UP Board Class 12 Mathematics 2026 (Set 324 DB) Question Paper with Solutions

General Instructions

(i) This booklet contains 9 questions, each provided with a complete, step-by-step solution.

(ii) It comprises 5 single-correct multiple-choice questions.

(iii) Attempt each question on your own before reviewing the given solution.

1(a). If $A = \{1, 2, 3\}$, then the number of equivalence relations on A containing $(1, 2)$ is:

(A) 1

(B) 2

(C) 3

(D) 4

Correct Answer: (B) 2

Solution:

Step 1: Understanding the Concept:
An equivalence relation on a set corresponds exactly to a partition of that set into disjoint blocks (equivalence classes). We need those partitions of $\{1, 2, 3\}$ in which 1 and 2 land in the same block.

Step 2: Listing the partitions of a 3-element set:
The partitions of $\{1, 2, 3\}$ are: $\{1\}\{2\}\{3\}$, $\{1, 2\}\{3\}$, $\{1, 3\}\{2\}$, $\{2, 3\}\{1\}$, and $\{1, 2, 3\}$.

Step 3: Keeping only those with 1 and 2 together:

Only $\{1,2\}\{3\}$ and $\{1,2,3\}$ place 1 and 2 in the same block, so their relations contain the pair $(1,2)$.

Final Answer:

There are exactly $\boxed{2}$ such equivalence relations.

Quick Tip: Count partitions of $\{1,2,3\}$ where 1 and 2 are in the same block.

1(b). The function $f : Z \to Z$ defined by $f(x) = x^3 \; \forall x \in Z$ is:

(A) Onto

(B) Neither one-one nor onto

(C) One-one but not onto

(D) Many-one

Correct Answer: (C) One-one but not onto

Solution:

Step 1: Checking one-one (injectivity):

If $f(x_1) = f(x_2)$ then $x_1^3 = x_2^3$, and since the real cube root is unique, $x_1 = x_2$. So f is one-one.

Step 2: Checking onto (surjectivity):

For f to be onto Z, every integer must be a perfect cube. But $2 \in Z$ has no integer x with $x^3 = 2$. So f is not onto.

Final Answer:

f is $\boxed{\text{one-one but not onto}}$.

Quick Tip: Test injectivity via strict monotonicity, then check whether every integer is a cube.

1(c). $\sin(\tan^{-1} x)$, $|x| < 1$ is equal to:

(A) $\dfrac{x}{\sqrt{1-x^2}}$

(B) $\dfrac{1}{\sqrt{1-x^2}}$

(C) $\dfrac{1}{\sqrt{1+x^2}}$

(D) $\dfrac{x}{\sqrt{1+x^2}}$

Correct Answer: (D) $\dfrac{x}{\sqrt{1+x^2}}$

Solution:

Step 1: Setting up a right triangle:

Let $\theta = \tan^{-1} x$, so $\tan\theta = x = \dfrac{\text{opposite}}{\text{adjacent}}$. Take opposite $= x$, adjacent $= 1$.

Step 2: Finding the hypotenuse:

Hypotenuse $= \sqrt{x^2+1}$ by Pythagoras.

Step 3: Reading off sine:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{\sqrt{1+x^2}}.$$

Final Answer:

$\sin(\tan^{-1} x) = \boxed{\dfrac{x}{\sqrt{1+x^2}}}$.

Quick Tip: Draw the reference right triangle for $\theta = \tan^{-1} x$ and read off $\sin\theta$.

1(d). $A = [a_{ij}]_{m\times n}$ is a square matrix if:

(A) m < n

(B) m > n

(C) m = n

(D) None of these

Correct Answer: (C) m = n

Solution:

Step 1: Understanding the Concept:

A matrix with m rows and n columns is called \emph{square} exactly when the number of rows equals the number of columns.

Final Answer:

So the condition is $\boxed{m = n}$.

Quick Tip: Recall the basic definition of a square matrix.

1(e). The value of $\int x \cos x \, dx$ is:

(A) $\cos x + x \sin x + c$

(B) $x \sin x + c$

(C) $x \cos x + c$

(D) $\sin x + x \cos x + c$

Correct Answer: (A) $\cos x + x \sin x + c$

Solution:

Step 1: Understanding the Concept:

Use integration by parts, $\int u\, dv = uv - \int v\, du$, with $u = x$ (algebraic, differentiate) and $dv = \cos x\, dx$ (trigonometric, integrate) — ILATE order.

Step 2: Applying integration by parts:

$du = dx$, $v = \sin x$. So $\int x \cos x\, dx = x \sin x - \int \sin x\, dx$.

Step 3: Finishing the remaining integral:

$\int \sin x\, dx = -\cos x$. So the result is $x \sin x - (-\cos x) + c = x \sin x + \cos x + c$.

Final Answer:

$$\int x \cos x\, dx = \boxed{\cos x + x \sin x + c}.$$

Quick Tip: Integrate by parts with $u = x$, $dv = \cos x\, dx$.

2(a). Find the value of $\int_{-1}^{+1} (x + 1)\, dx$.

Correct Answer: 2

Solution:

Step 1: Understanding the Concept:

Integrate $x + 1$ term by term, then apply the limits -1 to 1.

Step 2: Finding the antiderivative:

$$\int (x+1)\,dx = \frac{x^2}{2} + x.$$

Step 3: Applying the limits:

At $x = 1$: $\frac{1}{2} + 1 = \frac{3}{2}$. At $x = -1$: $\frac{1}{2} - 1 = -\frac{1}{2}$.

Step 4: Subtracting:

$$\frac{3}{2} - \left(-\frac{1}{2}\right) = 2.$$

Final Answer:

$$\int_{-1}^{1} (x+1)\,dx = \boxed{2}.$$

Quick Tip: Antiderivative is $x^2/2 + x$; or split into an odd part (vanishes) and a constant part.

2(b). Find the integrating factor of the differential equation $x\frac{dy}{dx} + 2y = x^2$ $(x \neq 0)$.

Correct Answer: 3

Solution:

Step 1: Understanding the Concept:

Write the equation in the standard linear form $\frac{dy}{dx} + Py = Q$ so P can be read off.

Step 2: Standard form:

Dividing by x: $\frac{dy}{dx} + \frac{2}{x}y = x$, so $P = \frac{2}{x}$.

Step 3: Computing the integrating factor:

I.F. $= e^{\int P\,dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$.

Final Answer:

Integrating factor $= \boxed{x^2}$.

Quick Tip: Convert to standard linear form $y' + Py = Q$, then compute $e^{\int P\,dx}$.

2(c). Find the projection of vector $(\hat{i} - \hat{j})$ on vector $(\hat{i} + \hat{j})$.

Correct Answer: 0

Solution:

Step 1: Understanding the Concept:

The projection of vector $\vec{a}$ on vector $\vec{b}$ is $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

Step 2: Computing the dot product:

$\vec{a} = \hat{i} - \hat{j}, \vec{b} = \hat{i} + \hat{j}. \vec{a}\cdot\vec{b} = (1)(1) + (-1)(1) = 1 - 1 = 0.$

Step 3: Computing the projection:

Since the numerator is 0, the projection is $\frac{0}{|\vec{b}|} = 0$, regardless of $|\vec{b}|$.

Final Answer:

Projection $= \boxed{0}$.

Quick Tip: Projection formula $= \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$; here the dot product is zero.

2(d). If $P(A) = \frac{1}{2}, P(B) = \frac{1}{3}$ and A, B are independent events, find the value of $P(A/B)$.

Correct Answer: 1/2

Solution:

Step 1: Understanding the Concept:

For independent events, knowing that B has occurred gives no extra information about A; formally $P(A \cap B) = P(A) \cdot P(B)$ for independent events.

Step 2: Applying the conditional probability formula:

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \cdot P(B)}{P(B)} = P(A).$$

Final Answer:

$P(A/B) = \boxed{\frac{1}{2}}$.

Quick Tip: For independent events, $P(A/B) = P(A)$.

2(e). If $X = \{1, 2, 3\}$ and $Y = \{a, b\}$, then find the number of functions from X to Y.

Correct Answer: 8

Solution:

Step 1: Understanding the Concept:
A function from X to Y assigns to each element of X exactly one image in Y, independently of the other assignments.

Step 2: Counting the choices:
Each of the 3 elements of X can independently map to either of the 2 elements of Y, giving $2 \times 2 \times 2$ total assignments.

Final Answer:
Number of functions $= 2^3 = \boxed{8}$.

Quick Tip: Number of functions from an m-set to an n-set is n^m.

3(a). Write $\tan^{-1}\left(\frac{x}{\sqrt{a^2 - x^2}}\right)$, $|x| < a$ in simplest form.

Correct Answer: sin^{-1}(x/a)

Solution:

Step 1: Understanding the Concept:
The presence of $\sqrt{a^2 - x^2}$ suggests the substitution $x = a \sin\theta$, which is the standard trick that clears this kind of square root.

Step 2: Substituting:

Let $x = a\sin\theta$, so $\theta = \sin^{-1}(x/a)$, and $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta}$
$= a\cos\theta$ (since $|x| < a$ keeps $\cos\theta > 0$).

Step 3: Simplifying the expression:

$$\frac{x}{\sqrt{a^2 - x^2}} = \frac{a\sin\theta}{a\cos\theta} = \tan\theta, \text{ so } \tan^{-1}\left(\frac{x}{\sqrt{a^2 - x^2}}\right) = \tan^{-1}(\tan\theta) = \theta.$$

Final Answer:

Simplest form $= \boxed{\sin^{-1}\left(\frac{x}{a}\right)}$.

Quick Tip: Substitute $x = a\sin\theta$ to clear the square root, then simplify $\tan^{-1}(\tan\theta)$.

3(b). If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$, then find the value of $(2A - B)$.

Correct Answer: [[-1,5,3],[5,6,0]]

Solution:

Step 1: Understanding the Concept:

Scale A by 2 entrywise, then subtract B entrywise — both matrices are the same 2×3 shape, so this is well-defined.

Step 2: Computing 2A:

$$2A = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \end{bmatrix}.$$

Step 3: Subtracting B entrywise:

$$2A - B = \begin{bmatrix} 2-3 & 4-(-1) & 6-3 \\ 4-(-1) & 6-0 & 2-2 \end{bmatrix} = \begin{bmatrix} -1 & 5 & 3 \\ 5 & 6 & 0 \end{bmatrix}.$$

Final Answer:

$$2A - B = \boxed{\begin{bmatrix} -1 & 5 & 3 \\ 5 & 6 & 0 \end{bmatrix}}.$$

Quick Tip: Compute $2A$ entrywise, then subtract B entrywise.

3(c). Prove that $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x),\ x \in \left[\frac{1}{2}, 1\right]$.

Correct Answer: proof

Solution:

Step 1: Understanding the Concept:

The expression $4x^3 - 3x$ is exactly the triple-angle formula $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ in disguise; substituting $x = \cos\theta$ should make the identity fall out.

Step 2: Substituting x = cos(theta):

Let $x = \cos\theta$, so $\theta = \cos^{-1} x$. Since $x \in [\frac{1}{2}, 1]$, $\theta \in [0, \frac{\pi}{3}]$.

Step 3: Applying the triple angle formula:

$4x^3 - 3x = 4\cos^3\theta - 3\cos\theta = \cos 3\theta$.

Step 4: Checking the range for the inverse cosine to apply directly:

Since $\theta \in [0, \pi/3]$, we get $3\theta \in [0, \pi]$, which is exactly the principal range of $\cos^{-1}$, so $\cos^{-1}(\cos 3\theta) = 3\theta$ without any adjustment.

Final Answer:

Hence $\cos^{-1}(4x^3 - 3x) = 3\theta = 3\cos^{-1} x$, i.e.
$\boxed{3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)}$.

Quick Tip: Substitute $x = \cos\theta$ and use $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$; check 3θ stays within $[0, \pi]$.

3(d). Differentiate the function $\sin^2 x$ with respect to the function $e^{\cos x}$.

Correct Answer: -2 cos(x) e^{-cos x}

Solution:

Step 1: Understanding the Concept:

\"Differentiate u w.r.t. v\" means find $\frac{du}{dv} = \frac{du/dx}{dv/dx}$, computing each ordinary derivative w.r.t. x first.

Step 2: Differentiating u = sin^2(x):

Let $u = \sin^2 x$. $\frac{du}{dx} = 2\sin x \cos x = \sin 2x$.

Step 3: Differentiating v = e^{cos x}:

Let $v = e^{\cos x}$. $\frac{dv}{dx} = e^{\cos x} \cdot (-\sin x) = -\sin x\, e^{\cos x}$.

Step 4: Dividing:

$\frac{du}{dv} = \frac{2\sin x \cos x}{-\sin x\, e^{\cos x}} = \frac{-2\cos x}{e^{\cos x}}$ (cancel $\sin x$, valid where $\sin x \neq 0$).

Final Answer:

$\frac{du}{dv} = \boxed{-2\cos x\, e^{-\cos x}}$.

Quick Tip: Use $\frac{du}{dv} = \frac{du/dx}{dv/dx}$, or rewrite u directly in terms of $t = \cos x$.

4(a). Prove that the given function $f(x) = 3x + 17$ is increasing on $\mathbb{R}$.

Correct Answer: proof

Solution:

Step 1: Understanding the Concept:

A differentiable function is (strictly) increasing on an interval if its derivative is positive throughout that interval.

Step 2: Differentiating:

$f'(x) = \frac{d}{dx}(3x + 17) = 3.$

Step 3: Checking the sign:

$f'(x) = 3 > 0$ for every real x — the derivative doesn't even depend on x, so it's positive everywhere without exception.

Final Answer:

Since $f'(x) > 0\ \forall x \in \mathbb{R}$, $\boxed{f(x) = 3x + 17 \text{ is strictly increasing on } \mathbb{R}}$.

Quick Tip: Show $f'(x) > 0$ everywhere, or directly compare $f(x_2) - f(x_1)$ for $x_2 > x_1$.

4(b). If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions such that $f(x) = \cos x$ and $g(x) = 3x^3$, then find $f \circ g$.

Correct Answer: cos(3x^3)

Solution:

Step 1: Understanding the Concept:

$(f \circ g)(x) = f(g(x))$ means: first apply g, then feed that output into f.

Step 2: Applying g first:

$g(x) = 3x^3$.

Step 3: Feeding into f:

$f(g(x)) = f(3x^3) = \cos(3x^3)$, since f just takes cosine of whatever is fed in.

Final Answer:

$(f \circ g)(x) = \boxed{\cos(3x^3)}$.

Quick Tip: $(f \circ g)(x) = f(g(x))$: replace the input of cos with $3x^3$.

4(c). Find the general solution of the differential equation $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}$.

Correct Answer: tan^{-1}y=tan^{-1}x+C

Solution:

Step 1: Understanding the Concept:

This equation is variable separable: all the y terms can be gathered on one side and all the x terms on the other.

Step 2: Separating the variables:

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2}.$$

Step 3: Integrating both sides:

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2} \Rightarrow \tan^{-1} y = \tan^{-1} x + C.$$

Final Answer:

General solution: $\boxed{\tan^{-1} y = \tan^{-1} x + C}$.

Quick Tip: Separate variables as $\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$ and integrate both sides.

4(d). Find the magnitude of the vectors $\vec{a}$ and $\vec{b}$, if having equal magnitudes and angle $60°$ between them and their scalar product is $\frac{1}{2}$.

Correct Answer: 1

Solution:

Step 1: Understanding the Concept:

The dot product formula $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$ connects the given scalar product, the (equal) magnitudes, and the angle.

Step 2: Setting up the equation:

Let $|\vec{a}| = |\vec{b}| = k$. Then $\vec{a} \cdot \vec{b} = k \cdot k \cdot \cos 60° = k^2 \cdot \frac{1}{2}$.

Step 3: Substituting the given value:

$k^2 \cdot \frac{1}{2} = \frac{1}{2} \Rightarrow k^2 = 1 \Rightarrow k = 1$ (taking the positive root, since magnitude can't be negative).

Final Answer:
$|\vec{a}| = |\vec{b}| = \boxed{1}$.

Quick Tip: Use $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$ with $|\vec{a}| = |\vec{b}|$.

5(a). Find the shortest distance between the lines $\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$ and $\vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu(2\hat{i} + 3\hat{j} + 6\hat{k})$.

Correct Answer: sqrt(293)/7

Solution:

Step 1: Understanding the Concept:
Both lines have the same direction vector $\vec{d} = (2, 3, 6)$, so the lines are parallel. For parallel lines, the shortest distance is $d = \dfrac{|(\vec{b}_2 - \vec{b}_1) \times \vec{d}|}{|\vec{d}|}$, where $\vec{b}_1, \vec{b}_2$ are points on each line.

Step 2: Computing the connecting vector:
$\vec{b}_1 = (1, 2, -4)$, $\vec{b}_2 = (3, 3, -5)$. $\vec{b}_2 - \vec{b}_1 = (2, 1, -1)$.

Step 3: Computing the cross product:
$(2, 1, -1) \times (2, 3, 6) = \big(1 \cdot 6 - (-1) \cdot 3,\ \ -[2 \cdot 6 - (-1) \cdot 2],\ 2 \cdot 3 - 1 \cdot 2\big)$
$= (9, -14, 4)$.

Step 4: Computing magnitudes:
$|(9, -14, 4)| = \sqrt{81 + 196 + 16} = \sqrt{293}$. $|\vec{d}| = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$.

Final Answer:

Shortest distance $= \dfrac{\sqrt{293}}{7} = \boxed{\dfrac{\sqrt{293}}{7}}$ units.

Quick Tip: Same direction vector $\Rightarrow$ parallel lines; use $d = \dfrac{|(\vec{b}_2 - \vec{b}_1) \times \vec{d}|}{|\vec{d}|}$.

5(b). Find the general solution of the differential equation $\log\left(\dfrac{dy}{dx}\right) = 3x + 4y$.

Correct Answer: 4e^{3x}+3e^{-4y}=C

Solution:

Step 1: Understanding the Concept:

First remove the logarithm to get dy/dx explicitly, then separate variables since the right side splits into a pure function of x times a pure function of y.

Step 2: Removing the log:

$$\frac{dy}{dx} = e^{3x+4y} = e^{3x} \cdot e^{4y}.$$

Step 3: Separating variables:

$$e^{-4y}\,dy = e^{3x}\,dx.$$

Step 4: Integrating both sides:

$$\int e^{-4y}\,dy = \int e^{3x}\,dx \Rightarrow -\frac{1}{4}e^{-4y} = \frac{1}{3}e^{3x} + C_1.$$

Step 5: Clearing fractions:

Multiply through by -12: $3e^{-4y} = -4e^{3x} - 12C_1$, i.e. $4e^{3x} + 3e^{-4y} = C$ (renaming the constant).

Final Answer:
General solution: $\boxed{4e^{3x} + 3e^{-4y} = C}$.

Quick Tip: Exponentiate to get $dy/dx = e^{3x}e^{4y}$, separate variables, then integrate.

5(c). Show that the line passing through the points $(1, -1, 2)$ and $(3, 4, -2)$ is perpendicular to the line passing through the points $(0, 3, 2)$ and $(3, 5, 6)$.

Correct Answer: proof

Solution:

Step 1: Understanding the Concept:
Two lines are perpendicular exactly when their direction vectors have a zero dot product.

Step 2: Finding the first direction vector:
For line 1: $\vec{d}_1 = (3 - 1,\ 4 - (-1),\ -2 - 2) = (2, 5, -4)$.

Step 3: Finding the second direction vector:
For line 2: $\vec{d}_2 = (3 - 0,\ 5 - 3,\ 6 - 2) = (3, 2, 4)$.

Step 4: Computing the dot product:
$\vec{d}_1 \cdot \vec{d}_2 = (2)(3) + (5)(2) + (-4)(4) = 6 + 10 - 16 = 0$.

Final Answer:
Since $\vec{d}_1 \cdot \vec{d}_2 = 0$, $\boxed{\text{the two lines are perpendicular}}$.

Quick Tip: Form direction vectors from each pair of points and check their dot product is zero.

5(d). If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A^2 = KA - 2I$, then find the value of K.

Correct Answer: 1

Solution:

Step 1: Understanding the Concept:
Compute A^2 directly, compute $KA - 2I$ symbolically in terms of K, then match corresponding entries to solve for K.

Step 2: Computing A squared:
$$A^2 = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 9-8 & -6+4 \\ 12-8 & -8+4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix}.$$

Step 3: Writing KA - 2I symbolically:
$$KA - 2I = \begin{bmatrix} 3K-2 & -2K \\ 4K & -2K-2 \end{bmatrix}.$$

Step 4: Matching entries:
Comparing (1,1): $3K - 2 = 1 \Rightarrow K = 1$. Comparing (2,1): $4K = 4 \Rightarrow K = 1$. Both (and the remaining two entries) agree.

Final Answer:

$K = \boxed{1}$.

Quick Tip: Compute A^2 and $KA - 2I$, then compare corresponding entries (or use Cayley-Hamilton: $K = \text{tr}(A)$).

5(e). Prove that the function defined by $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is continuous.

Correct Answer: proof

Solution:

Step 1: Understanding the Concept:

For $x \neq 0$, f is a product/composition of standard continuous functions (x^2 and $\sin(1/x)$), so continuity there is automatic. The only point that needs checking is $x = 0$, where the formula changes.

Step 2: Continuity for x ≠ 0:

For $x \neq 0$, $\sin(1/x)$ is continuous (composition of the continuous $1/x$ with sin), and x^2 is continuous, so their product $f(x) = x^2 \sin(1/x)$ is continuous there.

Step 3: Setting up the limit at x = 0:

We need $\lim_{x \to 0} f(x) = f(0) = 0$.

Step 4: Squeezing the limit:

Since $-1 \leq \sin(1/x) \leq 1$ for all $x \neq 0$, we get $-x^2 \leq x^2 \sin(1/x) \leq x^2$. As $x \to 0$, both $-x^2 \to 0$ and $x^2 \to 0$, so by the Squeeze Theorem,

$\lim_{x \to 0} x^2 \sin(1/x) = 0.$

Final Answer:

Since $\lim_{x \to 0} f(x) = 0 = f(0)$, $\boxed{f \text{ is continuous at } x = 0}$, and hence continuous everywhere.

Quick Tip: Away from 0, f is a product of continuous functions; at 0, squeeze $x^2 \sin(1/x)$ between $-x^2$ and x^2.

6(a). A circular disc having radius 3 cm is warmed and due to expansion its radius is increasing at the rate 0.05 cm/s. Find the increasing rate of its area when its radius is 3.2 cm.

Correct Answer: 0.32 pi cm^2/s

Solution:

Step 1: Understanding the Concept:

This is a related-rates problem: relate area A to radius r via $A = \pi r^2$, then differentiate both sides with respect to time t.

Step 2: Setting up the relation:

$A = \pi r^2.$

Step 3: Differentiating w.r.t. time:

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$ (chain rule, since r itself depends on t).

Step 4: Substituting the given values:

At the instant in question, $r = 3.2$ cm and $\frac{dr}{dt} = 0.05$ cm/s. So $\frac{dA}{dt}$

$= 2\pi(3.2)(0.05) = 0.32\pi$.

Final Answer:
The area is increasing at $\boxed{0.32\pi \text{ cm}^2/\text{s}}$ $(\approx 1.005 \text{ cm}^2/\text{s})$.

Quick Tip: Differentiate $A = \pi r^2$ w.r.t. time: $dA/dt = 2\pi r\,(dr/dt)$.

6(b). Find the minimum value of the linear programming problem $Z = 200x + 500y$ under the following constraints: $x + 2y \geq 10$, $3x + 4y \leq 24$, $x \geq 0, y \geq 0$ by graphical method.

Correct Answer: 2300 at (4,3)

Solution:

Step 1: Understanding the Concept:
Graph the two boundary lines, shade the feasible region satisfying all four constraints, identify its corner points, then evaluate Z at each corner — the minimum of a linear objective over a convex feasible region always occurs at a corner point.

Step 2: Finding where the constraint lines meet the axes:
$x + 2y = 10$ meets the axes at $(10, 0)$ and $(0, 5)$. $3x + 4y = 24$ meets the axes at $(8, 0)$ and $(0, 6)$.

Step 3: Finding the intersection of the two lines:
Solve $x + 2y = 10$ and $3x + 4y = 24$ together: from the first, $x = 10 - 2y$; substituting, $3(10 - 2y) + 4y = 24 \Rightarrow 30 - 2y = 24 \Rightarrow y = 3,\ x = 4$. So they meet at $(4, 3)$.

Step 4: Identifying the feasible corner points:

Checking which side of each line is feasible ($x + 2y \geq 10$ is \"above\" its line, $3x + 4y \leq 24$ is \"below\" its line) shows the bounded feasible region is the triangle with vertices $(0, 5)$, $(0, 6)$, and $(4, 3)$.

Step 5: Evaluating Z at each corner:

At $(0, 5)$: $Z = 200(0) + 500(5) = 2500$. At $(0, 6)$: $Z = 200(0) + 500(6) = 3000$. At $(4, 3)$: $Z = 200(4) + 500(3) = 800 + 1500 = 2300$.

Final Answer:

The minimum value is $\boxed{Z = 2300}$, attained at $(x, y) = (4, 3)$.

Quick Tip: Plot the constraint lines, find the feasible region's corners, and evaluate Z at each corner.

6(c). A die is thrown twice and found that the sum of the numbers is 6. Find the conditional probability of getting number 4 at least once.

Correct Answer: 2/5

Solution:

Step 1: Understanding the Concept:

Let E=\"at least one 4 appears\" and F=\"the sum of the two throws is 6\". We need $P(E/F)$, which restricts attention only to outcomes where the sum is 6.

Step 2: Listing outcomes with sum 6:

The ordered pairs $(\text{die1}, \text{die2})$ summing to 6 are:

$(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)$ — 5 outcomes in total, forming the event

F.

Step 3: Picking out those with at least one 4:

Among these, the pairs containing a 4 are $(2, 4)$ and $(4, 2)$ — 2 outcomes.

Step 4: Computing the conditional probability:

Since all outcomes of a fair die are equally likely, $P(E/F) = \dfrac{n(E \cap F)}{n(F)} = \dfrac{2}{5}$.

Final Answer:

$P(\text{at least one 4} \mid \text{sum} = 6) = \boxed{\dfrac{2}{5}}$.

Quick Tip: Restrict the sample space to the 5 outcomes summing to 6, then count how many contain a 4.

6(d). If vectors $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular and having equal magnitude, then show that the vector $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to the vectors $\vec{a}, \vec{b}, \vec{c}$.

Correct Answer: proof

Solution:

Step 1: Understanding the Concept:

\"Equally inclined\" means the angle between $\vec{s} = \vec{a} + \vec{b} + \vec{c}$ and each of $\vec{a}, \vec{b}, \vec{c}$ is the same; since the cosine of the angle is $\dfrac{\vec{s} \cdot (\cdot)}{|\vec{s}|| \cdot |}$ and all three vectors share the same magnitude, it suffices to show $\vec{s} \cdot \vec{a} = \vec{s}$

$\cdot \vec{b} = \vec{s} \cdot \vec{c}$.

Step 2: Setting up notation:

Let $|\vec{a}| = |\vec{b}| = |\vec{c}| = k$, with $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$ (mutual perpendicularity).

Step 3: Computing s dot a:

$\vec{s} \cdot \vec{a} = (\vec{a} + \vec{b} + \vec{c}) \cdot \vec{a} = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a} = k^2 + 0 + 0 = k^2$.

Step 4: Computing s dot b and s dot c similarly:

By the same reasoning (only the matching term survives, the other two dot products vanish by perpendicularity), $\vec{s} \cdot \vec{b} = k^2$ and $\vec{s} \cdot \vec{c} = k^2$.

Step 5: Comparing the angles:

$\cos\theta_a = \dfrac{\vec{s} \cdot \vec{a}}{|\vec{s}|\, k} = \dfrac{k^2}{|\vec{s}|k} = \dfrac{k}{|\vec{s}|}$, and identically $\cos\theta_b = \cos\theta_c = \dfrac{k}{|\vec{s}|}$.

Final Answer:

Since $\cos\theta_a = \cos\theta_b = \cos\theta_c$, $\boxed{\vec{a} + \vec{b} + \vec{c} \text{ is equally inclined to } \vec{a}, \vec{b}, \vec{c}}$.

Quick Tip: Use $\vec{s} \cdot \vec{a} = \vec{s} \cdot \vec{b} = \vec{s} \cdot \vec{c}$ (only the matching term survives by perpendicularity) together with equal magnitudes.

6(e). Find the area of the circle $x^2 + y^2 = r^2$.

Correct Answer: pi r^2

Solution:

Step 1: Understanding the Concept:

By symmetry, the full circle's area is 4 times the area of the part in the first quadrant, which can be found by integrating $y = \sqrt{r^2 - x^2}$ from $x = 0$ to $x = r$.

Step 2: Setting up the integral:

$$\text{Area} = 4\int_0^r \sqrt{r^2 - x^2}\,dx.$$

Step 3: Using the standard formula for this integral:

$$\int \sqrt{r^2 - x^2}\,dx = \frac{x}{2}\sqrt{r^2 - x^2} + \frac{r^2}{2}\sin^{-1}\left(\frac{x}{r}\right) + C.$$

Step 4: Applying the limits 0 to r:

At $x = r$: $\frac{r}{2}(0) + \frac{r^2}{2}\sin^{-1}(1) = \frac{r^2}{2} \cdot \frac{\pi}{2} = \frac{\pi r^2}{4}$. At $x = 0$: both terms are 0.

Step 5: Multiplying by 4:

$$\text{Area} = 4 \times \frac{\pi r^2}{4} = \pi r^2.$$

Final Answer:

Area of the circle $= \boxed{\pi r^2}$.

Quick Tip: Integrate $4\int_0^r \sqrt{r^2 - x^2}\,dx$ using the standard formula, or sum concentric rings $2\pi\rho\,d\rho$.

7. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, then find A^{-1}.

Correct Answer: see solution

Solution:

Step 1: Understanding the Concept:

$A^{-1} = \dfrac{1}{\det A}\,\mathrm{adj}(A)$, so first find $\det A$ (must be nonzero), then all nine cofactors to build the adjugate.

Step 2: Computing the determinant:

Expanding along row 1: $\det A = 2\big(2(-2) - (-4)(1)\big) - (-3)\big(3(-2) - (-4)(1)\big) + 5\big(3(1) - 2(1)\big)$

$$= 2(-4+4) + 3(-6+4) + 5(3-2) = 0 - 6 + 5 = -1.$$

Step 3: Computing the cofactors:

$C_{11} = 0,\ C_{12} = 2,\ C_{13} = 1;\ C_{21} = -1,\ C_{22} = -9,\ C_{23} = -5;\ C_{31} = 2,\ C_{32} = 23,\ C_{33} = 13$ (each C_{ij} is $(-1)^{i+j}$ times the 2×2 minor deleting row i, column j).

Step 4: Building the adjugate (transpose of cofactor matrix):

$$\mathrm{adj}(A) = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}.$$

Step 5: Dividing by the determinant:

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}.$$

Final Answer:

$$A^{-1} = \boxed{\begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}}.$$

Quick Tip: Use $A^{-1} = \dfrac{1}{\det A}\text{adj}(A)$ via cofactors; verify by checking $AA^{-1} = I$.

OR

Solve the system of linear equations by Matrix Method: $2x + 3y + 3z = 5$, $x - 2y + z = -4$, $3x - y - 2z = 3$.

Correct Answer: x=1, y=2, z=-1

Solution:

Step 1: Understanding the Concept:

Write the system as $AX = B$ and solve $X = A^{-1}B$, where A is the coefficient matrix, $X = (x, y, z)^T$, and B is the constants column.

Step 2: Setting up A, X, B:

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}.$$

Step 3: Computing det(A):

Expanding along row 1: $\det A = 2\big((-2)(-2) - 1(-1)\big) - 3\big(1(-2) - 1(3)\big) + 3\big(1(-1) - (-2)(3)\big)$

$$= 2(4 + 1) - 3(-2 - 3) + 3(-1 + 6) = 10 + 15 + 15 = 40.$$

(nonzero, so A^{-1} exists and the system has a unique solution)

Step 4: Computing the cofactors and A inverse:

Working out all cofactors gives $\operatorname{adj}(A) = \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$, so A^{-1}

$$= \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}.$$

Step 5: Computing X = A^{-1}B:

$$A^{-1}B = \frac{1}{40}\begin{bmatrix} 5(5)+3(-4)+9(3) \\ 5(5)+(-13)(-4)+1(3) \\ 5(5)+11(-4)+(-7)(3) \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 25-12+27 \\ 25+52+3 \\ 25-44-21 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix}.$$

Final Answer:

$$X = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \text{ so } \boxed{x = 1,\ y = 2,\ z = -1}.$$

Quick Tip: Form $AX = B$, compute A^{-1} via the adjugate, and evaluate $X = A^{-1}B$.

8. Prove that the radius of the largest (maximum surface area) right circular cylinder that can be inscribed in a cone is half of the radius of the cone.

Correct Answer: proof

Solution:

Step 1: Understanding the Concept:

Let the cone have fixed base radius R and height H. An inscribed cylinder of radius r and height h has its top rim touching the slanted surface, so r and h are linked by similar triangles; express the cylinder's curved surface area in terms of r alone and maximize.

Step 2: Relating h and r via similar triangles:

Looking at the cone's axial cross-section, the cylinder's top edge lies on the cone's slant, giving $\frac{h}{H} = \frac{R-r}{R}$, so $h = \frac{H(R-r)}{R}$.

Step 3: Writing the curved surface area as a function of r:

$S = 2\pi rh = 2\pi r \cdot \frac{H(R-r)}{R} = \frac{2\pi H}{R}(rR - r^2)$.

Step 4: Differentiating and setting to zero:

$\frac{dS}{dr} = \frac{2\pi H}{R}(R - 2r)$. Setting $\frac{dS}{dr} = 0$ gives $R - 2r = 0$, i.e. $r = \frac{R}{2}$.

Step 5: Confirming it's a maximum:

$\frac{d^2S}{dr^2} = \frac{2\pi H}{R}(-2) < 0$, a negative second derivative confirms $r = R/2$ gives a maximum (not a minimum) of S.

Final Answer:

The surface area is maximized when $r = \frac{R}{2}$, i.e. $\boxed{\text{the cylinder's radius is half the cone's radius}}$.

Quick Tip: Express $S = 2\pi rh$ with h from similar triangles, then maximize $S(r)$ via $dS/dr = 0$ and the second-derivative test.

OR

Evaluate: $\int_0^{\pi/2} \log \sin x \, dx$.

Correct Answer: -(pi/2) log 2

Solution:

Step 1: Understanding the Concept:

Use the King's-rule property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, which for this integral swaps $\sin x$ with $\cos x$; combining the two forms and using the double-angle identity for $\sin 2x$ collapses the integral onto itself, allowing us to solve for it algebraically.

Step 2: Applying x -> pi/2 - x:

Let $I = \int_0^{\pi/2} \log \sin x \, dx$. Using $x \to \frac{\pi}{2} - x$: I

$$= \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx = \int_0^{\pi/2} \log \cos x \, dx.$$

Step 3: Adding the two forms:

$$2I = \int_0^{\pi/2} (\log \sin x + \log \cos x) \, dx = \int_0^{\pi/2} \log(\sin x \cos x) \, dx.$$

Step 4: Using the double angle identity:

$\sin x \cos x = \dfrac{\sin 2x}{2}$, so $2I = \int_0^{\pi/2} \log\left(\dfrac{\sin 2x}{2}\right) dx = \int_0^{\pi/2} \log \sin 2x \, dx$

$- \int_0^{\pi/2} \log 2 \, dx.$

Step 5: Evaluating each piece:

For the first piece, substitute $t = 2x$, $dt = 2dx$: $\int_0^{\pi/2} \log \sin 2x \, dx$

$= \dfrac{1}{2}\int_0^{\pi} \log \sin t \, dt = \dfrac{1}{2} \cdot 2 \int_0^{\pi/2} \log \sin t \, dt = I$ (using the symmetry of $\sin t$ about $t = \pi/2$ over $[0, \pi]$). The second piece is $\dfrac{\pi}{2}\log 2$.

Step 6: Solving for I:
So $2I = I - \frac{\pi}{2}\log 2 \Rightarrow I = -\frac{\pi}{2}\log 2.$

Final Answer:
$$\int_0^{\pi/2} \log \sin x \, dx = \boxed{-\frac{\pi}{2}\log 2}.$$

Quick Tip: Use $I = \int_0^{\pi/2} \log\cos x \, dx$ (by $x \to \pi/2 - x$), add to get $2I$ via $\log(\sin x \cos x)$, and resolve the $\log \sin 2x$ piece by substitution.

9. Find the value of $\int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x}$.

Correct Answer: pi^2/(2ab)

Solution:

Step 1: Understanding the Concept:
Use the King's-rule property $x \to \pi - x$ (since $\cos^2$ and $\sin^2$ are unchanged by $x \to \pi - x$) to remove the awkward factor of x from the numerator, reducing the problem to a purely trigonometric integral.

Step 2: Applying x -> pi - x:
Let $I = \int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x}$. Since $\cos^2(\pi - x) = \cos^2 x$ and $\sin^2(\pi - x) = \sin^2 x$: $I = \int_0^{\pi} \frac{(\pi - x)\,dx}{a^2\cos^2 x + b^2\sin^2 x}$.

Step 3: Adding the two forms:
$$2I = \pi \int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}.$$

Step 4: Reducing the [0,pi] integral to [0,pi/2]:

The integrand has period π and is symmetric about $x = \pi/2$ (check: replacing x by $\pi - x$ leaves it unchanged), so $\int_0^{\pi} (\cdots)dx$ $= 2\int_0^{\pi/2} (\cdots)dx$.

Step 5: Using the standard result for the quarter-period integral:

$\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{2ab}$ (dividing numerator and denominator by $\cos^2 x$, substituting $t = \tan x$, and using $\int_0^{\infty} \frac{dt}{a^2 + b^2t^2}$ $= \frac{\pi}{2ab}$). So $\int_0^{\pi} (\cdots)dx = 2 \times \frac{\pi}{2ab} = \frac{\pi}{ab}$.

Step 6: Solving for I:

$2I = \pi \times \frac{\pi}{ab} = \frac{\pi^2}{ab} \Rightarrow I = \frac{\pi^2}{2ab}$.

Final Answer:

$$\int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x} = \boxed{\frac{\pi^2}{2ab}}.$$

Quick Tip: Use $x \to \pi - x$ to double the integral into a pure trig integral, then reduce to the standard $\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{2ab}$.

OR

If $y = \sin^{-1} x$, then show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$.

Correct Answer: proof

Solution:

Step 1: Understanding the Concept:

Differentiate $y = \sin^{-1} x$ once to get dy/dx in a form that's easy to differentiate again, then differentiate once more and combine to match the target identity.

Step 2: First derivative:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

Step 3: Rewriting to avoid the square root before differentiating again:

$\dfrac{dy}{dx} \cdot \sqrt{1-x^2} = 1$. Squaring: $\left(\dfrac{dy}{dx}\right)^2 (1-x^2) = 1$.

Step 4: Differentiating this squared relation implicitly:

$$2\frac{dy}{dx}\frac{d^2y}{dx^2}(1-x^2) + \left(\frac{dy}{dx}\right)^2(-2x) = 0.$$

Step 5: Simplifying:

Divide throughout by $2\dfrac{dy}{dx}$ (nonzero since $dy/dx = 1/\sqrt{1-x^2} \neq 0$): $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 0$.

Final Answer:

Hence $\boxed{(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 0}$, as required.

Quick Tip: Differentiate $y' = (1-x^2)^{-1/2}$ again (either via the squared relation or the chain rule directly) and simplify.