatrix}$$

$$A^3 - 23A - 40I = \begin{bmatrix} 63-23-40 & 46-46-0 & 69-69-0 \\ 69-69-0 & -6+46-40 & 23-23-0 \\ 92-92-0 & 46-46-0 & 63-23-40 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Final Answer:

Every entry of the computed expression is 0, so the matrix identity holds and this proves the required result.

$$\boxed{A^3 - 23A - 40I = O}$$

Quick Tip: Use the Cayley-Hamilton theorem: find the characteristic equation of A, then substitute A for x in that equation.

6(b). Prove that if A and B are independent events, then the probability of happening of at least one of A or B is $[1 - P(A')P(B')]$.

Correct Answer: —

Solution:

Step 1: Understanding What Is to Be Proved:

"At least one of A or B occurs" is the event $A \cup B$. We must show $P(A \cup B) = 1 - P(A')P(B')$, given that A and B are independent, meaning $P(A \cap B) = P(A)P(B)$.

Step 2: Use the Addition Theorem:

The probability of a union of two events is given by the addition theorem of probability.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Step 3: Substitute the Independence Condition:

Since A and B are independent, replace $P(A \cap B)$ with $P(A)P(B)$.

$$P(A \cup B) = P(A) + P(B) - P(A)P(B)$$

Step 4: Rewrite Using Complements:

Recall that $P(A') = 1 - P(A)$ and $P(B') = 1 - P(B)$, so their product can be expanded and compared with the expression above.

$$P(A')P(B') = (1 - P(A))(1 - P(B)) = 1 - P(A) - P(B) + P(A)P(B)$$

$$1 - P(A')P(B') = P(A) + P(B) - P(A)P(B)$$

The right side here is identical to the expression obtained for $P(A \cup B)$ in Step 3.

Final Answer:

Since both expressions equal $P(A) + P(B) - P(A)P(B)$, they must be equal to each other, proving the result.

$$\boxed{P(A \cup B) = 1 - P(A')P(B')}$$

Quick Tip: Use the addition theorem P(A union B) = P(A) + P(B) - P(A)P(B), then compare with 1 - P(A')P(B').

6(c). Prove that the height of the right circular cone of maximum volume inscribed in a sphere of radius r is $\frac{4r}{3}$.

Correct Answer: —

Solution:

Step 1: Setting up the geometry:

Let a right circular cone be inscribed in a sphere of radius r with center O.

Let h be the height of the cone and R be the radius of its base circle. The base circle and the apex both lie on the sphere, so the right triangle formed by R, r and $(h - r)$ gives the relation $R^2 = r^2 - (h - r)^2$.

Step 2: Writing volume as a function of one variable:

Simplify the relation to get $R^2 = 2rh - h^2$, valid for $0 < h < 2r$.

The volume of the cone is $V = \frac{1}{3}\pi R^2 h$.

Substitute R^2 to write V purely in terms of h:

$$V(h) = \frac{1}{3}\pi(2rh^2 - h^3), \quad 0 < h < 2r$$

Step 3: Differentiating and finding the critical point:

Differentiate V with respect to h.

$$\frac{dV}{dh} = \frac{1}{3}\pi(4rh - 3h^2) = \frac{1}{3}\pi h(4r - 3h)$$

Set $\frac{dV}{dh} = 0$. Since $h \neq 0$ for a real cone, $4r - 3h = 0$, so $h = \frac{4r}{3}$.

Step 4: Verifying it is a maximum:

Find the second derivative.

$$\frac{d^2V}{dh^2} = \frac{1}{3}\pi(4r - 6h)$$

At $h = \frac{4r}{3}$: $\frac{d^2V}{dh^2} = \frac{1}{3}\pi(4r - 8r) = -\frac{4\pi r}{3} < 0$, so V is maximum at this h.

Final Answer:

The volume is maximum when the height equals four-thirds of the radius.

$$\boxed{h = \frac{4r}{3}}$$

Quick Tip: Use R^2 = 2rh - h^2 from the sphere geometry, then maximize V(h) using calculus.

6(d). Find the area of the region bounded by $x = 0$, $x = 2\pi$ and the curve $y = \sin x$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

The curve $y = \sin x$ is positive on $(0, \pi)$ and negative on $(\pi, 2\pi)$.

Area bounded by a curve and the x-axis is always taken as a positive quantity, so the part below the axis must be taken with its absolute value.

Step 2: Splitting the integral at the sign change:

Since $\sin x = 0$ at $x = \pi$ inside $[0, 2\pi]$, split the area into two parts.

$$\text{Area} = \left|\int_0^{\pi} \sin x\, dx\right| + \left|\int_{\pi}^{2\pi} \sin x\, dx\right|$$

Step 3: Evaluating each integral:

Use the antiderivative $\int \sin x \, dx = -\cos x$.

$$\int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi} = -\cos\pi + \cos 0 = 1 + 1 = 2$$

$$\int_{\pi}^{2\pi} \sin x \, dx = [-\cos x]_{\pi}^{2\pi} = -\cos 2\pi + \cos\pi = -1 - 1 = -2$$

Taking the absolute value of the second result gives 2.

Final Answer:

Total area equals the sum of both positive areas.

$$\boxed{\text{Area} = 2 + 2 = 4 \text{ sq units}}$$

Quick Tip: Split the integral at x = pi since sin x changes sign there, and add the two positive areas.

6(e). Find the value of $\displaystyle\int_{-1}^{3/2} |x \sin \pi x| \, dx$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

To integrate $|x \sin \pi x|$, first find where $x \sin \pi x$ changes sign on $[-1, 3/2]$.

Simply integrating $x \sin \pi x$ directly would give a wrong answer wherever the function is negative, since the modulus flips its sign there.

Step 2: Finding the sign of $x \sin \pi x$ on each part:

$\sin \pi x = 0$ at $x = -1, 0, 1$ inside this interval.

On $(-1, 0)$: $x < 0$ and $\sin \pi x < 0$ (since $\pi x \in (-\pi, 0)$), so the product is positive.

On $(0, 1)$: $x > 0$ and $\sin \pi x > 0$, so the product is positive.

On $(1, 3/2)$: $x > 0$ but $\sin \pi x < 0$ (since $\pi x \in (\pi, 3\pi/2)$), so the product is negative.

So $|x \sin \pi x| = x \sin \pi x$ on $[-1, 1]$ and $|x \sin \pi x| = -x \sin \pi x$ on $[1, 3/2]$.

Step 3: Splitting and evaluating the integral:

$$I = \int_{-1}^{1} x \sin \pi x \, dx + \int_{1}^{3/2} (-x \sin \pi x) \, dx$$

Using integration by parts, $\int x \sin \pi x \, dx = -\dfrac{x \cos \pi x}{\pi} + \dfrac{\sin \pi x}{\pi^2} + C$.

Let $F(x) = -\dfrac{x \cos \pi x}{\pi} + \dfrac{\sin \pi x}{\pi^2}$.

$F(1) = \dfrac{1}{\pi}$, $F(-1) = -\dfrac{1}{\pi}$, $F(3/2) = -\dfrac{1}{\pi^2}$.

Step 4: Substituting the limits:

$$\int_{-1}^{1} x \sin \pi x \, dx = F(1) - F(-1) = \frac{1}{\pi} - \left(-\frac{1}{\pi}\right) = \frac{2}{\pi}$$

$$\int_{1}^{3/2} x \sin \pi x \, dx = F(3/2) - F(1) = -\frac{1}{\pi^2} - \frac{1}{\pi}$$

So the second piece contributes $\frac{1}{\pi^2} + \frac{1}{\pi}$ after reversing sign.

$$I = \frac{2}{\pi} + \frac{1}{\pi} + \frac{1}{\pi^2} = \frac{3}{\pi} + \frac{1}{\pi^2} = \frac{3\pi + 1}{\pi^2}$$

Final Answer:

The value of the definite integral after correctly handling the modulus is:

$$\boxed{I = \frac{3\pi + 1}{\pi^2}}$$

Quick Tip: Check the sign of x sin(pi x) separately on (-1,1) and (1,3/2) before removing the modulus.

7. Suppose that $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$. Find the matrix D such that $CD - AB = 0$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

The equation $CD - AB = 0$ means $CD = AB$.

Since C is a square matrix, if C is invertible, then $D = C^{-1}(AB)$.

Step 2: Computing the product AB:

$$AB = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 10-7 & 4-4 \\ 15+28 & 6+16 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix}$$

Step 3: Finding the inverse of C:

$\det C = 2 \times 8 - 5 \times 3 = 16 - 15 = 1$, so C is invertible.

Adjoint of C is obtained by swapping the diagonal entries and changing the sign of the off-diagonal entries.

$$C^{-1} = \frac{1}{\det C}\,\text{adj}(C) = \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$$

Step 4: Computing D:

$$D = C^{-1}(AB) = \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix} = \begin{bmatrix} 24-215 & 0-110 \\ -9+86 & 0+44 \end{bmatrix} = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$$

Final Answer:

This D satisfies CD=AB, so CD-AB=0.

$$\boxed{D = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}}$$

Quick Tip: From CD = AB, find D by multiplying the inverse of C with the product AB.

OR

Solve the following system of equations by matrix method: $\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4$, $\frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1$, $\frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

The equations are not linear in x, y, z because of the reciprocal terms. Substitute $u=\frac{1}{x}, v=\frac{1}{y}, w=\frac{1}{z}$ to convert the system into a linear system in u, v, w.

Step 2: Writing the linear system:

$$2u+3v+10w=4, \quad 4u-6v+5w=1, \quad 6u+9v-20w=2$$

In matrix form $MU=B$, where $M=\begin{bmatrix}2&3&10\\4&-6&5\\6&9&-20\end{bmatrix}$, $U=\begin{bmatrix}u\\v\\w\end{bmatrix}$, $B=\begin{bmatrix}4\\1\\2\end{bmatrix}$.

Step 3: Finding det(M):

Expand along the first row.

$$\det M=2[(-6)(-20)-5(9)]-3[4(-20)-5(6)]+10[4(9)-(-6)(6)]$$

$$=2(75)-3(-110)+10(72)=150+330+720=1200$$

Since $\det M\neq 0$, M is invertible and the system has a unique solution.

Step 4: Finding the adjoint and inverse:

Computing all cofactors gives:

$$\text{adj}(M) = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$M^{-1} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

Step 5: Solving for u, v, w:

$$U = M^{-1}B \implies u = \frac{75(4) + 150(1) + 75(2)}{1200} = \frac{600}{1200} = \frac{1}{2}$$

$$v = \frac{110(4) - 100(1) + 30(2)}{1200} = \frac{400}{1200} = \frac{1}{3}, \quad w = \frac{72(4) + 0(1) - 24(2)}{1200} = \frac{240}{1200} = \frac{1}{5}$$

Step 6: Converting back to x, y, z:

Since $u = \frac{1}{x}, v = \frac{1}{y}, w = \frac{1}{z}$:

$$x = \frac{1}{u} = 2, \quad y = \frac{1}{v} = 3, \quad z = \frac{1}{w} = 5$$

Final Answer:

These values satisfy all three original equations.

$$\boxed{x = 2,\ y = 3,\ z = 5}$$

Quick Tip: Substitute u = 1/x, v = 1/y, w = 1/z to make the system linear, then use the matrix inverse method.

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8. Prove that the function $y = Ae^{3x}\cos 4x + Be^{3x}\sin 4x$, where A, B are arbitrary constants, is the solution of the differential equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

The function $y = Ae^{3x}\cos 4x + Be^{3x}\sin 4x$ is given.

To prove it is a solution, differentiate y twice and put the values of y, dy/dx and d^2y/dx^2 into the left side of the equation.

If the left side reduces to zero, the function is a solution.

Step 2: Find dy/dx:

Differentiate each term using the product rule.

$$\frac{dy}{dx} = e^{3x}\left[(3A + 4B)\cos 4x + (3B - 4A)\sin 4x\right]$$

Step 3: Find d^2y/dx^2:

Differentiate dy/dx again using the product rule on the new expression.

$$\frac{d^2y}{dx^2} = e^{3x}\left[(-7A + 24B)\cos 4x + (-24A - 7B)\sin 4x\right]$$

Step 4: Substitute into the equation:

Put d^2y/dx^2, dy/dx and y into $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y$.

Collect the coefficient of $\cos 4x$: $(-7A + 24B) - 6(3A + 4B) + 25A$

$= (-7 - 18 + 25)A + (24 - 24)B = 0.$

Collect the coefficient of $\sin 4x$: $(-24A - 7B) - 6(3B - 4A) + 25B$

$= (-24 + 24)A + (-7 - 18 + 25)B = 0.$

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0$$

Final Answer:

The left side becomes zero for all A and B, so y satisfies the equation.

$$\boxed{y = Ae^{3x}\cos 4x + Be^{3x}\sin 4x \text{ is a solution of the equation}}$$

Quick Tip: Differentiate y twice and substitute into the equation; check the coefficients of cos4x and sin4x cancel to zero.

OR

Find $\int\left[\sqrt{\tan x} + \sqrt{\cot x}\right]dx.$

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

Write $\sqrt{\tan x}$ and $\sqrt{\cot x}$ in terms of sine and cosine so the two terms combine into one fraction.

This makes it possible to use a single substitution for the whole integral.

Step 2: Combine the terms:

$$\sqrt{\tan x} + \sqrt{\cot x} = \sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}} = \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}$$

Since $\sin x \cos x = \frac{\sin 2x}{2}$, rewrite the denominator.

$$\sqrt{\tan x} + \sqrt{\cot x} = \frac{\sqrt{2}(\sin x + \cos x)}{\sqrt{\sin 2x}}$$

Step 3: Choose the substitution:

Let $t = \sin x - \cos x$, so $dt = (\cos x + \sin x)dx$.

Also $t^2 = 1 - \sin 2x$, so $\sin 2x = 1 - t^2$.

The integral becomes:

$$\int \sqrt{2} \frac{dt}{\sqrt{1 - t^2}}$$

Step 4: Integrate:

Use the standard result $\int \frac{dt}{\sqrt{1 - t^2}} = \sin^{-1} t + C$.

$$\int \left[\sqrt{\tan x} + \sqrt{\cot x}\right] dx = \sqrt{2} \sin^{-1} t + C$$

Final Answer:

Substitute back $t = \sin x - \cos x$.

$$\boxed{\int \left[\sqrt{\tan x} + \sqrt{\cot x}\right] dx = \sqrt{2}\sin^{-1}(\sin x - \cos x) + C}$$

Quick Tip: Write both terms over a common denominator using sinx and cosx, then substitute $t = \sin x - \cos x$.

9. Find the particular solution of the differential equation $\frac{dy}{dx} + y\cot x = 4x\csc x$, $(x \neq 0)$, given that $y = 0$ when $x = \frac{\pi}{2}$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

The equation $\frac{dy}{dx} + y\cot x = 4x\csc x$ is a linear differential equation in y, of the form $\frac{dy}{dx} + Py = Q$ with $P = \cot x$ and $Q = 4x\csc x$.

Such equations are solved using an integrating factor.

Step 2: Find the integrating factor:

$$I.F. = e^{\int \cot x\, dx} = e^{\ln|\sin x|} = \sin x$$

Step 3: Write and integrate the solution:

Multiply both sides by $\sin x$; the left side becomes the derivative of $y \sin x$.

$$\frac{d}{dx}(y \sin x) = 4x \csc x \cdot \sin x = 4x$$

Integrate both sides with respect to x.

$$y \sin x = \int 4x \, dx = 2x^2 + C$$

Step 4: Apply the initial condition:

Put $x = \frac{\pi}{2}, y = 0$.

$$0 \cdot \sin \frac{\pi}{2} = 2\left(\frac{\pi}{2}\right)^2 + C \implies 0 = \frac{\pi^2}{2} + C \implies C = -\frac{\pi^2}{2}$$

Final Answer:

Substitute C back to get the particular solution.

$$\boxed{y = \frac{4x^2 - \pi^2}{2 \sin x}}$$

Quick Tip: This is linear in y. Use integrating factor $\sin x$, simplify $4x \csc x \cdot \sin x$, then apply the given condition.

OR

Find $\int \frac{x \sin^{-1} x}{\sqrt{1 - x^2}} dx$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

The integral $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$ has a product of $\sin^{-1}x$ and a function of x, so integration by parts is used.

Choose $u = \sin^{-1}x$ (easy to differentiate) and $dv = \frac{x}{\sqrt{1-x^2}}dx$ (easy to integrate).

Step 2: Find v and du:

For dv, let $w = 1 - x^2$, $dw = -2x\,dx$.

$$v = \int \frac{x}{\sqrt{1-x^2}}dx = -\sqrt{1-x^2}$$

$$du = \frac{1}{\sqrt{1-x^2}}dx$$

Step 3: Apply integration by parts:

Use $\int u\,dv = uv - \int v\,du$.

$$\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx = -\sqrt{1-x^2}\sin^{-1}x - \int \left(-\sqrt{1-x^2}\right)\frac{1}{\sqrt{1-x^2}}dx$$

$$= -\sqrt{1-x^2}\sin^{-1}x + \int 1\,dx$$

Step 4: Finish the integration:

$$\int 1\,dx = x$$

Final Answer:

Combine both parts to get the antiderivative.

$$\boxed{\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx = x - \sqrt{1-x^2}\,\sin^{-1}x + C}$$

Quick Tip: Use integration by parts with $u = \sin^{-1}x$ and $dv = \dfrac{x}{\sqrt{1-x^2}}dx$, so $v = -\sqrt{1-x^2}$.

Find $\displaystyle\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

Many integrals of the form $\int e^x[f(x)+f'(x)]dx$ simplify directly to $e^x f(x) + C$.

The plan is to rewrite $\dfrac{1+\sin x}{1+\cos x}$ in the form $f(x) + f'(x)$ for some function f.

Step 2: Use half-angle identities:

Use $1+\cos x = 2\cos^2(x/2)$ and $\sin x = 2\sin(x/2)\cos(x/2)$.

$$\frac{1+\sin x}{1+\cos x} = \frac{1+2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)} = \frac{1}{2\cos^2(x/2)} + \frac{\sin(x/2)}{\cos(x/2)}$$

$$= \frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}$$

Step 3: Identify f(x) and f'(x):

Let $f(x) = \tan\frac{x}{2}$.

Then $f'(x) = \frac{1}{2}\sec^2\frac{x}{2}$, which is exactly the other term found above.

So the integrand is $e^x[f(x) + f'(x)]$ with $f(x) = \tan(x/2)$.

Step 4: Apply the standard formula:

$$\int e^x[f(x) + f'(x)]dx = e^x f(x) + C$$

Final Answer:

Substitute $f(x) = \tan(x/2)$ into the formula.

$$\boxed{\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx = e^x\tan\frac{x}{2} + C}$$

Quick Tip: Use $1+\cos x = 2\cos^2(x/2)$ and $\sin x = 2\sin(x/2)\cos(x/2)$, then apply $\int e^x[f(x)+f'(x)]dx = e^x f(x) + C$.