UP Board Class 12 Mathematics 2026 (Set 324 DC) Question Paper with Solutions

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General Instructions

(i) This booklet contains 9 questions, each provided with a complete, step-by-step solution.

(ii) It comprises 5 single-correct multiple-choice questions.

(iii) Attempt each question on your own before reviewing the given solution.

1(a). The function $f : R \to R$ defined by $f(x) = 3x + 5,\ \forall x \in R$ is:

(A) f is one-one onto

(B) f is many-one onto

(C) f is one-one but not onto

(D) f is neither one-one nor onto

Correct Answer: (A) f is one-one onto

Solution:

Step 1: Understanding the Concept:

A function is one-one (injective) if different inputs always give different outputs, and onto (surjective) if every real number is hit by some input.

Step 2: Checking one-one:

Take $f(x_1) = f(x_2)$, so $3x_1 + 5 = 3x_2 + 5$, which gives $x_1 = x_2$. So f is one-one.

Step 3: Checking onto:

For any $y \in R$, solve $y = 3x + 5$ to get $x = \dfrac{y-5}{3}$, which is a real number for every real y. So every y has a pre-image, and f is onto.

Step 4: Why the other options are wrong:

Since f is both one-one and onto, options B, C and D are all ruled out.

Final Answer:

f is one-one onto.

$$\boxed{\text{one-one onto}}$$

Quick Tip: Check if f is strictly monotonic (gives one-one) and check the range covers all of R (gives onto).

1(b). The value of $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$ is:

(A) π

(B) $\dfrac{-\pi}{2}$

(C) 0

(D) $2\sqrt{3}$

Correct Answer: (B) $\dfrac{-\pi}{2}$

Solution:

Step 1: Key Formula or Approach:

Use the standard values $\tan^{-1}(\sqrt{3}) = \dfrac{\pi}{3}$ (range of $\tan^{-1}$ is $(-\pi/2, \pi/2)$), and for $\cot^{-1}$ (range $(0, \pi)$) use $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$.

Step 2: Evaluating each term:

$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$.

$\cot^{-1}(\sqrt{3}) = \frac{\pi}{6}$, so $\cot^{-1}(-\sqrt{3}) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Step 3: Subtracting:

$$\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3}) = \frac{\pi}{3} - \frac{5\pi}{6} = \frac{2\pi - 5\pi}{6} = \frac{-3\pi}{6} = \frac{-\pi}{2}$$

Final Answer:

The value is $-\pi/2$.

$$\boxed{-\frac{\pi}{2}}$$

Quick Tip: Use $\tan^{-1}(\sqrt{3})=\pi/3$ and $\cot^{-1}(-x)=\pi-\cot^{-1}(x)$ with $\cot^{-1}(\sqrt{3})=\pi/6$.

1(c). If A is any matrix and K is any constant, then $(KA)'$ is:

(A) $K'A'$

(B) $A'K'$

(C) KA'

(D) KA

Correct Answer: (C) KA'

Solution:

Step 1: Key Formula or Approach:

For a scalar K (a plain number, not a matrix) and a matrix A, the transpose of a scalar multiple obeys $(KA)' = KA'$ — the scalar just carries through the transpose unchanged.

Step 2: Why it works:

Transposing only swaps rows and columns of the matrix part; multiplying every entry by the constant K does not depend on position, so the order of "multiply by K" and "transpose" can be swapped freely.

Step 3: Why the other options are wrong:

$K'A'$ and $A'K'$ wrongly treat K as if it were a matrix needing its own transpose; KA forgets to transpose A at all.

Final Answer:

$(KA)' = KA'$.

$$\boxed{KA'}$$

Quick Tip: A scalar constant passes straight through a transpose: (KA)'=KA'.

1(d). If the radius of a circle is $r = 6$ cm, then the rate of change of Area with respect to radius is:

(A) 10π

(B) 12π

(C) 8π

(D) 11π

Correct Answer: (B) 12π

Solution:

Step 1: Key Formula or Approach:

The area of a circle is $A = \pi r^2$. "Rate of change of Area with respect to radius" means $\frac{dA}{dr}$.

Step 2: Differentiating:

$$\frac{dA}{dr} = 2\pi r$$

Step 3: Substituting $r = 6$:

$$\left.\frac{dA}{dr}\right|_{r=6} = 2\pi(6) = 12\pi$$

Final Answer:

The rate of change of Area with respect to radius at $r = 6$ cm is 12π.

$$\boxed{12\pi}$$

Quick Tip: Differentiate A=πr² to get dA/dr=2πr, then plug in r=6.

1(e). The value of $\int_0^{2/3} \frac{dx}{4+9x^2}$ is:

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{12}$

(C) $\frac{\pi}{24}$

(D) $\frac{\pi}{4}$

Correct Answer: (C) $\frac{\pi}{24}$

Solution:

Step 1: Key Formula or Approach:

Use the standard result $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$. Here write $4 + 9x^2 = 9\left(x^2 + \frac{4}{9}\right)$, so $a = \frac{2}{3}$ after rewriting in the form $9(x^2 + a^2)$.

Step 2: Rewriting the integrand:

$$\frac{1}{4+9x^2} = \frac{1}{9} \cdot \frac{1}{x^2 + (2/3)^2}$$

So $\int \frac{dx}{4+9x^2} = \frac{1}{9} \cdot \frac{1}{2/3}\tan^{-1}\left(\frac{x}{2/3}\right) + C = \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right) + C.$

Step 3: Applying the limits:

$$\left[\frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)\right]_0^{2/3} = \frac{1}{6}\left(\tan^{-1}(1) - \tan^{-1}(0)\right) = \frac{1}{6}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{24}$$

Final Answer:

The value of the definite integral is $\pi/24$.

$$\boxed{\frac{\pi}{24}}$$

Quick Tip: Write 4+9x²=9(x²+(2/3)²) and use ∫dx/(x²+a²)=(1/a)tan⁻¹(x/a).

2(a). Evaluate: $\displaystyle\int_{-\pi/2}^{\pi/2} (x^3 + x\cos x)\,dx$

Correct Answer: —

Solution:

Step 1: Key Formula or Approach:

For a function integrated over symmetric limits $[-a, a]$: if the function is odd, the integral is 0; if it is even, the integral is $2\int_0^a$. Check each term for odd/even behaviour first.

Step 2: Checking x^3:

$(-x)^3 = -x^3$, so x^3 is an odd function. Its integral over $[-\pi/2, \pi/2]$ is 0.

Step 3: Checking $x\cos x$:

x is odd and $\cos x$ is even, and odd $\times$ even = odd. So $(-x)\cos(-x)$ $= -x\cos x$, confirming $x\cos x$ is odd, and its integral over the symmetric interval is also 0.

Step 4: Adding the pieces:

$$\int_{-\pi/2}^{\pi/2} (x^3 + x\cos x)\,dx = 0 + 0 = 0$$

Final Answer:

The integral evaluates to 0.

$$\boxed{0}$$

Quick Tip: Both x^3 and $x \cdot \cos x$ are odd functions, so their integral over symmetric limits is 0.

2(b). Find the area of the region bounded by the curve $y = \sin x$, $x = 0$ and $x = 2\pi$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

The curve $y = \sin x$ lies above the x-axis on $[0, \pi]$ and below it on $[\pi, 2\pi]$. "Area" always means the actual enclosed region, taken as a positive quantity, so each part must be taken as $|\cdot|$ separately, not just one signed integral.

Step 2: Area on $[0, \pi]$:

$$\int_0^{\pi} \sin x\,dx = [-\cos x]_0^{\pi} = -\cos\pi + \cos 0 = 1 + 1 = 2$$

Step 3: Area on $[\pi, 2\pi]$:

$$\int_{\pi}^{2\pi} \sin x\, dx = [-\cos x]_{\pi}^{2\pi} = -\cos 2\pi + \cos \pi = -1 - 1 = -2$$

Since this piece is negative (curve below the axis), take its magnitude: $|-2| = 2$.

Step 4: Adding both pieces:

$$\text{Total area} = 2 + 2 = 4$$

Final Answer:

The area bounded by the curve is 4 square units.

$$\boxed{4 \text{ sq. units}}$$

Quick Tip: sin x is positive on [0,π] and negative on [π,2π]; add the absolute areas of both parts.

2(c). Find the order and degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0.$

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

Order is the highest derivative present. Degree is the power of the highest-order derivative, but degree is only defined if the equation can be written as a polynomial in ALL the derivatives that appear (no derivative sitting inside a trig, log, exponential, or root).

Step 2: Finding the order:

The highest derivative present is $\frac{d^2y}{dx^2}$, so the order is 2.

Step 3: Checking whether degree is defined:

The term $\sin\left(\frac{dy}{dx}\right)$ has a derivative, $\frac{dy}{dx}$, sitting inside a sine function. A sine of a derivative cannot be expanded as a finite power of that derivative, so the whole left side is not a polynomial in the derivatives.

Final Answer:

The order is 2 and the degree is not defined.

$$\boxed{\text{Order} = 2,\ \text{Degree not defined}}$$

Quick Tip: Order = highest derivative present. Degree needs the equation to be a polynomial in the derivatives; sin(dy/dx) breaks that.

2(d). Find the area of the parallelogram whose adjacent sides are given by $\vec{a} = 3\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$.

Correct Answer: —

Solution:

Step 1: Key Formula or Approach:

The area of a parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$ is $|\vec{a} \times \vec{b}|$.

Step 2: Computing the cross product:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix} = \hat{i}(1 \cdot 1 - 4 \cdot (-1)) - \hat{j}(3 \cdot 1 - 4 \cdot 1) + \hat{k}(3 \cdot (-1) - 1 \cdot 1)$$

$$= \hat{i}(1 + 4) - \hat{j}(3 - 4) + \hat{k}(-3 - 1) = 5\hat{i} + \hat{j} - 4\hat{k}$$

Step 3: Finding the magnitude:

$$|\vec{a} \times \vec{b}| = \sqrt{5^2 + 1^2 + (-4)^2} = \sqrt{25 + 1 + 16} = \sqrt{42}$$

Final Answer:

The area of the parallelogram is $\sqrt{42}$ square units.

$$\boxed{\sqrt{42}\text{ sq. units}}$$

Quick Tip: Area of parallelogram = |a × b|; compute the cross product then its magnitude.

2(e). If $A^2 = A$, then simplify $(I + A)^2 - 7A$, where A is a square matrix.

Correct Answer: —

Solution:

Step 1: Key Formula or Approach:

Expand $(I + A)^2$ using matrix algebra: $(I + A)^2 = I^2 + IA + AI + A^2$
$= I + 2A + A^2$ (since $IA = AI = A$ for the identity matrix).

Step 2: Using the given condition $A^2 = A$:

$$(I + A)^2 = I + 2A + A = I + 3A$$

Step 3: Subtracting $7A$:

$$(I + A)^2 - 7A = I + 3A - 7A = I - 4A$$

Final Answer:

The simplified expression is $I - 4A$.

$$\boxed{I - 4A}$$

Quick Tip: Expand (I+A)²=I+2A+A², then use A²=A to simplify before subtracting 7A.

3(a). If the ordered pairs $(2x - 3, -5)$ and $(x, y - 1)$ are equal, then find the value of x and y.

Correct Answer: —

Solution:

Step 1: Key Formula or Approach:

Two ordered pairs (a, b) and (c, d) are equal only when $a = c$ and $b = d$ — both coordinates must match separately.

Step 2: Matching the first coordinates:

$$2x - 3 = x \implies x = 3$$

Step 3: Matching the second coordinates:

$$-5 = y - 1 \implies y = -4$$

Final Answer:

$x = 3$ and $y = -4$.

$$\boxed{x = 3,\ y = -4}$$

Quick Tip: Equal ordered pairs mean the first entries are equal and the second entries are equal.

3(b). Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$.

Correct Answer: —

Solution:

Step 1: Key Formula or Approach:

A function f is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$, i.e. the left-hand limit, right-hand limit and the actual function value all agree.

Step 2: Finding $f(3)$:

$$f(3) = 2(3)^2 - 1 = 18 - 1 = 17$$

Step 3: Finding the limit as $x \to 3$:

Since $f(x) = 2x^2 - 1$ is a polynomial, it is continuous everywhere, so $\lim_{x \to 3} f(x) = 2(3)^2 - 1 = 17$ directly by substitution.

Step 4: Comparing:

The limit (17) equals $f(3)$ (17).

Final Answer:

$f(x)$ is continuous at $x = 3$, with $f(3) = 17$.

$$\boxed{\text{Continuous at } x = 3,\ f(3) = 17}$$

Quick Tip: Check if lim(x→3) f(x) equals f(3); polynomials are continuous everywhere.

3(c). Differentiate $\log(\log x)$, $x > 1$ with respect to x.

Correct Answer: —

Solution:

Step 1: Key Formula or Approach:

Use the chain rule: if $y = \log(u)$ where $u = \log x$, then $\frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx}$.

Step 2: Differentiating the outer function:

$$\frac{d}{dx}\log(\log x) = \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)$$

Step 3: Differentiating the inner function:

$\frac{d}{dx}(\log x) = \frac{1}{x}$. Substituting:

$$\frac{d}{dx}\log(\log x) = \frac{1}{\log x} \cdot \frac{1}{x} = \frac{1}{x\log x}$$

Final Answer:

The derivative is $\frac{1}{x\log x}$.

$$\boxed{\frac{1}{x\log x}}$$

Quick Tip: Chain rule: d/dx[log(log x)] = (1/log x)·(1/x).

3(d). Prove that on R, the function $f(x) = e^{2x}$ is increasing.

Correct Answer: —

Solution:

Step 1: Key Formula or Approach:
A differentiable function f is increasing on an interval if $f'(x) > 0$ for every x in that interval.

Step 2: Differentiating $f(x)$:

$$f'(x) = \frac{d}{dx}e^{2x} = 2e^{2x}$$

Step 3: Sign of $f'(x)$:
The exponential function e^{2x} is positive for every real x (an exponential is never zero or negative), and multiplying by 2 keeps it positive. So $f'(x) = 2e^{2x} > 0$ for all $x \in R$.

Final Answer:
Since $f'(x) > 0$ for every $x \in R$, $f(x) = e^{2x}$ is strictly increasing on R.

$$\boxed{f'(x) = 2e^{2x} > 0 \implies f \text{ is increasing on } R}$$

Quick Tip: Show f'(x)=2e^{2x} is always positive on R.

4(a). Prove that $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular. It is given that $\vec{a} \neq 0,\ \vec{b} \neq 0$.

Correct Answer: —

Solution:

Step 1: Key Formula or Approach:
Expand the dot product on the left using the distributive law: $(\vec{a} + \vec{b})$

$\cdot (\vec{a} + \vec{b}) = \vec{a} \cdot \vec{a} + 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} = |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2$.

Step 2: Setting up the "if and only if":

$$|\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 \iff 2\vec{a} \cdot \vec{b} = 0 \iff \vec{a} \cdot \vec{b} = 0$$

Step 3: Connecting to perpendicularity:

Since $\vec{a} \neq 0$ and $\vec{b} \neq 0$, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = 0$ forces $\cos\theta = 0$, i.e. $\theta = 90°$. So $\vec{a} \cdot \vec{b} = 0$ exactly when $\vec{a} \perp \vec{b}$.

Final Answer:

So $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$ holds if and only if $\vec{a} \perp \vec{b}$, which is what was to be proved.

$$\boxed{\vec{a} \cdot \vec{b} = 0 \iff \vec{a} \perp \vec{b}}$$

Quick Tip: Expand the dot product; the identity holds exactly when a·b=0, which means perpendicular.

4(b). Find the direction cosines of the x, y and z axes.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

Direction cosines of a line are the cosines of the angles it makes with the positive x, y and z axes. For a coordinate axis itself, this becomes very simple since it either lies along an axis or is perpendicular to it.

Step 2: Direction cosines of the x-axis:

The x-axis makes an angle of 0° with itself, and 90° with both the y-axis and z-axis. So its direction cosines are $\cos 0^\circ, \cos 90^\circ, \cos 90^\circ$, i.e. $(1, 0, 0)$.

Step 3: Direction cosines of the y-axis and z-axis:

By the same reasoning, the y-axis has direction cosines $(0, 1, 0)$ and the z-axis has direction cosines $(0, 0, 1)$.

Final Answer:

x-axis: $(1, 0, 0)$, y-axis: $(0, 1, 0)$, z-axis: $(0, 0, 1)$.

$$\boxed{(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1)}$$

Quick Tip: Each axis makes 0° with itself and 90° with the other two axes.

4(c). If $P(A) = \frac{7}{13}$, $P(B) = \frac{9}{13}$ and $P(A \cap B) = \frac{4}{13}$, then find $P(A/B)$.

Correct Answer: —

Solution:

Step 1: Key Formula or Approach:

Conditional probability is defined as $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$, valid whenever $P(B) > 0$.

Step 2: Substituting the given values:

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{4/13}{9/13}$$

Step 3: Simplifying the fraction:

The 13 in the numerator and denominator cancels:

$$P(A/B) = \frac{4}{9}$$

Final Answer:

$P(A/B) = \frac{4}{9}$.

$$\boxed{\frac{4}{9}}$$

Quick Tip: P(A/B) = P(A∩B)/P(B).

4(d). Solve: $\int \cos^2 x \, dx$

Correct Answer: —

Solution:

Step 1: Key Formula or Approach:

There is no direct antiderivative for $\cos^2 x$ as it stands, so first rewrite it using the double angle identity $\cos 2x = 2\cos^2 x - 1$, which gives $\cos^2 x = \frac{1 + \cos 2x}{2}$.

Step 2: Rewriting the integral:

$$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{1}{2}\int 1\,dx + \frac{1}{2}\int \cos 2x\,dx$$

Step 3: Integrating each piece:

$\int 1\,dx = x$, and $\int \cos 2x\,dx = \frac{\sin 2x}{2}$ (dividing by the derivative of the inner function $2x$).

Final Answer:

$$\int \cos^2 x\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$$

$$\boxed{\frac{x}{2} + \frac{\sin 2x}{4} + C}$$

Quick Tip: Use cos²x=(1+cos2x)/2, then integrate term by term.

5(a). Evaluate: $\int_0^1 \frac{\tan^{-1} x}{1+x^2}\,dx$

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

Notice that $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$. This means the integrand is of the form $\tan^{-1} x \cdot \frac{d}{dx}(\tan^{-1} x)$, which is perfectly set up for the substitution $t = \tan^{-1} x$.

Step 2: Substituting:

Let $t = \tan^{-1} x$, so $dt = \dfrac{dx}{1+x^2}$. The integral becomes:

$$\int t\,dt$$

Step 3: Changing the limits:

When $x = 0$, $t = \tan^{-1} 0 = 0$. When $x = 1$, $t = \tan^{-1} 1 = \dfrac{\pi}{4}$.

Step 4: Integrating and applying limits:

$$\int_0^{\pi/4} t\,dt = \left[\frac{t^2}{2}\right]_0^{\pi/4} = \frac{(\pi/4)^2}{2} - 0 = \frac{\pi^2}{32}$$

Final Answer:

The value of the integral is $\pi^2/32$.

$$\boxed{\frac{\pi^2}{32}}$$

Quick Tip: Substitute $t=\tan^{-1}x$, since $dt=dx/(1+x^2)$, turning it into $\int t\,dt$.

5(b). Three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a} + \vec{b} + \vec{c} = 0$. If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{c}| = 5$, then find the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$.

Correct Answer: —

Solution:

Step 1: Key Formula or Approach:

Since $\vec{a} + \vec{b} + \vec{c} = 0$, taking the dot product of both sides with itself gives $(\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) = 0$. Expanding this will bring out exactly the terms we need.

Step 2: Expanding the dot product:

$$|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$

Step 3: Substituting the magnitudes:

$|\vec{a}|^2 = 9,\ |\vec{b}|^2 = 16,\ |\vec{c}|^2 = 25$, so:

$$9 + 16 + 25 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$

$$50 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$

Step 4: Solving for the required sum:

$$\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = \frac{-50}{2} = -25$$

Final Answer:

The value is -25.

$$\boxed{-25}$$

Quick Tip: Square both sides of a+b+c=0 using the dot product with itself.

5(c). Find the angle between the lines $\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$ and $\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$.

Correct Answer: —

Solution:

Step 1: Key Formula or Approach:

The angle between two lines depends only on their direction vectors $\vec{d}_1$ and $\vec{d}_2$, using $\cos\theta = \dfrac{|\vec{d}_1 \cdot \vec{d}_2|}{|\vec{d}_1||\vec{d}_2|}$.

Step 2: Identifying the direction vectors:

$\vec{d}_1 = \hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{d}_2 = 3\hat{i} + 2\hat{j} + 6\hat{k}$.

Step 3: Computing the dot product and magnitudes:

$$\vec{d}_1 \cdot \vec{d}_2 = (1)(3) + (2)(2) + (2)(6) = 3 + 4 + 12 = 19$$

$$|\vec{d}_1| = \sqrt{1 + 4 + 4} = 3, \qquad |\vec{d}_2| = \sqrt{9 + 4 + 36} = \sqrt{49} = 7$$

Step 4: Computing $\cos\theta$:

$$\cos\theta = \frac{19}{3 \times 7} = \frac{19}{21}$$

Final Answer:

The angle between the lines is $\theta = \cos^{-1}\left(\frac{19}{21}\right)$.

$$\boxed{\theta = \cos^{-1}\left(\frac{19}{21}\right)}$$

Quick Tip: Use cosθ = |d1·d2|/(|d1||d2|) with the direction vectors of both lines.

5(d). Minimize $z = 3x + 2y$ under the following constraints by graphical method: $x + y \geq 8,\ x \geq 0,\ 3x + 5y \leq 15,\ y \geq 0$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

Before minimizing anything, the feasible region (the set of points satisfying every constraint at once) must actually exist. Graph each constraint line and check where all shaded regions overlap.

Step 2: Plotting the boundary lines:

$x + y = 8$ passes through $(8, 0)$ and $(0, 8)$; the constraint $x + y \geq 8$ is the region ON or ABOVE this line. $3x + 5y = 15$ passes through $(5, 0)$ and $(0, 3)$; the constraint $3x + 5y \leq 15$ is the region ON or BELOW this line.

Step 3: Checking for overlap:

On the line $3x + 5y = 15$ with $x, y \geq 0$, the largest possible value of $x + y$ occurs at a corner: at $(5, 0)$, $x + y = 5$; at $(0, 3)$, $x + y = 3$; anywhere in between is even smaller (since $3x + 5y \leq 15$ with $y \geq 0$ gives $3(x + y) \leq 3x + 5y \leq 15$, so $x + y \leq 5$).

Step 4: Comparing with the other constraint:

The first constraint needs $x + y \geq 8$, but the second constraint forces $x + y \leq 5$ in the first quadrant. Since $8 > 5$, no point can satisfy both together.

Final Answer:

The feasible region is empty, so this linear programming problem has no feasible solution, and z cannot be minimized.

$$\boxed{\text{No feasible solution exists}}$$

Quick Tip: Check if the region x+y≥8 overlaps with 3x+5y≤15 in the first quadrant; it does not.

5(e). If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, then find $(AB)^{-1}$.

Correct Answer: —

Solution:

Step 1: Key Formula or Approach:

First compute the product matrix AB, then use the 2×2 inverse formula: for $M = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, $M^{-1} = \dfrac{1}{ps - qr} \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$.

Step 2: Computing AB:

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(1)+3(-1) & 2(-2)+3(3) \\ 1(1)+(-4)(-1) & 1(-2)+(-4)(3) \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$$

Step 3: Finding the determinant of AB:

$$\det(AB) = (-1)(-14) - (5)(5) = 14 - 25 = -11$$

Step 4: Applying the inverse formula:

$$(AB)^{-1} = \frac{1}{-11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \begin{bmatrix} 14/11 & 5/11 \\ 5/11 & 1/11 \end{bmatrix}$$

Final Answer:

$$(AB)^{-1} = \begin{bmatrix} 14/11 & 5/11 \\ 5/11 & 1/11 \end{bmatrix}$$

$$\boxed{(AB)^{-1} = \begin{bmatrix} 14/11 & 5/11 \\ 5/11 & 1/11 \end{bmatrix}}$$

Quick Tip: Find AB first, then apply the 2x2 inverse formula; or use $(AB)^{-1}=B^{-1}A^{-1}$.

6(a). Write $\tan^{-1}\left(\dfrac{\cos x - \sin x}{\cos x + \sin x}\right)$, $\dfrac{-\pi}{4} < x < \dfrac{3\pi}{4}$ in the simplest form.

Correct Answer: —

Solution:

Step 1: Key Formula or Approach:

Divide the numerator and denominator inside the fraction by $\cos x$ to bring in $\tan x$, then recognise the tangent subtraction formula $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$ with $A = \pi/4$ (since $\tan(\pi/4) = 1$).

Step 2: Rewriting the fraction:

$$\frac{\cos x - \sin x}{\cos x + \sin x} = \frac{1 - \tan x}{1 + \tan x}$$

Step 3: Matching to the tangent subtraction identity:

$$\frac{1 - \tan x}{1 + \tan x} = \frac{\tan(\pi/4) - \tan x}{1 + \tan(\pi/4)\tan x} = \tan\left(\frac{\pi}{4} - x\right)$$

Step 4: Applying $\tan^{-1}$ and checking the domain:

$$\tan^{-1}\left(\tan\left(\frac{\pi}{4} - x\right)\right) = \frac{\pi}{4} - x$$

This step is valid because for $-\pi/4 < x < 3\pi/4$, the angle $\dfrac{\pi}{4} - x$ lies inside $(-\pi/2, \pi/2)$, the principal range of $\tan^{-1}$, so no adjustment is needed.

Final Answer:

The simplest form is $\dfrac{\pi}{4} - x$.

$$\boxed{\frac{\pi}{4} - x}$$

Quick Tip: Divide numerator and denominator by cos x to get (1−tanx)/(1+tanx)=tan(π/4−x).

6(b). If the function $f : \left[0, \frac{\pi}{2}\right] \to R$ is given by $f(x) = \cos x$ and the function $g : \left[0, \frac{\pi}{2}\right] \to R$ is given by $g(x) = \sin x$, then prove that f and g are one-one, but $f + g$ is not one-one.

Correct Answer: —

Solution:

Step 1: Proving $f(x) = \cos x$ is one-one:
On $\left[0, \frac{\pi}{2}\right]$, $\cos x$ is a strictly decreasing function (it steadily falls from $\cos 0 = 1$ to $\cos(\pi/2) = 0$). A strictly monotonic function can never take the same value twice, so f is one-one.

Step 2: Proving $g(x) = \sin x$ is one-one:
On the same interval, $\sin x$ is strictly increasing (from $\sin 0 = 0$ to $\sin(\pi/2) = 1$), so by the same reasoning g is one-one.

Step 3: Checking $f + g$:
$(f + g)(x) = \cos x + \sin x$. Evaluate it at the two endpoints:

$$(f + g)(0) = \cos 0 + \sin 0 = 1 + 0 = 1$$

$$(f+g)\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 0 + 1 = 1$$

Step 4: Drawing the conclusion:

Both $x = 0$ and $x = \pi/2$ give the same output, 1, for $f + g$, even though $0 \neq \pi/2$. Since two different inputs give the same output, $f + g$ is NOT one-one.

Final Answer:

f and g are individually one-one, but $(f+g)(0) = (f+g)(\pi/2) = 1$ shows $f + g$ is not one-one.

$$\boxed{f, g \text{ one-one; } f+g \text{ not one-one, since } (f+g)(0) = (f+g)(\pi/2) = 1}$$

Quick Tip: cos x is strictly decreasing and sin x strictly increasing on [0,π/2], so both are one-one; but f+g=cosx+sinx repeats a value at the two endpoints.

6(c). Is the function $f(x) = \begin{cases} x+5, & x \leq 1 \\ x-5, & x > 1 \end{cases}$ continuous at $x = 1$?

Correct Answer: —

Solution:

Step 1: Key Formula or Approach:

For a piecewise function at the joining point, check whether the left-hand limit (LHL), right-hand limit (RHL) and the function value all agree. If any two of these disagree, the function is discontinuous there.

Step 2: Finding $f(1)$:

Since $1 \leq 1$, use the first branch: $f(1) = 1 + 5 = 6$.

Step 3: Finding the left-hand limit:

For x slightly less than 1, use $f(x) = x + 5$:

$$\lim_{x \to 1^-} f(x) = 1 + 5 = 6$$

Step 4: Finding the right-hand limit:

For x slightly more than 1, use $f(x) = x - 5$:

$$\lim_{x \to 1^+} f(x) = 1 - 5 = -4$$

Step 5: Comparing:

LHL $= 6$ but RHL $= -4$; these are not equal, so the two-sided limit does not exist at $x = 1$.

Final Answer:

Since LHL $\neq$ RHL, $f(x)$ is NOT continuous at $x = 1$.

$$\boxed{\text{No, } f \text{ is discontinuous at } x = 1 \text{ (LHL} = 6 \neq \text{RHL} = -4)}$$

Quick Tip: Compare LHL, RHL and f(1); a mismatch means discontinuous.

6(d). Find the shortest distance between the lines $\vec{r} = \hat{i} + \hat{j} - \lambda(2\hat{i} + \hat{j} + \hat{k})$ and $\vec{r} = 2\hat{i} + \hat{j} + \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$.

Correct Answer: —

Solution:

Step 1: Key Formula or Approach:

For two skew lines $\vec{r} = \vec{a}_1 + \lambda\vec{d}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{d}_2$, the shortest distance is $\left|\dfrac{(\vec{a}_2 - \vec{a}_1)\cdot(\vec{d}_1 \times \vec{d}_2)}{|\vec{d}_1 \times \vec{d}_2|}\right|$.

Step 2: Reading off the data:

$\vec{a}_1 = \hat{i} + \hat{j}$, $\vec{d}_1 = 2\hat{i} + \hat{j} + \hat{k}$ (the minus sign just flips the direction, which does not change the line or the distance); $\vec{a}_2 = 2\hat{i} + \hat{j} + \hat{k}$, $\vec{d}_2 = 3\hat{i} - 5\hat{j} + 2\hat{k}$.

Step 3: Computing $\vec{d}_1 \times \vec{d}_2$:

$$\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 3 & -5 & 2 \end{vmatrix} = \hat{i}(1\cdot 2 - 1\cdot(-5)) - \hat{j}(2\cdot 2 - 1\cdot 3) + \hat{k}(2\cdot(-5) - 1\cdot 3)$$

$$= \hat{i}(2+5) - \hat{j}(4-3) + \hat{k}(-10-3) = 7\hat{i} - \hat{j} - 13\hat{k}$$

Step 4: Finding $|\vec{d}_1 \times \vec{d}_2|$ and $(\vec{a}_2 - \vec{a}_1)$:

$$|\vec{d}_1 \times \vec{d}_2| = \sqrt{49 + 1 + 169} = \sqrt{219}$$

$$\vec{a}_2 - \vec{a}_1 = (2-1)\hat{i} + (1-1)\hat{j} + (1-0)\hat{k} = \hat{i} + \hat{k}$$

Step 5: Dotting and dividing:

$$(\vec{a}_2 - \vec{a}_1) \cdot (\vec{d}_1 \times \vec{d}_2) = (1)(7) + (0)(-1) + (1)(-13) = 7 - 13 = -6$$

$$\text{Distance} = \left| \frac{-6}{\sqrt{219}} \right| = \frac{6}{\sqrt{219}}$$

Final Answer:

The shortest distance is $\dfrac{6}{\sqrt{219}}$ units.

$$\boxed{\frac{6}{\sqrt{219}} \text{ units}}$$

Quick Tip: Use the shortest-distance-between-skew-lines formula with the cross product of the direction vectors.

6(e). A family has two children. If it is known that at least one child is a boy, then find the probability of both children being boys.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

With two children, each equally likely to be a boy (B) or girl (G), the sample space of birth orders is $\{BB, BG, GB, GG\}$, each with probability 1/4. We need a conditional probability: given that "at least one is a boy" has already happened, what is the chance both are boys?

Step 2: Defining the events:

Let E = "both children are boys" = $\{BB\}$. Let F = "at least one child is a boy" = $\{BB, BG, GB\}$.

Step 3: Finding the probabilities:

$$P(F) = \frac{3}{4}, \qquad P(E \cap F) = P(\{BB\}) = \frac{1}{4}$$

(Note $E \cap F = E$, since if both are boys, at least one is automatically a boy too.)

Step 4: Applying the conditional probability formula:

$$P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Final Answer:

The probability that both children are boys, given at least one is a boy, is $\frac{1}{3}$.

$$\boxed{\frac{1}{3}}$$

Quick Tip: Sample space {BB,BG,GB,GG}; restrict to outcomes with at least one boy, then find the fraction that are BB.

7. Find the general solution of the differential equation $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x.$

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

The right side, divided by x, depends only on y/x: $\frac{dy}{dx} = \frac{y}{x} + \frac{1}{\cos(y/x)}$, which is a function of y/x alone. This makes the equation homogeneous, solvable with the substitution $y = vx$.

Step 2: Substituting $y = vx$:

With $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$. The original equation becomes:

$$x\cos(v)\left(v + x\frac{dv}{dx}\right) = vx\cos(v) + x$$

Step 3: Expanding and simplifying:

$$xv\cos(v) + x^2\cos(v)\frac{dv}{dx} = vx\cos(v) + x$$

The $xv\cos v$ terms on each side cancel:

$$x^2\cos(v)\frac{dv}{dx} = x \implies \cos(v)\frac{dv}{dx} = \frac{1}{x}$$

Step 4: Separating variables and integrating:

$$\cos(v)\,dv = \frac{dx}{x}$$

$$\int \cos v\,dv = \int \frac{dx}{x}$$

$$\sin v = \ln|x| + C$$

Step 5: Substituting back $v = y/x$:

$$\sin\left(\frac{y}{x}\right) = \ln|x| + C$$

Final Answer:

The general solution is $\sin(y/x) = \ln|x| + C$.

$$\boxed{\sin\left(\frac{y}{x}\right) = \ln|x| + C}$$

Quick Tip: Homogeneous equation: substitute y=vx, dy/dx=v+x dv/dx, then separate variables.

OR

Evaluate: $\int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x}$

Correct Answer: —

Solution:

Step 1: Key Formula or Approach:

Use the property $\int_0^{a} f(x)\,dx = \int_0^{a} f(a-x)\,dx$. Let I denote the given integral and apply this with $a = \pi$.

Step 2: Applying the property:

Let $I = \int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x}$. Replacing x by $\pi - x$: since $\cos(\pi - x) = -\cos x$ and $\sin(\pi - x) = \sin x$, and both appear squared, the denominator is unchanged:

$$I = \int_0^{\pi} \frac{(\pi - x)\,dx}{a^2\cos^2 x + b^2\sin^2 x}$$

Step 3: Adding the two forms of I:

$$2I = \int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x} + \int_0^{\pi} \frac{(\pi - x)\,dx}{a^2\cos^2 x + b^2\sin^2 x} = \pi \int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

Step 4: Evaluating $J = \int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$:

By the symmetry of $\cos^2 x$ and $\sin^2 x$ about $x = \pi/2$, $\int_0^{\pi} = 2\int_0^{\pi/2}$.

Using the standard result $\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{2ab}$ (divide numerator and denominator by $\cos^2 x$, substitute $t = \tan x$), we get J

$$= 2 \cdot \frac{\pi}{2ab} = \frac{\pi}{ab}.$$

Step 5: Solving for I:

$$2I = \pi \cdot \frac{\pi}{ab} = \frac{\pi^2}{ab} \implies I = \frac{\pi^2}{2ab}$$

Final Answer:

$$I = \frac{\pi^2}{2ab}$$

$$\boxed{\frac{\pi^2}{2ab}}$$

Quick Tip: Use ∫f(x)dx=∫f(a−x)dx over [0,π] to remove x, then evaluate the resulting symmetric integral.

8. Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

By symmetry, the ellipse is symmetric about both axes, so the total area is 4 times the area in the first quadrant.

Step 2: Expressing y in terms of x:

From $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, solving for the first-quadrant branch:

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

Step 3: Setting up the area integral:

$$\text{Area} = 4\int_0^a y\,dx = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

Step 4: Using the standard result $\int_0^a \sqrt{a^2 - x^2}\,dx = \frac{\pi a^2}{4}$:

$$\text{Area} = \frac{4b}{a} \cdot \frac{\pi a^2}{4} = \pi ab$$

Final Answer:

The area enclosed by the ellipse is πab square units.

$$\boxed{\pi ab}$$

Quick Tip: Area = 4×(first-quadrant area) = (4b/a)∫√(a²−x²)dx from 0 to a.

OR

Prove that the volume of the largest right circular cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

Set up the cone's height h and base radius r in terms of the sphere's radius R using the geometry of the inscribed cone, express the volume as a function of one variable, then maximize using calculus.

Step 2: Setting up the constraint:

For a cone inscribed in a sphere of radius R with its apex on the sphere and height h measured from the apex, the base circle's radius satisfies $r^2 = h(2R - h)$ (from the right-triangle relation inside the sphere), where $0 < h < 2R$.

Step 3: Writing the volume as a function of h:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h(2R - h)h = \frac{\pi}{3}(2Rh^2 - h^3)$$

Step 4: Differentiating and finding the critical point:

$$\frac{dV}{dh} = \frac{\pi}{3}(4Rh - 3h^2) = \frac{\pi h}{3}(4R - 3h)$$

Setting $\frac{dV}{dh} = 0$ (and rejecting $h = 0$, which gives no cone):

$$4R - 3h = 0 \implies h = \frac{4R}{3}$$

Step 5: Confirming it is a maximum:

$\frac{d^2V}{dh^2} = \frac{\pi}{3}(4R - 6h)$; at $h = 4R/3$, this is $\frac{\pi}{3}(4R - 8R) = \frac{\pi}{3}(-4R) < 0$, confirming a maximum.

Step 6: Computing the maximum volume:

At $h = \frac{4R}{3}$: $r^2 = \frac{4R}{3}\left(2R - \frac{4R}{3}\right) = \frac{4R}{3} \cdot \frac{2R}{3} = \frac{8R^2}{9}$.

$$V_{\text{max}} = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \cdot \frac{8R^2}{9} \cdot \frac{4R}{3} = \frac{32\pi R^3}{81}$$

Step 7: Comparing with the sphere's volume:

$$\frac{V_{\text{max}}}{V_{\text{sphere}}} = \frac{32\pi R^3/81}{(4/3)\pi R^3} = \frac{32}{81} \times \frac{3}{4} = \frac{96}{324} = \frac{8}{27}$$

Final Answer:

The largest inscribed cone has volume exactly $\frac{8}{27}$ of the sphere's volume, as required.

$$\boxed{V_{\text{max}} = \frac{8}{27} V_{\text{sphere}}}$$

Quick Tip: Use $r^2=h(2R-h)$ for the inscribed cone, maximize $V=(\pi/3)r^2h$ with calculus, then divide by sphere volume.

9. Solve the system of linear equations by Matrix Method: $2x - 3y + 5z = 11$, $3x + 2y - 4z = -5$, $x + y - 2z = -3$.

Correct Answer: —

Solution:

Step 1: Key Formula or Approach:

Write the system as $AX = B$ where $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$. If $\det(A) \neq 0$, the unique solution is $X = A^{-1}B$, which can be computed via Cramer's rule as $x = \dfrac{\det(A_x)}{\det(A)}$, etc.

Step 2: Computing $\det(A)$:

$$\det(A) = 2[2(-2) - (-4)(1)] - (-3)[3(-2) - (-4)(1)] + 5[3(1) - 2(1)]$$

$$= 2(-4 + 4) + 3(-6 + 4) + 5(3 - 2) = 0 - 6 + 5 = -1$$

Since $\det(A) = -1 \neq 0$, a unique solution exists.

Step 3: Finding x using A_x (replace column 1 with B):

$$\det(A_x) = \begin{vmatrix} 11 & -3 & 5 \\ -5 & 2 & -4 \\ -3 & 1 & -2 \end{vmatrix} = 11(-4 + 4) + 3(10 - 12) + 5(-5 + 6) = 0 - 6 + 5 = -1$$

$$x = \frac{\det(A_x)}{\det(A)} = \frac{-1}{-1} = 1$$

Step 4: Finding y using A_y (replace column 2 with B):

$$\det(A_y) = \begin{vmatrix} 2 & 11 & 5 \\ 3 & -5 & -4 \\ 1 & -3 & -2 \end{vmatrix} = 2(10-12) - 11(-6+4) + 5(-9+5) = -4 + 22 - 20 = -2$$

$$y = \frac{\det(A_y)}{\det(A)} = \frac{-2}{-1} = 2$$

Step 5: Finding z using A_z (replace column 3 with B):

$$\det(A_z) = \begin{vmatrix} 2 & -3 & 11 \\ 3 & 2 & -5 \\ 1 & 1 & -3 \end{vmatrix} = 2(-6+5) + 3(-9+5) + 11(3-2) = -2 - 12 + 11 = -3$$

$$z = \frac{\det(A_z)}{\det(A)} = \frac{-3}{-1} = 3$$

Step 6: Verifying:

$2(1) - 3(2) + 5(3) = 2 - 6 + 15 = 11\ \checkmark$; $3(1) + 2(2) - 4(3) = 3 + 4 - 12 = -5\ \checkmark$; $1 + 2 - 2(3) = 1 + 2 - 6 = -3\ \checkmark$.

Final Answer:

$x = 1,\ y = 2,\ z = 3.$

$$\boxed{x = 1,\ y = 2,\ z = 3}$$

Quick Tip: Write as AX=B, compute det(A), then use Cramer's rule or $A^{-1}B$.

OR

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, then show that $A^3 - 23A - 40I = 0$.

Correct Answer: —

Solution:

Step 1: Key Formula or Approach:

Use the Cayley-Hamilton theorem: every square matrix satisfies its own characteristic equation $\lambda^3 - (\text{tr } A)\lambda^2 + (\text{sum of principal } 2 \times 2 \text{ minors})\lambda - \det(A) = 0$, with A substituted for λ and I for the constant term.

Step 2: Finding the trace:

$$\text{tr}(A) = 1 + (-2) + 1 = 0$$

Step 3: Finding the sum of principal 2×2 minors:

$$M_{11} = \begin{vmatrix} -2 & 1 \\ 2 & 1 \end{vmatrix} = -2 - 2 = -4, \quad M_{22} = \begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix} = 1 - 12 = -11, \quad M_{33} = \begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = -2 - 6 = -8$$

$$\text{Sum} = -4 - 11 - 8 = -23$$

Step 4: Finding $\det(A)$:

$$\det(A) = 1[(-2)(1) - 1(2)] - 2[3(1) - 1(4)] + 3[3(2) - (-2)(4)]$$

$$= 1(-2-2) - 2(3-4) + 3(6+8) = -4 + 2 + 42 = 40$$

Step 5: Writing the characteristic equation:

$$\lambda^3 - (0)\lambda^2 + (-23)\lambda - 40 = 0 \implies \lambda^3 - 23\lambda - 40 = 0$$

Step 6: Applying Cayley-Hamilton:

By the Cayley-Hamilton theorem, A itself satisfies this equation:

$$A^3 - 23A - 40I = 0$$

Final Answer:

This proves the required identity.

$$\boxed{A^3 - 23A - 40I = 0}$$

Quick Tip: Use Cayley-Hamilton (characteristic equation of A) or verify directly by computing A^2 and A^3.