UP Board Class 12 Mathematics 2026 (Set 324 DA) Question Paper with Solutions

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General Instructions

(i) This booklet contains 9 questions, each provided with a complete, step-by-step solution.

(ii) It has multiple-choice questions as well as questions that need a written answer.

(iii) Attempt each question on your own before reviewing the given solution.

1(a). Let relation R be defined in the set N as follows:

$R = \{(a, b) : a = b - 2,\ b > 6\}$

For this relation R

(A) $(2, 4) \in R$

(B) $(3, 8) \in R$

(C) $(6, 8) \in R$

(D) $(8, 7) \in R$

Correct Answer: (C) $(6, 8) \in R$

Solution:

Step 1: Understand the relation.

The set R holds ordered pairs (a, b) of natural numbers. A pair belongs to R only if it passes two tests at once. The first number must equal the second number minus 2, that is $a = b - 2$. The second number must be greater than 6, that is $b > 6$.

Step 2: Check option (A), the pair $(2, 4)$.

Here $a = 2$ and $b = 4$. The condition $a = b - 2$ gives $4 - 2 = 2$, which is true. But $b = 4$ is not greater than 6, so the second test fails. So (A) is FALSE.

Step 3: Check option (B), the pair $(3, 8)$.
Here $a = 3$ and $b = 8$. We have $b > 6$, so that test passes. But $b - 2 = 8 - 2 = 6$, and $a = 3$ is not equal to 6. So (B) is FALSE.

Step 4: Check option (C), the pair $(6, 8)$.
Here $a = 6$ and $b = 8$. First, $b - 2 = 8 - 2 = 6 = a$, so $a = b - 2$ holds. Second, $b = 8 > 6$ also holds. Both tests pass, so $(6, 8) \in R$. So (C) is TRUE.

Step 5: Check option (D), the pair $(8, 7)$.
Here $a = 8$ and $b = 7$. We have $b = 7 > 6$, so that test passes. But $b - 2 = 7 - 2 = 5$, and $a = 8$ is not equal to 5. So (D) is FALSE.

Final Answer:
Only the pair $(6, 8)$ satisfies both $a = b - 2$ and $b > 6$, so option (C) is correct.

$$\boxed{(6, 8) \in R}$$

Quick Tip: A pair (a, b) is in R only if $a = b - 2$ and $b > 6$. Test each option against both conditions.

1(b). The value of $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$ is:

(A) π

(B) $\frac{-\pi}{2}$

(C) 0

(D) $2\sqrt{3}$

Correct Answer: (B) $\frac{-\pi}{2}$

Solution:

Step 1: Key Formula or Approach:

Use the standard values $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ (range of $\tan^{-1}$ is $(-\pi/2, \pi/2)$), and for $\cot^{-1}$ (range $(0, \pi)$) use $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$.

Step 2: Evaluating each term:

$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$.

$\cot^{-1}(\sqrt{3}) = \frac{\pi}{6}$, so $\cot^{-1}(-\sqrt{3}) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Step 3: Subtracting:

$$\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3}) = \frac{\pi}{3} - \frac{5\pi}{6} = \frac{2\pi - 5\pi}{6} = \frac{-3\pi}{6} = \frac{-\pi}{2}$$

Final Answer:

The value is $-\pi/2$.

$$\boxed{-\frac{\pi}{2}}$$

Quick Tip: Use $\tan^{-1}(\sqrt{3})=\pi/3$ and $\cot^{-1}(-x)=\pi-\cot^{-1}(x)$ with $\cot^{-1}(\sqrt{3})=\pi/6$.

1(c). If $|\vec{a}| = 2$, $|\vec{b}| = \sqrt{3}$ and $\vec{a} \cdot \vec{b} = \sqrt{6}$, then the angle between the vectors $\vec{a}$ and $\vec{b}$ is

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$

(D) $\frac{5\pi}{12}$

Correct Answer: (B) $\frac{\pi}{4}$

Solution:

Step 1: Recall the formula.

The dot product of two vectors is $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$, where θ is the angle between them. So $\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$.

Step 2: Put in the given values.

We are given $|\vec{a}| = 2$, $|\vec{b}| = \sqrt{3}$ and $\vec{a} \cdot \vec{b} = \sqrt{6}$.

$$\cos\theta = \frac{\sqrt{6}}{2 \times \sqrt{3}}$$

Step 3: Simplify.

Since $\sqrt{6} = \sqrt{2} \times \sqrt{3}$, the $\sqrt{3}$ cancels from top and bottom.

$$\cos\theta = \frac{\sqrt{2}\sqrt{3}}{2\sqrt{3}} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

Step 4: Find the angle.

We know $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, and the angle between two vectors lies in $[0, \pi]$. So $\theta = \frac{\pi}{4}$.

Step 5: Why the other options are wrong.

For $\theta = \frac{\pi}{6}$, $\cos\theta = \frac{\sqrt{3}}{2}$, which is about 0.866 and not 0.707.

For $\theta = \frac{\pi}{3}$, $\cos\theta = \frac{1}{2}$, which is not 0.707.

For $\theta = \frac{5\pi}{12}$, $\cos\theta$ is about 0.259, which is also not 0.707.

Final Answer:

The angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$, option (B).

$$\boxed{\frac{\pi}{4}}$$

Quick Tip: Use $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\sqrt{6}}{2\sqrt{3}} = \frac{1}{\sqrt{2}}$, so $\theta = \frac{\pi}{4}$.

1(d). The value of the integral $\int \cos^2 x\, dx$ is

(A) $\frac{1}{2}(x + \sin x \cos x) + C$

(B) $\frac{1}{2}(x - \sin x \cos x) + C$

(C) $\frac{x}{2} + \frac{\cos 2x}{4} + C$

(D) $\frac{x}{2} - \frac{\cos 2x}{4} + C$

Correct Answer: (A) $\frac{1}{2}(x+\sin x\cos x)+C$

Solution:

Step 1: Understanding the Concept.

We cannot integrate $\cos^2 x$ directly. The usual trick is to lower the power with the double angle identity, which turns the square into a plain cosine.

Step 2: Key Formula or Approach.

$$\cos 2x = 2\cos^2 x - 1 \quad \Rightarrow \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

Step 3: Rewrite and integrate.

$$\int \cos^2 x\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx = \frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)+C$$

The integral of $\cos 2x$ is $\frac{\sin 2x}{2}$, because we divide by the derivative of the inner function $2x$.

Step 4: Put it in the form of the options.

Since $\sin 2x = 2\sin x\cos x$, we get $\frac{\sin 2x}{2} = \sin x\cos x$. So

$$\int \cos^2 x\,dx = \frac{1}{2}(x+\sin x\cos x)+C$$

This is option (A). It also equals $\frac{x}{2}+\frac{\sin 2x}{4}+C$.

Step 5: Why the other options are wrong.

Differentiate each option and compare with $\cos^2 x$.

Option (B): the derivative is $\frac{1}{2}(1-\cos 2x) = \sin^2 x$, which is not $\cos^2 x$.

Option (C): the derivative is $\frac{1}{2} - \frac{\sin 2x}{2}$, which is not $\cos^2 x$.

Option (D): the derivative is $\frac{1}{2} + \frac{\sin 2x}{2}$, which is not $\cos^2 x$.

Final Answer:

The value of the integral is $\frac{1}{2}(x + \sin x \cos x) + C$, option (A).

$$\boxed{\frac{1}{2}(x + \sin x \cos x) + C}$$

Quick Tip: Write $\cos^2 x = \frac{1+\cos 2x}{2}$, integrate term by term and use $\sin 2x = 2\sin x \cos x$.

1(e). The degree of the differential equation $x\frac{d^2y}{dx^2} = \left(\frac{dy}{dx} + y\right)^{2/3}$ is

(A) 2

(B) 3

(C) 5

(D) 6

Correct Answer: (B) 3

Solution:

Step 1: Understanding the Concept.

The degree of a differential equation is the power of the highest order derivative, but only after the equation has been made a polynomial in

its derivatives. If a derivative sits under a fractional power or a radical, we must first remove that power.

Step 2: Key Formula or Approach.

First find the order, then clear the fractional power $\frac{2}{3}$ by raising both sides to a suitable whole number power.

Step 3: Find the order.

The highest derivative present is $\frac{d^2y}{dx^2}$, so the order is 2.

Step 4: Remove the fractional power.

The power on the right side is $\frac{2}{3}$. The denominator is 3, so we cube both sides.

$$\left(x\frac{d^2y}{dx^2}\right)^3 = \left[\left(\frac{dy}{dx}+y\right)^{2/3}\right]^3$$

$$x^3\left(\frac{d^2y}{dx^2}\right)^3 = \left(\frac{dy}{dx}+y\right)^2$$

Now the equation is a polynomial in the derivatives.

Step 5: Read the degree.

The highest order derivative is $\frac{d^2y}{dx^2}$, and it is raised to the power 3. So the degree is 3.

Step 6: Why the other options are wrong.

The number 2 is the order, not the degree, and also the power of

$\left(\frac{dy}{dx}+y\right)$, which is not the highest order derivative. The number 5 and 6 do not appear as a power of $\frac{d^2y}{dx^2}$ anywhere after clearing the fraction.

Final Answer:

The degree of the equation is 3, option (B).

$$\boxed{3}$$

Quick Tip: Cube both sides to remove the power $\frac{2}{3}$, giving $x^3(y'')^3 = (y'+y)^2$. The highest order derivative y'' has power 3.

2(a). Write $\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$, $|x|<a$ in simplest form.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

The presence of $\sqrt{a^2-x^2}$ suggests the substitution $x=a\sin\theta$, which is the standard trick that clears this kind of square root.

Step 2: Substituting:

Let $x=a\sin\theta$, so $\theta=\sin^{-1}(x/a)$, and $\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2\theta}$ $=a\cos\theta$ (since $|x|<a$ keeps $\cos\theta>0$).

Step 3: Simplifying the expression:

$$\frac{x}{\sqrt{a^2-x^2}}=\frac{a\sin\theta}{a\cos\theta}=\tan\theta, \text{ so } \tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)=\tan^{-1}(\tan\theta)=\theta.$$

Final Answer:

Simplest form = $\boxed{\sin^{-1}\left(\frac{x}{a}\right)}$.

Quick Tip: Substitute $x = a\sin\theta$ to clear the square root, then simplify $\tan^{-1}(\tan\theta)$.

2(b). If $e^y(x+1) = 1$, show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

The relation $e^y(x+1) = 1$ joins x and y. To show a statement about $\frac{d^2y}{dx^2}$ and $\left(\frac{dy}{dx}\right)^2$, we first need both derivatives in terms of x. A neat way is to write y itself as a function of x using logarithms.

Step 2: Key Formula or Approach:

1. If $e^y = k$, then $y = \log k$.
2. $\frac{d}{dx}\log(x+1) = \frac{1}{x+1}$.
3. $\frac{d}{dx}(x+1)^{-1} = -(x+1)^{-2}$.

Step 3: Write y as a function of x.

Divide both sides of $e^y(x+1) = 1$ by $(x+1)$. This is allowed because $e^y(x+1) = 1$ means $x+1$ is never zero.

$$e^y = \frac{1}{x+1}$$

Take the natural logarithm of both sides.

$$y = \log\left(\frac{1}{x+1}\right) = -\log(x+1)$$

Step 4: Find the first derivative.

Differentiate $y = -\log(x+1)$ with respect to x.

$$\frac{dy}{dx} = -\frac{1}{x+1}$$

Squaring this gives the right-hand side of the required result.

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{(x+1)^2}$$

Step 5: Find the second derivative.

Write $\frac{dy}{dx} = -(x+1)^{-1}$ and differentiate again.

$$\frac{d^2y}{dx^2} = -\left[-(x+1)^{-2}\right] = \frac{1}{(x+1)^2}$$

Step 6: Compare both sides.

From Step 4 and Step 5, both $\frac{d^2y}{dx^2}$ and $\left(\frac{dy}{dx}\right)^2$ equal $\frac{1}{(x+1)^2}$.

Final Answer:

The two expressions are equal for every x in the domain, so the statement is proved.

$$\boxed{\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2}$$

Quick Tip: Write $y = -\log(x+1)$ and differentiate twice. Both sides come out as $\frac{1}{(x+1)^2}$.

2(c). Find the general solution of $\frac{dy}{dx} = \sqrt{4-y^2}$, $(-2 < y < 2)$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

The equation $\frac{dy}{dx} = \sqrt{4-y^2}$ has only y on the right side. So the variables can be separated, with all y terms on one side and all x terms on the other. Then we integrate both sides.

Step 2: Key Formula or Approach:

1. Separate the variables: $\frac{dy}{\sqrt{4-y^2}} = dx$.

2. Standard integral: $\int \frac{dy}{\sqrt{a^2-y^2}} = \sin^{-1}\frac{y}{a} + C$.

Step 3: Separate the variables.

Since $-2 < y < 2$, the quantity $4 - y^2$ is positive, so we may divide by $\sqrt{4-y^2}$ safely.

$$\frac{dy}{\sqrt{4-y^2}} = dx$$

Step 4: Integrate both sides.

Write $4 = 2^2$ and use the standard integral with $a = 2$.

$$\int \frac{dy}{\sqrt{2^2-y^2}} = \int dx$$

$$\sin^{-1}\frac{y}{2} = x + C$$

Step 5: Write y in terms of x.

Take sine on both sides.

$$\frac{y}{2} = \sin(x + C) \quad \Rightarrow \quad y = 2\sin(x + C)$$

Here C is an arbitrary constant, so this is the general solution.

Final Answer:

The general solution is $\sin^{-1}\frac{y}{2} = x + C$, which can also be written as follows.

$$\boxed{y = 2\sin(x + C)}$$

Quick Tip: Separate the variables to get $\frac{dy}{\sqrt{4-y^2}} = dx$ and integrate both

sides.

2(d). Show that $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$.

Correct Answer: –

Solution:

Step 1: Understanding the Concept:

The denominator $a^2 + x^2$ is the form that goes with a tangent substitution, because $1 + \tan^2\theta = \sec^2\theta$ turns the sum into a single term. We take a to be a non-zero constant.

Step 2: Key Formula or Approach:

1. Put $x = a\tan\theta$, so $dx = a\sec^2\theta\, d\theta$.
2. Use $1 + \tan^2\theta = \sec^2\theta$.
3. At the end, $\tan\theta = \frac{x}{a}$ gives $\theta = \tan^{-1}\frac{x}{a}$.

Step 3: Substitute.

Replace x and dx in the integral.

$$\int \frac{dx}{a^2+x^2} = \int \frac{a\sec^2\theta\, d\theta}{a^2 + a^2\tan^2\theta}$$

Step 4: Simplify the denominator.

Take a^2 common and use the identity.

$$a^2 + a^2\tan^2\theta = a^2(1 + \tan^2\theta) = a^2\sec^2\theta$$

So the integral becomes

$$\int \frac{a\sec^2\theta}{a^2\sec^2\theta}\,d\theta = \frac{1}{a}\int d\theta$$

Step 5: Integrate and return to x.

$$\frac{1}{a}\int d\theta = \frac{\theta}{a} + C$$

Since $x = a\tan\theta$, we have $\theta = \tan^{-1}\frac{x}{a}$. Putting this back,

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Final Answer:

The integral equals the stated expression, so the result is shown.

$$\boxed{\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C}$$

Quick Tip: Put $x = a\tan\theta$, then $a^2 + x^2 = a^2\sec^2\theta$ and the integral reduces to $\frac{1}{a}\int d\theta$.

2(e). If $P(A) = \frac{6}{11}$, $P(B) = \frac{5}{11}$ and $P(A \cup B) = \frac{7}{11}$, find $P(A \mid B)$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

$P(A \mid B)$ is the probability of A given that B has already happened. It needs the probability of both events happening together, $P(A \cap B)$, which is not given directly. We get it from the union formula.

Step 2: Key Formula or Approach:

1. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
2. $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$, where $P(B) \neq 0$.

Step 3: Find P(A and B).

Put the given values in the union formula.

$$\frac{7}{11} = \frac{6}{11} + \frac{5}{11} - P(A \cap B)$$

$$P(A \cap B) = \frac{6}{11} + \frac{5}{11} - \frac{7}{11} = \frac{4}{11}$$

Step 4: Find the conditional probability.

Use the conditional probability formula with $P(B) = \dfrac{5}{11}$.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{4/11}{5/11} = \frac{4}{11} \times \frac{11}{5} = \frac{4}{5}$$

Final Answer:

Given that B has happened, the chance of A is four in five.

$$\boxed{P(A \mid B) = \frac{4}{5}}$$

Quick Tip: First get $P(A \cap B) = P(A) + P(B) - P(A \cup B) = \frac{4}{11}$, then divide by $P(B)$.

3(a). Is the function $f(x)$ defined by

$$f(x) = \begin{cases} x, & \text{if } x \leq 1 \\ 5, & \text{if } x > 1 \end{cases}$$

continuous at $x = 1$?

Correct Answer: —

Solution:

Step 1: Understanding the Concept.

A function f is continuous at $x = 1$ only when three numbers agree: the left hand limit, the right hand limit, and the value $f(1)$. If any two differ, the function is discontinuous there.

Step 2: Key Formula or Approach.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

Step 3: Value at the point.

The first rule applies for $x \leq 1$, so $x = 1$ is covered by it.

$$f(1) = 1$$

Step 4: Left hand limit.

For x slightly less than 1 we use $f(x) = x$.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$$

Step 5: Right hand limit.

For x slightly more than 1 we use $f(x) = 5$.

$$\lim_{x \to 1^+} f(x) = 5$$

Step 6: Compare.

The left hand limit is 1 and the right hand limit is 5. They are not equal, so the limit at $x = 1$ does not exist. The function makes a jump of $5 - 1 = 4$ at $x = 1$.

Final Answer:

The function is not continuous at $x = 1$.

$$\boxed{\text{No, } f \text{ is discontinuous at } x = 1 \text{ (LHL} = 1,\ \text{RHL} = 5)}$$

Quick Tip: Compare $f(1) = 1$, the left hand limit $\lim_{x \to 1^-} x = 1$ and the right hand limit 5. They are not all equal, so f is discontinuous at $x = 1$.

3(b). Prove that the function defined by $f : R \to \{x \in R : -1 < x < 1\}$, where $f(x) = \frac{2x}{1+|x|}$, $x \in R$ is one-one.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

A function f is one-one when different inputs always give different outputs. To prove it, we assume $f(x_1) = f(x_2)$ for two real numbers x_1, x_2 and show that this forces $x_1 = x_2$.

Step 2: Key Formula or Approach:

The modulus $|x|$ is defined in pieces: $|x| = x$ when $x \geq 0$ and $|x| = -x$ when $x < 0$. So we split the proof into cases by the sign of x_1 and x_2.

$$f(x) = \frac{2x}{1+|x|} = \begin{cases} \frac{2x}{1+x}, & x \geq 0 \\ \frac{2x}{1-x}, & x < 0 \end{cases}$$

Step 3: Case 1, both $x_1, x_2 \geq 0$.

Let $f(x_1) = f(x_2)$. Then

$$\frac{2x_1}{1+x_1} = \frac{2x_2}{1+x_2}$$

Cross-multiplying gives $2x_1(1+x_2) = 2x_2(1+x_1)$, that is

$$2x_1 + 2x_1x_2 = 2x_2 + 2x_1x_2$$

Cancel $2x_1x_2$ from both sides to get $2x_1 = 2x_2$, so $x_1 = x_2$.

Step 4: Case 2, both $x_1, x_2 < 0$.

Let $f(x_1) = f(x_2)$. Then

$$\frac{2x_1}{1 - x_1} = \frac{2x_2}{1 - x_2}$$

Cross-multiplying gives $2x_1(1 - x_2) = 2x_2(1 - x_1)$, that is

$$2x_1 - 2x_1x_2 = 2x_2 - 2x_1x_2$$

Again the product term cancels and $x_1 = x_2$.

Step 5: Case 3, one is non-negative and the other is negative.

Take $x_1 \geq 0$ and $x_2 < 0$. Then $f(x_1) = \frac{2x_1}{1 + x_1} \geq 0$, because the numerator is non-negative and the denominator is positive. But $f(x_2) = \frac{2x_2}{1 - x_2} < 0$, because the numerator is negative and the denominator is positive. A non-negative number can never equal a negative number, so $f(x_1) = f(x_2)$ is impossible here. This case never arises.

Final Answer:

In every possible case, $f(x_1) = f(x_2)$ gives $x_1 = x_2$. So f is one-one.

$$\boxed{f(x_1) = f(x_2) \Rightarrow x_1 = x_2, \text{ so } f \text{ is one-one}}$$

Quick Tip: Assume $f(x_1) = f(x_2)$ and split into cases by sign: both non-negative, both negative, or mixed (mixed is impossible because the signs of f differ). In the first two cases cross-multiplying gives $x_1 = x_2$.

3(c). Find the value of $\int \frac{x\,dx}{(x+1)(x+2)}$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept.

The integrand is a proper rational function, because the numerator has degree 1 and the denominator has degree 2. Such a function is integrated by splitting it into partial fractions.

Step 2: Key Formula or Approach.

Write $\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$ and find A and B.

Step 3: Find A and B.

Multiply both sides by $(x+1)(x+2)$:

$$x = A(x+2) + B(x+1)$$

Put $x = -1$: $-1 = A(1)$, so $A = -1$.

Put $x = -2$: $-2 = B(-1)$, so $B = 2$.

$$\frac{x}{(x+1)(x+2)} = -\frac{1}{x+1} + \frac{2}{x+2}$$

Step 4: Integrate each term.

$$\int \frac{x\,dx}{(x+1)(x+2)} = -\int \frac{dx}{x+1} + 2\int \frac{dx}{x+2} = -\log|x+1| + 2\log|x+2| + C$$

Final Answer:

$$\boxed{2\log|x+2| - \log|x+1| + C}$$

Quick Tip: Split into partial fractions: $\frac{x}{(x+1)(x+2)} = -\frac{1}{x+1} + \frac{2}{x+2}$, then integrate each term.

3(d). Find the position vectors of the points R_1 and R_2, which divide the line joining the points $P(\hat{i} + 2\hat{j} - \hat{k})$ and $Q(-\hat{i} + \hat{j} + \hat{k})$,
(i) internally, (ii) externally in the ratio 2 : 1 respectively.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:
We are given two points by their position vectors and a ratio. We must find the point that cuts the segment PQ in that ratio, once from inside the segment and once from outside it. The section formula for vectors does exactly this.

Step 2: Key Formula or Approach:
Let $\vec{p}$ and $\vec{q}$ be the position vectors of P and Q, and let a point R divide PQ in the ratio $m : n$.
Internal division:

$$\vec{r} = \frac{m\vec{q} + n\vec{p}}{m+n}$$

External division:

$$\vec{r} = \frac{m\vec{q} - n\vec{p}}{m - n}$$

Step 3: Write down the data.

$$\vec{p} = \hat{i} + 2\hat{j} - \hat{k}, \qquad \vec{q} = -\hat{i} + \hat{j} + \hat{k}, \qquad m = 2,\ n = 1$$

Step 4: Part (i), internal division at R_1.

Put $m = 2$ and $n = 1$ in the internal formula.

$$\vec{r_1} = \frac{2(-\hat{i} + \hat{j} + \hat{k}) + 1(\hat{i} + 2\hat{j} - \hat{k})}{2 + 1}$$

Open the brackets in the numerator.

$$\vec{r_1} = \frac{-2\hat{i} + 2\hat{j} + 2\hat{k} + \hat{i} + 2\hat{j} - \hat{k}}{3}$$

Collect the $\hat{i}$, $\hat{j}$ and $\hat{k}$ terms separately.

$$\vec{r_1} = \frac{-\hat{i} + 4\hat{j} + \hat{k}}{3} = -\frac{1}{3}\hat{i} + \frac{4}{3}\hat{j} + \frac{1}{3}\hat{k}$$

Step 5: Part (ii), external division at R_2.

Put $m = 2$ and $n = 1$ in the external formula. The denominator is $m - n = 2 - 1 = 1$.

$$\vec{r_2} = \frac{2(-\hat{i} + \hat{j} + \hat{k}) - 1(\hat{i} + 2\hat{j} - \hat{k})}{2 - 1}$$

$$\vec{r_2} = -2\hat{i} + 2\hat{j} + 2\hat{k} - \hat{i} - 2\hat{j} + \hat{k}$$

$$\vec{r_2} = -3\hat{i} + 0\,\hat{j} + 3\hat{k}$$

Final Answer:

The internal point has position vector $-\frac{1}{3}\hat{i} + \frac{4}{3}\hat{j} + \frac{1}{3}\hat{k}$ and the external point has position vector $-3\hat{i} + 3\hat{k}$.

$$\boxed{\vec{r_1} = \frac{1}{3}(-\hat{i} + 4\hat{j} + \hat{k}), \qquad \vec{r_2} = -3\hat{i} + 3\hat{k}}$$

Quick Tip: Use the section formula: internal $\frac{m\vec{q}+n\vec{p}}{m+n}$, external $\frac{m\vec{q}-n\vec{p}}{m-n}$ with $m = 2, n = 1$.

4(a). Find matrix AB if $A = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

Two matrices can be multiplied only when the number of columns of the first equals the number of rows of the second. Here A is of order 2×3 and B is of order 3×2. The inner numbers match (both are 3), so AB exists and has order 2×2.

Step 2: Key Formula or Approach:

The entry in row i and column j of AB is found by multiplying row i of A with column j of B, term by term, and adding the products.

Step 3: Write the product with the matrices side by side.

$$AB = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$$

Step 4: Find each entry.

Row 1 of A with column 1 of B: $3(2) + (-1)(1) + 3(3) = 6 - 1 + 9 = 14$.

Row 1 of A with column 2 of B: $3(-3) + (-1)(0) + 3(1) = -9 + 0 + 3 = -6$.

Row 2 of A with column 1 of B: $(-1)(2) + 0(1) + 2(3) = -2 + 0 + 6 = 4$.

Row 2 of A with column 2 of B: $(-1)(-3) + 0(0) + 2(1) = 3 + 0 + 2 = 5$.

Step 5: Assemble the answer.

$$AB = \begin{bmatrix} 14 & -6 \\ 4 & 5 \end{bmatrix}$$

Final Answer:

The product is a 2×2 matrix.

$$\boxed{AB = \begin{bmatrix} 14 & -6 \\ 4 & 5 \end{bmatrix}}$$

Quick Tip: Check the orders: 2×3 times 3×2 gives 2×2. Multiply each row of A with each column of B.

4(b). Find the intervals in which the function f given by $f(x) = x^2 - 4x + 6$ is
(i) increasing, (ii) decreasing.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:
A function is increasing where its slope is positive and decreasing where its slope is negative. The slope of f is its derivative $f'(x)$, so we study the sign of $f'(x)$.

Step 2: Differentiate the function.

$$f(x) = x^2 - 4x + 6$$

$$f'(x) = 2x - 4 = 2(x - 2)$$

Step 3: Find the point where the slope is zero.
Put $f'(x) = 0$.

$$2(x - 2) = 0 \Rightarrow x = 2$$

This point splits the real line into two intervals, $(-\infty, 2)$ and $(2, \infty)$.

Step 4: Part (i), where f is increasing.
We need $f'(x) > 0$, that is $2(x-2) > 0$, which gives $x > 2$. On $(2, \infty)$ the slope is positive. So f is increasing on $(2, \infty)$.

Step 5: Part (ii), where f is decreasing.
We need $f'(x) < 0$, that is $2(x-2) < 0$, which gives $x < 2$. On $(-\infty, 2)$ the slope is negative. So f is decreasing on $(-\infty, 2)$.

Final Answer:
The function increases for $x > 2$ and decreases for $x < 2$.

$$\boxed{\text{(i) increasing on } (2, \infty); \quad \text{(ii) decreasing on } (-\infty, 2)}$$

Quick Tip: Find $f'(x) = 2x - 4$. It is positive for $x > 2$ and negative for $x < 2$.

4(c). Find the general solution of the differential equation $\frac{y\,dx - x\,dy}{y} = 0$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept.
The given equation can be rearranged so that all terms in x sit with dx and all terms in y sit with dy. This is the variable separable method.

Step 2: Separate the variables.
The fraction is zero only when its numerator is zero, because $y \neq 0$. So

$$y\,dx - x\,dy = 0 \quad \Rightarrow \quad y\,dx = x\,dy$$

Divide both sides by xy:

$$\frac{dx}{x} = \frac{dy}{y}$$

Step 3: Integrate both sides.

$$\int \frac{dx}{x} = \int \frac{dy}{y} \quad \Rightarrow \quad \log|x| = \log|y| + \log|c|$$

Here we wrote the constant of integration as $\log|c|$ so that the logs combine neatly.

Step 4: Combine the logs.

$$\log|x| = \log|cy| \quad \Rightarrow \quad x = cy$$

Final Answer:

The general solution is a family of straight lines through the origin.

$$\boxed{x = cy}$$

Quick Tip: Cancel y to get $y\,dx = x\,dy$, separate as $\frac{dx}{x} = \frac{dy}{y}$ and integrate to get $x = cy$.

4(d). If a leap year is selected at random, then what is the probability that it will contain 53 Tuesdays?

Correct Answer: –

Solution:

Step 1: Understanding the Concept:

A week has 7 days. A leap year has 366 days, which is 52 full weeks and 2 extra days. The 52 full weeks give exactly 52 Tuesdays. So the year has 53 Tuesdays only when one of the 2 extra days is a Tuesday.

Step 2: Key Formula or Approach:

$$P(E) = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

Step 3: List the possible pairs of extra days.

The two extra days are consecutive days of the week. Starting from the first day of the year, the pair can be one of the following 7 equally likely cases.

(1) Sunday and Monday

(2) Monday and Tuesday

(3) Tuesday and Wednesday

(4) Wednesday and Thursday

(5) Thursday and Friday

(6) Friday and Saturday

(7) Saturday and Sunday

So the total number of outcomes is 7.

Step 4: Count the favourable cases.

Tuesday appears in case (2) and case (3). So the number of favourable

outcomes is 2.

Step 5: Find the probability.

$$P(53 \text{ Tuesdays}) = \frac{2}{7}$$

Final Answer:

The probability that a leap year has 53 Tuesdays is $\frac{2}{7}$.

$$\boxed{\frac{2}{7}}$$

Quick Tip: A leap year has 52 weeks and 2 extra days. Count how many of the 7 possible pairs of extra days contain a Tuesday.

5(a). Find the value of the determinant $\begin{vmatrix} 7 & -9 & 11 \\ 13 & 15 & -8 \\ -6 & 9 & 7 \end{vmatrix}$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

The value of a 3×3 determinant is found by expanding along a row or a column. Each element is multiplied by its minor (the 2×2 determinant left after deleting its row and column), with the signs $+, -, +$ along the first row.

Step 2: Key Formula or Approach:

Expanding along the first row:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Step 3: Substitute the values.

Let Δ be the given determinant.

$$\Delta = 7\big[(15)(7) - (9)(-8)\big] - (-9)\big[(13)(7) - (-6)(-8)\big] + 11\big[(13)(9) - (-6)(15)\big]$$

Step 4: Work out each bracket.

First bracket: $105 + 72 = 177$.

Second bracket: $91 - 48 = 43$.

Third bracket: $117 + 90 = 207$.

Step 5: Multiply and add.

$$\Delta = 7(177) + 9(43) + 11(207)$$

$$\Delta = 1239 + 387 + 2277$$

$$\Delta = 3903$$

Final Answer:

The value of the determinant is 3903.

$$\boxed{3903}$$

Quick Tip: Expand along the first row: $7(177) + 9(43) + 11(207)$. Check the signs carefully.

5(b). Prove that $\tan^{-1} x + \tan^{-1} \frac{2x}{1-x^2} = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$, $|x| < \frac{1}{\sqrt{3}}$.

Correct Answer: —

Solution:

Step 1: Understanding the Question:

We must prove an identity for $|x| < \frac{1}{\sqrt{3}}$. The right side has $3x - x^3$ and $1 - 3x^2$, which are the numerator and denominator of the formula for $\tan 3\theta$ in terms of $\tan\theta$. The left side has $\frac{2x}{1-x^2}$, which is the formula for $\tan 2\theta$. So the substitution $x = \tan\theta$ is the natural choice.

Step 2: Key Formulas:

1. $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$.
2. $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$.
3. $\tan^{-1}(\tan\alpha) = \alpha$ only when $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Step 3: Put $x = \tan\theta$ and fix the range of θ.

Let $x = \tan\theta$, so $\theta = \tan^{-1} x$. Since $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$, we get $-\frac{\pi}{6} < \theta < \frac{\pi}{6}$.

Then 2θ lies in $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ and 3θ lies in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Both are inside the principal range of $\tan^{-1}$, and this is exactly why the condition on x is given.

Step 4: Simplify the second term on the left.

$$\frac{2x}{1-x^2} = \frac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta$$

Because $2\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}(\tan 2\theta) = 2\theta$.

Step 5: Add the two terms.

$$\text{LHS} = \tan^{-1} x + \tan^{-1} \frac{2x}{1-x^2} = \theta + 2\theta = 3\theta$$

Step 6: Simplify the right side.

$$\frac{3x-x^3}{1-3x^2} = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta} = \tan 3\theta$$

Since $3\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we get $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) = \tan^{-1}(\tan 3\theta) = 3\theta$.

Final Answer:

Both sides equal 3θ, so the identity is proved.

$$\boxed{\tan^{-1} x + \tan^{-1} \frac{2x}{1-x^2} = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)}$$

Quick Tip: Put $x = \tan\theta$. The condition gives 3θ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so both sides

reduce to 3θ.

5(c). Find $\frac{dy}{dx}$ of $(\cos x)^{\tan x} + x^y = 0$.

Correct Answer: —

Solution:

Step 1: Understanding the Question:

The equation $(\cos x)^{\tan x} + x^y = 0$ has two terms where both the base and the exponent change. Neither can be differentiated with the power rule alone, so we take the logarithm of each term separately.

Step 2: Key Approach:

Let $u = (\cos x)^{\tan x}$ and $v = x^y$. The equation becomes $u + v = 0$, so

$$\frac{du}{dx} + \frac{dv}{dx} = 0$$

Step 3: Differentiate $u = (\cos x)^{\tan x}$.

Take logarithms: $\log u = \tan x \cdot \log(\cos x)$.

Differentiate both sides with the product rule:

$$\frac{1}{u}\frac{du}{dx} = \sec^2 x \cdot \log(\cos x) + \tan x \cdot \frac{-\sin x}{\cos x} = \sec^2 x \cdot \log(\cos x) - \tan^2 x$$

$$\frac{du}{dx} = (\cos x)^{\tan x}\left[\sec^2 x \log(\cos x) - \tan^2 x\right]$$

Step 4: Differentiate $v = x^y$.

Take logarithms: $\log v = y \log x$. Here y is a function of x, so the product rule gives

$$\frac{1}{v}\frac{dv}{dx} = \frac{dy}{dx} \cdot \log x + y \cdot \frac{1}{x}$$

$$\frac{dv}{dx} = x^y \left[\log x \frac{dy}{dx} + \frac{y}{x}\right]$$

Step 5: Substitute in $\frac{du}{dx} + \frac{dv}{dx} = 0$.

$$(\cos x)^{\tan x}\left[\sec^2 x \log(\cos x) - \tan^2 x\right] + x^y \log x \frac{dy}{dx} + y\, x^{y-1} = 0$$

Step 6: Solve for $\frac{dy}{dx}$.

Move the terms without $\frac{dy}{dx}$ to the right side and divide by $x^y \log x$:

$$\frac{dy}{dx} = -\frac{(\cos x)^{\tan x}\left[\sec^2 x \log(\cos x) - \tan^2 x\right] + y\, x^{y-1}}{x^y \log x}$$

Final Answer:

The derivative is

$$\boxed{\frac{dy}{dx} = -\frac{(\cos x)^{\tan x}\left[\sec^2 x \log(\cos x) - \tan^2 x\right] + y\, x^{y-1}}{x^y \log x}}$$

Quick Tip: Write $u = (\cos x)^{\tan x}$ and $v = x^y$, take logs of each, differentiate, and use $u' + v' = 0$.

5(d). If the position vectors of points A, B, C and D are respectively $\hat{i} + \hat{j} + \hat{k}$, $2\hat{i} + 5\hat{j}$, $3\hat{i} + 2\hat{j} - 3\hat{k}$ and $\hat{i} - 6\hat{j} - \hat{k}$, then find the angle between the lines AB and CD. Prove that AB and CD are collinear.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

The vector along the line joining two points is found by subtracting the position vector of the starting point from that of the end point. The angle between two lines is the angle between their direction vectors, found using the dot product formula $\cos\theta = \dfrac{\vec{AB} \cdot \vec{CD}}{|\vec{AB}||\vec{CD}|}$.

Step 2: Find the Vectors AB and CD:

Position vectors are $A = \hat{i} + \hat{j} + \hat{k}$, $B = 2\hat{i} + 5\hat{j}$, $C = 3\hat{i} + 2\hat{j} - 3\hat{k}$, $D = \hat{i} - 6\hat{j} - \hat{k}$.

Subtract to get the direction of each line.

$$\vec{AB} = B - A = \hat{i} + 4\hat{j} - \hat{k}, \qquad \vec{CD} = D - C = -2\hat{i} - 8\hat{j} + 2\hat{k}$$

Step 3: Compute the Angle:

Find the magnitudes and the dot product of the two vectors.

$$|\vec{AB}| = \sqrt{1 + 16 + 1} = 3\sqrt{2}, \qquad |\vec{CD}| = \sqrt{4 + 64 + 4} = 6\sqrt{2}$$

$$\vec{AB} \cdot \vec{CD} = (1)(-2) + (4)(-8) + (-1)(2) = -36$$

$$\cos\theta = \frac{-36}{3\sqrt{2} \times 6\sqrt{2}} = \frac{-36}{36} = -1 \implies \theta = 180^\circ$$

Step 4: Prove AB and CD are Collinear:

Compare the two vectors found in Step 2.

$$\vec{CD} = -2\hat{i} - 8\hat{j} + 2\hat{k} = -2(\hat{i} + 4\hat{j} - \hat{k}) = -2\vec{AB}$$

Since $\vec{CD}$ is a scalar multiple of $\vec{AB}$, the two vectors are parallel, which means lines AB and CD are collinear.

The scalar -2 is negative, so the vectors point in opposite senses, which is exactly why the angle between them came out as 180° in Step 3.

Final Answer:

The angle between AB and CD is 180 degrees, and CD = -2 AB proves the lines are collinear.

$$\boxed{\theta = 180^\circ, \quad \vec{CD} = -2\vec{AB}}$$

Quick Tip: Find vectors AB and CD by subtraction, then check if one is a scalar multiple of the other.

5(e). Find the particular solution of $2xy + y^2 - 2x^2\frac{dy}{dx} = 0$, given that $y = 2$, when $x = 1$.

Correct Answer: —

Solution:

Step 1: Put the equation in the form $\frac{dy}{dx} = f(x, y)$.

From $2xy + y^2 - 2x^2\frac{dy}{dx} = 0$ we get

$$\frac{dy}{dx} = \frac{2xy + y^2}{2x^2}$$

Step 2: Identify the type.

Every term in the numerator and denominator has degree 2. So the right side depends only on $\frac{y}{x}$, and the equation is homogeneous. We use the substitution $y = vx$.

Step 3: Substitute $y = vx$.

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. The right side becomes

$$\frac{2x(vx) + v^2x^2}{2x^2} = \frac{2v + v^2}{2} = v + \frac{v^2}{2}$$

So

$$v + x\frac{dv}{dx} = v + \frac{v^2}{2} \quad \Rightarrow \quad x\frac{dv}{dx} = \frac{v^2}{2}$$

Step 4: Separate the variables.

$$\frac{dv}{v^2} = \frac{dx}{2x}$$

Step 5: Integrate both sides.

$$-\frac{1}{v} = \frac{1}{2}\log|x| + C$$

Now put back $v = \frac{y}{x}$:

$$-\frac{x}{y} = \frac{1}{2}\log|x| + C$$

Step 6: Use the condition $y = 2$ when $x = 1$.

$$-\frac{1}{2} = \frac{1}{2}\log 1 + C$$

Since $\log 1 = 0$, we get $C = -\frac{1}{2}$.

Step 7: Write the particular solution.

$$-\frac{x}{y} = \frac{1}{2}\log|x| - \frac{1}{2} \quad \Rightarrow \quad \frac{x}{y} = \frac{1 - \log|x|}{2}$$

So $y = \dfrac{2x}{1 - \log|x|}$. Near $x = 1$ the value of x is positive, so $|x| = x$.

Final Answer:

The particular solution is

$$\boxed{y = \frac{2x}{1 - \log x}}$$

Quick Tip: The equation is homogeneous. Put $y = vx$ to get $\frac{dv}{v^2} = \frac{dx}{2x}$, integrate, then use $y(1) = 2$.

6(a). Find the shortest distance between the lines $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$.

Correct Answer: —

Solution:

Step 1: Understanding the Question:

The two lines are not parallel (their direction ratios are different), and they may not meet. We need the length of the shortest segment joining them, which is the shortest distance between two skew lines.

Step 2: Key Formula:

For lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, the shortest distance is

$$d = \frac{\left|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)\right|}{\left|\vec{b}_1 \times \vec{b}_2\right|}$$

Step 3: Read the point and direction of each line.

Line 1: $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$. It passes through $(-1, -1, -1)$, so $\vec{a}_1 = -\hat{i} - \hat{j} - \hat{k}$, and $\vec{b}_1 = 7\hat{i} - 6\hat{j} + \hat{k}$.

Line 2: $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$. It passes through $(3, 5, 7)$, so $\vec{a}_2 = 3\hat{i} + 5\hat{j} + 7\hat{k}$, and $\vec{b}_2 = \hat{i} - 2\hat{j} + \hat{k}$.

Step 4: Find $\vec{a}_2 - \vec{a}_1$.

$$\vec{a}_2 - \vec{a}_1 = (3+1)\hat{i} + (5+1)\hat{j} + (7+1)\hat{k} = 4\hat{i} + 6\hat{j} + 8\hat{k}$$

Step 5: Find $\vec{b}_1 \times \vec{b}_2$.

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= \hat{i}[(-6)(1) - (1)(-2)] - \hat{j}[(7)(1) - (1)(1)] + \hat{k}[(7)(-2) - (-6)(1)]$$

$$= -4\hat{i} - 6\hat{j} - 8\hat{k}$$

Its magnitude is $\sqrt{16 + 36 + 64} = \sqrt{116} = 2\sqrt{29}$.

Step 6: Find the dot product.

$$(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 4(-4) + 6(-6) + 8(-8) = -16 - 36 - 64 = -116$$

Step 7: Put the values in the formula.

$$d = \frac{|-116|}{\sqrt{116}} = \sqrt{116} = 2\sqrt{29}$$

Final Answer:

The shortest distance between the two lines is $2\sqrt{29}$ units, which is about 10.77 units.

$$\boxed{2\sqrt{29} \text{ units}}$$

Quick Tip: Use $d = \dfrac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$ with $\vec{b}_1 \times \vec{b}_2 = (-4, -6, -8)$.

6(b). Minimize $Z = 10(x - 7y + 190)$ subject to the constraints $x + y \leq 8,\ x + y \geq 4,\ x \leq 5,\ y \leq 5,\ x \geq 0,\ y \geq 0,$ by graphical method.

Correct Answer: —

Solution:

Step 1: Understanding the Question:

We must minimize $Z = 10(x - 7y + 190)$ over the region given by the six constraints. In a linear programming problem with a bounded feasible region, the minimum is reached at a corner point.

Step 2: Simplify the objective function.

$$Z = 10x - 70y + 1900$$

Step 3: Draw the lines of the constraints.

Replace each inequality by an equation:

$x + y = 8$ passes through $(8, 0)$ and $(0, 8)$.

$x + y = 4$ passes through $(4, 0)$ and $(0, 4)$.

$x = 5$ is a vertical line and $y = 5$ is a horizontal line.

The conditions $x \geq 0,\ y \geq 0$ keep us in the first quadrant.

Step 4: Mark the feasible region.

$x + y \leq 8$ takes the side of the origin, $x + y \geq 4$ takes the side away from the origin, $x \leq 5$ is to the left of the line $x = 5$, and $y \leq 5$ is below the line $y = 5$. Together they give a bounded six-sided region (a hexagon).

Step 5: Find the corner points.

1. $x + y = 4$ with $y = 0$: $(4, 0)$.
2. $x = 5$ with $y = 0$: $(5, 0)$.
3. $x = 5$ with $x + y = 8$: $(5, 3)$.
4. $y = 5$ with $x + y = 8$: $(3, 5)$.
5. $y = 5$ with $x = 0$: $(0, 5)$.
6. $x + y = 4$ with $x = 0$: $(0, 4)$.

Step 6: Calculate Z at each corner point.

Corner point	$Z = 10x - 70y + 1900$
$(4, 0)$	1940
$(5, 0)$	1950
$(5, 3)$	1740
$(3, 5)$	1580
$(0, 5)$	1550
$(0, 4)$	1620

Step 7: Pick the smallest value.

The least value is 1550, reached at $(0, 5)$. The region is bounded, so this is the minimum.

Final Answer:

The minimum value of Z is 1550 at the point $x = 0,\ y = 5$.

$$\boxed{Z_{\text{min}} = 1550 \text{ at } (0, 5)}$$

Quick Tip: Corner points: $(4, 0), (5, 0), (5, 3), (3, 5), (0, 5), (0, 4)$. Evaluate $Z = 10x - 70y + 1900$ at each and take the smallest.

6(c). Two bags I and II are given. Bag I contains 3 red and 4 black balls whereas bag II contains 5 red and 6 black balls. One ball has been drawn out at random from any one bag, which is of red colour. What is the probability that this ball has been drawn out from bag II ?

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

We are told the ball drawn is red, and we must find the chance that it came from bag II. This is a "reverse" probability, so we use Bayes' theorem.

Step 2: Key Formula or Approach:

Let E_1 be the event "bag I is chosen", E_2 the event "bag II is chosen" and R the event "the ball drawn is red". Bayes' theorem gives

$$P(E_2 \mid R) = \frac{P(E_2)\,P(R \mid E_2)}{P(E_1)\,P(R \mid E_1) + P(E_2)\,P(R \mid E_2)}$$

Step 3: Write down the prior probabilities.

The ball is drawn from "any one bag" chosen at random, so both bags are equally likely.

$$P(E_1) = P(E_2) = \frac{1}{2}$$

Step 4: Write down the likelihoods.

Bag I has 3 red and 4 black balls, 7 balls in all, so

$$P(R \mid E_1) = \frac{3}{7}$$

Bag II has 5 red and 6 black balls, 11 balls in all, so

$$P(R \mid E_2) = \frac{5}{11}$$

Step 5: Substitute in Bayes' theorem.

$$P(E_2 \mid R) = \frac{\frac{1}{2} \times \frac{5}{11}}{\frac{1}{2} \times \frac{3}{7} + \frac{1}{2} \times \frac{5}{11}}$$

The common factor $\frac{1}{2}$ cancels from the top and bottom.

$$P(E_2 \mid R) = \frac{\frac{5}{11}}{\frac{3}{7} + \frac{5}{11}}$$

Add the fractions in the denominator using the common denominator 77.

$$\frac{3}{7} + \frac{5}{11} = \frac{33 + 35}{77} = \frac{68}{77}$$

$$P(E_2 \mid R) = \frac{5}{11} \times \frac{77}{68} = \frac{35}{68}$$

Final Answer:

The probability that the red ball was drawn from bag II is $\frac{35}{68}$.

$$\boxed{\frac{35}{68}}$$

Quick Tip: Use Bayes' theorem with $P(E_1) = P(E_2) = \frac{1}{2}$, $P(R \mid E_1) = \frac{3}{7}$ and $P(R \mid E_2) = \frac{5}{11}$.
The answer is $\frac{35}{68}$.

6(d). The order of the differential equation $x\left(\frac{d^3y}{dx^3}\right)^2 + \sin x\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^4 + y^5 = 0$ is

(A) 2

(B) 3

(C) 5

(D) 6

Correct Answer: (B) 3

Solution:

Step 1: Definition of order:

The order of a differential equation is the order of the highest derivative appearing in it, regardless of the power it is raised to.

Step 2: Scanning the equation:

The derivatives present are $\frac{d^3y}{dx^3}$ (third order), $\frac{d^2y}{dx^2}$ (second order) and $\frac{dy}{dx}$ (first order).

Step 3: Picking the highest:

The highest-order derivative present is $\frac{d^3y}{dx^3}$.

Final Answer:

$$\boxed{\text{Order} = 3}$$

Quick Tip: Order = highest derivative level present, ignoring the powers on each term.

6(e). For the vectors $\vec{a}$ and $\vec{b}$, prove that $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept.

This is the triangle inequality for vectors: one side of a triangle is

never longer than the sum of the other two. To prove it, we compare the squares of both sides. Squaring is safe here because both $|\vec{a}+\vec{b}|$ and $|\vec{a}|+|\vec{b}|$ are non-negative.

Step 2: Key Formula or Approach.

1. $|\vec{v}|^2 = \vec{v}\cdot\vec{v}$ for any vector $\vec{v}$.
2. $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$, where θ is the angle between the vectors.

Step 3: Expand the left side.

$$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = |\vec{a}|^2 + 2\,\vec{a}\cdot\vec{b} + |\vec{b}|^2$$

Step 4: Bound the dot product.

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta \le |\vec{a}|\,|\vec{b}|$$

This is true because $\cos\theta \le 1$ and $|\vec{a}|\,|\vec{b}| \ge 0$.

Step 5: Use the bound.

$$|\vec{a}+\vec{b}|^2 \le |\vec{a}|^2 + 2\,|\vec{a}|\,|\vec{b}| + |\vec{b}|^2 = \left(|\vec{a}|+|\vec{b}|\right)^2$$

Step 6: Take square roots.

Both sides are non-negative, so taking the positive square root keeps the inequality.

$$|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$$

Equality holds when $\cos\theta = 1$, that is when the two vectors point the same way.

Final Answer:

Hence the result is proved.

$$\boxed{|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|}$$

Quick Tip: Expand $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2$, use $\vec{a}\cdot\vec{b} \le |\vec{a}||\vec{b}|$ to get $(|\vec{a}| + |\vec{b}|)^2$, then take square roots.

7(a). Solve the following system of equations by matrix method: $\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4, \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1, \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2.$

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

The equations are not linear in x, y, z because of the reciprocal terms. Substitute $u = \frac{1}{x}, v = \frac{1}{y}, w = \frac{1}{z}$ to convert the system into a linear system in u, v, w.

Step 2: Writing the linear system:

$$2u + 3v + 10w = 4, \quad 4u - 6v + 5w = 1, \quad 6u + 9v - 20w = 2$$

In matrix form $MU = B$, where $M = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$, $U = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$.

Step 3: Finding det(M):

Expand along the first row.

$$\det M = 2[(-6)(-20) - 5(9)] - 3[4(-20) - 5(6)] + 10[4(9) - (-6)(6)]$$

$$= 2(75) - 3(-110) + 10(72) = 150 + 330 + 720 = 1200$$

Since $\det M \neq 0$, M is invertible and the system has a unique solution.

Step 4: Finding the adjoint and inverse:

Computing all cofactors gives:

$$\text{adj}(M) = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$M^{-1} = \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

Step 5: Solving for u, v, w:

$$U = M^{-1}B \implies u = \frac{75(4) + 150(1) + 75(2)}{1200} = \frac{600}{1200} = \frac{1}{2}$$

$$v = \frac{110(4) - 100(1) + 30(2)}{1200} = \frac{400}{1200} = \frac{1}{3}, \quad w = \frac{72(4) + 0(1) - 24(2)}{1200} = \frac{240}{1200} = \frac{1}{5}$$

Step 6: Converting back to x, y, z:

Since $u = \frac{1}{x}, v = \frac{1}{y}, w = \frac{1}{z}$:

$$x = \frac{1}{u} = 2, \quad y = \frac{1}{v} = 3, \quad z = \frac{1}{w} = 5$$

Final Answer:

These values satisfy all three original equations.

$$\boxed{x = 2,\ y = 3,\ z = 5}$$

Quick Tip: Substitute u = 1/x, v = 1/y, w = 1/z to make the system linear, then use the matrix inverse method.

OR

Find the inverse of the following matrix :

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$$

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

A square matrix A has an inverse only when $|A| \neq 0$. When it exists, the inverse is found with the adjoint (the transpose of the cofactor matrix).

Step 2: Key Formula or Approach:

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$$

where $\operatorname{adj} A$ is the transpose of the matrix of cofactors C_{ij} $= (-1)^{i+j} M_{ij}$.

Step 3: Find the determinant of A.

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$$

Expand along the first row.

$$|A| = 0\,(2 \cdot 1 - 3 \cdot 1) - 1\,(1 \cdot 1 - 3 \cdot 3) + 2\,(1 \cdot 1 - 2 \cdot 3)$$

$$|A| = 0 - 1(1 - 9) + 2(1 - 6) = 8 - 10 = -2$$

Since $|A| = -2 \neq 0$, the inverse exists.

Step 4: Find all nine cofactors.

$$C_{11} = +(2 \cdot 1 - 3 \cdot 1) = -1, \quad C_{12} = -(1 \cdot 1 - 3 \cdot 3) = 8, \quad C_{13} = +(1 \cdot 1 - 2 \cdot 3) = -5$$

$$C_{21} = -(1 \cdot 1 - 2 \cdot 1) = 1, \quad C_{22} = +(0 \cdot 1 - 2 \cdot 3) = -6, \quad C_{23} = -(0 \cdot 1 - 1 \cdot 3) = 3$$

$$C_{31} = +(1 \cdot 3 - 2 \cdot 2) = -1, \quad C_{32} = -(0 \cdot 3 - 2 \cdot 1) = 2, \quad C_{33} = +(0 \cdot 2 - 1 \cdot 1) = -1$$

Step 5: Write the adjoint.

The cofactor matrix is $\begin{pmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{pmatrix}$. The adjoint is its transpose:

$$\operatorname{adj} A = \begin{pmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{pmatrix}$$

Step 6: Divide by the determinant.

$$A^{-1} = \frac{1}{-2} \begin{pmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

Step 7: Quick check.

Multiply the first row of A by the first column of A^{-1}: $0 \cdot \frac{1}{2} + 1 \cdot (-4) + 2 \cdot \frac{5}{2} = -4 + 5 = 1$. Multiply the first row of A by the second column: $0 + 3 - 3 = 0$. These match the identity matrix.

Final Answer:

The inverse of A is

$$\boxed{A^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{pmatrix}}$$

Quick Tip: Find $|A| = -2$, then the cofactors, then use $A^{-1} = \frac{1}{|A|}\text{adj}\, A$.

Result: $A^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$.

8(a). Prove that the height of the cylinder of given surface and maximum volume is equal to the diameter of its base.

Correct Answer: —

Solution:

Step 1: Understanding the Concept:

The surface area of the cylinder is fixed, and we want the shape that gives the largest volume. We write the volume in one variable (the radius), find where its derivative is zero, and then confirm it is a maximum.

Step 2: Key Formula or Approach:

Let the radius be r, the height h and the (given, constant) total surface area S.

$$S = 2\pi r^2 + 2\pi rh, \qquad V = \pi r^2 h$$

Step 3: Express h in terms of r and S.

From the surface area formula,

$$2\pi rh = S - 2\pi r^2 \quad \Rightarrow \quad h = \frac{S - 2\pi r^2}{2\pi r}$$

Step 4: Write V as a function of r alone.

$$V = \pi r^2 \cdot \frac{S - 2\pi r^2}{2\pi r} = \frac{r(S - 2\pi r^2)}{2} = \frac{Sr - 2\pi r^3}{2}$$

Step 5: Differentiate and set the derivative to zero.

Here S is a constant.

$$\frac{dV}{dr} = \frac{S - 6\pi r^2}{2}$$

For a maximum or minimum, $\frac{dV}{dr} = 0$, so

$$S = 6\pi r^2$$

Step 6: Find the height at this radius.

Put $S = 6\pi r^2$ in $S = 2\pi r^2 + 2\pi rh$:

$$6\pi r^2 = 2\pi r^2 + 2\pi rh \quad \Rightarrow \quad 4\pi r^2 = 2\pi rh \quad \Rightarrow \quad h = 2r$$

Step 7: Check that it is a maximum.

Differentiate once more:

$$\frac{d^2V}{dr^2} = -6\pi r$$

Since $r > 0$, $\frac{d^2V}{dr^2} < 0$. So V is maximum when $S = 6\pi r^2$, that is when $h = 2r$.

Final Answer:

At maximum volume the height $h = 2r$, which is the diameter of the base. Hence the height of the cylinder of given surface area and maximum volume equals the diameter of its base.

$$\boxed{h = 2r}$$

Quick Tip: Write $V = \frac{Sr - 2\pi r^3}{2}$ using $S = 2\pi r^2 + 2\pi rh$.
Setting $\frac{dV}{dr} = 0$ gives $S = 6\pi r^2$, so $h = 2r$.

OR

Evaluate: $\int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x}$

Correct Answer: —

Solution:

Step 1: Key Formula or Approach:

Use the property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. Let I denote the given integral and apply this with $a = \pi$.

Step 2: Applying the property:

Let $I = \int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x}$. Replacing x by $\pi - x$: since $\cos(\pi$

$-x) = -\cos x$ and $\sin(\pi - x) = \sin x$, and both appear squared, the denominator is unchanged:

$$I = \int_0^{\pi} \frac{(\pi - x)\,dx}{a^2\cos^2 x + b^2\sin^2 x}$$

Step 3: Adding the two forms of I:

$$2I = \int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x} + \int_0^{\pi} \frac{(\pi - x)\,dx}{a^2\cos^2 x + b^2\sin^2 x} = \pi\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

Step 4: Evaluating $J = \int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$:

By the symmetry of $\cos^2 x$ and $\sin^2 x$ about $x = \pi/2$, $\int_0^{\pi} = 2\int_0^{\pi/2}$.

Using the standard result $\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{2ab}$ (divide numerator and denominator by $\cos^2 x$, substitute $t = \tan x$), we get $J = 2 \cdot \frac{\pi}{2ab} = \frac{\pi}{ab}$.

Step 5: Solving for I:

$$2I = \pi \cdot \frac{\pi}{ab} = \frac{\pi^2}{ab} \implies I = \frac{\pi^2}{2ab}$$

Final Answer:

$$I = \frac{\pi^2}{2ab}$$

$$\boxed{\frac{\pi^2}{2ab}}$$

Quick Tip: Use $\int f(x)dx = \int f(a-x)dx$ over $[0,\pi]$ to remove x, then evaluate the resulting symmetric integral.

9(a). Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Correct Answer: –

Solution:

Step 1: Understanding the Concept:

By symmetry, the ellipse is symmetric about both axes, so the total area is 4 times the area in the first quadrant.

Step 2: Expressing y in terms of x:

From $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, solving for the first-quadrant branch:

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

Step 3: Setting up the area integral:

$$\text{Area} = 4\int_0^a y\,dx = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

Step 4: Using the standard result $\int_0^a \sqrt{a^2 - x^2}\, dx = \frac{\pi a^2}{4}$:

$$\text{Area} = \frac{4b}{a} \cdot \frac{\pi a^2}{4} = \pi ab$$

Final Answer:

The area enclosed by the ellipse is πab square units.

$$\boxed{\pi ab}$$

Quick Tip: Area = 4×(first-quadrant area) = (4b/a)∫√(a²−x²)dx from 0 to a.

OR

Find the value of $\int \frac{x}{(x^2+1)(x-1)}\, dx$.

Correct Answer: —

Solution:

Step 1: Understanding the Concept.

The integrand is a proper rational function whose denominator has a linear factor $(x-1)$ and an irreducible quadratic factor (x^2+1). We split it into partial fractions of the matching form and integrate term by term.

Step 2: Key Formula or Approach.

$$\frac{x}{(x^2+1)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

Step 3: Find A, B and C.

Multiply both sides by $(x^2+1)(x-1)$:

$$x = A(x^2+1) + (Bx+C)(x-1)$$

Put $x = 1$: $1 = 2A$, so $A = \frac{1}{2}$.

Expand the right side: $x = (A+B)x^2 + (C-B)x + (A-C)$.

Compare the coefficient of x^2: $A + B = 0$, so $B = -\frac{1}{2}$.

Compare the constant term: $A - C = 0$, so $C = \frac{1}{2}$.

Check the coefficient of x: $C - B = \frac{1}{2} + \frac{1}{2} = 1$, which matches.

$$\frac{x}{(x^2+1)(x-1)} = \frac{1}{2(x-1)} + \frac{-x+1}{2(x^2+1)}$$

Step 4: Integrate each part.

$$\int \frac{x\,dx}{(x^2+1)(x-1)} = \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{x\,dx}{x^2+1} + \frac{1}{2}\int \frac{dx}{x^2+1}$$

For the middle term, put $x^2 + 1 = t$, so $x\,dx = \frac{dt}{2}$ and $\int \frac{x\,dx}{x^2+1}$ $= \frac{1}{2}\log(x^2+1)$.

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log(x^2+1) + \frac{1}{2}\tan^{-1}x + C$$

Final Answer:

$$\boxed{\frac{1}{2}\log|x-1| - \frac{1}{4}\log(x^2+1) + \frac{1}{2}\tan^{-1}x + C}$$

Quick Tip: Use partial fractions $\frac{x}{(x^2+1)(x-1)} = \frac{1/2}{x-1} + \frac{(-x+1)/2}{x^2+1}$ and integrate each term.