Bihar Board Class 12 Mathematics Set C 2026

Question Paper with Solutions

Conducted by the Bihar School Examination Board (BSEB)

Time Allowed: 3 hrs 15 min **Maximum Marks: 100**

General Instructions

(i) This question booklet is divided into two sections: **Section A** and **Section B**. The paper has **138 questions** (100 + 30 + 8) and carries **100 marks**.

(ii) In **Section A**, there are 100 objective type questions, out of which **any 50 questions** are to be answered, each carrying **1 mark**. If more than 50 questions are answered, only the first 50 answers are evaluated. The answers are marked on the OMR answer sheet.

(iii) In **Section B**, there are 30 short answer type questions, out of which **any 15 questions** are to be answered, each carrying **2 marks**.

(iv) Apart from these, Section B has 8 long answer type questions, out of which **any 4 questions** are to be answered, each carrying **5 marks**.

(v) Candidates must enter the 10-digit Question Booklet Serial No. on the OMR answer sheet and should give their answers in their own words as far as practicable.

(vi) Figures in the right-hand margin indicate full marks. 15 minutes of extra time have been allotted to read the questions carefully.

Section A: Objective Type Questions (any 50 to be answered, 1 mark each)

1. $\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx =$ [1 mark]

(A) $2(e-1)$

(B) $e-1$

(C) $2(e+1)$

(D) $e+1$

Correct Answer: (A) $2(e-1)$

Solution:

Concept:

- The integrand $\frac{e^{\sqrt{x}}}{\sqrt{x}}$ contains $\sqrt{x}$ inside an exponent and also in the denominator.
- This suggests the substitution $u = \sqrt{x}$.

Step 1: Substitute Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}\,dx$, so $\frac{dx}{\sqrt{x}} = 2\,du$.
Change the limits: when $x = 0$, $u = 0$; when $x = 1$, $u = 1$.

$$\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int_0^1 2e^u\,du$$

Step 2: Integrate

$$2\Big[e^u\Big]_0^1 = 2\big(e^1 - e^0\big) = 2(e-1)$$

Step 3: Why the other options are wrong Option (B) $e-1$ forgets the factor 2 that comes from $\frac{dx}{\sqrt{x}} = 2\,du$.
Options (C) $2(e+1)$ and (D) $e+1$ add $e^0 = 1$ instead of subtracting it at the lower limit. A definite integral always takes upper value minus lower value.

Step 4: Conclusion The integral equals $2(e-1)$, which is option (A).

Quick Tip to Solve: Put $u = \sqrt{x}$, so that $\frac{dx}{\sqrt{x}} = 2\,du$, and change the limits.

2. $\int_0^1 xe^x\,dx =$ [1 mark]

(A) 1

(B) 0

(C) 2

(D) -1

Correct Answer: (A) 1

Solution:

Concept:

- The integrand xe^x is a product of a polynomial and an exponential, so we use integration by parts:

- We pick $u = x$ because it becomes simpler on differentiation, and $dv = e^x\,dx$.

Step 1: Apply integration by parts With $u = x$ and $dv = e^x\,dx$, we get $du = dx$ and $v = e^x$.

$$\int_0^1 xe^x\,dx = \Big[xe^x\Big]_0^1 - \int_0^1 e^x\,dx$$

Step 2: Evaluate each part

$$\Big[xe^x\Big]_0^1 = 1 \cdot e - 0 = e$$

$$\int_0^1 e^x\,dx = e - 1$$

$$\int_0^1 xe^x\,dx = e - (e-1) = 1$$

Step 3: Why the other options are wrong The integrand xe^x is positive on $(0, 1]$, so the integral cannot be 0 or negative. This rules out (B) and (D).
Also $xe^x \le e$ on $[0, 1]$, and in fact much smaller for most of the interval, so the area cannot reach 2. This rules out (C).

Step 4: Conclusion The integral equals 1, which is option (A).

Quick Tip to Solve: Use integration by parts with $u = x$ and $dv = e^x\,dx$, then put in the limits 0 and 1.

3. $\int_1^4 \frac{dx}{\sqrt{x}} =$ [1 mark]

(A) 1

(B) -2

(C) 2

(D) -1

Correct Answer: (C) 2

Solution:

Concept:

- Rewrite $\frac{1}{\sqrt{x}}$ as a power: $x^{-1/2}$.
- Then use the power rule $\int x^n\,dx = \frac{x^{n+1}}{n+1}$ for $n \neq -1$.

Step 1: Find the antiderivative Here $n = -\frac{1}{2}$, so $n + 1 = \frac{1}{2}$.

$$\int x^{-1/2}\,dx = \frac{x^{1/2}}{1/2} = 2\sqrt{x}$$

Step 2: Apply the limits

$$\int_1^4 \frac{dx}{\sqrt{x}} = \Big[2\sqrt{x}\Big]_1^4 = 2\sqrt{4} - 2\sqrt{1} = 4 - 2 = 2$$

Step 3: Why the other options are wrong Option (A) 1 comes from writing the antiderivative as $\sqrt{x}$, which drops the factor $\frac{1}{n+1} = 2$.

Options (B) -2 and (D) -1 are negative. The integrand $\frac{1}{\sqrt{x}}$ is positive and the limits go upward from 1 to 4, so the integral must be positive.

Step 4: Conclusion The integral equals 2, which is option (C).

Quick Tip to Solve: Write $\frac{1}{\sqrt{x}}$ as $x^{-1/2}$, use the power rule, then put in the limits 1 and 4.

---

4. $\int_0^a \sqrt{a^2 - x^2}\, dx =$ [1 mark]

(A) $\frac{\pi}{4}$

(B) $\frac{a^2}{4}$

(C) $\frac{\pi a^2}{4}$

(D) π

Correct Answer: (C) $\frac{\pi a^2}{4}$

Solution:

Concept:

- We have to evaluate a definite integral of $\sqrt{a^2 - x^2}$ from 0 to a, where a is a positive constant.
- The integrand has the form of a circle, so a trigonometric substitution works well.

Step 1: Key Formula or Approach Put $x = a\sin\theta$. Then $dx = a\cos\theta\, d\theta$ and $\sqrt{a^2 - x^2} = a\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$. We also use $\cos^2\theta = \frac{1 + \cos 2\theta}{2}$.

Step 2: Change the limits When $x = 0$, $\sin\theta = 0$, so $\theta = 0$.
When $x = a$, $\sin\theta = 1$, so $\theta = \frac{\pi}{2}$.

Step 3: Rewrite and integrate

$$I = \int_0^{\pi/2} a\cos\theta \cdot a\cos\theta\, d\theta = a^2 \int_0^{\pi/2} \cos^2\theta\, d\theta = \frac{a^2}{2}\int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta$$

$$I = \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{a^2}{2}\left[\left(\frac{\pi}{2} + 0\right) - (0 + 0)\right] = \frac{\pi a^2}{4}$$

Step 4: Why the other options are wrong Option (1), $\frac{\pi}{4}$, is the value only when $a = 1$. It has no a in it, but the integral depends on a.

Option (2), $\frac{a^2}{4}$, leaves out the factor π that comes from the limit $\frac{\pi}{2}$.

Option (4), π, has no a and is the wrong size.

Step 5: Conclusion The integral equals $\frac{\pi a^2}{4}$, which is option (3).

Quick Tip to Solve: The curve $y = \sqrt{a^2 - x^2}$ is the upper half of a circle of radius a. The integral from 0 to a is one quarter of the circle's area.

5. $\int_{-2}^{2} |x|\, dx =$ [1 mark]

(A) 4

(B) 3

(C) 2

(D) 0

Correct Answer: (A) 4

Solution:

Concept:

- The function $|x|$ equals x when $x \geq 0$ and $-x$ when $x < 0$.
- So the integral cannot be done in one piece. We split the interval at $x = 0$, where the rule changes.

Step 1: Split the integral

$$\int_{-2}^{2} |x|\, dx = \int_{-2}^{0} (-x)\, dx + \int_{0}^{2} x\, dx$$

Step 2: Evaluate each part First part: $\int_{-2}^{0}(-x)\, dx = [-\frac{x^2}{2}]_{-2}^{0} = 0 - (-\frac{4}{2}) = 2.$

Second part: $\int_{0}^{2} x\, dx = [\frac{x^2}{2}]_{0}^{2} = \frac{4}{2} - 0 = 2.$

$$\int_{-2}^{2} |x|\, dx = 2 + 2 = 4$$

Step 3: Why the other options are wrong Option (B) 3 and option (D) 0 do not match the sum $2 + 2$. Option (D) comes from forgetting the modulus: $\int_{-2}^{2} x\, dx = 0$ because x is an odd function, but $|x|$ is never negative, so its integral cannot be 0.
Option (C) 2 is only one of the two equal halves. The area on the left of the y-axis is also 2, so both halves must be added.

Step 4: Conclusion The value of the integral is 4, which is option (A).

Quick Tip to Solve: Split the interval at $x = 0$ since $|x|$ changes its rule there, or use the fact that $|x|$ is an even function.

6. The order and degree of the differential equation $(\frac{d^2s}{dt^2})^2 + (\frac{ds}{dt})^3 + 4 = 0$ **is** [1 mark]

(A) order = 2, degree = 1

(B) order = 2, degree = 2

(C) order = 1, degree = 2

(D) order = 1, degree = 1

Correct Answer: (B) order = 2, degree = 2

Solution:

Step 1: Recall order and degree The order of a differential equation is the order of the highest derivative in it.
The degree is the power of that highest-order derivative, once the equation is written as a polynomial in the derivatives.

Step 2: Find the order The derivatives present are $\frac{d^2s}{dt^2}$ and $\frac{ds}{dt}$. The highest one is the second derivative.
So the order is 2.

Step 3: Find the degree The equation $(\frac{d^2s}{dt^2})^2+(\frac{ds}{dt})^3+4=0$ is already a polynomial in the derivatives, with no roots or fractions of derivatives.
The highest-order derivative $\frac{d^2s}{dt^2}$ appears with power 2. The cube on $\frac{ds}{dt}$ does not matter, because it belongs to a lower-order derivative.
So the degree is 2.

Step 4: Check the options Option (A) says degree 1, which ignores the square on the second derivative. Option (C) gives order 1, but a second derivative is present. Option (D) is wrong on both counts. Only option (B), order = 2 and degree = 2, matches.

Step 5: Conclusion The equation has order 2 and degree 2, which is option (B).

Quick Tip to Solve: Order is the highest derivative present. Degree is the power of that highest-order derivative.

7. The integrating factor of the differential equation $(1+x^2)\frac{dy}{dx}+y=e^{\tan^{-1}x}$ is [1 mark]

(A) $e^{\tan^{-1}x}$

(B) $e^{\sin^{-1}x}$

(C) $\tan^{-1}x$

(D) $\sin^{-1}x$

Correct Answer: (A) $e^{\tan^{-1}x}$

Solution:

Step 1: Write the equation in standard linear form A first-order linear equation has the form $\frac{dy}{dx}+Py=Q$, and its integrating factor is $e^{\int P\,dx}$.
The given equation is $(1+x^2)\frac{dy}{dx}+y=e^{\tan^{-1}x}$. The coefficient of $\frac{dy}{dx}$ must be 1, so we divide every term by $1+x^2$.

$$\frac{dy}{dx}+\frac{1}{1+x^2}y=\frac{e^{\tan^{-1}x}}{1+x^2}$$

Comparing, $P=\frac{1}{1+x^2}$.

Step 2: Integrate P

$$\int P\,dx=\int\frac{dx}{1+x^2}=\tan^{-1}x$$

No constant is needed here, since any one integrating factor is enough.

Step 3: Write the integrating factor

$$\text{I.F.} = e^{\int P\,dx} = e^{\tan^{-1} x}$$

Step 4: Check the options Option (B) $e^{\sin^{-1} x}$ would come from $P = \dfrac{1}{\sqrt{1-x^2}}$, which is not our P.
Options (C) and (D) are plain functions, but an integrating factor is always an exponential $e^{\int P\,dx}$. They also leave out the exponential.
So only option (A) fits.

Step 5: Conclusion The integrating factor is $e^{\tan^{-1} x}$, option (A).

Quick Tip to Solve: Divide by $1 + x^2$ to get $P = \dfrac{1}{1+x^2}$, then use I.F. $= e^{\int P\,dx}$.

---

8. The solution of differential equation $\dfrac{dy}{dx} = e^{x+y}$ is [1 mark]

(A) $e^x + e^{-y} = k$

(B) $e^x + e^y = k$

(C) $e^{-x} + e^y = k$

(D) $e^{-x} + e^{-y} = k$

Correct Answer: (A) $e^x + e^{-y} = k$

Solution:

Concept:

- The equation has x and y mixed only through a product after we split the exponent.
- Such an equation can be written as (function of y) dy = (function of x) dx.
- This is the variable separable method.

Step 1: Key Formula or Approach 1. Use the law of exponents $e^{x+y} = e^x \cdot e^y$.
2. Move all y terms to the left and all x terms to the right, then integrate both sides.
3. Use $\int e^{ax}\,dx = \dfrac{e^{ax}}{a} + C$.

Step 2: Separate the variables. Split the exponent:

$$\frac{dy}{dx} = e^x \cdot e^y$$

Divide both sides by e^y and multiply by dx:

$$e^{-y}\,dy = e^x\,dx$$

Step 3: Integrate both sides. For the left side take $a = -1$, so $\int e^{-y}\,dy = -e^{-y}$. The right side gives $\int e^x\,dx = e^x$.

$$-e^{-y} = e^x + C$$

Bring both exponential terms to one side:

$$e^x + e^{-y} = -C$$

Since $-C$ is just another arbitrary constant, write it as k:

$$e^x + e^{-y} = k$$

Step 4: Why the other options are wrong. Option (B) $e^x + e^y = k$: differentiating gives $e^x + e^y \frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -e^{x-y}$, which is not e^{x+y}.

Option (C) $e^{-x} + e^y = k$: differentiating gives $-e^{-x} + e^y \frac{dy}{dx} = 0$, so $\frac{dy}{dx} = e^{-x-y}$, not e^{x+y}.

Option (D) $e^{-x} + e^{-y} = k$: differentiating gives $-e^{-x} - e^{-y} \frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -e^{y-x}$, not e^{x+y}.

Option (A): differentiating gives $e^x - e^{-y} \frac{dy}{dx} = 0$, so $\frac{dy}{dx} = e^{x+y}$, which matches the given equation.

Step 5: Conclusion The general solution is $e^x + e^{-y} = k$, which is option (A).

Quick Tip to Solve: Write $e^{x+y} = e^x e^y$ and separate the variables to get $e^{-y}\,dy = e^x\,dx$. Then integrate both sides.

---

9. The solution of differential equation $x\frac{dy}{dx} = \cot y$ is [1 mark]

(A) $x \cos y = k$

(B) $x \tan y = k$

(C) $x \sec y = k$

(D) $x \sin y = k$

Correct Answer: (A) $x \cos y = k$

Solution:

Step 1: Separate the variables The equation is $x\frac{dy}{dx} = \cot y$. We keep all y terms with dy and all x terms with dx.

$$\frac{dy}{\cot y} = \frac{dx}{x} \quad \Rightarrow \quad \tan y\,dy = \frac{dx}{x}$$

Step 2: Integrate both sides We use $\int \tan y\,dy = -\ln|\cos y|$ and $\int \frac{dx}{x} = \ln|x|$.

$$-\ln|\cos y| = \ln|x| + C$$

Step 3: Combine the logarithms Move the log of cosine to the right side.

$$0 = \ln|x| + \ln|\cos y| + C \quad \Rightarrow \quad \ln|x \cos y| = -C$$

Taking the exponential, $x \cos y = k$, where $k = \pm e^{-C}$ is an arbitrary constant.

Step 4: Check by differentiating Differentiate $x \cos y = k$ with respect to x: $\cos y - x \sin y \frac{dy}{dx} = 0$.

So $x\frac{dy}{dx} = \frac{\cos y}{\sin y} = \cot y$, which is the given equation.

The other options give $x\frac{dy}{dx} = -\sin y \cos y$ for (B), $-\cot y$ for (C) and $-\tan y$ for (D). None of these match, so they are wrong.

Step 5: Conclusion The solution is $x \cos y = k$, option (A).

Quick Tip to Solve: Separate the variables to get $\tan y\, dy = \dfrac{dx}{x}$ and integrate both sides.

10. If the operation $*$ is defined as $a * b = a + 2b$, then $(2 * 3) * 4$ is [1 mark]

(A) 30

(B) 20

(C) 16

(D) 15

Correct Answer: (C) 16

Solution:

Concept:

- The symbol $*$ is not ordinary multiplication. It is defined by the rule $a * b = a + 2b$, so we add the first number to twice the second.
- We evaluate the bracket first, then use that result as the first number of the second operation.

Step 1: Evaluate the bracket

$$2 * 3 = 2 + 2(3) = 2 + 6 = 8$$

Step 2: Apply the operation again Now we need $8 * 4$.

$$8 * 4 = 8 + 2(4) = 8 + 8 = 16$$

Step 3: Check the options The result is 16, which is option (C).
Options (A) 30, (B) 20 and (D) 15 are not equal to 16. If we mistakenly used the other bracket, $2 * (3 * 4) = 2 + 2(3 + 8) = 24$, which is also not an option. This shows the operation is not associative, so the bracket order matters.

Step 4: Conclusion $(2 * 3) * 4 = 16$, option (C).

Quick Tip to Solve: Work out the bracket $2 * 3$ first using $a + 2b$, then apply the rule again with 4.

11. $\tan^{-1}\left(-\dfrac{1}{\sqrt{3}}\right) =$ [1 mark]

(A) $\dfrac{\pi}{3}$

(B) $\dfrac{\pi}{6}$

(C) $-\dfrac{\pi}{3}$

(D) $-\dfrac{\pi}{6}$

Correct Answer: (D) $-\dfrac{\pi}{6}$

Solution:

Concept:

- The principal value branch of $\tan^{-1} x$ has range $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.
- So the answer must be an angle in this open interval whose tangent is the given number.

Step 1: Key Formula or Approach $\tan^{-1}(-x) = -\tan^{-1}(x)$, because tangent is an odd function. Also $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.

Step 2: Detailed Explanation Pull the minus sign outside:

$$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Since $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ and $\frac{\pi}{6}$ lies in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have $\tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$.

$$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

Step 3: Why the other options are wrong Option (1), $\frac{\pi}{3}$: $\tan\frac{\pi}{3} = \sqrt{3}$, not $-\frac{1}{\sqrt{3}}$.

Option (2), $\frac{\pi}{6}$: the magnitude is right but the sign is wrong, since $\tan\frac{\pi}{6} = +\frac{1}{\sqrt{3}}$.

Option (3), $-\frac{\pi}{3}$: $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$, which is not $-\frac{1}{\sqrt{3}}$.

Step 4: Conclusion The value is $-\frac{\pi}{6}$, which is option (4).

Quick Tip to Solve: Use $\tan^{-1}(-x) = -\tan^{-1}x$ and $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.

---

12. $2\tan^{-1}\frac{1}{3} =$ [1 mark]

(A) $\tan^{-1}\frac{3}{2}$

(B) $\tan^{-1}\frac{3}{4}$

(C) $\tan^{-1}\frac{4}{3}$

(D) $\tan^{-1}\frac{2}{3}$

Correct Answer: (B) $\tan^{-1}\frac{3}{4}$

Solution:

Concept:

- We have to combine $2\tan^{-1}x$ into a single inverse tangent.
- There is a standard double-angle identity for this.

Step 1: Key Formula or Approach For $|x| < 1$:

$$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Step 2: Detailed Explanation Here $x = \frac{1}{3}$, and $|x| < 1$, so the identity applies.

$$\frac{2x}{1-x^2} = \frac{2/3}{1-\frac{1}{9}} = \frac{2/3}{8/9} = \frac{2}{3} \times \frac{9}{8} = \frac{3}{4}$$

$$2\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{3}{4}$$

Step 3: Why the other options are wrong Option (1), $\tan^{-1}\frac{3}{2}$: this is the reciprocal of $\frac{2}{3}$, which does not come out of the identity.
Option (3), $\tan^{-1}\frac{4}{3}$: this is the reciprocal of the correct $\frac{3}{4}$. It would be the angle complementary to the right one, not the angle itself.
Option (4), $\tan^{-1}\frac{2}{3}$: this is just $2x$, which comes from forgetting the denominator $1 - x^2$.

Step 4: Conclusion Using the double-angle identity, the value is $\tan^{-1}\frac{3}{4}$, option (2).

Quick Tip to Solve: Use $2\tan^{-1} x = \tan^{-1}\frac{2x}{1-x^2}$ for $|x| < 1$.

13. $x \in R, \cot(\tan^{-1} x + \cot^{-1} x) =$ [1 mark]

(A) 1
(B) $\frac{1}{2}$
(C) 0
(D) $\frac{1}{3}$

Correct Answer: (C) 0

Solution:

Concept:

- The question asks for $\cot$ of a sum of two inverse trigonometric functions.
- Before touching the cotangent, we first simplify the angle inside the bracket.

Step 1: Key Formula or Approach For every real number x, the standard identity is

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}.$$

This holds for all $x \in R$, because $\cot^{-1} x$ is the complement of $\tan^{-1} x$.

Step 2: Detailed Explanation Replace the angle inside the bracket using the identity:

$$\cot\left(\tan^{-1} x + \cot^{-1} x\right) = \cot\frac{\pi}{2}.$$

Now $\cot\frac{\pi}{2} = \frac{\cos(\pi/2)}{\sin(\pi/2)} = \frac{0}{1} = 0$.
The value does not depend on x at all.

Step 3: Check the options. Option (A) 1 is the value of $\cot\frac{\pi}{4}$, but our angle is $\frac{\pi}{2}$, not $\frac{\pi}{4}$.

Options (B) $\frac{1}{2}$ and (D) $\frac{1}{3}$ are not values of the cotangent at any standard angle that can come out of this identity.
Only option (C) 0 matches.

Step 4: Conclusion The sum $\tan^{-1} x + \cot^{-1} x$ is $\frac{\pi}{2}$, and $\cot \frac{\pi}{2} = 0$. This is option (C).

Quick Tip to Solve: Use the identity tan^-1 x + cot^-1 x = pi/2 for every real x, then evaluate cot(pi/2).

14. $\sin(\cos^{-1} 3/5) =$ [1 mark]

(A) $\frac{3}{4}$

(B) $\frac{4}{5}$

(C) $\frac{3}{5}$

(D) $\frac{5}{4}$

Correct Answer: (B) $\frac{4}{5}$

Solution:

Concept:

- $\cos^{-1} \frac{3}{5}$ is an angle.
- Let us call it θ.
- We want the sine of that angle, so we first find the angle's cosine and then get its sine.

Step 1: Key Formula or Approach Let $\theta = \cos^{-1} \frac{3}{5}$. Then $\cos\theta = \frac{3}{5}$ with θ in $[0, \pi]$.
Since $\frac{3}{5}$ is positive, θ lies in $[0, \frac{\pi}{2}]$, where sine is non-negative. We use $\sin^2\theta + \cos^2\theta = 1$.

Step 2: Detailed Explanation

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{9}{25} = \frac{16}{25}.$$

Taking the positive root (because θ is in the first quadrant):

$$\sin\theta = \frac{4}{5}.$$

So $\sin\left(\cos^{-1} \frac{3}{5}\right) = \frac{4}{5}$.

Step 3: Check the options. (A) $\frac{3}{4}$ is the value of $\tan\theta$ upside down: $\tan\theta = \frac{4}{3}$, so $\frac{3}{4}$ is $\cot\theta$, not $\sin\theta$. It is wrong.
(B) $\frac{4}{5}$ is exactly what we found, so it is correct.
(C) $\frac{3}{5}$ is $\cos\theta$, the value we started with, not the sine. It is wrong.
(D) $\frac{5}{4}$ is greater than 1, and a sine can never exceed 1. It is wrong.

Step 4: Conclusion The sine of the angle whose cosine is $\frac{3}{5}$ is $\frac{4}{5}$. This is option (B).

Quick Tip to Solve: Let the angle be theta with cos theta = 3/5, then use sin ^ 2 theta + cos ^ 2 theta = 1 (or a 3-4-5 triangle).

15. $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right) =$ [1 mark]

(A) π

(B) $\frac{2\pi}{3}$

(C) $\frac{3\pi}{4}$

(D) $\frac{\pi}{2}$

Correct Answer: (C) $\frac{3\pi}{4}$

Solution:

Concept:

- Each inverse trigonometric function returns an angle in a fixed range called the principal value branch.
- We find each of the three angles separately and then add them.

Step 1: Key Formula or Approach 1. $\tan^{-1} x$ takes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
2. $\cos^{-1} x$ takes values in $[0, \pi]$, and $\cos^{-1}(-x) = \pi - \cos^{-1} x$.
3. $\sin^{-1} x$ takes values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and $\sin^{-1}(-x) = -\sin^{-1} x$.

Step 2: Detailed Explanation First term: $\tan\frac{\pi}{4} = 1$, so $\tan^{-1}(1) = \frac{\pi}{4}$.
Second term: $\cos^{-1}\left(-\frac{1}{2}\right) = \pi - \cos^{-1}\frac{1}{2} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.
Third term: $\sin^{-1}\left(-\frac{1}{2}\right) = -\sin^{-1}\frac{1}{2} = -\frac{\pi}{6}$.
Now add the three values:

$$\frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6} = \frac{3\pi + 8\pi - 2\pi}{12} = \frac{9\pi}{12} = \frac{3\pi}{4}.$$

Step 3: Check the options. (A) π equals $\frac{12\pi}{12}$, which is not $\frac{9\pi}{12}$.
(B) $\frac{2\pi}{3}$ is only the second term.
(D) $\frac{\pi}{2}$ equals $\frac{6\pi}{12}$, which is not $\frac{9\pi}{12}$.
Only (C) $\frac{3\pi}{4}$ equals our sum.

Step 4: Conclusion The sum of the three principal values is $\frac{3\pi}{4}$, option (C).

Quick Tip to Solve: Find each principal value separately: pi/4, 2pi/3 and -pi/6, then add them.

16. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) =$ [1 mark]

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{3}$

(C) $\frac{5\pi}{6}$

(D) $\frac{7\pi}{6}$

Correct Answer: (C) $\frac{5\pi}{6}$

Solution:

Concept:

- The rule $\cos^{-1}(\cos\theta) = \theta$ is true only when θ lies in the principal range $[0, \pi]$ of $\cos^{-1}$.
- Here $\frac{7\pi}{6}$ is bigger than π, so the rule cannot be applied directly.

Step 1: Key Formula or Approach Rewrite $\cos\frac{7\pi}{6}$ as the cosine of an angle that does lie in $[0, \pi]$. We use $\cos(2\pi - \theta) = \cos\theta$, or $\cos(\pi + \alpha) = -\cos\alpha$.

Step 2: Detailed Explanation Since $2\pi - \frac{7\pi}{6} = \frac{5\pi}{6}$, we have

$$\cos\frac{7\pi}{6} = \cos\left(2\pi - \frac{5\pi}{6}\right) = \cos\frac{5\pi}{6}.$$

(Check: both equal $-\frac{\sqrt{3}}{2}$.)

The angle $\frac{5\pi}{6}$ is in $[0, \pi]$, so

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\frac{5\pi}{6}\right) = \frac{5\pi}{6}.$$

Step 3: Check the options. (D) $\frac{7\pi}{6}$ is the tempting answer, but it is outside the range $[0, \pi]$, so $\cos^{-1}$ can never return it.

(A) $\frac{\pi}{6}$ and (B) $\frac{\pi}{3}$ have cosines $\frac{\sqrt{3}}{2}$ and $\frac{1}{2}$, which are positive, while $\cos\frac{7\pi}{6}$ is negative.

Only (C) $\frac{5\pi}{6}$ has the right cosine and lies in the range.

Step 4: Conclusion The principal value is $\frac{5\pi}{6}$, option (C).

Quick Tip to Solve: 7pi/6 is outside [0, pi]. Rewrite cos(7pi/6) as cos(5pi/6) before taking the inverse.

17. $\tan^{-1}2 + \tan^{-1}3 =$ [1 mark]

(A) $-\frac{\pi}{4}$

(B) $\frac{\pi}{4}$

(C) $\frac{3\pi}{4}$

(D) π

Correct Answer: (C) $\frac{3\pi}{4}$

Solution:

Concept:

- We add two inverse tangents.
- The usual addition formula has a condition on the product xy, and ignoring that condition is the most common mistake in this type of question.

Step 1: Key Formula or Approach For $x, y > 0$:

$$\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1} \dfrac{x+y}{1-xy}, & xy < 1 \\ \pi + \tan^{-1} \dfrac{x+y}{1-xy}, & xy > 1 \end{cases}$$

Step 2: Detailed Explanation Here $x = 2$ and $y = 3$, so $xy = 6$, which is greater than 1. We must use the second case.

$$\frac{x+y}{1-xy} = \frac{2+3}{1-6} = \frac{5}{-5} = -1.$$

$$\tan^{-1} 2 + \tan^{-1} 3 = \pi + \tan^{-1}(-1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

Sanity check: $\tan^{-1} 2$ is about 63.4° and $\tan^{-1} 3$ is about 71.6°. Their sum is about 135°, which is $\dfrac{3\pi}{4}$.

Step 3: Check the options. (A) $-\dfrac{\pi}{4}$ is what we get by using $\tan^{-1} \dfrac{x+y}{1-xy}$ blindly without adding π. It is wrong because both angles are positive, so their sum cannot be negative.
(B) $\dfrac{\pi}{4}$ is less than $\tan^{-1} 2$ alone, so it cannot be the sum.
(D) π is too large because each angle is less than $\dfrac{\pi}{2}$ and their sum is below π.

Step 4: Conclusion $\tan^{-1} 2 + \tan^{-1} 3 = \dfrac{3\pi}{4}$, option (C).

Quick Tip to Solve: Here xy = 6 is greater than 1, so add pi to tan^-1 of (x+y)/(1-xy) = −1.

18. If $|x| \leq 1$, then $\tan(\cos^{-1} x) =$ [1 mark]

(A) $\dfrac{\sqrt{1-x^2}}{x}$

(B) $\dfrac{x}{1+x^2}$

(C) $\dfrac{\sqrt{1+x^2}}{x}$

(D) $\sqrt{1-x^2}$

Correct Answer: (A) $\dfrac{\sqrt{1-x^2}}{x}$

Solution:

Concept:

- We need $\tan(\cos^{-1} x)$.
- Let $\theta = \cos^{-1} x$.
- By the definition of the inverse cosine, θ lies in $[0, \pi]$ and $\cos\theta = x$.

Step 1: Key Formula or Approach 1. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.
2. $\sin^2\theta + \cos^2\theta = 1$.

Step 2: Find sin of theta From $\sin^2\theta = 1 - \cos^2\theta = 1 - x^2$ we get $\sin\theta = \pm\sqrt{1-x^2}$.
On $[0, \pi]$ the sine is never negative, so we take the positive root:

$$\sin\theta = \sqrt{1-x^2}$$

Step 3: Find tan of theta

$$\tan(\cos^{-1}x) = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1-x^2}}{x}$$

A quick check with $x = \frac{1}{2}$: $\cos^{-1}\frac{1}{2} = 60°$, $\tan 60° = \sqrt{3}$, and $\dfrac{\sqrt{1-1/4}}{1/2} = \sqrt{3}$. They match.

Step 4: Why the other options are wrong (B) $\dfrac{x}{1+x^2}$: at $x = \frac{1}{2}$ it equals 0.4, not $\sqrt{3}$.
(C) $\dfrac{\sqrt{1+x^2}}{x}$: it has $+x^2$ under the root, but the identity gives $1 - x^2$. At $x = \frac{1}{2}$ it is about 2.24.
(D) $\sqrt{1-x^2}$: this is $\sin(\cos^{-1}x)$, not the tangent, because the division by $\cos\theta = x$ is missing.

Step 5: Conclusion The value is $\dfrac{\sqrt{1-x^2}}{x}$, which is option (A).

Quick Tip to Solve: Put $\theta = \cos^{-1}x$, so $\cos\theta = x$ and $\sin\theta = \sqrt{1-x^2}$. Then $\tan\theta = \sin\theta/\cos\theta$.

19. $\tan^{-1}\dfrac{x}{y} - \tan^{-1}\dfrac{x-y}{x+y} =$ [1 mark]

(A) $-\dfrac{3\pi}{4}$
(B) $\dfrac{\pi}{2}$
(C) $\dfrac{\pi}{4}$
(D) $\dfrac{\pi}{3}$

Correct Answer: (C) $\dfrac{\pi}{4}$

Solution:

Concept:

- The question asks for the difference of two inverse tangents.
- Such a difference collapses into a single inverse tangent with a standard formula.

Step 1: Key Formula or Approach For $AB > -1$:

$$\tan^{-1}A - \tan^{-1}B = \tan^{-1}\frac{A-B}{1+AB}$$

Here $A = \dfrac{x}{y}$ and $B = \dfrac{x-y}{x+y}$.

Step 2: Find A - B

$$A - B = \frac{x}{y} - \frac{x-y}{x+y} = \frac{x(x+y) - y(x-y)}{y(x+y)} = \frac{x^2+xy-xy+y^2}{y(x+y)} = \frac{x^2+y^2}{y(x+y)}$$

Step 3: Find 1 + AB

$$1 + AB = 1 + \frac{x(x-y)}{y(x+y)} = \frac{y(x+y) + x(x-y)}{y(x+y)} = \frac{xy + y^2 + x^2 - xy}{y(x+y)} = \frac{x^2 + y^2}{y(x+y)}$$

Step 4: Combine The two fractions are equal, so

$$\frac{A - B}{1 + AB} = 1$$

$$\tan^{-1}\frac{x}{y} - \tan^{-1}\frac{x-y}{x+y} = \tan^{-1} 1 = \frac{\pi}{4}$$

This holds for all x, y with $y(x+y) > 0$, the usual range in which the formula applies. Check with $x = 3, y = 2$: $\tan^{-1} 1.5 - \tan^{-1} 0.2 = 0.9828 - 0.1974 = 0.7854 = \pi/4$.

Step 5: Why the other options are wrong The result is a fixed number, $\pi/4 \approx 0.785$. Option (A) $-3\pi/4$ is negative, (B) $\pi/2 \approx 1.571$ and (D) $\pi/3 \approx 1.047$ are both different from 0.785.

Step 6: Conclusion The expression equals $\frac{\pi}{4}$, option (C).

Quick Tip to Solve: Use $\tan^{-1} A - \tan^{-1} B = \tan^{-1}\frac{A-B}{1+AB}$. The numerator and denominator turn out to be equal, so the fraction is 1.

20. $|x| \le 1, \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) =$ [1 mark]

(A) $2\cos^{-1} x$

(B) $2\sin^{-1} x$

(C) $2\tan^{-1} x$

(D) $\tan^{-1} 2x$

Correct Answer: (C) $2\tan^{-1} x$

Solution:

Concept:

- The fraction $\frac{1-x^2}{1+x^2}$ looks like the double-angle formula for cosine written in terms of tangent.
- So we substitute $x = \tan\theta$.

Step 1: Key Formula or Approach

$$\cos 2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$$

Also $\cos^{-1}(\cos\phi) = \phi$ only when $\phi \in [0, \pi]$.

Step 2: Substitute Let $x = \tan\theta$, so $\theta = \tan^{-1} x$. For $0 \le x \le 1$ we get $0 \le \theta \le \frac{\pi}{4}$.

$$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta)$$

Step 3: Remove the inverse cosine Since $0 \le 2\theta \le \frac{\pi}{2}$, the angle 2θ lies inside $[0, \pi]$. So $\cos^{-1}(\cos 2\theta) =$

2θ.

$$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\theta = 2\tan^{-1} x$$

For $-1 \le x < 0$ the same working gives $-2\tan^{-1} x$. The printed condition $|x| \le 1$ is slightly loose, and the expected answer is the standard one for $x \ge 0$.

Step 4: Why the other options are wrong Test $x = 1$: the left side is $\cos^{-1} 0 = \frac{\pi}{2}$.
(A) $2\cos^{-1} 1 = 0$, wrong. (B) $2\sin^{-1} 1 = \pi$, wrong. (D) $\tan^{-1} 2 \approx 1.107$, wrong.
(C) $2\tan^{-1} 1 = \frac{\pi}{2}$, correct.

Step 5: Conclusion The expression equals $2\tan^{-1} x$, option (C).

Quick Tip to Solve: Put $x = \tan\theta$. Then $\frac{1-x^2}{1+x^2} = \cos 2\theta$, so the inverse cosine gives 2θ.

21. The angle between the straight lines $\frac{x-2}{2} = \frac{y-1}{7} = \frac{z+3}{-3}$ and $\frac{x+2}{-1} = \frac{y-4}{2} = \frac{z-5}{4}$ is [1 mark]

(A) $\frac{\pi}{2}$
(B) 0
(C) $\frac{\pi}{6}$
(D) $\frac{\pi}{4}$

Correct Answer: (A) $\frac{\pi}{2}$

Solution:

Concept:

- For a line written as $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$, the denominators a, b, c are its direction ratios.
- The angle between two lines is the angle between their direction vectors.

Step 1: Key Formula or Approach If the direction ratios are (a_1, b_1, c_1) and (a_2, b_2, c_2), then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Step 2: Read off the direction ratios First line: $\frac{x-2}{2} = \frac{y-1}{7} = \frac{z+3}{-3}$, so the direction ratios are $(2, 7, -3)$.
Second line: $\frac{x+2}{-1} = \frac{y-4}{2} = \frac{z-5}{4}$, so the direction ratios are $(-1, 2, 4)$.

Step 3: Find the numerator

$$a_1a_2 + b_1b_2 + c_1c_2 = (2)(-1) + (7)(2) + (-3)(4) = -2 + 14 - 12 = 0$$

The magnitudes are $\sqrt{4+49+9} = \sqrt{62}$ and $\sqrt{1+4+16} = \sqrt{21}$. Both are non-zero, so the denominator is not zero.

Step 4: Find the angle

$$\cos\theta = \frac{0}{\sqrt{62}\sqrt{21}} = 0 \Rightarrow \theta = \frac{\pi}{2}$$

Step 5: Why the other options are wrong Option (B), angle 0, would mean the lines are parallel. That

needs proportional ratios, but $\frac{2}{-1} = -2$ and $\frac{7}{2} = 3.5$ are not equal.

Option (C), $\frac{\pi}{6}$, needs $\cos\theta = \frac{\sqrt{3}}{2}$, and option (D), $\frac{\pi}{4}$, needs $\cos\theta = \frac{1}{\sqrt{2}}$. We got $\cos\theta = 0$, so neither fits.

Step 6: Conclusion The dot product of the direction ratios is zero, so the lines are perpendicular. This is option (A).

Quick Tip to Solve: Take the denominators of each line as its direction ratios and find their dot product. If it is zero, the lines are perpendicular.

22. The distance of the plane $x + 2y - 2z = 9$ from the point $(2, 3, -5)$ is [1 mark]

(A) 1

(B) 2

(C) 3

(D) 4

Correct Answer: (C) 3

Solution:

Concept:

- The distance of a point from a plane is the length of the perpendicular dropped from the point to the plane.

Step 1: Key Formula or Approach For the plane $Ax + By + Cz + D = 0$ and the point (x_0, y_0, z_0),

$$\text{distance} = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Step 2: Write the plane in standard form The plane $x + 2y - 2z = 9$ becomes $x + 2y - 2z - 9 = 0$. So $A = 1, B = 2, C = -2, D = -9$.

Step 3: Put the point in the numerator For $(2, 3, -5)$:

$$Ax_0 + By_0 + Cz_0 + D = (1)(2) + (2)(3) + (-2)(-5) - 9 = 2 + 6 + 10 - 9 = 9$$

Step 4: Find the denominator and the distance

$$\sqrt{A^2 + B^2 + C^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$$

$$\text{distance} = \frac{|9|}{3} = 3$$

Step 5: Why the other options are wrong A value of 1 comes from dividing by 9 instead of by $\sqrt{9} = 3$, which is the common slip of forgetting the square root in the denominator.
The values 2 and 4 do not come from the correct numerator 9 and denominator 3, so they cannot be the distance.

Step 6: Conclusion The point is at distance 3 units from the plane. This is option (C).

Quick Tip to Solve: Use the point-to-plane distance formula after writing the plane as $x+2y-2z-9=0$.

23. If two planes $x-4y+\lambda z+3=0$ and $2x+2y+3z=5$ are perpendicular to each other then $\lambda=$ [1 mark]

(A) 1

(B) 2

(C) 3

(D) 4

Correct Answer: (B) 2

Solution:

Concept:

- Two planes are perpendicular when their normal vectors are perpendicular.
- The normal of $ax+by+cz+d=0$ is (a,b,c).

Step 1: Key Formula or Approach For planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$, perpendicularity means

$$a_1a_2+b_1b_2+c_1c_2=0$$

Step 2: Read off the normals Plane 1: $x-4y+\lambda z+3=0$ has normal $(1,-4,\lambda)$.
Plane 2: $2x+2y+3z-5=0$ has normal $(2,2,3)$.

Step 3: Apply the condition

$$(1)(2)+(-4)(2)+(\lambda)(3)=0$$

$$2-8+3\lambda=0\Rightarrow 3\lambda=6\Rightarrow\lambda=2$$

Step 4: Why the other options are wrong Put each other value in the dot product $2-8+3\lambda$.
For $\lambda=1$ it is -3. For $\lambda=3$ it is 3. For $\lambda=4$ it is 6.
None of these is zero, so the planes would not be perpendicular.

Step 5: Conclusion The planes are perpendicular only when $\lambda=2$. This is option (B).

Quick Tip to Solve: Perpendicular planes have perpendicular normals, so the dot product of $(1,-4,\lambda)$ and $(2,2,3)$ must be zero.

24. If the line $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ is parallel to the plane $a_2x+b_2y+c_2z+d=0$, then [1 mark]

(A) $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

(B) $a_1x+b_1y+c_1z+d=0$

(C) $a_1a_2+b_1b_2+c_1c_2=0$

(D) none of these

Correct Answer: (C) $a_1a_2+b_1b_2+c_1c_2=0$

Solution:

Concept:

- A line is described by its direction ratios, which give the direction of the line.
- A plane is described by the direction ratios of its normal, which is the vector perpendicular to the plane.
- A line is parallel to a plane when the line runs along the plane, so it must be at right angles to the plane's normal.

Step 1: Key Formula or Approach 1. Line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ has direction vector $\vec{d} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$.
2. Plane $a_2x + b_2y + c_2z + d = 0$ has normal vector $\vec{n} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$.
3. Two vectors are perpendicular when their dot product is 0.

Step 2: Apply the condition. Line parallel to plane means $\vec{d} \perp \vec{n}$, so $\vec{d} \cdot \vec{n} = 0$.

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

This is option 3.

Step 3: Why the other options are wrong. Option 1 ($\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$) says $\vec{d}$ is parallel to $\vec{n}$. That means the line is perpendicular to the plane, which is the opposite of parallel.
Option 2 ($a_1x + b_1y + c_1z + d = 0$) is the equation of some plane. It is not a relation between the coefficients of the line and the plane, so it cannot be the condition.
Option 4 (none of these) is wrong because option 3 is correct.

Step 4: Conclusion The line is parallel to the plane when the dot product of the direction vector and the normal vector is zero, which is option 3.

Quick Tip to Solve: A line parallel to a plane is perpendicular to the plane's normal, so the dot product of the direction ratios and the normal ratios is zero.

---

25. The equation of the plane whose intercepts on the x, y and z-axes are -2, 3 and 4 respectively, will be [1 mark]

(A) $6x - 4y - 3z + 12 = 0$

(B) $6x + 4y + 3z + 12 = 0$

(C) $6x - 4y - 3z = 0$

(D) none of these

Correct Answer: (A) $6x - 4y - 3z + 12 = 0$

Solution:

Concept:

- A plane that cuts the axes at $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ has a simple standard equation called the intercept form.

Step 1: Key Formula or Approach

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Step 2: Substitute the intercepts Here $a = -2$, $b = 3$, $c = 4$. So

$$\frac{x}{-2} + \frac{y}{3} + \frac{z}{4} = 1$$

Step 3: Clear the fractions The LCM of 2, 3 and 4 is 12. Multiply both sides by 12:

$$-6x + 4y + 3z = 12$$

Move everything to one side and change all signs:

$$6x - 4y - 3z + 12 = 0$$

Step 4: Why the other options are wrong Option (B), $6x + 4y + 3z + 12 = 0$, meets the x-axis correctly at $x = -2$, but on the y-axis it gives $y = -3$ and on the z-axis $z = -4$. The signs of these intercepts are wrong.
Option (C), $6x - 4y - 3z = 0$, passes through the origin, so it has no proper intercepts at all.
Option (D) is wrong because option (A) is correct.

Step 5: Conclusion The plane is $6x - 4y - 3z + 12 = 0$. This is option (A).

Quick Tip to Solve: Use the intercept form $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ with $a = -2, b = 3, c = 4$ and multiply by 12.

26. If the direction cosines of two straight lines are l_1, m_1, n_1 and l_2, m_2, n_2 then the cosine of the angle between the lines will be [1 mark]

(A) $(l_1 + m_1 + n_1)(l_2 + m_2 + n_2)$

(B) $\frac{l_1}{l_2} + \frac{m_1}{m_2} + \frac{n_1}{n_2}$

(C) $l_1l_2 + m_1m_2 + n_1n_2$

(D) none of these

Correct Answer: (C) $l_1l_2 + m_1m_2 + n_1n_2$

Solution:

Concept:

- Direction cosines l, m, n of a line are the cosines of the angles it makes with the x, y and z axes.
- They satisfy $l^2 + m^2 + n^2 = 1$, so (l, m, n) is a unit vector along the line.

Step 1: Key Formula or Approach For direction ratios (a_1, b_1, c_1) and (a_2, b_2, c_2),

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Step 2: Apply it to direction cosines Direction cosines are direction ratios whose squares add to 1. So here $a_1 = l_1,\ b_1 = m_1,\ c_1 = n_1$ and $a_2 = l_2,\ b_2 = m_2,\ c_2 = n_2$.
Both denominators become $\sqrt{1} = 1$.

Step 3: Write the result

$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$$

Step 4: Why the other options are wrong Option (A), $(l_1 + m_1 + n_1)(l_2 + m_2 + n_2)$, multiplies the

sums of the cosines. This has no geometric meaning here. For two lines along the x-axis, it gives 1, but for a line along the x-axis and one along the y-axis it also gives 1, though the angle is 90 degrees.
Option (B), $\frac{l_1}{l_2} + \frac{m_1}{m_2} + \frac{n_1}{n_2}$, is not even defined when a cosine is zero, and for two identical lines it gives 3 instead of 1.
Option (D) is wrong because option (C) is correct.

Step 5: Conclusion The cosine of the angle between the lines is $l_1l_2 + m_1m_2 + n_1n_2$. This is option (C).

Quick Tip to Solve: Direction cosines are the components of a unit vector, so the cosine of the angle is just the dot product of the two sets.

27. If operation 'o' is defined as $(a\,o\,b) = a^3 + b^3$, then $4\,o\,(1\,o\,2) =$ [1 mark]

(A) 729

(B) 793

(C) 783

(D) 792

Correct Answer: (B) 793

Solution:

Concept:

- The symbol 'o' is a new operation defined by the rule $a\,o\,b = a^3 + b^3$.
- It is not ordinary multiplication or composition.
- We work out the bracket first, as in any expression.

Step 1: Key Formula or Approach Use the rule twice: first on the inner pair $(1\,o\,2)$, then on 4 and the value we get.

Step 2: Detailed Explanation Inner bracket:

$$1\,o\,2 = 1^3 + 2^3 = 1 + 8 = 9.$$

Now the outer step:

$$4\,o\,9 = 4^3 + 9^3 = 64 + 729 = 793.$$

Step 3: Why the other options are wrong Option (A) $729 = 9^3$ forgets to add $4^3 = 64$.
Option (C) 783 and option (D) 792 come from small slips in adding $64 + 729$. The correct sum is 793.

Step 4: Conclusion The value of $4\,o\,(1\,o\,2)$ is 793, which is option (B).

Quick Tip to Solve: First work out $1\,o\,2 = 1^3 + 2^3 = 9$, then $4\,o\,9 = 4^3 + 9^3$.

28. If $n(A) = 4$ and $n(B) = 2$, then $n(A \times B)$ is? [1 mark]

(A) 6

(B) 8

(C) 16

(D) none of these

Correct Answer: (B) 8

Solution:

Quick Tip to Solve: The cardinality of the Cartesian product $A \times B$ is simply the product of the cardinalities of sets A and B.

29. $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix} =$ [1 mark]

(A) $a^2+b^2+c^2+d^2$

(B) $a^2-b^2-c^2-d^2$

(C) $a^2-b^2+c^2+d^2$

(D) $a^2+b^2+c^2-d^2$

Correct Answer: (A) $a^2+b^2+c^2+d^2$

Solution:

Step 1: Recall the rule for a 2 by 2 determinant A determinant $\begin{vmatrix} p & q \\ r & s \end{vmatrix}$ equals $ps-qr$. Here the entries are complex numbers, so we multiply them with the rule $i^2=-1$.

Step 2: Multiply the main diagonal The main diagonal gives $(a+ib)(a-ib)$. This is of the form $(x+y)(x-y)=x^2-y^2$ with $x=a$ and $y=ib$.

$$(a+ib)(a-ib)=a^2-i^2b^2=a^2+b^2$$

Step 3: Multiply the other diagonal The other diagonal gives $(c+id)(-c+id)$. Write it as $(id+c)(id-c)$, which is $(id)^2-c^2$.

$$(c+id)(-c+id)=i^2d^2-c^2=-d^2-c^2=-(c^2+d^2)$$

Step 4: Subtract the two products The determinant is the main diagonal product minus the other diagonal product.

$$\Delta=(a^2+b^2)-\left[-(c^2+d^2)\right]=a^2+b^2+c^2+d^2$$

Step 5: Check the options Option (1) is $a^2+b^2+c^2+d^2$, which matches our result exactly.
Option (2) has minus signs on b^2, c^2, d^2. Those signs would come from forgetting that $i^2=-1$ and from dropping the minus in front of the second product. So option (2) is wrong.
Option (3) keeps $-b^2$, but $(a+ib)(a-ib)$ gives $+b^2$. So option (3) is wrong.
Option (4) has $-d^2$. But subtracting $-(c^2+d^2)$ adds d^2, so the sign of d^2 is plus and option (4) is wrong.

Step 6: Conclusion The determinant equals the sum of the four squares, which is option (1).

Quick Tip to Solve: Expand as $ps-qr$ and use $i^2=-1$. The first product is a^2+b^2 and the second is $-(c^2+d^2)$.

30. The maximum value of $Z=4x+y$ subject to the constraints $x+y\le 50,\ x\ge 0,\ y\ge 0$ is [1 mark]

(A) 50

(B) 250

(C) 0

(D) none of these

Correct Answer: (D) none of these

Solution:

Concept:

- The constraints $x \geq 0$ and $y \geq 0$ keep us in the first quadrant. The line $x + y = 50$ cuts the axes at $(50, 0)$ and $(0, 50)$.
- The region $x + y \leq 50$ is the side containing the origin. So the feasible region is the triangle with vertices $(0, 0), (50, 0), (0, 50)$.
- It is bounded, so the maximum of a linear function sits at a corner.

Step 1: Use the corner point method Evaluate $Z = 4x + y$ at each vertex.
At $(0, 0)$: $Z = 0$.
At $(50, 0)$: $Z = 4(50) + 0 = 200$.
At $(0, 50)$: $Z = 0 + 50 = 50$.

Step 2: Pick the largest value The values are $0, 200, 50$. The largest is 200, at $(50, 0)$.

Step 3: Compare with the options Option (1), 50, is the value at $(0, 50)$ only. It is not the maximum.
Option (2), 250, would need $4x + y = 250$, but even with $x = 50$ we get $Z = 200$, and no point of the triangle reaches 250.
Option (3), 0, is the minimum, found at the origin.
The true maximum, 200, is not listed, so option (4) is correct.

Step 4: Conclusion The maximum value of Z is 200. It is not among the first three options, so the answer is option (4), none of these.

Quick Tip to Solve: Draw the triangle with vertices $(0, 0), (50, 0), (0, 50)$ and test $4x + y$ at each corner.

31. $\begin{vmatrix} 23 & 12 & 11 \\ 36 & 10 & 26 \\ 63 & 26 & 37 \end{vmatrix} =$ [1 mark]

(A) 1

(B) -1

(C) 0

(D) 2

Correct Answer: (C) 0

Solution:

Concept:

- We have to find the value of a 3×3 determinant whose entries look unfriendly.
- Before expanding, we look for a relation between rows or columns, because that can make the work very short.

Step 1: Key Formula or Approach 1. Adding or subtracting multiples of other columns to one column does not change the value of a determinant.
2. A determinant with a column (or row) of all zeros is equal to 0.

Step 2: Spot the relation between the columns Look at the first column and compare it with the sum

of the other two columns.

$$12+11=23, \qquad 10+26=36, \qquad 26+37=63$$

So every entry of C_1 equals the sum of the matching entries of C_2 and C_3. That is, $C_1 = C_2 + C_3$.

Step 3: Apply the column operation Apply $C_1 \to C_1 - C_2 - C_3$. The value of the determinant stays the same.

$$\begin{vmatrix} 23 & 12 & 11 \\ 36 & 10 & 26 \\ 63 & 26 & 37 \end{vmatrix} = \begin{vmatrix} 23-12-11 & 12 & 11 \\ 36-10-26 & 10 & 26 \\ 63-26-37 & 26 & 37 \end{vmatrix} = \begin{vmatrix} 0 & 12 & 11 \\ 0 & 10 & 26 \\ 0 & 26 & 37 \end{vmatrix}$$

The first column is now all zeros, so the determinant is 0.

Step 4: Why the other options are wrong Option (1), 1, and option (2), -1, would need a nonzero determinant. But a determinant with a zero column cannot be nonzero.
Option (4), 2, is also nonzero, so it is wrong for the same reason. Only option (3), 0, fits.

Step 5: Conclusion The determinant equals zero because $C_1 = C_2 + C_3$. This is option (3).

Quick Tip to Solve: Check whether the first column equals the sum of the other two columns: $12 + 11 = 23$, $10 + 26 = 36$, $26 + 37 = 63$. Then apply $C_1 \to C_1 - C_2 - C_3$ to get a zero column.

---

32. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} =$ [1 mark]

(A) 1

(B) 0

(C) -1

(D) $a+b+c$

Correct Answer: (B) 0

Solution:

Concept:

- A determinant does not change if we add a multiple of one column to another column.
- If one column becomes all zeros, the determinant is zero.

Step 1: Key Formula or Approach Apply the operation $C_1 \to C_1 + C_2 + C_3$, which leaves the value unchanged.

Step 2: Detailed Explanation Add the three entries of each row and put the sum in the first column:
Row 1: $(a-b)+(b-c)+(c-a)=0$.
Row 2: $(b-c)+(c-a)+(a-b)=0$.
Row 3: $(c-a)+(a-b)+(b-c)=0$.

$$\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix}$$

The first column is all zeros. Expanding along it gives $\Delta = 0$.

Step 3: Why the other options are wrong The determinant is identically zero for every a, b, c. Options

(A) 1 and (C) -1 are non-zero constants. Option (D) $a+b+c$ is not zero in general, for example it equals 3 when $a=b=c=1$.

Step 4: Conclusion The determinant is 0, option (B).

Quick Tip to Solve: Add all three columns into the first column. Each row of the first column then becomes 0, so the determinant is zero.

33. $\begin{vmatrix} x & 4 \\ 2 & 2x \end{vmatrix} = 0 \Rightarrow x =$ [1 mark]

(A) ± 2

(B) ± 1

(C) ± 3

(D) 0

Correct Answer: (A) ± 2

Solution:

Concept:

- The value of a 2×2 determinant is the product of the main diagonal minus the product of the other diagonal.
- We set this value equal to zero and solve for x.

Step 1: Key Formula or Approach

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Step 2: Expand the determinant Here $a = x,\ b = 4,\ c = 2,\ d = 2x$.

$$\begin{vmatrix} x & 4 \\ 2 & 2x \end{vmatrix} = x \cdot 2x - 4 \cdot 2 = 2x^2 - 8$$

Step 3: Solve the equation

$$2x^2 - 8 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

Step 4: Why the other options are wrong Put each option into $2x^2 - 8$. For $x = \pm 1$ we get $2 - 8 = -6$. For $x = \pm 3$ we get $18 - 8 = 10$. For $x = 0$ we get -8. None of these is zero, so only $x = \pm 2$ works.

Step 5: Conclusion The solutions are $x = \pm 2$, option (A).

Quick Tip to Solve: Expand the determinant as $ad - bc = 2x^2 - 8$ and set it equal to zero.

34. $\begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix} =$ [1 mark]

(A) 1

(B) 0

(C) -1

(D) $\dfrac{1}{2}$

Correct Answer: (B) 0

Solution:

Concept:

- The value of a 2 x 2 determinant is the product of the main diagonal entries minus the product of the other diagonal entries.
- We also use the identity for cosine of a sum of two angles.

Step 1: Key Formula or Approach 1. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

2. $\cos(A + B) = \cos A \cos B - \sin A \sin B$.

Step 2: Expand the determinant Here $a = \cos 15^\circ$, $b = \sin 15^\circ$, $c = \sin 75^\circ$, $d = \cos 75^\circ$. So

$$\begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix} = \cos 15^\circ \cos 75^\circ - \sin 15^\circ \sin 75^\circ$$

Step 3: Use the cosine of a sum The expression has the form $\cos A \cos B - \sin A \sin B$ with $A = 15^\circ$ and $B = 75^\circ$. So

$$\cos 15^\circ \cos 75^\circ - \sin 15^\circ \sin 75^\circ = \cos(15^\circ + 75^\circ) = \cos 90^\circ = 0$$

Step 4: Why the other options are wrong Option (A) 1 would need the value to equal $\cos 0^\circ$, which would happen only for $\cos(A - B)$ with equal angles. Option (C) -1 would be $\cos 180^\circ$. Option (D) $\frac{1}{2}$ would be $\cos 60^\circ$. None of these angles equals $15^\circ + 75^\circ = 90^\circ$, so only 0 is correct.

Step 5: Conclusion The determinant equals $\cos 90^\circ = 0$, which is option (B).

Quick Tip to Solve: Expand the determinant and use $\cos(A + B) = \cos A \cos B - \sin A \sin B$ with $A = 15^\circ$, $B = 75^\circ$.

35. $1 - P(A' \cap B') =$ [1 mark]

(A) $P(A \cap B)$

(B) $P(A \cup B)$

(C) $P(A)$

(D) $P(B)$

Correct Answer: (B) $P(A \cup B)$

Solution:

Concept:

- A' and B' are the complements of the events A and B.
- The set $A' \cap B'$ is the set of outcomes that are in neither A nor B.

Step 1: Key Formula or Approach De Morgan's law says $A' \cap B' = (A \cup B)'$. Also, for any event E, $P(E') = 1 - P(E)$.

Step 2: Apply De Morgan's law Replace the intersection of complements by the complement of the union:

$$P(A' \cap B') = P\big((A \cup B)'\big)$$

Step 3: Use the complement rule Now

$$P\big((A \cup B)'\big) = 1 - P(A \cup B)$$

So

$$1 - P(A' \cap B') = 1 - \big[1 - P(A \cup B)\big] = P(A \cup B)$$

Step 4: Why the other options are wrong $P(A \cap B)$ counts only the outcomes in both events, which is smaller than or equal to $P(A \cup B)$ and is not equal in general. $P(A)$ and $P(B)$ each count only one event, so they miss outcomes that lie in the other event. Only the union covers 'at least one of A or B'.

Step 5: Conclusion Not 'neither A nor B' means 'at least one of A or B', which is option (B).

Quick Tip to Solve: Use De Morgan's law: $A' \cap B' = (A \cup B)'$, then $P(E') = 1 - P(E)$.

36. If A, B and C are three independent events then $P(ABC) =$ [1 mark]

(A) $P(A) + P(B) + P(C)$

(B) $P(A) - P(B) - P(C)$

(C) $P(A) \cdot P(B) \cdot P(C)$

(D) None of these

Correct Answer: (C) $P(A) \cdot P(B) \cdot P(C)$

Solution:

Concept:

- Events are independent when the occurrence of one does not change the chance of the others.
- For three events, we need independence of every pair and of the triple.

Step 1: Key Formula or Approach For two independent events, $P(A \cap B) = P(A)\,P(B)$. For three independent events the same idea extends to $P(A \cap B \cap C)$.

Step 2: Build up to three events Treat $A \cap B$ as one event. Since C is independent of $A \cap B$,

$$P(ABC) = P\big((A \cap B) \cap C\big) = P(A \cap B)\,P(C)$$

But $P(A \cap B) = P(A)\,P(B)$, so

$$P(ABC) = P(A)\,P(B)\,P(C)$$

Step 3: Check option (A) A sum of three probabilities can exceed 1, for example $0.5 + 0.5 + 0.5 = 1.5$, but $P(ABC)$ can never exceed 1. So (A) is wrong.

Step 4: Check option (B) A difference such as $P(A) - P(B) - P(C)$ can be negative, and a probability is never negative. So (B) is wrong.

Step 5: Check option (D) 'None of these' is wrong because option (C) is correct.

Step 6: Conclusion For independent events the probability of all happening together is the product, option (C).

Quick Tip to Solve: For independent events, the probability that all occur together is the product of the separate probabilities.

37. $P(A) = \frac{7}{13},\ P(B) = \frac{9}{13},\ P(A \cap B) = \frac{4}{13} \Rightarrow P(A/B) =$ [1 mark]

(A) $\frac{4}{9}$

(B) $\frac{4}{7}$

(C) $\frac{12}{13}$

(D) $\frac{1}{6}$

Correct Answer: (A) $\frac{4}{9}$

Solution:

Concept:

- $P(A/B)$ is the conditional probability of A given that B has already happened.
- The sample space shrinks to the outcomes of B.

Step 1: Key Formula or Approach

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0$$

Step 2: Substitute the values We are given $P(A \cap B) = \frac{4}{13}$ and $P(B) = \frac{9}{13}$. So

$$P(A/B) = \frac{4/13}{9/13}$$

The common factor 13 cancels:

$$P(A/B) = \frac{4}{9}$$

Step 3: Why the other options are wrong $\frac{4}{7}$ is $P(B/A)$, because it divides by $P(A) = \frac{7}{13}$ instead of $P(B)$. $\frac{12}{13}$ and $\frac{1}{6}$ do not come from any correct use of the formula. The value $P(A)$ is not needed for this question.

Step 4: Conclusion The conditional probability is $\frac{4}{9}$, option (A).

Quick Tip to Solve: Use $P(A/B) = \frac{P(A \cap B)}{P(B)}$.

38. **If the odds against the event A is $3 : 7$ then $P(A) =$** [1 mark]

(A) $\frac{3}{10}$

(B) $\frac{7}{10}$

(C) $\frac{3}{7}$

(D) $\frac{7}{3}$

Correct Answer: (B) $\frac{7}{10}$

Solution:

Concept:

- Odds compare the chance that an event does not happen with the chance that it does. 'Odds against A are m : n' means the ratio of unfavourable cases to favourable cases is m : n.

Step 1: Key Formula or Approach If the odds against A are $m : n$, then

$$P(A) = \frac{n}{m+n}, \quad P(A') = \frac{m}{m+n}$$

Step 2: Read the given odds The odds against A are $3 : 7$, so $m = 3$ unfavourable and $n = 7$ favourable. The total number of parts is $3 + 7 = 10$.

Step 3: Find the probability

$$P(A) = \frac{7}{10}$$

Check: $P(A') = \frac{3}{10}$, and $\frac{3}{10} + \frac{7}{10} = 1$.

Step 4: Why the other options are wrong $\frac{3}{10}$ is $P(A')$, the probability that A does not happen. It comes from mixing up 'against' with 'in favour'. $\frac{3}{7}$ is the odds ratio itself, not a probability. $\frac{7}{3}$ is bigger than 1, so it cannot be a probability.

Step 5: Conclusion $P(A) = \frac{7}{10}$, option (B).

Quick Tip to Solve: If the odds against are $m : n$, then $P(A) = \frac{n}{m+n}$.

---

39. $\frac{d}{dx}(\log x^n) =$ [1 mark]

(A) $\frac{1}{x^n}$

(B) n

(C) $\frac{1}{x}$

(D) $\frac{n}{x}$

Correct Answer: (D) $\frac{n}{x}$

Solution:

Concept:

- We need the derivative of $\log x^n$ with respect to x.
- Here n is a constant and $\log$ is the natural logarithm, taken for $x > 0$.

Step 1: Key Formula or Approach 1. Power rule of logarithms: $\log x^n = n \log x$.
2. $\frac{d}{dx}(\log x) = \frac{1}{x}$.
3. A constant multiplier comes out of the derivative.

Step 2: Detailed Explanation First bring the power down using the log rule.

$$\log x^n = n \log x$$

Now differentiate. Since n is a constant, it stays as a multiplier.

$$\frac{d}{dx}(n\log x) = n\cdot\frac{d}{dx}(\log x) = n\cdot\frac{1}{x} = \frac{n}{x}$$

Step 3: Why the other options are wrong Option (A) $\frac{1}{x^n}$ comes from writing $\frac{1}{u}$ for $u = x^n$ and forgetting to multiply by $\frac{du}{dx}$.
Option (B) n has no x in it, but the derivative of a logarithm always carries $\frac{1}{x}$.
Option (C) $\frac{1}{x}$ drops the factor n that was brought down by the log rule.
Only option (D) $\frac{n}{x}$ matches the working.

Step 4: Conclusion The derivative is $\frac{n}{x}$, which is option (D), the fourth option.

Quick Tip to Solve: Use $\log x^n = n\log x$ first, then differentiate $\log x$.

---

40. $\frac{d^2}{dx^2}(\sin 2x) =$ [1 mark]

(A) $4\sin 2x$

(B) $4\cos^2 2x$

(C) $-4\sin 2x$

(D) $2\sin 4x$

Correct Answer: (C) $-4\sin 2x$

Solution:

Concept:

- The second derivative means we differentiate $\sin 2x$ once, then differentiate the result again.
- Each time the inner function $2x$ needs the chain rule.

Step 1: Key Formula or Approach 1. $\frac{d}{dx}(\sin kx) = k\cos kx$.
2. $\frac{d}{dx}(\cos kx) = -k\sin kx$.

Step 2: First derivative

$$\frac{d}{dx}(\sin 2x) = \cos 2x\cdot 2 = 2\cos 2x$$

Step 3: Second derivative Differentiate $2\cos 2x$ again. The constant 2 stays and $\cos 2x$ gives $-2\sin 2x$.

$$\frac{d^2}{dx^2}(\sin 2x) = 2\cdot(-\sin 2x)\cdot 2 = -4\sin 2x$$

Step 4: Why the other options are wrong Option (A) $4\sin 2x$ has the right size but the wrong sign. Differentiating $\cos$ brings a minus sign.
Option (B) $4\cos^2 2x$ is not a multiple of $\sin 2x$, so it cannot come from two differentiations of $\sin 2x$.
Option (D) $2\sin 4x$ is what we would get by wrongly doubling the angle instead of multiplying by 2.
Only option (C) matches.

Step 5: Conclusion The second derivative is $-4\sin 2x$, option (C), the third option.

Quick Tip to Solve: Differentiate twice with the chain rule. Each step multiplies by 2, and cos brings a minus sign.

41. $\vec{i} \times \vec{k} =$ [1 mark]

(A) 1

(B) $\vec{k}$

(C) $\vec{j}$

(D) $-\vec{j}$

Correct Answer: (D) $-\vec{j}$

Solution:

Concept:

- The cross product of two vectors is a vector.
- For the unit vectors $\vec{i}, \vec{j}, \vec{k}$, the cross products follow a cyclic pattern.

Step 1: Key Formula or Approach Going in the order $\vec{i} \to \vec{j} \to \vec{k} \to \vec{i}$ gives a positive sign:

$$\vec{i} \times \vec{j} = \vec{k}, \quad \vec{j} \times \vec{k} = \vec{i}, \quad \vec{k} \times \vec{i} = \vec{j}$$

Reversing the order flips the sign, because $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.

Step 2: Detailed Explanation The product asked is $\vec{i} \times \vec{k}$. This is the reverse of $\vec{k} \times \vec{i} = \vec{j}$. So

$$\vec{i} \times \vec{k} = -(\vec{k} \times \vec{i}) = -\vec{j}$$

Step 3: Why the other options are wrong. (A) 1 is a number, but a cross product is always a vector, so it cannot be 1.
(B) $\vec{k}$ is the value of $\vec{i} \times \vec{j}$, and the cross product of $\vec{i}$ and $\vec{k}$ must be perpendicular to both, so it cannot be $\vec{k}$.
(C) $\vec{j}$ is the value of $\vec{k} \times \vec{i}$. Here the order is reversed, so the sign is opposite.

Step 4: Conclusion $\vec{i} \times \vec{k} = -\vec{j}$, which is option (D).

Quick Tip to Solve: Use $\vec{k} \times \vec{i} = \vec{j}$ and reverse the order, which changes the sign.

42. $\vec{a} \cdot (\vec{a} \times \vec{a}) =$ [1 mark]

(A) 1

(B) 0

(C) $\vec{a}$

(D) -1

Correct Answer: (B) 0

Solution:

Concept:

- The cross product $\vec{a} \times \vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$.
- Its magnitude is $|\vec{a}||\vec{b}| \sin\theta$, where θ is the angle between the vectors.

Step 1: Key Formula or Approach First find $\vec{a} \times \vec{a}$ using the magnitude formula. Then take the dot product of $\vec{a}$ with that result.

Step 2: Find $\vec{a} \times \vec{a}$. A vector makes an angle $\theta = 0^\circ$ with itself. So

$$|\vec{a} \times \vec{a}| = |\vec{a}||\vec{a}| \sin 0^\circ = 0$$

A vector with zero length is the zero vector, so $\vec{a} \times \vec{a} = \vec{0}$.

Step 3: Take the dot product. The dot product of any vector with the zero vector is the number zero.

$$\vec{a} \cdot (\vec{a} \times \vec{a}) = \vec{a} \cdot \vec{0} = 0$$

Step 4: Why the other options are wrong. (A) 1 and (D) -1 are not possible, since the result is zero. (C) $\vec{a}$ is a vector, but a dot product always gives a scalar, so it cannot be $\vec{a}$.

Step 5: Conclusion The value is 0, which is option (B).

Quick Tip to Solve: A vector crossed with itself gives the zero vector, and any dot product with the zero vector is 0.

43. If \[\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k} \quad \text{and} \quad \vec{b} = \hat{i} + \lambda \hat{j} - 3\hat{k} \] are perpendicular to each other, then the value of λ is: [1 mark]

(A) -3

(B) -6

(C) -9

(D) -1

Correct Answer: (C) -9

Solution:

Quick Tip to Solve: For two vectors to be perpendicular, their dot product must equal zero. Always expand the dot product carefully and solve for the unknown variable.

44. The projection of the vector $\vec{i} + 3\vec{j} + 7\vec{k}$ on the vector $2\vec{i} - 3\vec{j} + 6\vec{k}$ is [1 mark]

(A) 5

(B) 25

(C) 6

(D) none of these

Correct Answer: (A) 5

Solution:

Concept:

- The projection of a vector $\vec{a}$ on a vector $\vec{b}$ is the length of the shadow of $\vec{a}$ along the direction of $\vec{b}$.

Step 1: Key Formula or Approach

$$\text{Projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

In the printed English question, $\vec{a} = \vec{i} + 3\vec{j} + 7\vec{k}$ and $\vec{b} = 2\vec{i} - 3\vec{j} + 6\vec{k}$.

Step 2: Find the dot product. Multiply matching components and add.

$$\vec{a} \cdot \vec{b} = (1)(2) + (3)(-3) + (7)(6) = 2 - 9 + 42 = 35$$

Step 3: Find the magnitude of $\vec{b}$.

$$|\vec{b}| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$$

Step 4: Divide.

$$\text{Projection} = \frac{35}{7} = 5$$

Step 5: Check the options. (A) 5 matches our result.
(B) 25 and (C) 6 do not equal 5.
(D) "none of these" is wrong because option (A) is correct.

Step 6: Conclusion The projection is 5, which is option (A).

Quick Tip to Solve: Use projection $= \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$ with $\vec{a} \cdot \vec{b} = 35$ and $|\vec{b}| = 7$.

45. $|\vec{a}| = 2,\ |\vec{b}| = 3,\ \vec{a} \cdot \vec{b} = 4 \Rightarrow |\vec{a} - \vec{b}| =$ [1 mark]

(A) 5

(B) $\sqrt{5}$

(C) 4

(D) 2

Correct Answer: (B) $\sqrt{5}$

Solution:

Concept:

- We are given the lengths of two vectors and their dot product, and we must find the length of their difference.
- The length of a vector is linked to its dot product with itself: $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$. So we first find $|\vec{a} - \vec{b}|^2$ and take the square root at the end.

Step 1: Key Formula Expanding the dot product of the difference with itself:

$$|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{b} + |\vec{b}|^2$$

The two middle terms are equal because the dot product is commutative.

Step 2: Substitute the values Here $|\vec{a}|^2 = 2^2 = 4$, $|\vec{b}|^2 = 3^2 = 9$ and $\vec{a} \cdot \vec{b} = 4$.

$$|\vec{a} - \vec{b}|^2 = 4 - 2(4) + 9 = 4 - 8 + 9 = 5$$

Taking the positive square root, because a length cannot be negative:

$$|\vec{a} - \vec{b}| = \sqrt{5}$$

Step 3: Why the other options are wrong Option (1), the number 5, is the value of $|\vec{a} - \vec{b}|^2$. It forgets the final square root.
Option (3), the number 4, is just the given dot product $\vec{a} \cdot \vec{b}$, which is not the length asked for.
Option (4), the number 2, is only $|\vec{a}|$. No step of the working gives this value.

Step 4: Conclusion The length of the difference vector is the square root of 5, which is option (2).

Quick Tip to Solve: Use $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{b} + |\vec{b}|^2$, then take the square root.

46. If $\vec{a} = \vec{i} - \vec{j} + 2\vec{k}$ and $\vec{b} = 2\vec{i} + 3\vec{j} - 4\vec{k}$ then $|\vec{a} \times \vec{b}| =$ [1 mark]

(A) $\sqrt{174}$

(B) $\sqrt{87}$

(C) $\sqrt{93}$

(D) none of these

Correct Answer: (C) $\sqrt{93}$

Solution:

Concept:

- The cross product $\vec{a} \times \vec{b}$ is a new vector, perpendicular to both.
- We first find that vector, then find its length.

Step 1: Key Formula For $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Step 2: Expand the determinant

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 2 \\ 2 & 3 & -4 \end{vmatrix}$$

The $\vec{i}$ coefficient is $(-1)(-4) - (2)(3) = 4 - 6 = -2$.
The $\vec{j}$ coefficient is $-[(1)(-4) - (2)(2)] = -(-4 - 4) = 8$.
The $\vec{k}$ coefficient is $(1)(3) - (-1)(2) = 3 + 2 = 5$.

$$\vec{a} \times \vec{b} = -2\vec{i} + 8\vec{j} + 5\vec{k}$$

Step 3: Find the magnitude

$$|\vec{a} \times \vec{b}| = \sqrt{(-2)^2 + 8^2 + 5^2} = \sqrt{4 + 64 + 25} = \sqrt{93}$$

Step 4: Check the options Option (1), $\sqrt{174}$, equals $|\vec{a}||\vec{b}| = \sqrt{6}\sqrt{29}$. That is the largest the cross product could be, reached only when the vectors are perpendicular. Here they are not, since $\vec{a} \cdot \vec{b} =$

$2 - 3 - 8 = -9 \neq 0$.
Option (2), $\sqrt{87}$, does not come out of any correct step.
Option (4) is wrong because option (3) is correct.

Step 5: Conclusion The magnitude of the cross product is the square root of 93, which is option (3).

Quick Tip to Solve: Expand the 3x3 determinant to get $\vec{a} \times \vec{b} = -2\vec{i} + 8\vec{j} + 5\vec{k}$, then find its length.

47. If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, then [1 mark]

(A) $|\vec{a}| = |\vec{b}|$

(B) $\vec{a} \parallel \vec{b}$

(C) $\vec{a} \perp \vec{b}$

(D) none of these

Correct Answer: (C) $\vec{a} \perp \vec{b}$

Solution:

Concept:

- The magnitude of a vector is linked to the dot product by $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$.
- Since both sides of the given equation are lengths (never negative), we may square both sides without losing any information.

Step 1: Key Formula or Approach 1. $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + 2\,\vec{a} \cdot \vec{b} + |\vec{b}|^2$.
2. $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{b} + |\vec{b}|^2$.

Step 2: Square the given condition.

$$|\vec{a} + \vec{b}|^2 = |\vec{a} - \vec{b}|^2$$

Expand both sides with the two formulas above.

$$|\vec{a}|^2 + 2\,\vec{a} \cdot \vec{b} + |\vec{b}|^2 = |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{b} + |\vec{b}|^2$$

Step 3: Cancel and simplify. The terms $|\vec{a}|^2$ and $|\vec{b}|^2$ are the same on both sides, so they cancel. This leaves

$$4\,\vec{a} \cdot \vec{b} = 0 \quad \Rightarrow \quad \vec{a} \cdot \vec{b} = 0$$

A zero dot product of two non-zero vectors means they are at right angles, so $\vec{a} \perp \vec{b}$.

Step 4: Why the other options are wrong. Option 1 ($|\vec{a}| = |\vec{b}|$): take $\vec{a} = \vec{b} = \vec{i}$. The lengths are equal, but $|\vec{a} + \vec{b}| = 2$ and $|\vec{a} - \vec{b}| = 0$, so the condition fails. Equal lengths are not needed.
Option 2 ($\vec{a} \parallel \vec{b}$): for parallel non-zero vectors $\vec{a} \cdot \vec{b} \neq 0$, so the condition cannot hold. The same example $\vec{a} = \vec{b} = \vec{i}$ shows it fails.
Option 4 (none of these) is wrong because option 3 is correct.

Step 5: Conclusion The two diagonals $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ have equal length only when $\vec{a}$ and $\vec{b}$ are perpendicular, which is option 3.

Quick Tip to Solve: Square both sides and expand with $|\vec{a} \pm \vec{b}|^2 = |\vec{a}|^2 \pm 2\vec{a} \cdot \vec{b} + |\vec{b}|^2$. Only the dot product term survives.

48. $2\vec{j} \cdot (-3)\vec{k} =$ [1 mark]

(A) 6

(B) -6

(C) 0

(D) $-6\vec{i}$

Correct Answer: (C) 0

Solution:

Concept:

- The dot product of two vectors is a number (a scalar).
- It equals $|\vec{u}||\vec{v}|\cos\theta$, where θ is the angle between them.

Step 1: Key Formula $\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors along three mutually perpendicular axes. So the dot product of two different ones is zero:

$$\vec{j}\cdot\vec{k} = (1)(1)\cos 90^\circ = 0$$

Scalar multipliers can be pulled out of a dot product.

Step 2: Detailed Explanation

$$2\vec{j}\cdot(-3)\vec{k} = (2)(-3)\,(\vec{j}\cdot\vec{k}) = (-6)(0) = 0$$

Step 3: Why the other options are wrong Options (1) and (2), 6 and -6, would be right only if $\vec{j}\cdot\vec{k}$ were 1 or -1. That happens only for parallel vectors, but $\vec{j}$ and $\vec{k}$ are perpendicular.
Option (4), $-6\vec{i}$, is a vector. A dot product always gives a scalar, so a vector answer cannot be right. (It looks like the cross product $\vec{j}\times\vec{k} = \vec{i}$ was used instead.)

Step 4: Conclusion Since the two vectors are perpendicular, their dot product is zero. This is option (3).

Quick Tip to Solve: Pull out the scalars: $(2)(-3)(\vec{j}\cdot\vec{k})$, and recall that $\vec{j}$ and $\vec{k}$ are perpendicular.

49. The direction ratios of a straight line are $2, 6, -3$. Then its direction cosines are [1 mark]

(A) $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}$

(B) $\frac{2}{7}, \frac{-6}{7}, \frac{3}{7}$

(C) $\frac{2}{7}, \frac{6}{7}, \frac{-3}{7}$

(D) none of these

Correct Answer: (C) $\frac{2}{7}, \frac{6}{7}, \frac{-3}{7}$

Solution:

Concept:

- Direction ratios a, b, c are any numbers proportional to the direction cosines l, m, n of a line.
- To get the direction cosines, we divide each ratio by the same number so that $l^2 + m^2 + n^2 = 1$.

Step 1: Key Formula

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

Step 2: Detailed Explanation Here $a = 2,\ b = 6,\ c = -3$.

$$\sqrt{a^2+b^2+c^2} = \sqrt{4+36+9} = \sqrt{49} = 7$$

Dividing each ratio by 7:

$$l = \frac{2}{7}, \quad m = \frac{6}{7}, \quad n = \frac{-3}{7}$$

Check: $\frac{4}{49} + \frac{36}{49} + \frac{9}{49} = 1$. This matches the rule for direction cosines.

Step 3: Why the other options are wrong Option (1): the numbers 1, 2, 3 are not in the ratio $2 : 6 : -3$, and $\frac{1+4+9}{49} = \frac{14}{49} \neq 1$.

Option (2): its squares do add to 1, but the ratio $2 : -6 : 3$ is not proportional to $2 : 6 : -3$. The signs of the second and third numbers are changed.

Option (4) is wrong because option (3) is correct.

Step 4: Conclusion The direction cosines are 2/7, 6/7 and $-3/7$ (the other valid set is the same with all signs reversed, which is not listed). This is option (3).

Quick Tip to Solve: Divide each direction ratio by $\sqrt{2^2+6^2+(-3)^2} = 7$.

50. If a line makes angles α, β and γ with the positive directions of x, y and z axes respectively, then [1 mark]

(A) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

(B) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 4$

(C) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 2$

(D) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 1$

Correct Answer: (A) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Solution:

Concept:

- The cosines of the angles a line makes with the three axes are called its direction cosines.
- We write $l = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma$.

Step 1: Key Formula Take a unit vector $\hat{r}$ along the line. Its components along the axes are exactly l, m, n, so $\hat{r} = l\vec{i} + m\vec{j} + n\vec{k}$.

A unit vector has length 1, so

$$l^2 + m^2 + n^2 = 1$$

Step 2: Apply to the angles

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

This is option (1).

Step 3: Check the sine options Use $\sin^2\theta = 1 - \cos^2\theta$ for each angle and add:

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 3 - (\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 3 - 1 = 2$$

So the sine sum is 2. It is not 4 (option 2) and not 1 (option 4).
Option (3) claims the cosine sum is 2, but we showed it is 1.

Step 4: Conclusion The squares of the direction cosines add up to 1. This is option (1).

Quick Tip to Solve: The direction cosines l, m, n are the components of a unit vector, so $l^2+m^2+n^2 = 1$.

---

51. $\int \dfrac{1-\cos 2x}{1+\cos 2x}\,dx =$ [1 mark]

(A) $\tan x + x + k$

(B) $\tan x - x + k$

(C) $x - \tan^2 x + k$

(D) $\tan\dfrac{x}{2} + k$

Correct Answer: (B) $\tan x - x + k$

Solution:

Concept:

- The integrand is a ratio of two expressions in $\cos 2x$.
- A double-angle identity turns both the numerator and the denominator into squares of a single sine or cosine, so the fraction simplifies a lot before we integrate.

Step 1: Key Formula or Approach 1. $1 - \cos 2x = 2\sin^2 x$.
2. $1 + \cos 2x = 2\cos^2 x$.
3. $\dfrac{\sin^2 x}{\cos^2 x} = \tan^2 x$ and $\tan^2 x = \sec^2 x - 1$.
4. $\int \sec^2 x\,dx = \tan x + k$.

Step 2: Simplify the integrand. Put the identities from Step 2 into the fraction.

$$\frac{1-\cos 2x}{1+\cos 2x} = \frac{2\sin^2 x}{2\cos^2 x} = \tan^2 x$$

The factor 2 cancels, and the rest is $\tan^2 x$.

Step 3: Integrate. We cannot integrate $\tan^2 x$ directly, so rewrite it as $\sec^2 x - 1$.

$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \int \sec^2 x\,dx - \int 1\,dx$$

$$= \tan x - x + k$$

Check by differentiating: $\dfrac{d}{dx}(\tan x - x) = \sec^2 x - 1 = \tan^2 x$, which is the simplified integrand.

Step 4: Why the other options are wrong. Option (1), $\tan x + x + k$: its derivative is $\sec^2 x + 1$, which is $\tan^2 x + 2$. The sign of x is wrong.
Option (3), $x - \tan^2 x + k$: its derivative is $1 - 2\tan x \sec^2 x$, which has nothing to do with $\tan^2 x$.

Option (4), $\tan\frac{x}{2}+k$: its derivative is $\frac{1}{2}\sec^2\frac{x}{2}$, which is not $\tan^2 x$.

Step 5: Conclusion The integral equals $\tan x - x + k$, which is option (2).

Quick Tip to Solve: Write $1-\cos 2x = 2\sin^2 x$ and $1+\cos 2x = 2\cos^2 x$ to get $\tan^2 x$, then use $\tan^2 x = \sec^2 x - 1$.

---

52. $\int \frac{dx}{x-1} =$ [1 mark]

(A) $\log|x+1|+k$

(B) $-\log|x+1|+k$

(C) $\log|x-1|+k$

(D) $\log x + k$

Correct Answer: (C) $\log|x-1|+k$

Solution:

Concept:

- The integrand $\frac{1}{x-1}$ has the form $\frac{1}{\text{linear expression}}$.
- Its antiderivative is a logarithm.

Step 1: Key Formula or Approach 1. $\int \frac{dt}{t} = \log|t|+k$.

2. A substitution $t = x-1$ has $dt = dx$, so nothing extra appears.

Step 2: Integrate Let $t = x-1$. Then $dt = dx$.

$$\int \frac{dx}{x-1} = \int \frac{dt}{t} = \log|t|+k = \log|x-1|+k$$

The modulus is needed because $x-1$ can be negative, and a logarithm is defined only for positive numbers.

Step 3: Why the other options are wrong Option (A) $\log|x+1|$ differentiates to $\frac{1}{x+1}$, which has the wrong sign inside the denominator.

Option (B) $-\log|x+1|$ differentiates to $-\frac{1}{x+1}$, wrong in both sign and denominator.

Option (D) $\log x$ differentiates to $\frac{1}{x}$, which is missing the shift by 1.

Step 4: Conclusion The integral is $\log|x-1|+k$, which is option (C).

Quick Tip to Solve: Substitute $t = x-1$, so $dt = dx$, and use $\int \frac{dt}{t} = \log|t|+k$.

---

53. $\int \log x\, dx =$ [1 mark]

(A) $\frac{1}{x}+k$

(B) $x\log x + k$

(C) $x\log x - x + k$

(D) $x\log x + x + k$

Correct Answer: (C) $x\log x - x + k$

Solution:

Concept:

- We need the integral of $\log x$. There is no standard formula for it, and it is not a product of two functions.
- We can still treat it as a product by writing $\log x = \log x \cdot 1$, and then use integration by parts. Here $\log x$ means the natural logarithm.

Step 1: Key Formula Integration by parts says $\int u\,dv = uv - \int v\,du$. We choose u as the function that becomes simpler when differentiated, and dv as the part that we can integrate easily.
So we take $u = \log x$ and $dv = dx$.

Step 2: Find du and v Differentiating u gives $du = \frac{1}{x}\,dx$.
Integrating dv gives $v = x$.

Step 3: Apply the formula

$$\int \log x\,dx = x\log x - \int x \cdot \frac{1}{x}\,dx$$

The x in the numerator cancels the x in the denominator, so

$$\int \log x\,dx = x\log x - \int 1\,dx = x\log x - x + k$$

Step 4: Check the answer by differentiating $\frac{d}{dx}(x\log x - x) = \log x + x \cdot \frac{1}{x} - 1 = \log x + 1 - 1 = \log x$.
This is the original integrand, so the result is correct. The correct option is (C).

Step 5: Why the other options are wrong Option (A): $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$, not $\log x$. It is also the integral of a different function.
Option (B): $\frac{d}{dx}(x\log x) = \log x + 1$. The extra 1 is left over, so the term $-x$ is missing.
Option (D): $\frac{d}{dx}(x\log x + x) = \log x + 2$, which is not $\log x$. The sign of x was taken wrongly.

Step 6: Conclusion Using integration by parts with $u = \log x$ and $dv = dx$, the integral is $x\log x - x + k$.
This is option (C).

Quick Tip to Solve: Write $\log x = \log x \cdot 1$ and use integration by parts with $u = \log x$, $dv = dx$.

54. $\int \cos\sqrt{x}\,dx =$ [1 mark]

(A) $\sin\sqrt{x} + \cos\sqrt{x} + k$

(B) $\frac{1}{2}\left(\sqrt{x}\sin\sqrt{x} - \cos\sqrt{x}\right) + k$

(C) $2\left(\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right) + k$

(D) $\sin\sqrt{x} + k$

Correct Answer: (C) $2\left(\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right) + k$

Solution:

Concept:

- The integrand $\cos\sqrt{x}$ has a square root inside the cosine.
- Direct integration is not possible, so we first remove the root with a substitution.
- After that, the integral becomes a product of two functions, which we handle by parts.

Step 1: Substitute Let $\sqrt{x} = t$. Then $x = t^2$ and $dx = 2t\,dt$.

$$\int \cos\sqrt{x}\,dx = \int \cos t \cdot 2t\,dt = 2\int t\cos t\,dt$$

Step 2: Integrate by parts Use $\int u\,dv = uv - \int v\,du$ with $u = t$ and $dv = \cos t\,dt$.
Then $du = dt$ and $v = \sin t$.

$$\int t\cos t\,dt = t\sin t - \int \sin t\,dt = t\sin t + \cos t$$

Step 3: Put back the limits of the substitution Multiply by the factor 2 and replace t by $\sqrt{x}$:

$$\int \cos\sqrt{x}\,dx = 2\big(t\sin t + \cos t\big) + k = 2\big(\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\big) + k$$

This is option (C).

Step 4: Check by differentiating $\dfrac{d}{dx}\left[2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x}\right] = 2\left[\dfrac{\sin\sqrt{x}}{2\sqrt{x}} + \dfrac{\sqrt{x}\cos\sqrt{x}}{2\sqrt{x}}\right] - \dfrac{2\sin\sqrt{x}}{2\sqrt{x}} = \cos\sqrt{x}$. The $\sin\sqrt{x}/\sqrt{x}$ terms cancel, which confirms the answer.

Step 5: Why the other options are wrong Option (A): its derivative is $\dfrac{\cos\sqrt{x} - \sin\sqrt{x}}{2\sqrt{x}}$, which is not $\cos\sqrt{x}$.

Option (B): it has the factor $\dfrac{1}{2}$ in place of 2 and a wrong sign on the cosine. Its derivative is $\dfrac{\sin\sqrt{x}}{2\sqrt{x}} + \dfrac{\cos\sqrt{x}}{4}$.

Option (D): the derivative of $\sin\sqrt{x}$ is $\dfrac{\cos\sqrt{x}}{2\sqrt{x}}$. The factor $2\sqrt{x}$ in the denominator means it does not equal $\cos\sqrt{x}$. This is the mistake of treating $\sqrt{x}$ as if it were x.

Step 6: Conclusion After substituting $\sqrt{x} = t$ and using integration by parts, the integral is $2(\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}) + k$. This is option (C).

Quick Tip to Solve: Put $\sqrt{x} = t$ so that $dx = 2t\,dt$, then integrate $2t\cos t$ by parts.

55. $\int e^x\left(\tan^{-1}x + \dfrac{1}{1+x^2}\right)dx =$ [1 mark]

(A) $e^x\tan^{-1}x + k$

(B) $e^x \cdot \dfrac{1}{1+x^2} + k$

(C) $e^x + k$

(D) $\tan^{-1}x + k$

Correct Answer: (A) $e^x\tan^{-1}x + k$

Solution:

Concept:

- The integrand has the form $e^x\,[\,f(x) + f'(x)\,]$.
- Here the first function inside the bracket is $\tan^{-1}x$, and the second is its derivative $\dfrac{1}{1+x^2}$.
- This pattern has a ready formula.

Step 1: Key Formula

$$\int e^x [f(x) + f'(x)]dx = e^x f(x) + k$$

We can prove it quickly by parts: $\int e^x f\,dx = e^x f - \int e^x f'\,dx$. So $\int e^x f\,dx + \int e^x f'\,dx = e^x f$.

Step 2: Identify f(x) Take $f(x) = \tan^{-1} x$. Then $f'(x) = \dfrac{1}{1+x^2}$.
This matches the bracket exactly, so the formula applies.

Step 3: Write the result

$$\int e^x \left(\tan^{-1} x + \frac{1}{1+x^2}\right)dx = e^x \tan^{-1} x + k$$

This is option (A).

Step 4: Why the other options are wrong Option (B): $\dfrac{d}{dx}\left(\dfrac{e^x}{1+x^2}\right) = \dfrac{e^x(x-1)^2}{(1+x^2)^2}$, which is not the integrand. This is the mistake of using $f'(x)$ as the answer in place of $f(x)$.
Option (C): the derivative of e^x is e^x, which has no $\tan^{-1} x$ or $\dfrac{1}{1+x^2}$ terms.
Option (D): the derivative of $\tan^{-1} x$ is $\dfrac{1}{1+x^2}$ only, and the factor e^x is missing.

Step 5: Conclusion The integrand is of the form $e^x[f(x) + f'(x)]$ with $f(x) = \tan^{-1} x$, so the integral is $e^x \tan^{-1} x + k$. This is option (A).

Quick Tip to Solve: Use $\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + k$ with $f(x) = \tan^{-1} x$.

56. $\int \dfrac{dx}{x^2 - a^2} =$ [1 mark]

(A) $\dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + k$

(B) $\dfrac{1}{2a}\log\left|\dfrac{x-a}{x+a}\right| + k$

(C) $\dfrac{1}{2a}\log\left|\dfrac{a+x}{a-x}\right| + k$

(D) $\dfrac{1}{a}\log\left|\dfrac{x-a}{x+a}\right| + k$

Correct Answer: (B) $\dfrac{1}{2a}\log\left|\dfrac{x-a}{x+a}\right| + k$

Solution:

Concept:

- The denominator $x^2 - a^2$ is a difference of two squares, so it factorises as $(x-a)(x+a)$.
- When the denominator splits into two different linear factors, partial fractions is the natural tool.

Step 1: Split into partial fractions Write $\dfrac{1}{x^2-a^2} = \dfrac{A}{x-a} + \dfrac{B}{x+a}$.
Multiply both sides by $(x-a)(x+a)$: $1 = A(x+a) + B(x-a)$.
Put $x = a$: $1 = 2aA$, so $A = \dfrac{1}{2a}$.
Put $x = -a$: $1 = -2aB$, so $B = -\dfrac{1}{2a}$.

$$\frac{1}{x^2-a^2} = \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right]$$

Step 2: Integrate each part Each term is of the form $\int \frac{dx}{x \pm a} = \log|x \pm a|$.

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\left[\log|x - a| - \log|x + a|\right] + k$$

Using $\log p - \log q = \log \frac{p}{q}$,

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x - a}{x + a}\right| + k$$

This is option (B).

Step 3: Why the other options are wrong Option (A): $\frac{1}{a}\tan^{-1}\frac{x}{a}$ is the integral of $\frac{1}{x^2 + a^2}$, with a plus sign in the denominator.

Option (C): $\frac{1}{2a}\log\left|\frac{a + x}{a - x}\right|$ is the integral of $\frac{1}{a^2 - x^2}$. It differs from the answer by a minus sign, because the fraction inside the log is flipped.

Option (D): its derivative is $\frac{2}{x^2 - a^2}$, which is double the required integrand. The factor $\frac{1}{2a}$ has been replaced by $\frac{1}{a}$.

Step 4: Conclusion By partial fractions, $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x - a}{x + a}\right| + k$. This is option (B).

Quick Tip to Solve: Factorise $x^2 - a^2 = (x - a)(x + a)$ and split $\frac{1}{x^2 - a^2}$ into partial fractions.

57. $\int \sin^2\frac{x}{2}\,dx =$ [1 mark]

(A) $\frac{1}{2}x - \frac{1}{2}\sin x + k$

(B) $\frac{1}{2}x - \frac{1}{2}\cos x + k$

(C) $\frac{1}{2}\sin x + k$

(D) $-\frac{1}{2}\sin x + k$

Correct Answer: (A) $\frac{1}{2}x - \frac{1}{2}\sin x + k$

Solution:

Concept:

- We cannot integrate $\sin^2\frac{x}{2}$ directly, because the sine is squared.
- The standard way is to lower the power with a double-angle identity.

Step 1: Key Formula The identity is $\cos 2\theta = 1 - 2\sin^2\theta$. It can be rearranged as $\sin^2\theta = \frac{1 - \cos 2\theta}{2}$.

Here $\theta = \frac{x}{2}$, so $2\theta = x$.

Step 2: Rewrite the integrand

$$\sin^2\frac{x}{2} = \frac{1 - \cos x}{2}$$

$$\int \sin^2 \frac{x}{2}\,dx = \frac{1}{2}\int (1-\cos x)\,dx$$

Step 3: Integrate We know $\int 1\,dx = x$ and $\int \cos x\,dx = \sin x$.

$$\int \sin^2 \frac{x}{2}\,dx = \frac{1}{2}(x - \sin x) + k = \frac{1}{2}x - \frac{1}{2}\sin x + k$$

This is option (A).

Step 4: Why the other options are wrong Option (B): its derivative is $\frac{1}{2} + \frac{1}{2}\sin x$. The sign is wrong, and $\cos x$ should not appear in the answer, because $\int \cos x\,dx = \sin x$.
Option (C): its derivative is $\frac{1}{2}\cos x$, which is not $\sin^2 \frac{x}{2}$.
Option (D): its derivative is $-\frac{1}{2}\cos x$, also not equal to $\sin^2 \frac{x}{2}$. Both (C) and (D) miss the term $\frac{x}{2}$ that comes from the constant 1 in $1 - \cos x$.

Step 5: Conclusion Using $\sin^2 \frac{x}{2} = \frac{1-\cos x}{2}$, the integral is $\frac{1}{2}x - \frac{1}{2}\sin x + k$. This is option (A).

Quick Tip to Solve: Write $\sin^2 \frac{x}{2} = \frac{1-\cos x}{2}$ and integrate term by term.

---

58. $\int \frac{\tan(\sin^{-1} x)}{\sqrt{1-x^2}}\,dx =$ [1 mark]

(A) $\log|\sec(\sin^{-1} x)| + k$

(B) $\log|\cos(\sin^{-1} x)| + k$

(C) $\tan(\sin^{-1} x) + k$

(D) $\log|\sin^{-1} x| + k$

Correct Answer: (A) $\log|\sec(\sin^{-1} x)| + k$

Solution:

Concept:

- The integrand has the form (function of $\sin^{-1} x$) times $\frac{1}{\sqrt{1-x^2}}$.
- We know that $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, so the extra factor is exactly the derivative of the inner function.
- This is the signal to use substitution.

Step 1: Key Formula or Approach 1. Put $u = \sin^{-1} x$.
2. Then $du = \frac{dx}{\sqrt{1-x^2}}$.
3. $\int \tan u\,du = \log|\sec u| + k$.

Step 2: Substitute and integrate After the substitution the integral becomes

$$\int \tan u\,du = \int \frac{\sin u}{\cos u}\,du$$

Now put $t = \cos u$, so $dt = -\sin u\, du$:

$$\int \frac{\sin u}{\cos u}\, du = -\int \frac{dt}{t} = -\log|t| + k = -\log|\cos u| + k = \log|\sec u| + k$$

Step 3: Go back to x Replace u by $\sin^{-1} x$:

$$\int \frac{\tan(\sin^{-1} x)}{\sqrt{1-x^2}}\, dx = \log|\sec(\sin^{-1} x)| + k$$

This matches option (A).

Step 4: Why the other options are wrong Option (B): $\log|\cos(\sin^{-1} x)|$ equals $-\log|\sec(\sin^{-1} x)|$, so it has the opposite sign. Its derivative is $-\dfrac{\tan(\sin^{-1} x)}{\sqrt{1-x^2}}$, not the integrand.

Option (C): the derivative of $\tan(\sin^{-1} x)$ is $\dfrac{\sec^2(\sin^{-1} x)}{\sqrt{1-x^2}}$. This is a different function.

Option (D): the derivative of $\log|\sin^{-1} x|$ is $\dfrac{1}{\sin^{-1} x\sqrt{1-x^2}}$. This has no tangent in it, so it is wrong.

Step 5: Conclusion The integral equals $\log|\sec(\sin^{-1} x)| + k$, which is option (A).

Quick Tip to Solve: Put $u = \sin^{-1} x$. Then $du = \dfrac{dx}{\sqrt{1-x^2}}$ and the integral becomes $\int \tan u\, du$.

59. $\int \dfrac{dx}{e^x + e^{-x}} =$ [1 mark]

(A) $\cot^{-1}(e^x) + k$

(B) $\tan^{-1}(e^x) + k$

(C) $\log|e^x + 1| + k$

(D) $\sin^{-1}(e^x) + k$

Correct Answer: (B) $\tan^{-1}(e^x) + k$

Solution:

Concept:

- The denominator mixes e^x and e^{-x}.
- Multiplying the top and bottom by e^x turns the denominator into a polynomial in e^x, and then a substitution $t = e^x$ works.

Step 1: Key Formula or Approach 1. $e^x \cdot e^{-x} = 1$.

2. $\int \dfrac{dt}{1+t^2} = \tan^{-1} t + k$.

Step 2: Detailed Explanation Multiply numerator and denominator by e^x:

$$\frac{1}{e^x + e^{-x}} = \frac{e^x}{e^{2x} + 1}$$

Put $t = e^x$, so $dt = e^x\, dx$. The integral becomes

$$\int \frac{dt}{t^2 + 1} = \tan^{-1} t + k = \tan^{-1}(e^x) + k$$

This is option (B).

Step 3: Why the other options are wrong Option (A): $\frac{d}{dx}\cot^{-1}(e^x) = -\frac{e^x}{1+e^{2x}}$, which has the wrong sign.
Option (C): $\frac{d}{dx}\log|e^x+1| = \frac{e^x}{e^x+1}$, which is not the integrand.
Option (D): $\frac{d}{dx}\sin^{-1}(e^x) = \frac{e^x}{\sqrt{1-e^{2x}}}$, which has a square root and is not the integrand.
Step 4: Conclusion The integral is $\tan^{-1}(e^x)+k$, which is option (B).

Quick Tip to Solve: Multiply the top and bottom by e^x to get $\frac{e^x}{e^{2x}+1}$, then put $t = e^x$.

60. $\int_0^1 \frac{dx}{1+x^2} =$ [1 mark]

(A) $\frac{\pi}{2}$
(B) $\frac{\pi}{3}$
(C) $\frac{\pi}{4}$
(D) π

Correct Answer: (C) $\frac{\pi}{4}$

Solution:

Concept:

- This is a standard definite integral.
- The function $\frac{1}{1+x^2}$ is the derivative of $\tan^{-1}x$, so we can evaluate it with the fundamental theorem of calculus.

Step 1: Key Formula or Approach

$$\int \frac{dx}{1+x^2} = \tan^{-1}x + k$$

and $\int_a^b f(x)\,dx = F(b) - F(a)$.

Step 2: Detailed Explanation Apply the limits 0 and 1:

$$\int_0^1 \frac{dx}{1+x^2} = \Big[\tan^{-1}x\Big]_0^1 = \tan^{-1}1 - \tan^{-1}0$$

We know $\tan\frac{\pi}{4} = 1$, so $\tan^{-1}1 = \frac{\pi}{4}$. Also $\tan^{-1}0 = 0$. So the value is $\frac{\pi}{4}$.

Step 3: Why the other options are wrong Option (A), $\frac{\pi}{2}$, is the value when the upper limit is ∞ instead of 1. Option (B), $\frac{\pi}{3}$, would need $\tan^{-1}x = \frac{\pi}{3}$, which happens at $x = \sqrt{3}$, not at $x = 1$. Option (D), π, is larger than the area under the curve allows: the integrand is at most 1 on $[0,1]$, so the integral cannot exceed 1, while $\pi > 3$.

Step 4: Conclusion The value of the integral is $\frac{\pi}{4}$, option (C).

Quick Tip to Solve: The antiderivative of $\frac{1}{1+x^2}$ is $\tan^{-1}x$. Put in the limits 0 and 1.

61. **If** $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \cdots}}}$ **then** $\frac{dy}{dx} =$ [1 mark]

(A) $\frac{1}{2y-1}$

(B) $\frac{\cos x}{2y-1}$

(C) $\frac{\sin x}{2y-1}$

(D) $\frac{2y-1}{\cos x}$

Correct Answer: (B) $\frac{\cos x}{2y-1}$

Solution:

Concept:

- The expression for y is an infinite nested square root in which the same pattern $\sqrt{\sin x + \cdots}$ repeats forever.
- Because the pattern repeats, the part under the first root is $\sin x + y$ again.

Step 1: Key Formula or Approach 1. Remove the nesting: the inner expression equals y itself, so $y = \sqrt{\sin x + y}$.
2. Square both sides and differentiate implicitly with respect to x.

Step 2: Remove the infinite root

$$y = \sqrt{\sin x + y} \quad \Rightarrow \quad y^2 = \sin x + y$$

Step 3: Differentiate both sides with respect to x y is a function of x, so $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$.

$$2y\frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

Bring the terms with $\frac{dy}{dx}$ to one side.

$$(2y-1)\frac{dy}{dx} = \cos x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\cos x}{2y-1}$$

Step 4: Why the other options are wrong Option (1), $\frac{1}{2y-1}$, loses the factor $\cos x$. That factor comes from $\frac{d}{dx}(\sin x)$.
Option (3), $\frac{\sin x}{2y-1}$, uses $\sin x$ in the numerator, but the derivative of $\sin x$ is $\cos x$.
Option (4), $\frac{2y-1}{\cos x}$, is the reciprocal of the correct answer.

Step 5: Conclusion $\frac{dy}{dx} = \frac{\cos x}{2y-1}$, which is option (2).

Quick Tip to Solve: Because the root repeats forever, $y = \sqrt{\sin x + y}$, so $y^2 = \sin x + y$. Differentiate both sides with respect to x.

62. **If** $x^n + y^n = a^n$ **then** $\frac{dy}{dx} =$ [1 mark]

(A) $-\frac{x^{n-1}}{y^{n-1}}$

(B) $\frac{x^{n-1}}{y^{n-1}}$

(C) $-\frac{y^{n-1}}{x^{n-1}}$

(D) nx^{n-1}

Correct Answer: (A) $-\frac{x^{n-1}}{y^{n-1}}$

Solution:

Concept:

- The equation ties x and y together without solving for y.
- This is an implicit function.
- We differentiate both sides with respect to x and treat y as a function of x, so the chain rule gives an extra factor $\frac{dy}{dx}$ for every term in y.

Step 1: Key Formula or Approach 1. $\frac{d}{dx}(x^n) = nx^{n-1}$.

2. $\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$.

3. a is a constant, so $\frac{d}{dx}(a^n) = 0$.

Step 2: Differentiate both sides

$$nx^{n-1} + ny^{n-1}\frac{dy}{dx} = 0$$

Divide by n:

$$x^{n-1} + y^{n-1}\frac{dy}{dx} = 0$$

Step 3: Solve for dy/dx Move x^{n-1} to the right side and divide by y^{n-1}:

$$\frac{dy}{dx} = -\frac{x^{n-1}}{y^{n-1}}$$

Step 4: Why the other options are wrong Option (B) drops the minus sign. It would come from moving x^{n-1} across without changing its sign.
Option (C) turns the fraction upside down, so it uses y^{n-1} on top and x^{n-1} at the bottom. The factor $\frac{dy}{dx}$ multiplies y^{n-1}, so y^{n-1} must go in the denominator.
Option (D) is only $\frac{d}{dx}(x^n)$. It ignores the y term completely.

Step 5: Conclusion Differentiating implicitly gives option (A).

Quick Tip to Solve: Differentiate both sides with respect to x, treating y as a function of x, then solve for dy/dx.

63. **If** $x = a(1 - \cos\theta),\ y = a(\theta + \sin\theta)$, **then** $\frac{dy}{dx} =$ [1 mark]

(A) $\tan\frac{\theta}{2}$

collegedunia

(B) $-\tan\frac{\theta}{2}$

(C) $\cot\frac{\theta}{2}$

(D) $-\cot\frac{\theta}{2}$

Correct Answer: (C) $\cot\frac{\theta}{2}$

Solution:

Concept:

- Both x and y are given in terms of a parameter θ.
- We cannot differentiate y with respect to x directly, so we differentiate each with respect to θ and divide.

Step 1: Key Formula or Approach

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

We also use $1+\cos\theta = 2\cos^2\frac{\theta}{2}$ and $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$.

Step 2: Differentiate x and y with respect to theta

$$\frac{dx}{d\theta} = a\sin\theta$$

$$\frac{dy}{d\theta} = a(1+\cos\theta)$$

Step 3: Divide

$$\frac{dy}{dx} = \frac{a(1+\cos\theta)}{a\sin\theta} = \frac{1+\cos\theta}{\sin\theta}$$

Replace the numerator and denominator with half-angle forms:

$$\frac{dy}{dx} = \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

Step 4: Why the other options are wrong Option (A) $\tan\frac{\theta}{2}$ is the reciprocal. It would come from $\frac{dx}{dy}$, not $\frac{dy}{dx}$.
Option (B) is the reciprocal with a wrong sign.
Option (D) has the right function but the wrong sign. The signs of $\frac{dy}{d\theta} = a(1+\cos\theta)$ and $\frac{dx}{d\theta} = a\sin\theta$ are both positive for small positive θ, so the ratio cannot be negative there.

Step 5: Conclusion The derivative is $\cot\frac{\theta}{2}$, option (C).

Quick Tip to Solve: Find dx/d(theta) and dy/d(theta), divide, and simplify with the half-angle forms of 1 + cos(theta) and sin(theta).

64. If $y = x^x$ then $\frac{dy}{dx} =$ [1 mark]

(A) $x^x(\log x + 1)$

(B) $\log x$

(C) $(\log x + 1)$

(D) nx^{n-1}

Correct Answer: (A) $x^x(\log x + 1)$

Solution:

Concept:

- Here the base and the power both contain x.
- The power rule $\frac{d}{dx}x^n = nx^{n-1}$ needs a constant power, and the exponential rule $\frac{d}{dx}a^x = a^x \log a$ needs a constant base.
- Neither fits, so we use logarithmic differentiation.

Step 1: Key Formula or Approach Take the natural log of both sides, use $\log m^p = p \log m$, then differentiate with the product rule and the chain rule.

Step 2: Take the logarithm

$$\log y = x \log x$$

Step 3: Differentiate both sides Left side: $\frac{1}{y}\frac{dy}{dx}$. Right side by the product rule:

$$\frac{d}{dx}(x \log x) = x \cdot \frac{1}{x} + \log x \cdot 1 = 1 + \log x$$

So

$$\frac{1}{y}\frac{dy}{dx} = 1 + \log x$$

Step 4: Solve for dy/dx Multiply by $y = x^x$:

$$\frac{dy}{dx} = x^x(\log x + 1)$$

Step 5: Why the other options are wrong Option (B) $\log x$ and option (C) $\log x + 1$ leave out the factor $y = x^x$ (and (B) also drops the 1). The slope of x^x at $x = 2$, for example, is $4(\log 2 + 1)$, not $\log 2 + 1$. Option (D) nx^{n-1} is the power rule, which is not valid here because the power is not a constant. It also contains n, which is not in the question.

Step 6: Conclusion The derivative is $x^x(\log x + 1)$, option (A).

Quick Tip to Solve: Take the log of both sides to get log y = x log x, then differentiate and multiply back by y.

65. $\frac{d}{dx}\sin^{-1}(3x - 4x^3) =$ [1 mark]

(A) $\frac{3}{\sqrt{1-x^2}}$

(B) $\frac{-3}{\sqrt{1-x^2}}$

(C) $\frac{1}{\sqrt{1-x^2}}$

(D) $\dfrac{-1}{\sqrt{1-x^2}}$

Correct Answer: (A) $\dfrac{3}{\sqrt{1-x^2}}$

Solution:

Concept:

- The expression $3x - 4x^3$ looks like the triple angle formula $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.
- So a substitution $x = \sin\theta$ turns the inverse sine into a simple multiple of θ.

Step 1: Key Formula or Approach 1. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.
2. $\sin^{-1}(\sin\phi) = \phi$ when $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.
3. $\dfrac{d}{dx}\sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}}$.

Step 2: Substitute Let $x = \sin\theta$, so $\theta = \sin^{-1} x$. Then

$$\sin^{-1}(3x - 4x^3) = \sin^{-1}(\sin 3\theta) = 3\theta = 3\sin^{-1} x$$

This step holds when 3θ stays in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, that is when $-\frac{1}{2} \le x \le \frac{1}{2}$. This is the standard range assumed for this question.

Step 3: Differentiate

$$\frac{d}{dx}(3\sin^{-1} x) = \frac{3}{\sqrt{1-x^2}}$$

Step 4: Why the other options are wrong Option (C) is the derivative of $\sin^{-1} x$ alone. It forgets the factor 3 that comes from 3θ.
Options (B) and (D) carry a negative sign. Direct differentiation gives $\dfrac{3(1-4x^2)}{\sqrt{1-x^2}\,|1-4x^2|}$, which is negative only when $|x| > \frac{1}{2}$. In the usual range $|x| < \frac{1}{2}$ the sign is positive, so (B) and (D) are not the expected answers.

Step 5: Conclusion For $|x| < \frac{1}{2}$ the derivative is $\dfrac{3}{\sqrt{1-x^2}}$, option (A).

Quick Tip to Solve: Put x = sin(theta) so that 3x - 4x^3 = sin(3 theta), then differentiate 3 sin^(−1) x.

66. $\dfrac{d}{dx}(\sin 3x \cdot \cos 5x) =$ [1 mark]

(A) $4\cos 8x$

(B) $4\cos 8x - \cos 2x$

(C) $\cos 2x$

(D) $\cos 2x - 4\cos 8x$

Correct Answer: (B) $4\cos 8x - \cos 2x$

Solution:

Concept:

- We must differentiate the product $\sin 3x \cdot \cos 5x$.

- The product rule works, and the answer can then be tidied with product-to-sum formulas so that it matches one of the options.

Step 1: Key Formula or Approach 1. Product rule: $\frac{d}{dx}(uv) = u'v + uv'$.
2. $\frac{d}{dx}\sin ax = a\cos ax$ and $\frac{d}{dx}\cos ax = -a\sin ax$.
3. $2\cos A\cos B = \cos(A-B) + \cos(A+B)$ and $2\sin A\sin B = \cos(A-B) - \cos(A+B)$.

Step 2: Apply the product rule Take $u = \sin 3x$ and $v = \cos 5x$. Then $u' = 3\cos 3x$ and $v' = -5\sin 5x$.

$$\frac{d}{dx}(\sin 3x\cos 5x) = 3\cos 3x\cos 5x - 5\sin 3x\sin 5x$$

Step 3: Convert each product into a sum

$$3\cos 3x\cos 5x = \frac{3}{2}(\cos 2x + \cos 8x)$$

$$5\sin 3x\sin 5x = \frac{5}{2}(\cos 2x - \cos 8x)$$

Subtract the second from the first:

$$\frac{3}{2}\cos 2x + \frac{3}{2}\cos 8x - \frac{5}{2}\cos 2x + \frac{5}{2}\cos 8x = 4\cos 8x - \cos 2x$$

Step 4: Compare with the options Option (A) $4\cos 8x$ leaves out the $-\cos 2x$ term, so it is wrong.
Option (B) $4\cos 8x - \cos 2x$ is exactly our result, so it is correct.
Option (C) $\cos 2x$ has neither the $\cos 8x$ term nor the right sign, so it is wrong.
Option (D) $\cos 2x - 4\cos 8x$ is the negative of the correct answer, a sign slip, so it is wrong.

Step 5: Conclusion The derivative is $4\cos 8x - \cos 2x$, which is option (B).

Quick Tip to Solve: Apply the product rule to $\sin 3x \cdot \cos 5x$, then use $2\cos A\cos B$ and $2\sin A\sin B$ to combine the terms into $\cos 8x$ and $\cos 2x$.

---

67. $\frac{d}{dx}(\cos x^3) =$ [1 mark]

(A) $-3x^2\sin x^3$

(B) $\sin x^3$

(C) $3x^2\sin x^3$

(D) $3x^2$

Correct Answer: (A) $-3x^2\sin x^3$

Solution:

Concept:

- The expression $\cos x^3$ means cosine of x^3, so it is a function inside another function.
- We use the chain rule.

Step 1: Key Formula or Approach 1. Chain rule: $\frac{d}{dx}f(g(x)) = f'(g(x))\cdot g'(x)$.
2. $\frac{d}{dx}\cos t = -\sin t$ and $\frac{d}{dx}x^n = nx^{n-1}$.

Step 2: Differentiate the outer and inner parts The outer function is cosine, and the inner function is

$g(x) = x^3$.
The derivative of the outer function, kept at x^3, is $-\sin x^3$.
The derivative of the inner function is $3x^2$.

$$\frac{d}{dx}(\cos x^3) = -\sin x^3 \cdot 3x^2 = -3x^2 \sin x^3$$

Step 3: Why the other options are wrong Option (B) $\sin x^3$ drops both the minus sign and the factor $3x^2$.
Option (C) $3x^2 \sin x^3$ has the right factors but the wrong sign, because the derivative of cosine is negative.
Option (D) $3x^2$ is only the derivative of the inner function, so the outer part is missing.

Step 4: Conclusion The derivative is $-3x^2 \sin x^3$, which is option (A).

Quick Tip to Solve: Use the chain rule: differentiate $\cos$ to get $-\sin x^3$, then multiply by the derivative of x^3, which is $3x^2$.

68. If $f : R \to R$, where $f(x) = 3x - 5$, then $f^{-1}(x) =$ [1 mark]

(A) $\frac{1}{3}(x + 5)$

(B) $\frac{1}{3}(x - 5)$

(C) $3x - 5$

(D) none of these

Correct Answer: (A) $\frac{1}{3}(x + 5)$

Solution:

Concept:

- The inverse of a function undoes what the function does.
- If $y = f(x)$, then the inverse gives back x from y.
- To find it we solve the equation $y = f(x)$ for x.

Step 1: Key Formula or Approach 1. Write $y = 3x - 5$.
2. Solve for x in terms of y; that expression is $f^{-1}(y)$.
3. Replace y by x to write the answer in the usual form.

Step 2: Solve for x

$$y = 3x - 5$$

Add 5 to both sides:

$$y + 5 = 3x$$

Divide by 3:

$$x = \frac{y + 5}{3}$$

So $f^{-1}(y) = \frac{1}{3}(y + 5)$, and writing the variable as x gives $f^{-1}(x) = \frac{1}{3}(x + 5)$.

Step 3: Check the answer If f^{-1} is right, then $f\left(f^{-1}(x)\right) = x$.

$$f\left(\frac{x+5}{3}\right) = 3 \cdot \frac{x+5}{3} - 5 = x + 5 - 5 = x$$

Step 4: Why the other options are wrong Option (B) $\frac{1}{3}(x-5)$ gives $f\left(f^{-1}(x)\right) = x - 10$, so it does not undo f.
Option (C) $3x - 5$ is the function f itself, not its inverse.
Option (D) is wrong because option (A) is correct. Also f is strictly increasing and takes every real value, so it is one-one and onto and an inverse does exist.

Step 5: Conclusion The inverse function is $f^{-1}(x) = \frac{1}{3}(x+5)$, which is option (A).

Quick Tip to Solve: Put $y = 3x - 5$ and solve for x. Then swap the letters x and y to read off $f^{-1}(x)$.

69. $\int (x+2)\,dx =$ [1 mark]

(A) $(x+2)^3 + k$

(B) $\frac{x^2}{2} + k$

(C) $\frac{x^2}{2} + 2x + k$

(D) $\log(x+2) + k$

Correct Answer: (C) $\frac{x^2}{2} + 2x + k$

Solution:

Concept:

- We have to find the indefinite integral of a simple polynomial.
- Integration is the reverse of differentiation, so the answer is a function whose derivative is $x + 2$, plus an arbitrary constant k.

Step 1: Key Formula or Approach 1. Sum rule: $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$.

2. Power rule: $\int x^n\,dx = \frac{x^{n+1}}{n+1} + k$ for $n \neq -1$.

3. Constant rule: $\int c\,dx = cx + k$.

Step 2: Detailed Explanation Split the integral into two parts.

$$\int (x+2)\,dx = \int x\,dx + \int 2\,dx$$

Apply the power rule to the first part with $n = 1$.

$$\int x\,dx = \frac{x^2}{2}$$

Apply the constant rule to the second part.

$$\int 2\,dx = 2x$$

Add both results and write one constant k at the end.

$$\int (x+2)\,dx = \frac{x^2}{2} + 2x + k$$

Check by differentiating: $\frac{d}{dx}\left(\frac{x^2}{2} + 2x + k\right) = x + 2$, which is the original integrand.

Step 3: Why the other options are wrong. Option (1), $(x+2)^3 + k$: its derivative is $3(x+2)^2$, not $x+2$.

Option (2), $\frac{x^2}{2} + k$: its derivative is only x, so the term 2 is missing.

Option (4), $\log(x+2) + k$: its derivative is $\frac{1}{x+2}$. This is the integral of $\frac{1}{x+2}$, not of $x+2$.

Step 4: Conclusion The integral is $\frac{x^2}{2} + 2x + k$, which is option (3).

Quick Tip to Solve: Integrate term by term: use the power rule on x and the constant rule on 2, then add the constant k.

---

70. $\int \frac{\cos 2x}{(\sin x + \cos x)^2}\,dx =$ [1 mark]

(A) $2\log(\sin x + \cos x) + k$

(B) $\log(\sin x + \cos x) + k$

(C) $\log(\sin x - \cos x) + k$

(D) $-\frac{1}{\sin x + \cos x} + k$

Correct Answer: (B) $\log(\sin x + \cos x) + k$

Solution:

Concept:

- The numerator is $\cos 2x$ and the denominator is the square of $\sin x + \cos x$.
- The numerator can be factorised so that a common factor cancels, which leaves a simple integrand.

Step 1: Key Formula or Approach 1. $\cos 2x = \cos^2 x - \sin^2 x = (\cos x - \sin x)(\cos x + \sin x)$.

2. $\int \frac{dt}{t} = \log|t| + k$.

3. The derivative of $\sin x + \cos x$ is $\cos x - \sin x$, which will appear in the numerator.

Step 2: Simplify the integrand

$$\frac{\cos 2x}{(\sin x + \cos x)^2} = \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\sin x + \cos x)^2} = \frac{\cos x - \sin x}{\sin x + \cos x}$$

Step 3: Substitute Let $t = \sin x + \cos x$. Then $dt = (\cos x - \sin x)\,dx$.

$$\int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx = \int \frac{dt}{t} = \log|t| + k = \log|\sin x + \cos x| + k$$

Step 4: Why the other options are wrong Option (A) has an extra factor 2. Differentiating it gives $2\,\frac{\cos x - \sin x}{\sin x + \cos x}$, double the integrand.

Option (C) $\log|\sin x - \cos x|$ differentiates to $\frac{\cos x + \sin x}{\sin x - \cos x}$, which is not the integrand.

Option (D) $-\dfrac{1}{\sin x+\cos x}$ differentiates to $\dfrac{\cos x-\sin x}{(\sin x+\cos x)^2}$. The numerator is $\cos x-\sin x$, not $\cos 2x$, so it is wrong.

Step 5: Conclusion The integral equals $\log(\sin x+\cos x)+k$ (the printed form of option (B), with the modulus understood).

Quick Tip to Solve: Write $\cos 2x=(\cos x-\sin x)(\cos x+\sin x)$, cancel one factor, then put $t=\sin x+\cos x$.

---

71. $\dfrac{d}{dx}(e^{x-a})=$ [1 mark]

(A) e^{x-a}

(B) $(x-a)e^{x-a}$

(C) e^{x}

(D) $-e^{x-a}$

Correct Answer: (A) e^{x-a}

Solution:

Concept:

- We differentiate e^{x-a} with respect to x.
- Here a is a constant, so $x-a$ is a simple shift of x.

Step 1: Key Formula or Approach Chain rule for the exponential: $\dfrac{d}{dx}(e^{u})=e^{u}\cdot\dfrac{du}{dx}$.

Step 2: Detailed Explanation Put $u=x-a$. Since a is a constant, $\dfrac{du}{dx}=1-0=1$.

$$\frac{d}{dx}\left(e^{x-a}\right)=e^{x-a}\cdot 1=e^{x-a}$$

Step 3: Why the other options are wrong Option (B) $(x-a)e^{x-a}$ treats the exponent like a power rule, but e^{u} does not give an extra factor u.
Option (C) e^{x} loses the constant a, which stays in the exponent.
Option (D) $-e^{x-a}$ adds a minus sign, but the derivative of $x-a$ is $+1$.
Only option (A) is correct.

Step 4: Conclusion The derivative is e^{x-a}, option (A), the first option.

Quick Tip to Solve: The derivative of e^{u} is $e^{u}\,u'$, and here $u=x-a$ has derivative 1.

---

72. $\dfrac{d}{dx}\sqrt{x^2+ax+1}=$ [1 mark]

(A) $\dfrac{x+a}{2\sqrt{x^2+ax+1}}$

(B) $\dfrac{2x+a}{2\sqrt{x^2+ax+1}}$

(C) $\dfrac{2x+a}{\sqrt{x^2+ax+1}}$

(D) $\dfrac{1}{2\sqrt{x^2+ax+1}}$

Correct Answer: (B) $\frac{2x+a}{2\sqrt{x^2+ax+1}}$

Solution:

Concept:

- The function is a square root of a quadratic, so it is a composite function.
- The outer function is the square root and the inner function is x^2+ax+1.

Step 1: Key Formula or Approach 1. $\frac{d}{dx}\sqrt{u} = \frac{1}{2\sqrt{u}} \cdot \frac{du}{dx}$.

2. Put $u = x^2+ax+1$, with a a constant.

Step 2: Find the derivative of the inner part

$$\frac{du}{dx} = 2x+a$$

Step 3: Apply the chain rule

$$\frac{d}{dx}\sqrt{x^2+ax+1} = \frac{1}{2\sqrt{x^2+ax+1}} \cdot (2x+a) = \frac{2x+a}{2\sqrt{x^2+ax+1}}$$

Step 4: Why the other options are wrong Option (A) has $x+a$ on top, which is a wrong derivative of the inner quadratic.

Option (C) has no 2 in the denominator, so the $\frac{1}{2}$ from the square root rule is missing.

Option (D) has only 1 on top, so the factor $\frac{du}{dx}$ is missing.

Only option (B) matches.

Step 5: Conclusion The derivative is $\frac{2x+a}{2\sqrt{x^2+ax+1}}$, option (B), the second option.

Quick Tip to Solve: Use the chain rule: the derivative of $\sqrt{u}$ is $\frac{u'}{2\sqrt{u}}$, with $u = x^2+ax+1$.

---

73. $\frac{d}{dx}(\sin x^2) =$ [1 mark]

(A) $2x\cos x^2$

(B) $\cos x^2$

(C) $x^2\cos x^2$

(D) $x\cos x^2$

Correct Answer: (A) $2x\cos x^2$

Solution:

Concept:

- The function $\sin x^2$ means $\sin(x^2)$, the sine of the square of x.
- The outer function is sine and the inner function is x^2.

Step 1: Key Formula or Approach Chain rule: $\frac{d}{dx}\sin u = \cos u \cdot \frac{du}{dx}$, with $u = x^2$.

Step 2: Detailed Explanation The inner derivative is $\frac{du}{dx} = 2x$.

$$\frac{d}{dx}(\sin x^2) = \cos x^2 \cdot 2x = 2x \cos x^2$$

Step 3: Why the other options are wrong Option (B) $\cos x^2$ forgets the inner derivative $2x$.
Option (C) $x^2 \cos x^2$ multiplies by x^2, which is the inner function, not its derivative.
Option (D) $x \cos x^2$ loses the 2 that comes from differentiating x^2.
Only option (A) is correct.

Step 4: Conclusion The derivative is $2x \cos x^2$, option (A), the first option.

Quick Tip to Solve: Use the chain rule with inner function x^2: the derivative is $\cos x^2$ times $2x$.

---

74. $\frac{d}{dx}\sqrt{\cot x} =$ [1 mark]

(A) $\frac{1}{2\sqrt{\cot x}}$

(B) $\sqrt{\text{cosec}^2 x}$

(C) $\frac{-\text{cosec}^2 x}{2\sqrt{\cot x}}$

(D) $\frac{\text{cosec}^2 x}{2\sqrt{\cot x}}$

Correct Answer: (C) $\frac{-\text{cosec}^2 x}{2\sqrt{\cot x}}$

Solution:

Concept:

- We need the derivative of a square root of a function.
- The function inside the root is $\cot x$, so this is a composite function: an outer square root and an inner cotangent.

Step 1: Key Formula or Approach 1. Chain rule: $\frac{d}{dx}\sqrt{u} = \frac{1}{2\sqrt{u}} \cdot \frac{du}{dx}$.

2. $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$.

Step 2: Differentiate the outer function Take $u = \cot x$. The square root of u has derivative $\frac{1}{2\sqrt{u}}$ with respect to u. So the outer part gives $\frac{1}{2\sqrt{\cot x}}$.

Step 3: Multiply by the derivative of the inner function The chain rule says we must also multiply by $\frac{du}{dx} = -\text{cosec}^2 x$.

$$\frac{d}{dx}\sqrt{\cot x} = \frac{1}{2\sqrt{\cot x}} \times (-\text{cosec}^2 x) = \frac{-\text{cosec}^2 x}{2\sqrt{\cot x}}$$

This matches option (C).

Step 4: Why the other options are wrong Option (A) $\frac{1}{2\sqrt{\cot x}}$ stops after the outer derivative. It forgets to multiply by the derivative of $\cot x$.
Option (B) $\sqrt{\text{cosec}^2 x}$ is just $|\text{cosec}\, x|$. It does not come from any correct differentiation of the root.

Option (D) has the right size but the wrong sign. The derivative of $\cot x$ is negative, so the answer must carry a minus sign.

Step 5: Conclusion The derivative is $\frac{-\text{cosec}^2 x}{2\sqrt{\cot x}}$, which is option (C).

Quick Tip to Solve: Use the chain rule: the derivative of $\sqrt{u}$ is $\frac{u'}{2\sqrt{u}}$, and $\frac{d}{dx}\cot x = -\text{cosec}^2 x$.

75. $\frac{d}{dx}(\tan^{-1}\sqrt{x} + \cot^{-1}\sqrt{x}) =$ [1 mark]

(A) $\frac{\pi}{2}$

(B) 0

(C) 1

(D) π

Correct Answer: (B) 0

Solution:

Concept:

- We must differentiate a sum of two inverse trigonometric functions, both with the same argument $\sqrt{x}$.
- Before differentiating, it is worth checking whether the sum simplifies.

Step 1: Key Formula or Approach For every real number t in the domain, the two inverse functions add up to a constant:

$$\tan^{-1} t + \cot^{-1} t = \frac{\pi}{2}$$

Step 2: Apply the identity Here $t = \sqrt{x}$, which is a valid real number for $x \geq 0$. So

$$\tan^{-1}\sqrt{x} + \cot^{-1}\sqrt{x} = \frac{\pi}{2}$$

Step 3: Differentiate the constant The expression is just the number $\frac{\pi}{2}$, which does not change with x. The derivative of a constant is zero.

$$\frac{d}{dx}\left(\frac{\pi}{2}\right) = 0$$

This is option (B).

Step 4: Why the other options are wrong Option (A) $\frac{\pi}{2}$ is the value of the sum, not its derivative. Option (C) 1 and option (D) π are not the derivative of any constant, so they cannot be right.

Step 5: Conclusion The sum is constant, so its derivative is zero, which is option (B).

Quick Tip to Solve: Recall that $\tan^{-1} t + \cot^{-1} t = \frac{\pi}{2}$, a constant, so its derivative is zero.

76. $\frac{d}{dx}\left(2\tan^{-1} x\right) =$ [1 mark]

(A) $\frac{1}{1+x^2}$

(B) $\frac{2}{1+x^2}$

(C) $\frac{1}{2(1+x^2)}$

(D) $\frac{1}{1-x^2}$

Correct Answer: (B) $\frac{2}{1+x^2}$

Solution:

Concept:

- We have to differentiate $2\tan^{-1}x$ with respect to x.
- The constant 2 multiplies an inverse trigonometric function, so this is a standard derivative with a constant multiple.

Step 1: Key Formula or Approach 1. Constant multiple rule: $\frac{d}{dx}\left(kf(x)\right) = k\frac{d}{dx}f(x)$.

2. Standard result: $\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$.

Step 2: Detailed Explanation Take the constant 2 outside the derivative.

$$\frac{d}{dx}\left(2\tan^{-1}x\right) = 2\frac{d}{dx}\left(\tan^{-1}x\right)$$

Now put in the standard derivative.

$$= 2\cdot\frac{1}{1+x^2} = \frac{2}{1+x^2}$$

Step 3: Why the other options are wrong Option (1), $\frac{1}{1+x^2}$, is the derivative of $\tan^{-1}x$ alone. It forgets the factor 2.

Option (3), $\frac{1}{2(1+x^2)}$, divides by 2 instead of multiplying. A constant factor stays as a factor, it does not move to the denominator.

Option (4), $\frac{1}{1-x^2}$, has the wrong sign in the denominator. It looks like the derivative of an inverse hyperbolic function, not of $\tan^{-1}x$.

Step 4: Conclusion The derivative is $\frac{2}{1+x^2}$, which is option (2).

Quick Tip to Solve: Take the constant 2 out and use $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$.

77. $\frac{d}{dx}(\cos\sqrt{x}) =$ [1 mark]

(A) $\sin\sqrt{x}$

(B) $\frac{-\sin\sqrt{x}}{2\sqrt{x}}$

(C) $\frac{\sin\sqrt{x}}{2\sqrt{x}}$

(D) $\frac{1}{2\sqrt{x}}$

Correct Answer: (B) $\frac{-\sin\sqrt{x}}{2\sqrt{x}}$

Solution:

Concept:

- The function $\cos\sqrt{x}$ is a cosine whose angle is itself a function of x.
- So we need the chain rule with the cosine as the outer function and $\sqrt{x}$ as the inner function.

Step 1: Key Formula or Approach 1. $\frac{d}{dx}\cos u = -\sin u \cdot \frac{du}{dx}$.
2. $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$.

Step 2: Apply the chain rule Let $u = \sqrt{x}$. The outer derivative is $-\sin\sqrt{x}$. We multiply it by the inner derivative $\frac{1}{2\sqrt{x}}$.

$$\frac{d}{dx}\cos\sqrt{x} = -\sin\sqrt{x} \times \frac{1}{2\sqrt{x}} = \frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

This is option (B).

Step 3: Why the other options are wrong Option (A) $\sin\sqrt{x}$ has the wrong sign and ignores the inner derivative.
Option (C) $\frac{\sin\sqrt{x}}{2\sqrt{x}}$ includes the inner derivative but misses the minus sign that comes from differentiating cosine.
Option (D) $\frac{1}{2\sqrt{x}}$ is only the derivative of $\sqrt{x}$. It drops the trigonometric part completely.

Step 4: Conclusion The derivative is $\frac{-\sin\sqrt{x}}{2\sqrt{x}}$, which is option (B).

Quick Tip to Solve: Apply the chain rule: $\frac{d}{dx}\cos u = -\sin u \cdot u'$ with $u = \sqrt{x}$ and $u' = \frac{1}{2\sqrt{x}}$.

---

78. $\frac{d}{dx}\{\lim_{x\to 0}\frac{x^5 - a^5}{x - a}\} =$ [1 mark]

(A) a

(B) 0

(C) $5a^4$

(D) 5

Correct Answer: (B) 0

Solution:

Concept:

- The curly bracket holds a limit, and we differentiate that whole bracket with respect to x.
- So the first job is to find out what the limit equals.
- Here a is a constant (assume $a \neq 0$).

Step 1: Key Formula or Approach 1. A limit as x tends to a fixed number is a fixed number. It no longer depends on x.
2. The derivative of a constant is zero.

Step 2: Evaluate the limit as printed As printed, $x \to 0$. The expression is continuous at $x = 0$ because

the denominator $x - a$ is not zero there (since $a \neq 0$). So we can substitute directly.

$$\lim_{x \to 0} \frac{x^5 - a^5}{x - a} = \frac{0 - a^5}{0 - a} = a^4$$

Step 3: Differentiate the result The bracket equals a^4, which is a constant in terms of x.

$$\frac{d}{dx}\left(a^4\right) = 0$$

This is option (B).

Step 4: Check the other options If the limit had been taken as $x \to a$, the value would be $5a^4$, the derivative of x^5 at a. Even then it is a constant, so its derivative is still 0.
Option (C) $5a^4$ is only that limit value, not the derivative of the bracket.
Option (A) a and option (D) 5 do not come from any correct step.

Step 5: Conclusion The bracket is a constant, so its derivative is zero, which is option (B).

Quick Tip to Solve: First find the limit. It is a constant that does not depend on x, and the derivative of a constant is zero.

79. $\frac{d}{dx}(\cot^{-1} x) =$ [1 mark]

(A) $\frac{1}{1+x^2}$

(B) $\frac{-1}{1+x^2}$

(C) $\frac{1}{x}$

(D) $\frac{-1}{x}$

Correct Answer: (B) $\frac{-1}{1+x^2}$

Solution:

Concept:

- We need the standard derivative of the inverse cotangent function.

Step 1: Key Formula or Approach The standard result is $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$. It has the same size as the derivative of $\tan^{-1} x$, but with a minus sign. This is because $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$, a constant, so the two derivatives must cancel.

Step 2: Derive it from the identity Write $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$ and differentiate.

$$\frac{d}{dx}(\cot^{-1} x) = 0 - \frac{1}{1+x^2} = \frac{-1}{1+x^2}$$

This is option (B).

Step 3: Why the other options are wrong Option (A) $\frac{1}{1+x^2}$ is the derivative of $\tan^{-1} x$. It has the wrong sign for $\cot^{-1} x$.
Option (C) $\frac{1}{x}$ is the derivative of $\log x$.

Option (D) $\frac{-1}{x}$ is the derivative of $-\log x$. Neither of them involves an inverse trigonometric function.

Step 4: Conclusion The derivative of $\cot^{-1} x$ is $\frac{-1}{1+x^2}$, which is option (B).

Quick Tip to Solve: Recall the standard result for the derivative of $\cot^{-1} x$: it is the negative of the derivative of $\tan^{-1} x$.

80. The equation of the tangent to the curve $y = x^2 + 4x + 1$ at the point $x = 3$ is [1 mark]

(A) $x + 10y = 8$

(B) $10x + y = 8$

(C) $10x - y = 8$

(D) $x - 10y = 8$

Correct Answer: (C) $10x - y = 8$

Solution:

Concept:

- The slope of the tangent to a curve $y = f(x)$ at a point is the value of the derivative $\frac{dy}{dx}$ at that point.
- The tangent also passes through the point of contact on the curve.
- So we need the point and the slope.

Step 1: Key Formula or Approach 1. Point of contact: put $x = 3$ in the curve to get y.
2. Slope: $m = \left.\frac{dy}{dx}\right|_{x=3}$.
3. Tangent: $y - y_1 = m(x - x_1)$.

Step 2: Find the point of contact At $x = 3$: $y = 3^2 + 4(3) + 1 = 9 + 12 + 1 = 22$.
So the point of contact is $(3, 22)$.

Step 3: Find the slope Differentiate the curve with respect to x:

$$\frac{dy}{dx} = 2x + 4$$

At $x = 3$:

$$m = 2(3) + 4 = 10$$

Step 4: Write the tangent Use $y - 22 = 10(x - 3)$.

$$y - 22 = 10x - 30$$

$$10x - y = 8$$

Step 5: Why the other options are wrong Option (A) $x + 10y = 8$ has slope $-\frac{1}{10}$, which is not 10.
Option (B) $10x + y = 8$ has slope -10, the wrong sign. At $x = 3$ it gives $y = -22$, so it misses the point $(3, 22)$.
Option (D) $x - 10y = 8$ has slope $\frac{1}{10}$, the reciprocal of the needed slope. It does not pass through $(3, 22)$ either.

Only option (C) has slope 10 and satisfies $10(3) - 22 = 8$.

Step 6: Conclusion The tangent at $x = 3$ is $10x - y = 8$, which is option (C).

Quick Tip to Solve: Find the point on the curve at x = 3, take the slope as dy/dx there, then use y - y1 = m(x - x1).

81. $\int_0^1 \frac{x^3\,dx}{1+x^8} =$ [1 mark]

(A) $\frac{\pi}{2}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{8}$

(D) $\frac{\pi}{16}$

Correct Answer: (D) $\frac{\pi}{16}$

Solution:

Concept:

- Here $x^8 = (x^4)^2$ and the numerator contains x^3, which is the derivative of x^4 up to a constant.
- So the substitution $t = x^4$ reduces the integral to the standard form $\frac{1}{1+t^2}$.

Step 1: Key Formula or Approach 1. $t = x^4$, so $dt = 4x^3\,dx$ and $x^3\,dx = \frac{dt}{4}$.
2. $x^8 = t^2$.
3. Limits: $x = 0 \Rightarrow t = 0$ and $x = 1 \Rightarrow t = 1$.

Step 2: Detailed Explanation Substitute everything:

$$\int_0^1 \frac{x^3\,dx}{1+x^8} = \frac{1}{4}\int_0^1 \frac{dt}{1+t^2}$$

$$= \frac{1}{4}\left[\tan^{-1} t\right]_0^1 = \frac{1}{4}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{16}$$

This is option (D).

Step 3: Why the other options are wrong Option (B), $\frac{\pi}{4}$, comes from forgetting the factor $\frac{1}{4}$ that appears when x^3dx is replaced by $\frac{dt}{4}$. Option (C), $\frac{\pi}{8}$, comes from using $\frac{1}{2}$ instead of $\frac{1}{4}$. Option (A), $\frac{\pi}{2}$, is too big: the integrand is at most $\frac{1}{2}$ on $[0, 1]$, so the integral is below 0.5, while $\frac{\pi}{2} \approx 1.57$.

Step 4: Conclusion The value is $\frac{\pi}{16}$, option (D).

Quick Tip to Solve: Put $t = x^4$. Then $x^3dx = \frac{dt}{4}$ and $x^8 = t^2$, giving $\frac{1}{4}\int_0^1 \frac{dt}{1+t^2}$.

82. $\int_1^e \frac{(\log x)^2}{x}\,dx =$ [1 mark]

(A) $\frac{1}{3}$

(B) $\frac{1}{3}e^3$

(C) $\frac{1}{3}(e^3 - 1)$

(D) e^3

Correct Answer: (A) $\frac{1}{3}$

Solution:

Concept:

- The integrand is a power of $\log x$ multiplied by $\frac{1}{x}$, and $\frac{1}{x}$ is the derivative of $\log x$.
- So we substitute $t = \log x$.

Step 1: Key Formula or Approach 1. $t = \log x$, so $dt = \frac{dx}{x}$.
2. Limits: $x = 1 \Rightarrow t = \log 1 = 0$ and $x = e \Rightarrow t = \log e = 1$.
3. $\int t^2\, dt = \frac{t^3}{3}$.

Step 2: Detailed Explanation The integral becomes

$$\int_1^e \frac{(\log x)^2}{x}\, dx = \int_0^1 t^2\, dt = \left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

This is option (A).

Step 3: Why the other options are wrong Options (B) and (D) contain e^3, which would come from wrongly putting $x = e$ into x^3. The limits must be changed to the t-values 0 and 1, so no e remains. Option (C), $\frac{e^3 - 1}{3}$, is the value of $\int_1^e x^2\, dx$, which is a different integral.

Step 4: Conclusion The value is $\frac{1}{3}$, option (A).

Quick Tip to Solve: Put $t = \log x$. Then $dt = \frac{dx}{x}$ and the limits become 0 and 1.

---

83. $A = \begin{pmatrix} 4 & 2 & 3 \end{pmatrix} \Rightarrow A' =$ [1 mark]

(A) $\begin{pmatrix} 4 & 2 & 3 \end{pmatrix}$

(B) $\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$

(C) $\begin{pmatrix} 3 & 2 & 4 \end{pmatrix}$

(D) $\begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$

Correct Answer: (D) $\begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$

collegedunia

Solution:

Concept:

- The transpose of a matrix A, written A', is the matrix we get when rows become columns and columns become rows.
- If A has order $m \times n$, then A' has order $n \times m$. The entry in row i, column j of A moves to row j, column i of A'.

Step 1: Key Formula or Approach $(A')_{ji} = A_{ij}$. So the first row of A becomes the first column of A', keeping the entries in the same order from left to right and top to bottom.

Step 2: Detailed Explanation Here $A = \begin{pmatrix} 4 & 2 & 3 \end{pmatrix}$ has just one row and three columns, so its order is 1×3.
Its transpose must therefore have order 3×1, which is a column matrix.
Write the single row as a column, keeping the order $4, 2, 3$:

$$A' = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$$

Step 3: Why the other options are wrong Option (1) is A itself, a row matrix of order 1×3. Transposing must change the order, so it cannot be A'.
Option (2) is a column matrix but the entries run in the order $3, 2, 4$. That is the reverse of the row, so it is not the transpose.
Option (3) is a row matrix with entries reversed. It has the wrong order and the wrong order of entries.
Only option (4) has order 3×1 with the entries $4, 2, 3$ in the original sequence.

Step 4: Conclusion Writing the row $4, 2, 3$ as a column gives A'. This is option (4).

Quick Tip to Solve: The transpose turns rows into columns. A 1 x 3 row matrix becomes a 3 x 1 column with the same entries in the same order.

84. $\int_0^{\pi/4} \frac{e^{\tan x}}{\cos^2 x}\, dx =$ [1 mark]

(A) $e - 1$

(B) $e + 1$

(C) $\frac{1}{e} + 1$

(D) $\frac{1}{e} - 1$

Correct Answer: (A) $e - 1$

Solution:

Concept:

- The integrand has the form $e^{f(x)} \cdot f'(x)$ once we notice that the derivative of $\tan x$ is $\sec^2 x = \frac{1}{\cos^2 x}$.
- This is a standard case for the substitution method.

Step 1: Key Formula or Approach Put $t = \tan x$. Then $dt = \sec^2 x\, dx = \frac{dx}{\cos^2 x}$.
Change the limits too: when $x = 0$, $t = \tan 0 = 0$. When $x = \frac{\pi}{4}$, $t = \tan \frac{\pi}{4} = 1$.

Step 2: Detailed Explanation The integral becomes much simpler in terms of t:

$$\int_0^{\pi/4} \frac{e^{\tan x}}{\cos^2 x}\, dx = \int_0^1 e^t\, dt$$

The antiderivative of e^t is e^t, so

$$\int_0^1 e^t\, dt = \left[e^t\right]_0^1 = e^1 - e^0 = e - 1$$

Step 3: Why the other options are wrong Option (2), $e + 1$, comes from adding $e^0 = 1$ instead of subtracting it at the lower limit.
Options (3) and (4) contain $\frac{1}{e}$. That would come from evaluating e^{-1}, but t goes from 0 to 1, never to -1.
Also $e - 1 \approx 1.718 > 0$ as it must be, because the integrand is positive. Option (4) is negative, so it is impossible.

Step 4: Conclusion The value of the integral is $e - 1$, which is option (1).

Quick Tip to Solve: Put $t = \tan x$ so that $dt = \frac{dx}{\cos^2 x}$, then integrate e^t from 0 to 1.

85. $\int_0^{\pi/2} \cos^2 x\, dx =$ [1 mark]

(A) $\frac{\pi}{2}$

(B) π

(C) $\frac{\pi}{4}$

(D) 1

Correct Answer: (C) $\frac{\pi}{4}$

Solution:

Concept:

- We cannot integrate $\cos^2 x$ directly.
- The usual trick is to lower the power with the double angle identity.

Step 1: Key Formula or Approach $\cos 2x = 2\cos^2 x - 1$, so $\cos^2 x = \frac{1 + \cos 2x}{2}$.

Step 2: Detailed Explanation Replace $\cos^2 x$ in the integral:

$$\int_0^{\pi/2} \cos^2 x\, dx = \frac{1}{2}\int_0^{\pi/2} (1 + \cos 2x)\, dx$$

Integrate term by term. The antiderivative of 1 is x, and that of $\cos 2x$ is $\frac{\sin 2x}{2}$:

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2}$$

At $x = \frac{\pi}{2}$: $\frac{\pi}{2} + \frac{\sin \pi}{2} = \frac{\pi}{2} + 0$. At $x = 0$: $0 + 0 = 0$.

$$= \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$$

Step 3: Why the other options are wrong Option (1), $\frac{\pi}{2}$, forgets the factor $\frac{1}{2}$ from the identity.
Option (2), π, is twice that again, so it is not possible.
Option (4), 1, is the value of $\int_0^{\pi/2} \cos x \, dx$, the integral with power 1, not power 2.
Since $\cos^2 x \leq 1$ on an interval of length $\frac{\pi}{2}$, the answer must be below $\frac{\pi}{2}$, which also removes options (1) and (2).

Step 4: Conclusion The integral equals $\frac{\pi}{4}$, which is option (3).

Quick Tip to Solve: Use $\cos^2 x = \frac{1 + \cos 2x}{2}$ and integrate term by term from 0 to $\frac{\pi}{2}$.

86. $\int_0^{\pi/3} \cos^3 x \, dx =$ [1 mark]

(A) $\frac{3\sqrt{3}}{8}$

(B) $\frac{\sqrt{3}}{8}$

(C) $\frac{3}{8}$

(D) $\frac{1}{8}$

Correct Answer: (A) $\frac{3\sqrt{3}}{8}$

Solution:

Concept:

- An odd power of cosine can be integrated by saving one factor of $\cos x$ and writing the rest in terms of $\sin x$.

Step 1: Key Formula or Approach Use $\cos^2 x = 1 - \sin^2 x$, so $\cos^3 x = (1 - \sin^2 x) \cos x$.

Then put $t = \sin x$, so $dt = \cos x \, dx$. The limits change: $x = 0 \Rightarrow t = 0$ and $x = \frac{\pi}{3} \Rightarrow t = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Step 2: Detailed Explanation

$$\int_0^{\pi/3} \cos^3 x \, dx = \int_0^{\sqrt{3}/2} (1 - t^2) \, dt = \left[t - \frac{t^3}{3}\right]_0^{\sqrt{3}/2}$$

Now $t^3 = \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8}$, so $\frac{t^3}{3} = \frac{\sqrt{3}}{8}$.

$$= \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8} = \frac{4\sqrt{3} - \sqrt{3}}{8} = \frac{3\sqrt{3}}{8}$$

Step 3: Why the other options are wrong Option (2), $\frac{\sqrt{3}}{8}$, is only the subtracted term $\frac{t^3}{3}$, so the first term was lost.

Option (3), $\frac{3}{8}$, and option (4), $\frac{1}{8}$, have no $\sqrt{3}$. The limit $\sin\frac{\pi}{3}$ always brings a $\sqrt{3}$ into the answer, so they cannot be right.
The value $\frac{3\sqrt{3}}{8} \approx 0.65$ is also reasonable, because $\cos^3 x$ falls from 1 to $\frac{1}{8}$ over an interval of length about 1.05.

Step 4: Conclusion The integral equals $\frac{3\sqrt{3}}{8}$, which is option (1).

Quick Tip to Solve: Write $\cos^3 x = (1-\sin^2 x)\cos x$ and substitute $t = \sin x$, with limits 0 to $\frac{\sqrt{3}}{2}$.

87. $\int_0^{2\pi} |\sin x|\, dx =$ [1 mark]

(A) 2

(B) 4

(C) 1

(D) 3

Correct Answer: (B) 4

Solution:

Concept:

- The function $|\sin x|$ is not given by one formula on $[0, 2\pi]$.
- We must remove the modulus by checking where $\sin x$ is positive and where it is negative.

Step 1: Key Formula or Approach On $[0, \pi]$, $\sin x \geq 0$, so $|\sin x| = \sin x$.
On $[\pi, 2\pi]$, $\sin x \leq 0$, so $|\sin x| = -\sin x$.
Split the integral at $x = \pi$.

Step 2: Detailed Explanation

$$\int_0^{2\pi} |\sin x|\, dx = \int_0^{\pi} \sin x\, dx + \int_{\pi}^{2\pi} (-\sin x)\, dx$$

First part: $\left[-\cos x\right]_0^{\pi} = -\cos\pi + \cos 0 = 1 + 1 = 2$.
Second part: $\left[\cos x\right]_{\pi}^{2\pi} = \cos 2\pi - \cos\pi = 1 - (-1) = 2$.

$$\text{Total} = 2 + 2 = 4$$

Step 3: Why the other options are wrong Option (1), 2, is the area of just one hump of the sine curve. There are two humps on $[0, 2\pi]$, one above the axis and one below.
Without the modulus, $\int_0^{2\pi} \sin x\, dx = 0$, because the two humps cancel. The modulus makes both areas count as positive.
Options (3) and (4), with values 1 and 3, do not come from any step of the correct working.

Step 4: Conclusion The integral equals 4, which is option (2).

Quick Tip to Solve: Split the range at π, where $\sin x$ changes sign, and take $|\sin x| = \sin x$ then $-\sin x$.

88. $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx =$ [1 mark]

(A) π
(B) $\frac{\pi}{2}$
(C) 0
(D) $\frac{\pi}{4}$

Correct Answer: (D) $\frac{\pi}{4}$

Solution:

Concept:

- The integrand $\frac{\sin x}{\sin x + \cos x}$ has no simple antiderivative by direct methods, so we use a property of definite integrals.
- The property is $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. It lets us replace x by $\frac{\pi}{2} - x$ without changing the value.

Step 1: Apply the property Let $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx$.
Replace x by $\frac{\pi}{2} - x$. Since $\sin\left(\frac{\pi}{2} - x\right) = \cos x$ and $\cos\left(\frac{\pi}{2} - x\right) = \sin x$, we get

$$I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx$$

Step 2: Add the two forms of I Adding the two expressions for I gives

$$2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2}$$

So $I = \frac{\pi}{4}$.

Step 3: Why the other options are wrong On $[0, \frac{\pi}{2}]$ the integrand lies between 0 and 1, and it is positive. So the integral is positive and smaller than the interval length $\frac{\pi}{2}$.
Option (C) 0 is impossible because the integrand is positive. Option (A) π and option (B) $\frac{\pi}{2}$ are too large; $\frac{\pi}{2}$ would be the value only if the integrand were always 1.
So only option (D) is possible.

Step 4: Conclusion The integral equals $\frac{\pi}{4}$, which is option (D).

Quick Tip to Solve: Replace x by $\frac{\pi}{2} - x$, add the two integrals and the integrand becomes 1.

89. $\int_{-\pi}^{\pi} \sin^5 x\,dx =$ [1 mark]

(A) $\frac{3\pi}{4}$
(B) 2π
(C) $\frac{5\pi}{6}$
(D) 0

Correct Answer: (D) 0

Solution:

Concept:

- A function f is odd if $f(-x) = -f(x)$. For an odd function, the area to the left of the y-axis cancels the area to the right.
- So $\int_{-a}^{a} f(x)\,dx = 0$ for every odd function f.

Step 1: Check that the integrand is odd Let $f(x) = \sin^5 x$. Then

$$f(-x) = \sin^5(-x) = (-\sin x)^5 = -\sin^5 x = -f(x)$$

An odd power of an odd function is odd, so f is odd. The limits $-\pi$ and π are symmetric about 0.

Step 2: Evaluate

$$\int_{-\pi}^{\pi} \sin^5 x\,dx = 0$$

Step 3: Why the other options are wrong Options (A) $\frac{3\pi}{4}$, (B) 2π and (C) $\frac{5\pi}{6}$ are all non-zero. They would need the positive and negative parts of $\sin^5 x$ to leave a net area, but the two halves are exact mirror images with opposite signs.
Even over $[0, \pi]$ alone the integral is $\frac{16}{15}$, which is not any of these values, so none of them can come from the full interval either.

Step 4: Conclusion The integrand is odd and the interval is symmetric, so the integral is 0. This is option (D).

Quick Tip to Solve: Check whether $\sin^5 x$ is odd or even, then use the symmetric limits $-\pi$ to π.

90. $\int_0^a \frac{\sqrt{x}}{\sqrt{a-x}+\sqrt{x}}\,dx =$ [1 mark]

(A) a

(B) $\frac{a}{2}$

(C) $2a$

(D) $3a$

Correct Answer: (B) $\frac{a}{2}$

Solution:

Concept:

- The limits are 0 and a, and the integrand has the pair $\sqrt{x}$ and $\sqrt{a-x}$.
- This is the usual sign that the king property of definite integrals will help.

Step 1: Key Formula or Approach For a definite integral over $[0, a]$, replacing x by $a - x$ does not change its value:

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

Adding the original integral and the changed one often makes the integrand collapse to 1.

Step 2: Apply the property. Let $I = \int_0^a \frac{\sqrt{x}}{\sqrt{a-x}+\sqrt{x}}\,dx$. Put $x \to a - x$. Then $\sqrt{x}$ becomes $\sqrt{a-x}$,

and $\sqrt{a-x}$ becomes $\sqrt{x}$. So

$$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}\,dx$$

Step 3: Add the two forms of I. The two denominators are the same, so the numerators add directly:

$$2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\,dx = \int_0^a 1\,dx = a$$

So $I = \frac{a}{2}$.

Step 4: Why the other options are wrong. Option (A), a, is the value of $2I$, which is what we get before dividing by 2.
Option (C), $2a$, and option (D), $3a$, are bigger than a. That is not possible, because the integrand is always between 0 and 1 on $[0, a]$, so the integral cannot exceed the interval length a.

Step 5: Conclusion The value of the integral is half of a, which is option (B).

Quick Tip to Solve: Use $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, then add the two forms so the integrand becomes 1.

91. If $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$**, then** $A^3 =$ [1 mark]

(A) $3A$

(B) $4A$

(C) $2A$

(D) none of these

Correct Answer: (B) $4A$

Solution:

Concept:

- We are asked for the cube of a 2 by 2 matrix.
- The cube means $A^3 = A^2 \cdot A$, so we first find A^2 by ordinary matrix multiplication (row times column).

Step 1: Key Formula or Approach For matrices, $(AB)_{ij} = \sum_k a_{ik}b_{kj}$. We find A^2, then multiply that result by A once more, and compare the answer with the given options.

Step 2: Find A squared Take $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

$$A^2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1\cdot 1+(-1)(-1) & 1\cdot(-1)+(-1)\cdot 1 \\ (-1)\cdot 1+1\cdot(-1) & (-1)(-1)+1\cdot 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = 2A$$

So A^2 is just twice A. This shortcut makes the next step easy.

Step 3: Find A cubed Use $A^3 = A^2 \cdot A = (2A)\cdot A = 2A^2$.

Put $A^2 = 2A$ again: $A^3 = 2(2A) = 4A$.

$$A^3 = 4A = \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}$$

Step 4: Check the options Option (A) $3A$ is wrong because the multiplier we found is 4, not 3.
Option (B) $4A$ matches our result, so it is correct.
Option (C) $2A$ is the value of A^2, not of A^3, so it is wrong.
Option (D) "none of these" is wrong because option (B) is right.

Step 5: Conclusion The cube of the matrix equals four times the matrix, which is option (B).

Quick Tip to Solve: Hint:
Multiply the matrix by itself first. You will see that A^2 is a constant times A, then use it again for A^3.

92. **If** $A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$ **and** $A + A' = I$, **then** $\alpha =$ [1 mark]

(A) π
(B) $\frac{\pi}{3}$
(C) $\frac{3\pi}{2}$
(D) $\frac{\pi}{6}$

Correct Answer: (B) $\frac{\pi}{3}$

Solution:

Concept:

- The transpose A' is formed by swapping the rows and columns of A.
- We add A and A' entry by entry and compare the sum with the identity matrix I.

Step 1: Key Formula or Approach Two matrices are equal only when every matching entry is equal.
So we make the sum equal to $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and read off the condition on α.

Step 2: Write the transpose and add

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \quad A' = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$A + A' = \begin{pmatrix} 2\cos\alpha & \sin\alpha - \sin\alpha \\ -\sin\alpha + \sin\alpha & 2\cos\alpha \end{pmatrix} = \begin{pmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{pmatrix}$$

The off-diagonal entries are always 0, so only the diagonal gives a condition.

Step 3: Solve for alpha Setting $A + A' = I$ gives $2\cos\alpha = 1$, so $\cos\alpha = \frac{1}{2}$.
The standard angle with cosine $\frac{1}{2}$ is $\alpha = \frac{\pi}{3}$.

Step 4: Check the options Option (A) π: $\cos\pi = -1$, so the diagonal would be -2, not 1. Wrong.
Option (B) $\frac{\pi}{3}$: $\cos\frac{\pi}{3} = \frac{1}{2}$, so the diagonal is 1. Correct.

Option (C) $\frac{3\pi}{2}$: $\cos\frac{3\pi}{2} = 0$, so the diagonal would be 0. Wrong.

Option (D) $\frac{\pi}{6}$: $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, so the diagonal would be $\sqrt{3}$. Wrong.

Step 5: Conclusion The condition reduces to $\cos\alpha = \frac{1}{2}$, which holds for option (B).

Quick Tip to Solve: Hint:
Add A and its transpose. The sine terms cancel and you are left with $2\cos\alpha$ on the diagonal.

93. If $A = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$ **then** $\text{adj}\, A =$ [1 mark]

(A) $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$

(B) $\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}$

(C) $\begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$

(D) none of these

Correct Answer: (A) $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$

Solution:

Concept:

- The adjoint of a square matrix is the transpose of its cofactor matrix.
- For a 2×2 matrix we find the four cofactors, arrange them in a matrix, and then transpose.

Step 1: Key Formula or Approach 1. Cofactor $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the minor (the entry left after deleting row i and column j).
2. $\text{adj}\, A = (\text{cofactor matrix})^T$.

Step 2: Find the cofactors of A Here $a_{11} = 3,\ a_{12} = -5,\ a_{21} = -1,\ a_{22} = 2$.

$$C_{11} = (+1)(2) = 2, \qquad C_{12} = (-1)(-1) = 1$$

$$C_{21} = (-1)(-5) = 5, \qquad C_{22} = (+1)(3) = 3$$

So the cofactor matrix is $\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$.

Step 3: Transpose the cofactor matrix

$$\text{adj}\, A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}^T = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$$

Step 4: Why the other options are wrong Option (2), $\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}$, has the right first diagonal entry 2 but

the wrong second one, 5. The diagonal of the adjoint must be 2 and 3, and the off-diagonal entries must be 5 and 1.
Option (3), $\begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$, has the wrong diagonal entries (1 and 5). The first diagonal entry must be 2, the old a_{22}, and the second must be 3, the old a_{11}.
Option (4), none of these, is wrong because option (1) is correct.

Step 5: Conclusion $\text{adj}\, A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$, which is option (1).

Quick Tip to Solve: For a 2×2 matrix, swap the diagonal entries and change the sign of the off-diagonal entries.

---

94. **If** $A = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$ **and** $B = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ **then** $AB =$ [1 mark]

(A) $\begin{pmatrix} 30 \end{pmatrix}$

(B) $\begin{pmatrix} 10 \end{pmatrix}$

(C) $\begin{pmatrix} 20 \end{pmatrix}$

(D) $\begin{pmatrix} 40 \end{pmatrix}$

Correct Answer: (A) $\begin{pmatrix} 30 \end{pmatrix}$

Solution:

Concept:

- We multiply a row matrix of order 1×4 by a column matrix of order 4×1.
- The number of columns of the first (4) equals the number of rows of the second (4), so the product is defined.

Step 1: Key Formula or Approach The order of the product is 1×1, a single entry. That entry is the sum of the products of matching elements:

$$AB = [\, a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 \,]$$

Step 2: Multiply

$$AB = [\, 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 + 4 \cdot 4 \,] = [\, 1 + 4 + 9 + 16 \,] = [\, 30 \,]$$

So $AB = \begin{pmatrix} 30 \end{pmatrix}$.

Step 3: Check the options Option (A) [30] matches the result, so it is correct.
Option (B) [10] is the plain sum $1+2+3+4$, which comes from adding the elements and not multiplying matching ones. Wrong.
Option (C) [20] and option (D) [40] do not come from any correct step. Wrong.

Step 4: Conclusion The product is the 1×1 matrix with entry 30, which is option (A).

Quick Tip to Solve: **Hint:**
A row matrix times a column matrix gives one number: multiply matching entries and add them.

95. $A = [a_{ij}]_{m\times n}$ **is a square matrix if** [1 mark]

(A) $m = n$

(B) $m < n$

(C) $m > n$

(D) none of these

Correct Answer: (A) $m = n$

Solution:

Concept:

- The matrix $A = [a_{ij}]_{m\times n}$ has m rows and n columns.
- The subscript $m \times n$ is called the order of the matrix.

Step 1: Key Formula or Approach A square matrix is a matrix whose number of rows is equal to its number of columns.

Step 2: Detailed Explanation For A to be square we need the number of rows to be the same as the number of columns.

$$m = n$$

For example, a matrix of order 3×3 is square, while one of order 2×3 is a rectangular matrix.

Step 3: Why the other options are wrong. If $m < n$, the matrix has fewer rows than columns, so it is a horizontal rectangular matrix.
If $m > n$, the matrix has more rows than columns, so it is a vertical rectangular matrix.
Neither of these is square, and "none of these" is wrong because option (A) is correct.

Step 4: Conclusion The matrix is square when $m = n$, which is option (A).

Quick Tip to Solve: A square matrix has the same number of rows and columns, so $m = n$.

96. $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \Rightarrow A + A' =$ [1 mark]

(A) $\begin{pmatrix} 6 & -3 \\ -3 & -2 \end{pmatrix}$

(B) $\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}$

(C) $\begin{pmatrix} 6 & 3 \\ -3 & -2 \end{pmatrix}$

(D) $\begin{pmatrix} 6 & 3 \\ -3 & 2 \end{pmatrix}$

Correct Answer: (A) $\begin{pmatrix} 6 & -3 \\ -3 & -2 \end{pmatrix}$

Solution:

Concept:

- The transpose A' of a matrix is obtained by writing its rows as columns.
- We then add A and A' by adding the entries in matching positions.

Step 1: Key Formula or Approach If $A = [a_{ij}]$, then $A' = [a_{ji}]$, and $(A + A')_{ij} = a_{ij} + a_{ji}$.

Step 2: Find the transpose Row 1 of A is $(3, -4)$, so it becomes column 1 of A'. Row 2 $(1, -1)$ becomes column 2.

$$A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \quad \Rightarrow \quad A' = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$$

Step 3: Add the two matrices

$$A + A' = \begin{pmatrix} 3+3 & -4+1 \\ 1+(-4) & -1+(-1) \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ -3 & -2 \end{pmatrix}$$

Step 4: Check the options Option (A) is $\begin{pmatrix} 6 & -3 \\ -3 & -2 \end{pmatrix}$, which matches. Correct.

Option (B) has $+3$ off the diagonal and $+2$ in the corner, so the signs are wrong.
Option (C) has 3 and -3 off the diagonal. The sum $A + A'$ is always symmetric, so those two must be equal. Wrong.
Option (D) has the same symmetry problem and also a wrong corner entry $+2$. Wrong.

Step 5: Conclusion The sum is the symmetric matrix with rows $(6, -3)$ and $(-3, -2)$, which is option (A).

Quick Tip to Solve: Hint:
Swap rows and columns to get A', then add entry by entry. The sum is always a symmetric matrix.

97. $f : A \to B$ **will be an into function if** [1 mark]

(A) $f(A) \subset B$

(B) $f(A) = B$

(C) $f(A) \supset B$

(D) none of these

Correct Answer: (A) $f(A) \subset B$

Solution:

Concept:

- For a function $f : A \to B$, the set A is the domain and B is the codomain.
- The range (or image) is $f(A)$, the set of values f actually takes.
- A function is "onto" when the range fills the whole codomain, and "into" when it does not.

Step 1: Key Formula or Approach An into function is one where $f(A)$ is a proper subset of B. This means at least one element of B is not the image of any element of A.

Step 2: Apply the definition Every value $f(x)$ lies in B, so $f(A) \subseteq B$ always holds. The function is into exactly when $f(A)$ is a smaller part of B, written $f(A) \subset B$.
For example, $f : \{1, 2\} \to \{1, 2, 3\}$ with $f(1) = 1,\ f(2) = 2$ has $f(A) = \{1, 2\} \subset B$, and the element 3

has no pre-image.

Step 3: Check the options Option (A) $f(A) \subset B$ is the definition of an into function. Correct.
Option (B) $f(A) = B$ is the definition of an onto function, the opposite case. Wrong.
Option (C) $f(A) \supset B$ would mean the range is larger than the codomain, which is impossible because $f(A)$ is always inside B. Wrong.
Option (D) "none of these" is wrong because option (A) is right.

Step 4: Conclusion The function is into when its range is a proper subset of the codomain, option (A).

Quick Tip to Solve: **Hint:**
An into function misses at least one element of the codomain, so its range is a proper subset of B.

98. **What type of a relation is "less than" in the set of real numbers?** [1 mark]

(A) Only symmetric

(B) Only transitive

(C) Only reflexive

(D) Equivalence

Correct Answer: (B) Only transitive

Solution:

Concept:

- A relation R on a set is called reflexive if every element is related to itself. It is symmetric if aRb always gives bRa.
- It is transitive if aRb and bRc always give aRc. An equivalence relation has all three properties.

Step 1: Key Formula or Approach Here aRb means $a < b$, where a, b are real numbers. We test the three properties one by one with real numbers.

Step 2: Test reflexivity. For any real number a, the statement $a < a$ is false. For example, $3 < 3$ is not true. So the relation is NOT reflexive.

Step 3: Test symmetry. Take $a = 2$ and $b = 5$. Then $2 < 5$ is true, but $5 < 2$ is false. So aRb does not give bRa, and the relation is NOT symmetric.

Step 4: Test transitivity. Suppose $a < b$ and $b < c$. Then $a < b < c$, so $a < c$ must hold. This is true for every choice of real numbers, so the relation IS transitive.

Step 5: Check the options. (A) Only symmetric: wrong, the relation is not symmetric at all.
(B) Only transitive: right, transitivity is the single property that holds.
(C) Only reflexive: wrong, $a < a$ fails.
(D) Equivalence: wrong, an equivalence relation needs reflexive, symmetric and transitive together, and two of these fail.

Step 6: Conclusion The relation "less than" is only transitive. This is option (B).

Quick Tip to Solve: Test the three properties: $a < a$ is false, and $a < b$ does not give $b < a$, but $a < b < c$ gives $a < c$.

99. $|3\vec{i} + 4\vec{j} + 7\vec{k}| =$ [1 mark]

(A) $\sqrt{14}$

collegedunia

(B) $\sqrt{74}$

(C) $\sqrt{61}$

(D) $\sqrt{94}$

Correct Answer: (B) $\sqrt{74}$

Solution:

Concept:

- The magnitude (length) of a vector $x\vec{i}+y\vec{j}+z\vec{k}$ is the distance of the point (x, y, z) from the origin.

Step 1: Key Formula or Approach

$$|x\vec{i}+y\vec{j}+z\vec{k}| = \sqrt{x^2+y^2+z^2}$$

Step 2: Detailed Explanation Here $x = 3$, $y = 4$ and $z = 7$. Put these values in the formula.

$$|3\vec{i}+4\vec{j}+7\vec{k}| = \sqrt{3^2+4^2+7^2}$$

Now square each number.

$$= \sqrt{9+16+49} = \sqrt{74}$$

Step 3: Check the options. (A) $\sqrt{14}$ would come from $1+4+9$, which is the wrong set of squares.
(C) $\sqrt{61}$ would need $9+16+36$, so it uses 6 instead of 7.
(D) $\sqrt{94}$ would need the squares to add to 94, which they do not.
Only $\sqrt{74}$ matches the sum $9+16+49 = 74$.

Step 4: Conclusion The magnitude is $\sqrt{74}$, which is option (B).

Quick Tip to Solve: Use $|x\vec{i}+y\vec{j}+z\vec{k}| = \sqrt{x^2+y^2+z^2}$ and add $9+16+49$.

100. $\vec{i}\cdot(\vec{j}\times\vec{k}) =$ [1 mark]

(A) 1

(B) 0

(C) -1

(D) $\vec{i}$

Correct Answer: (A) 1

Solution:

Concept:

- The vectors $\vec{i}, \vec{j}, \vec{k}$ are the unit vectors along the x, y and z axes.
- They are at right angles to each other and form a right-handed system.
- The question asks for a dot product of $\vec{i}$ with a cross product.

Step 1: Key Formula or Approach 1. For the right-handed unit vectors, $\vec{i}\times\vec{j} = \vec{k},\ \vec{j}\times\vec{k} = \vec{i},\ \vec{k}\times\vec{i} = \vec{j}$.
2. The dot product of a unit vector with itself is 1, that is $\vec{i}\cdot\vec{i} = |\vec{i}|^2 = 1$.

Step 2: Work out the cross product first. The bracket is solved before the dot product, so we start with $\vec{j}\times\vec{k}$.

$$\vec{j}\times\vec{k} = \vec{i}$$

Step 3: Take the dot product. Put this value back in the expression.

$$\vec{i} \cdot (\vec{j} \times \vec{k}) = \vec{i} \cdot \vec{i} = 1$$

Step 4: Why the other options are wrong. Option 2 (0) would be the answer for $\vec{i} \cdot \vec{j}$ or $\vec{i} \cdot \vec{k}$, where the two unit vectors are perpendicular. Here the second vector is $\vec{i}$ itself, so the dot product is not zero.
Option 3 (-1) would come from the wrong order $\vec{i} \cdot (\vec{k} \times \vec{j})$, because $\vec{k} \times \vec{j} = -\vec{i}$. The order given is $\vec{j} \times \vec{k}$, so the sign is positive.
Option 4 ($\vec{i}$) is a vector. A dot product of two vectors is always a scalar, so a vector cannot be the answer.

Step 5: Conclusion The value of $\vec{i} \cdot (\vec{j} \times \vec{k})$ is 1, which is option 1.

Quick Tip to Solve: Solve the bracket first using $\vec{j} \times \vec{k} = \vec{i}$, then use $\vec{i} \cdot \vec{i} = 1$.

Section B: Short Answer Type Questions (any 15 to be answered, 2 marks each)

SA-1. Solve : $\int_1^2 \frac{\log x}{x}\, dx$. [2 marks]

Solution:

Concept:

- We have to evaluate a definite integral whose integrand is a logarithm divided by x. Here $\log x$ means the natural logarithm $\log_e x$.
- The derivative of $\log x$ is $\frac{1}{x}$, and that factor already sits in the integrand. So a substitution will work.

Step 1: Substitute Let $t = \log x$. Then $dt = \frac{1}{x}\, dx$.
Change the limits too, so we do not have to go back to x:
when $x = 1$, $t = \log 1 = 0$, and when $x = 2$, $t = \log 2$.

Step 2: Integrate The integral becomes

$$\int_0^{\log 2} t\, dt = \left[\frac{t^2}{2}\right]_0^{\log 2}$$

Put in the limits:

$$\frac{(\log 2)^2}{2} - \frac{0^2}{2} = \frac{(\log 2)^2}{2}$$

Step 3: Conclusion The value of the integral is half the square of $\log 2$. Numerically it is about 0.24.

Quick Tip to Solve: Put $t = \log x$ so that $dt = dx/x$, and change the limits to 0 and $\log 2$.

SA-2. Find the value of $\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx$. [2 marks]

Solution:

Concept:

- The integrand cannot be integrated directly.

- But the limits are 0 and $\frac{\pi}{2}$, and $\cos$ and $\sin$ swap into each other when x is replaced by $\frac{\pi}{2} - x$.
- So we use the property of definite integrals.

Step 1: Key Formula

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

Here $a = \frac{\pi}{2}$. Also $\cos\left(\frac{\pi}{2} - x\right) = \sin x$ and $\sin\left(\frac{\pi}{2} - x\right) = \cos x$.

Step 2: Apply the property Let

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx \quad \cdots(1)$$

Replace x by $\frac{\pi}{2} - x$:

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx \quad \cdots(2)$$

Step 3: Add (1) and (2)

$$2I = \int_0^{\pi/2} \frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2}$$

So $I = \frac{\pi}{4}$.

Step 4: Conclusion The integral equals one quarter of π.

Quick Tip to Solve: Use $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ with $a = \pi/2$, then add the two integrals.

SA-3. Solve : $\frac{dy}{dx} + 2y\tan x = \sin x$. [2 marks]

Solution:

Step 1: Identify the type The equation has the form $\frac{dy}{dx} + Py = Q$, a linear differential equation of first order.

Here $P = 2\tan x$ and $Q = \sin x$.

Step 2: Find the integrating factor The integrating factor is I.F. $= e^{\int P\,dx}$.

$$\int 2\tan x\,dx = 2\log|\sec x| = \log\sec^2 x$$

So I.F. $= e^{\log\sec^2 x} = \sec^2 x$.

Step 3: Write the solution form The general solution is $y \cdot \text{I.F.} = \int Q \cdot \text{I.F.}\,dx + C$. So

$$y\sec^2 x = \int \sin x\sec^2 x\,dx + C$$

Step 4: Integrate the right side Write $\sin x\sec^2 x = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \sec x\tan x$.

Since $\int \sec x\tan x\,dx = \sec x$, we get

$$y\sec^2 x = \sec x + C$$

collegedunia

Step 5: Solve for y Multiply both sides by $\cos^2 x$:

$$y = \cos x + C\cos^2 x$$

A quick check: differentiating gives $y' = -\sin x - 2C\sin x\cos x$, and $2y\tan x = 2\sin x + 2C\sin x\cos x$. Their sum is $\sin x$, as required.

Step 6: Conclusion The general solution of the equation is given below, where C is an arbitrary constant.

Quick Tip to Solve: It is linear: find the integrating factor $e^{\int 2\tan x\,dx} = \sec^2 x$, then integrate $\sin x\sec^2 x$.

SA-4. **Prove that** $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$. [2 marks]

Solution:

Step 1: Plan We know $\tan^{-1}1 = \frac{\pi}{4}$. So we only need to show that $\tan^{-1}2 + \tan^{-1}3 = \frac{3\pi}{4}$.

Step 2: Use the addition formula carefully The formula is $\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\frac{x+y}{1-xy}$ when $x, y > 0$ and $xy > 1$. Here $x = 2,\ y = 3$, so $xy = 6 > 1$ and this form applies.

$$\tan^{-1}2 + \tan^{-1}3 = \pi + \tan^{-1}\frac{2+3}{1-6} = \pi + \tan^{-1}(-1)$$

Since $\tan^{-1}(-1) = -\frac{\pi}{4}$,

$$\tan^{-1}2 + \tan^{-1}3 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Step 3: Add the first term

$$\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \frac{\pi}{4} + \frac{3\pi}{4} = \pi$$

Step 4: Conclusion The sum equals π, as required.

Quick Tip to Solve: Use $\tan^{-1}1 = \pi/4$. For $\tan^{-1}2 + \tan^{-1}3$ the product is more than 1, so the formula carries an extra π.

SA-5. **Find the value of** $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$. [2 marks]

Solution:

Step 1: Combine the doubled term Use $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$, valid for $|x| < 1$. With $x = \frac{1}{3}$, which is less than 1,

$$2\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{2/3}{1-1/9} = \tan^{-1}\frac{2/3}{8/9} = \tan^{-1}\frac{3}{4}$$

Step 2: Add the remaining term Now $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7}$. The product $\frac{3}{4}\cdot\frac{1}{7} = \frac{3}{28} < 1$, so we can use

$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \dfrac{x+y}{1-xy}$.

$$\tan^{-1} \frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{28}} = \tan^{-1} \frac{25/28}{25/28} = \tan^{-1} 1$$

Step 3: Read off the value $\tan^{-1} 1 = \dfrac{\pi}{4}$.

Step 4: Conclusion The value is $\dfrac{\pi}{4}$.

Quick Tip to Solve: First turn $2\tan^{-1}\dfrac{1}{3}$ into $\tan^{-1}\dfrac{3}{4}$, then add $\tan^{-1}\dfrac{1}{7}$ with the sum formula.

SA-6. Prove that $\sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3) = 15$. [2 marks]

Solution:

Step 1: Name the angles Let $\theta = \tan^{-1} 2$, so $\tan\theta = 2$. Let $\phi = \cot^{-1} 3$, so $\cot\phi = 3$. We must find $\sec^2\theta + \operatorname{cosec}^2\phi$.

Step 2: Use the Pythagorean identities The identities are $\sec^2\theta = 1 + \tan^2\theta$ and $\operatorname{cosec}^2\phi = 1 + \cot^2\phi$.

Step 3: Substitute

$$\sec^2\theta = 1 + 2^2 = 5$$

$$\operatorname{cosec}^2\phi = 1 + 3^2 = 10$$

Step 4: Add

$$\sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3) = 5 + 10 = 15$$

Step 5: Conclusion The left side equals 15, so the statement is proved.

Quick Tip to Solve: Put $\tan\theta = 2$ and $\cot\phi = 3$, then use $\sec^2 = 1 + \tan^2$ and $\operatorname{cosec}^2 = 1 + \cot^2$.

SA-7. Find the distance between the planes $2x - y + 3z + 4 = 0$ and $6x - 3y + 9z - 3 = 0$. [2 marks]

Solution:

Concept:

- The distance between two parallel planes is the gap between them measured along their common normal.
- The formula needs both planes to have the same coefficients of x, y, z.

Step 1: Key Formula or Approach For parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$,

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

Step 2: Make the coefficients equal. The first plane is $2x - y + 3z + 4 = 0$.

In the second plane $6x - 3y + 9z - 3 = 0$, every term is a multiple of 3. Divide by 3:

$$2x - y + 3z - 1 = 0$$

Now the normals agree, so the planes are parallel.

Step 3: Put the values in the formula. Here $a = 2,\ b = -1,\ c = 3,\ d_1 = 4,\ d_2 = -1$.

$$D = \frac{|4 - (-1)|}{\sqrt{4 + 1 + 9}} = \frac{5}{\sqrt{14}}$$

Step 4: Conclusion The distance between the planes is $\frac{5}{\sqrt{14}}$ units, about 1.34 units.

Quick Tip to Solve: Divide the second plane by 3 to get $2x - y + 3z - 1 = 0$, then use $D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$.

SA-8. Find the value of p so that the straight lines $\frac{x-1}{-1} = \frac{y+3}{p} = \frac{z-7}{2}$ and $\frac{x+1}{p} = \frac{y}{2} = \frac{z+6}{-3}$ are perpendicular to each other. [2 marks]

Solution:

Concept:

- A line written as $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ has direction ratios a, b, c.
- Two lines are perpendicular when their direction vectors have dot product zero.

Step 1: Key Formula or Approach

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

Step 2: Read the direction ratios. From the first line $\frac{x-1}{-1} = \frac{y+3}{p} = \frac{z-7}{2}$, the direction ratios are $(-1,\ p,\ 2)$.

From the second line $\frac{x+1}{p} = \frac{y}{2} = \frac{z+6}{-3}$, the direction ratios are $(p,\ 2,\ -3)$.

Step 3: Apply the condition.

$$(-1)(p) + (p)(2) + (2)(-3) = 0$$

$$-p + 2p - 6 = 0$$

$$p - 6 = 0$$

So $p = 6$.

Step 4: Conclusion The lines are perpendicular when $p = 6$.

Quick Tip to Solve: Take the denominators as direction ratios and set $a_1a_2 + b_1b_2 + c_1c_2 = 0$. This gives $p - 6 = 0$.

SA-9. If $f : R \to R;\ f(x) = (3 - x^3)^{\frac{1}{3}}$, then prove that $(f \circ f)(x) = x$. [2 marks]

Solution:

Concept:

- The notation $(f \circ f)(x)$ means $f(f(x))$. We put $f(x)$ into f again and simplify.
- Here $f(x) = (3 - x^3)^{\frac{1}{3}}$ is the real cube root of $3 - x^3$. A real number always has exactly one real cube root, so f is defined for every real x.

Step 1: Write the composition Replace the input of f by $f(x)$:

$$(f \circ f)(x) = f(f(x)) = \left(3 - \{f(x)\}^3\right)^{\frac{1}{3}}$$

Step 2: Find the cube of f(x) Cubing a cube root gives back the number under it:

$$\{f(x)\}^3 = \left[(3 - x^3)^{\frac{1}{3}}\right]^3 = 3 - x^3$$

Step 3: Substitute and simplify Put this value into the expression from Step 2:

$$f(f(x)) = \left(3 - (3 - x^3)\right)^{\frac{1}{3}} = (x^3)^{\frac{1}{3}}$$

The real cube root of x^3 is x for every real x, positive or negative, because $t \mapsto t^3$ is one-one on R. So $f(f(x)) = x$.

Step 4: Conclusion We have shown that $(f \circ f)(x) = x$ for all real x. So f is its own inverse.

Quick Tip to Solve: Write $(f \circ f)(x) = f(f(x))$ and cube the inner $f(x)$.
The cube and the cube root cancel, and $3 - (3 - x^3) = x^3$.

SA-10. Integrate : $\int \frac{dx}{x^2 + 6x + 13}$. [2 marks]

Solution:

Concept:

- The denominator is a quadratic that does not factor over the real numbers.
- The standard way is to complete the square and use the arctangent integral.

Step 1: Key Formula or Approach

$$\int \frac{dx}{u^2 + k^2} = \frac{1}{k} \tan^{-1} \frac{u}{k} + C$$

Step 2: Complete the square

$$x^2 + 6x + 13 = (x^2 + 6x + 9) + 4 = (x + 3)^2 + 2^2$$

So the integral is $\int \frac{dx}{(x + 3)^2 + 2^2}$.

Step 3: Integrate Put $u = x + 3$, so $du = dx$ and $k = 2$.

$$\int \frac{du}{u^2 + 2^2} = \frac{1}{2}\tan^{-1}\frac{u}{2} + C$$

Step 4: Go back to x Replace u by $x + 3$.

$$\int \frac{dx}{x^2 + 6x + 13} = \frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + C$$

Step 5: Conclusion The integral is an arctangent with the shifted variable.

Quick Tip to Solve: Complete the square: $x^2+6x+13 = (x+3)^2+4$, then use the arctangent integral.

SA-11. Find the value of $\int_0^a \frac{x\,dx}{\sqrt{a^2+x^2}}$. [2 marks]

Solution:

Concept:

- The integrand has x on top and $\sqrt{a^2+x^2}$ below.
- The derivative of $a^2 + x^2$ is $2x$, so a substitution removes the root.
- We take $a > 0$.

Step 1: Key Formula or Approach Put $t = a^2 + x^2$. Then $dt = 2x\,dx$, so $x\,dx = \frac{dt}{2}$.

Step 2: Change the limits When $x = 0$, $t = a^2$.
When $x = a$, $t = 2a^2$.

Step 3: Integrate

$$\int_0^a \frac{x\,dx}{\sqrt{a^2+x^2}} = \frac{1}{2}\int_{a^2}^{2a^2} t^{-1/2}\,dt = \frac{1}{2}\left[2\sqrt{t}\right]_{a^2}^{2a^2} = \left[\sqrt{t}\right]_{a^2}^{2a^2}$$

Step 4: Put in the limits

$$\sqrt{2a^2} - \sqrt{a^2} = a\sqrt{2} - a = a(\sqrt{2} - 1)$$

Step 5: Conclusion The value of the definite integral is $a(\sqrt{2} - 1)$.

Quick Tip to Solve: Put $t = a^2 + x^2$, or notice that $\sqrt{a^2+x^2}$ has derivative $\frac{x}{\sqrt{a^2+x^2}}$.

SA-12. Solve : $\log_e\left(\frac{dy}{dx}\right) = ax + by$. [2 marks]

Solution:

Concept:

- The equation has $\log_e \frac{dy}{dx}$ on the left.
- We first remove the logarithm by taking the exponential, and then check if the variables can be separated.

Step 1: Key Formula or Approach If $\log_e P = Q$, then $P = e^Q$. Also $e^{m+n} = e^m \cdot e^n$.

Step 2: Remove the logarithm

$$\frac{dy}{dx} = e^{ax+by} = e^{ax} \cdot e^{by}$$

Step 3: Separate the variables Keep all y terms with dy and all x terms with dx.

$$e^{-by}\, dy = e^{ax}\, dx$$

Step 4: Integrate both sides

$$\int e^{-by}\, dy = \int e^{ax}\, dx$$

$$-\frac{e^{-by}}{b} = \frac{e^{ax}}{a} + c$$

Here $a \neq 0$ and $b \neq 0$, and c is the constant of integration.

Step 5: Tidy the result Multiply by $-ab$ and write $C = -abc$:

$$a\, e^{-by} = -b\, e^{ax} + C \quad \Rightarrow \quad a\, e^{-by} + b\, e^{ax} = C$$

Step 6: Conclusion The general solution of the equation is

Quick Tip to Solve: Write $\frac{dy}{dx} = e^{ax}e^{by}$ and separate the variables.

SA-13. Prove that the value of the determinant $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 0.$ [2 marks]

Solution:

Concept:

- A determinant is zero when one of its rows is a constant multiple of another row.
- Equivalently, a row operation that makes a whole row of zeros leaves the value unchanged, and a row of zeros gives determinant 0.

Step 1: Compare the first and third rows Row 1 is $(102,\ 18,\ 36)$ and row 3 is $(17,\ 3,\ 6)$.
Divide each entry of row 1 by the matching entry of row 3:

$$\frac{102}{17} = 6, \quad \frac{18}{3} = 6, \quad \frac{36}{6} = 6$$

All three ratios are the same, so $R_1 = 6R_3$.

Step 2: Apply the row operation Replace R_1 by $R_1 - 6R_3$. Adding a multiple of one row to another row does not change the determinant.

$$\Delta = \begin{vmatrix} 102-102 & 18-18 & 36-36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

Now the first row is entirely zero. Expanding along that row, every term has a factor 0, so $\Delta = 0$.

Step 3: Conclusion The first row is 6 times the third row, so the determinant vanishes. Hence the

given determinant is 0, as required.

Quick Tip to Solve: Compare row 1 with row 3. Check whether one is a constant multiple of the other.

SA-14. Prove that $\begin{vmatrix} 0 & a & b \\ c & 0 & b \\ c & a & 0 \end{vmatrix} = 2abc$**.** [2 marks]

Solution:

Concept:

- We expand the 3 by 3 determinant along the first row.
- For a row with entries p, q, r the expansion is $p\,M_{11} - q\,M_{12} + r\,M_{13}$, where M_{1j} is the 2 by 2 minor left after deleting row 1 and column j.

Step 1: Write the expansion The first row is $(0,\ a,\ b)$, so

$$\Delta = 0\begin{vmatrix} 0 & b \\ a & 0 \end{vmatrix} - a\begin{vmatrix} c & b \\ c & 0 \end{vmatrix} + b\begin{vmatrix} c & 0 \\ c & a \end{vmatrix}$$

Step 2: Evaluate the minors The first term is 0 because it is multiplied by 0.

$$\begin{vmatrix} c & b \\ c & 0 \end{vmatrix} = c \cdot 0 - b \cdot c = -bc$$

$$\begin{vmatrix} c & 0 \\ c & a \end{vmatrix} = c \cdot a - 0 \cdot c = ac$$

Step 3: Combine

$$\Delta = 0 - a(-bc) + b(ac) = abc + abc = 2abc$$

This is the required value, so the result is proved.

Step 4: Conclusion Expanding along the first row gives $abc + abc$, so the determinant equals $2abc$.

Quick Tip to Solve: Expand the determinant along the first row. The first term vanishes because the entry is 0.

SA-15. Find the value of x**, when** $\begin{vmatrix} 2x & 8 \\ 4 & x \end{vmatrix} = 0$**.** [2 marks]

Solution:

Concept:

- The value of a 2 by 2 determinant $\begin{vmatrix} p & q \\ r & s \end{vmatrix}$ is $ps - qr$.
- We expand the given determinant and set it equal to 0.

Step 1: Expand the determinant

$$\begin{vmatrix} 2x & 8 \\ 4 & x \end{vmatrix} = (2x)(x) - (8)(4) = 2x^2 - 32$$

Step 2: Form and solve the equation Set the value equal to 0:

$$2x^2 - 32 = 0$$

Divide both sides by 2:

$$x^2 = 16$$

Take the square root of both sides. Both the positive and the negative root work:

$$x = 4 \quad \text{or} \quad x = -4$$

Step 3: Check For $x = 4$: $2(4)(4) - 8 \cdot 4 = 32 - 32 = 0$.
For $x = -4$: $2(-4)(-4) - 8 \cdot 4 = 32 - 32 = 0$.
Both values satisfy the equation.

Step 4: Conclusion The determinant is zero for two values of x.

Quick Tip to Solve: Expand the 2 by 2 determinant as $ps - qr$, set it to zero and solve the quadratic in x.

SA-16. Find the slope at the point $(1, \sqrt{2})$ of the curve $x^2 + y^2 = 3$. [2 marks]

Solution:

Concept:

- The slope of a curve at a point is the value of $\frac{dy}{dx}$ at that point.
- The curve is given implicitly, so we differentiate both sides with respect to x and treat y as a function of x.

Step 1: Check that the point lies on the curve Put $x = 1,\ y = \sqrt{2}$: $1^2 + (\sqrt{2})^2 = 1 + 2 = 3$. The point is on the curve.

Step 2: Differentiate implicitly

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(3)$$

$$2x + 2y\frac{dy}{dx} = 0$$

The chain rule gives the factor $\frac{dy}{dx}$ for the y^2 term. Solving for the derivative:

$$\frac{dy}{dx} = -\frac{x}{y}$$

Step 3: Substitute the point At $(1, \sqrt{2})$:

$$\frac{dy}{dx}\Big|_{(1,\sqrt{2})} = -\frac{1}{\sqrt{2}}$$

Step 4: Conclusion The slope of the curve at the given point is $-\frac{1}{\sqrt{2}}$, which equals $-\frac{\sqrt{2}}{2}$.

Quick Tip to Solve: Differentiate $x^2 + y^2 = 3$ implicitly to get $dy/dx = -x/y$, then substitute the point.

SA-17. Find $\frac{dy}{dx}$ when $x = at^2,\ y = 2at$. [2 marks]

Solution:

Concept:

- For a curve given in parametric form $x = f(t),\ y = g(t)$, the chain rule gives
- So we differentiate x and y separately with respect to the parameter t, then divide.

Step 1: Differentiate x and y with respect to t

$$\frac{dx}{dt} = \frac{d}{dt}(at^2) = 2at$$

$$\frac{dy}{dt} = \frac{d}{dt}(2at) = 2a$$

Step 2: Divide

$$\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$$

The common factor 2a cancels, valid for $a \neq 0,\ t \neq 0$.

Step 3: Conclusion The derivative of y with respect to x is 1/t.

Quick Tip to Solve: Use $dy/dx = (dy/dt)/(dx/dt)$. Differentiate x and y with respect to t first.

SA-18. If $y = \sec\{\tan(\sqrt{x})\}$, then find $\frac{dy}{dx}$. [2 marks]

Solution:

Concept:

- We have a function made of several layers: $\sqrt{x}$ sits inside $\tan$, and $\tan$ sits inside $\sec$.
- To differentiate such a function we use the chain rule, working from the outside layer to the inside layer.

Step 1: Key Formula or Approach 1. $\frac{d}{dx}\sec u = \sec u \tan u \cdot \frac{du}{dx}$.

2. $\frac{d}{dx}\tan v = \sec^2 v \cdot \frac{dv}{dx}$.

3. $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$.

Step 2: Differentiate the outer layer Put $u = \tan(\sqrt{x})$, so $y = \sec u$. Then

$$\frac{dy}{dx} = \sec u \tan u \cdot \frac{du}{dx} = \sec\{\tan(\sqrt{x})\}\ \tan\{\tan(\sqrt{x})\} \cdot \frac{d}{dx}\tan(\sqrt{x}).$$

Step 3: Differentiate the middle and inner layers For the middle layer, $\frac{d}{dx}\tan(\sqrt{x}) = \sec^2(\sqrt{x}) \cdot \frac{d}{dx}\sqrt{x}$.
The inner layer gives $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$.
So $\frac{d}{dx}\tan(\sqrt{x}) = \frac{\sec^2(\sqrt{x})}{2\sqrt{x}}$.

Step 4: Combine the results Multiply the three factors together:

$$\frac{dy}{dx} = \sec\{\tan(\sqrt{x})\}\ \tan\{\tan(\sqrt{x})\} \cdot \frac{\sec^2(\sqrt{x})}{2\sqrt{x}}$$

Step 5: Conclusion Using the chain rule layer by layer, we get

Quick Tip to Solve: Use the chain rule three times: differentiate $\sec$, then $\tan$, then $\sqrt{x}$, and multiply the results.

SA-19. If A and B are two events such that $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{3}$ and $P(A \cup B) = \frac{1}{2}$ then prove that A and B are independent events. [2 marks]

Solution:

Concept:

- Two events A and B are independent when the chance of both happening together equals the product of their separate chances.
- So we must show $P(A \cap B) = P(A)\,P(B)$.

Step 1: Key Formula or Approach The addition rule for any two events is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

We know $P(A \cup B)$, $P(A)$ and $P(B)$, so this rule gives $P(A \cap B)$.

Step 2: Find $P(A \cap B)$. Rearrange the addition rule:

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$P(A \cap B) = \frac{1}{4} + \frac{1}{3} - \frac{1}{2} = \frac{3 + 4 - 6}{12} = \frac{1}{12}$$

Step 3: Find the product $P(A)\,P(B)$.

$$P(A)\,P(B) = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$$

Step 4: Compare. We got $P(A \cap B) = \frac{1}{12}$ and $P(A)\,P(B) = \frac{1}{12}$. These are equal, so the condition for independence holds.

Step 5: Conclusion Since $P(A \cap B) = P(A)\,P(B)$, the events A and B are independent.

Quick Tip to Solve: Find $P(A \cap B)$ from $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, then compare it with $P(A)P(B)$.

SA-20. Two dice are thrown. Find the probability that the numbers appeared have the sum 8 if the second die always exhibits 4. [2 marks]

Solution:

Concept:

- This is a conditional probability question. We want the chance that the sum is 8, given that the second die shows 4.
- Let E be the event "sum is 8" and F the event "second die shows 4". We need $P(E \mid F)$.

Step 1: Key Formula or Approach

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

Two dice give $6 \times 6 = 36$ equally likely outcomes.

Step 2: Count the outcomes of F. The second die is 4, and the first die can be anything from 1 to 6.

$$F = \{(1,4), (2,4), (3,4), (4,4), (5,4), (6,4)\}$$

So $P(F) = \frac{6}{36} = \frac{1}{6}$.

Step 3: Count the outcomes of $E \cap F$. We need a sum of 8 with the second die equal to 4, so the first die must be 4. Only $(4,4)$ works.

So $P(E \cap F) = \frac{1}{36}$.

Step 4: Divide.

$$P(E \mid F) = \frac{1/36}{6/36} = \frac{1}{6}$$

Step 5: Conclusion The required probability is $\frac{1}{6}$.

Quick Tip to Solve: Use $P(E \mid F) = \frac{P(E \cap F)}{P(F)}$. Only $(4,4)$ gives sum 8 among the six outcomes with second die 4.

SA-21. Maximize $Z = 3x + 2y$
subject to $3x + y \leq 15$
$x \geq 0,\ y \geq 0$ [2 marks]

Solution:

Concept:

- In the graphical method of linear programming we draw the feasible region, find its corner points, and evaluate the objective function Z at each corner.
- The maximum or minimum value of Z on a bounded feasible region always occurs at a corner point.

Step 1: Draw the feasible region. The constraint $3x + y \leq 15$ lies on the origin side of the line $3x + y = 15$. This line meets the x-axis at $(5,0)$ and the y-axis at $(0,15)$.
The conditions $x \geq 0$ and $y \geq 0$ keep us in the first quadrant. So the feasible region is the triangle with vertices $O(0,0)$, $A(5,0)$ and $B(0,15)$. This triangle is bounded, so the maximum exists.

Step 2: Evaluate Z at each corner point. At $O(0,0)$: $Z = 3(0) + 2(0) = 0$.
At $A(5,0)$: $Z = 3(5) + 2(0) = 15$.

At $B(0,15)$: $Z = 3(0) + 2(15) = 30$.

Step 3: Pick the largest value. The largest of $0, 15, 30$ is 30, and it occurs at $B(0,15)$.

Step 4: Conclusion The maximum value of Z is 30, reached at $x = 0,\ y = 15$.

Quick Tip to Solve: Draw the triangle cut off by $3x + y \leq 15$ in the first quadrant and compare Z at its three corners.

SA-22. Integrate : $\int \frac{dx}{1+\cos x}$. [2 marks]

Solution:

Concept:

- The integrand $\frac{1}{1+\cos x}$ has no direct standard integral.
- We first change the denominator using a half-angle identity so that it becomes a perfect square.

Step 1: Key Formula or Approach 1. $1 + \cos x = 2\cos^2\frac{x}{2}$.

2. $\int \sec^2(ax)\,dx = \frac{\tan(ax)}{a} + C$.

Step 2: Apply the identity

$$\int \frac{dx}{1+\cos x} = \int \frac{dx}{2\cos^2\frac{x}{2}} = \frac{1}{2}\int \sec^2\frac{x}{2}\,dx.$$

Step 3: Integrate Since $\int \sec^2\frac{x}{2}\,dx = \frac{\tan(x/2)}{1/2} = 2\tan\frac{x}{2}$, we get

$$\frac{1}{2}\cdot 2\tan\frac{x}{2} + C = \tan\frac{x}{2} + C.$$

Step 4: Conclusion

Quick Tip to Solve: Write $1 + \cos x = 2\cos^2\frac{x}{2}$, so the integral becomes $\frac{1}{2}\int \sec^2\frac{x}{2}\,dx$.

SA-23. Integrate : $\int \sec^4 x\,dx$. [2 marks]

Solution:

Concept:

- We have to integrate an even power of secant, $\sec^4 x$.
- It is not a standard integral, so we split the power and use an identity that turns the integral into a power of $\tan x$.

Step 1: Key Formula or Approach 1. $\sec^4 x = \sec^2 x \cdot \sec^2 x$.

2. $\sec^2 x = 1 + \tan^2 x$.

3. $\frac{d}{dx}(\tan x) = \sec^2 x$, so the substitution $t = \tan x$ works.

Step 2: Rewrite the integrand Keep one $\sec^2 x$ as it is and replace the other by $1 + \tan^2 x$.

$$\int \sec^4 x \, dx = \int (1 + \tan^2 x) \sec^2 x \, dx$$

Step 3: Substitute Put $t = \tan x$. Then $dt = \sec^2 x \, dx$, and the integral becomes a simple polynomial in t.

$$\int (1 + t^2) \, dt = t + \frac{t^3}{3} + C$$

Step 4: Go back to x Replace t by $\tan x$.

$$\int \sec^4 x \, dx = \tan x + \frac{\tan^3 x}{3} + C$$

Step 5: Conclusion The integral of $\sec^4 x$ is $\tan x + \frac{1}{3} \tan^3 x$ plus a constant.

Quick Tip to Solve: Write $\sec^4 x = (1 + \tan^2 x) \sec^2 x$ and put $t = \tan x$.

SA-24. Integrate : $\int \frac{\sin^{-1} x}{\sqrt{1 - x^2}} \, dx$. [2 marks]

Solution:

Concept:

- The integrand has $\sin^{-1} x$ on top and $\sqrt{1 - x^2}$ at the bottom.
- The derivative of $\sin^{-1} x$ is exactly $\frac{1}{\sqrt{1 - x^2}}$, so a substitution will work.

Step 1: Key Formula or Approach $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$. So we put $t = \sin^{-1} x$.

Step 2: Substitute Let $t = \sin^{-1} x$. Then $dt = \frac{dx}{\sqrt{1 - x^2}}$. The integral becomes

$$\int t \, dt = \frac{t^2}{2} + C$$

Step 3: Go back to x Replace t by $\sin^{-1} x$.

$$\int \frac{\sin^{-1} x}{\sqrt{1 - x^2}} \, dx = \frac{(\sin^{-1} x)^2}{2} + C$$

Step 4: Conclusion The integral equals half the square of $\sin^{-1} x$, plus a constant.

Quick Tip to Solve: Put $t = \sin^{-1} x$; then $dt = \frac{dx}{\sqrt{1 - x^2}}$.

SA-25. If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$**, then prove that** $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$. [2 marks]

Solution:

Concept:

collegedunia

- We are given that the three vectors add to the zero vector.
- We must show that the three cross products $\vec{a} \times \vec{b}$, $\vec{b} \times \vec{c}$ and $\vec{c} \times \vec{a}$ are all equal.

Step 1: Facts we will use 1. The cross product of a vector with itself is zero: $\vec{u} \times \vec{u} = \vec{0}$.
2. The cross product is anti-commutative: $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.
3. The cross product distributes over addition.

Step 2: Show that a x b = b x c From the given condition, $\vec{a} = -\vec{b} - \vec{c}$.
Take the cross product of this equation with $\vec{b}$ on the right:

$$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{b}) - (\vec{c} \times \vec{b}) = \vec{0} + \vec{b} \times \vec{c} = \vec{b} \times \vec{c}$$

Step 3: Show that b x c = c x a Now write $\vec{b} = -\vec{c} - \vec{a}$ and take the cross product with $\vec{c}$ on the right:

$$\vec{b} \times \vec{c} = -(\vec{c} \times \vec{c}) - (\vec{a} \times \vec{c}) = \vec{0} + \vec{c} \times \vec{a} = \vec{c} \times \vec{a}$$

Step 4: Join the two results From the last two steps, $\vec{a} \times \vec{b} = \vec{b} \times \vec{c}$ and $\vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

Step 5: Conclusion All three cross products are equal, so the statement is proved.

Quick Tip to Solve: Write $\vec{a} = -\vec{b} - \vec{c}$, cross with $\vec{b}$, and use $\vec{b} \times \vec{b} = \vec{0}$.

SA-26. Find the area of the parallelogram whose adjacent sides are vectors $\hat{i}+2\hat{j}+3\hat{k}$ and $-3\hat{i}-2\hat{j}+\hat{k}$. [2 marks]

Solution:

Step 1: Key Formula If $\vec{a}$ and $\vec{b}$ are adjacent sides of a parallelogram, its area is the magnitude of their cross product:

$$\text{Area} = |\vec{a} \times \vec{b}|$$

Step 2: Write the vectors Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = -3\hat{i} - 2\hat{j} + \hat{k}$.

Step 3: Find the cross product Expand the determinant:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -3 & -2 & 1 \end{vmatrix}$$

$$= \hat{i}(2 \cdot 1 - 3 \cdot (-2)) - \hat{j}(1 \cdot 1 - 3 \cdot (-3)) + \hat{k}(1 \cdot (-2) - 2 \cdot (-3))$$

$$= \hat{i}(2 + 6) - \hat{j}(1 + 9) + \hat{k}(-2 + 6) = 8\hat{i} - 10\hat{j} + 4\hat{k}$$

Step 4: Find the magnitude

$$|\vec{a} \times \vec{b}| = \sqrt{8^2 + (-10)^2 + 4^2} = \sqrt{64 + 100 + 16} = \sqrt{180} = 6\sqrt{5}$$

Step 5: Conclusion The area of the parallelogram is $6\sqrt{5}$ square units.

Quick Tip to Solve: Area of the parallelogram is $|\vec{a} \times \vec{b}|$. The cross product is $8\hat{i} - 10\hat{j} + 4\hat{k}$.

SA-27. Find the sine of the angle between the two vectors $3\hat{i} + \hat{j} + 2\hat{k}$ and $2\hat{i} - 2\hat{j} + 4\hat{k}$. [2 marks]

Solution:

Concept:

- The sine of the angle between two vectors comes from their cross product.
- The length of $\vec{a} \times \vec{b}$ equals $|\vec{a}||\vec{b}| \sin\theta$.

Step 1: Key Formula or Approach

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$$

Here $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + 4\hat{k}$.

Step 2: Find the cross product. Write the determinant and expand along the first row.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix}$$

$$= \hat{i}(1 \cdot 4 - 2 \cdot (-2)) - \hat{j}(3 \cdot 4 - 2 \cdot 2) + \hat{k}(3 \cdot (-2) - 1 \cdot 2)$$

$$= 8\hat{i} - 8\hat{j} - 8\hat{k}$$

Step 3: Find the magnitudes.

$$|\vec{a} \times \vec{b}| = \sqrt{64 + 64 + 64} = \sqrt{192} = 8\sqrt{3}$$

$$|\vec{a}| = \sqrt{9 + 1 + 4} = \sqrt{14}, \qquad |\vec{b}| = \sqrt{4 + 4 + 16} = \sqrt{24} = 2\sqrt{6}$$

So $|\vec{a}||\vec{b}| = 2\sqrt{84} = 4\sqrt{21}$.

Step 4: Put the values in the formula.

$$\sin\theta = \frac{8\sqrt{3}}{4\sqrt{21}} = \frac{2\sqrt{3}}{\sqrt{21}} = \frac{2}{\sqrt{7}}$$

Step 5: Conclusion The sine of the angle between the two vectors is $\frac{2}{\sqrt{7}}$.

Quick Tip to Solve: Use $\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$. The cross product is $8\hat{i} - 8\hat{j} - 8\hat{k}$.

SA-28. **Find $\frac{dy}{dx}$ when $x^{\sin y} = y^{\sin x}$.** [2 marks]

Solution:

Concept:

- The variable y appears in the exponent on the left and in the base on the right, so we cannot solve for y directly.
- Both sides have a variable in the exponent, which is the case for logarithmic differentiation.

Step 1: Key Formula or Approach 1. Take $\log$ of both sides so that $\log a^b = b \log a$ brings the exponents down.
2. Differentiate both sides with respect to x using the product rule and the chain rule (treat y as a function of x).

Step 2: Take logarithms Given $x^{\sin y} = y^{\sin x}$. Taking natural logs on both sides,

$$\sin y \cdot \log x = \sin x \cdot \log y.$$

Step 3: Differentiate both sides with respect to x Left side (product rule, with $\frac{d}{dx}\sin y = \cos y \cdot \frac{dy}{dx}$):

$$\frac{d}{dx}(\sin y \log x) = \cos y \cdot \frac{dy}{dx} \cdot \log x + \frac{\sin y}{x}.$$

Right side (product rule):

$$\frac{d}{dx}(\sin x \log y) = \cos x \cdot \log y + \sin x \cdot \frac{1}{y}\frac{dy}{dx}.$$

Equating the two,

$$\cos y \ \log x \ \frac{dy}{dx} + \frac{\sin y}{x} = \cos x \ \log y + \frac{\sin x}{y}\frac{dy}{dx}.$$

Step 4: Collect the dy/dx terms Move all terms with $\frac{dy}{dx}$ to the left and the rest to the right:

$$\frac{dy}{dx}(\cos y \ \log x - \frac{\sin x}{y}) = \cos x \ \log y - \frac{\sin y}{x}.$$

Multiply the bracket and the right side by xy to clear the fractions:

$$\frac{dy}{dx}\ (xy \cos y \ \log x - x \sin x) = xy \cos x \ \log y - y \sin y.$$

Step 5: Conclusion Dividing by the bracket,

Quick Tip to Solve: Take logarithms of both sides first, then differentiate implicitly using the product rule on each side.

SA-29. Find the inverse matrix of the following matrix : $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ [2 marks]

Solution:

Concept:

- We need the inverse of the 2×2 matrix $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$.
- An inverse exists only when $|A| \neq 0$.

Step 1: Key Formula or Approach For a square matrix, $A^{-1} = \frac{1}{|A|}\text{adj}\ A$. For a 2×2 matrix, the adjoint is found by swapping the two diagonal entries and changing the sign of the other two.

Step 2: Find the determinant

$$|A| = (4)(3) - (1)(2) = 12 - 2 = 10.$$

Since $|A| = 10 \neq 0$, the inverse exists.

Step 3: Find the cofactors and the adjoint Cofactors: $C_{11} = 3,\ C_{12} = -2,\ C_{21} = -1,\ C_{22} = 4$. The adjoint is the transpose of the cofactor matrix:

$$\operatorname{adj} A = \begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix}^T = \begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}.$$

Step 4: Divide by the determinant

$$A^{-1} = \frac{1}{10}\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{10} & -\frac{1}{10} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix}.$$

Check: the first row of A times the first column of A^{-1} gives $4 \cdot \frac{3}{10} + 1 \cdot \left(-\frac{1}{5}\right) = \frac{6}{5} - \frac{1}{5} = 1$, as it should.

Step 5: Conclusion

Quick Tip to Solve: Use $A^{-1} = \frac{1}{|A|}\operatorname{adj} A$. Here $|A| = 12 - 2 = 10$, and the adjoint comes from swapping the diagonal and negating the off-diagonal entries.

SA-30. **If matrix $A = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 7 \\ 2 & -4 \end{pmatrix}$, then find the value of $(4A - B)$.** [2 marks]

Solution:

Concept:

- We are given two 2×2 matrices A and B and must find $4A - B$.
- This needs two operations: multiplying a matrix by a number (scalar multiplication) and subtracting two matrices of the same order.

Step 1: Key Formula or Approach 1. To find kA, multiply every entry of A by k.
2. To find $P - Q$, subtract the entries of Q from the entries of P in the same positions.

Step 2: Find 4A

$$4A = 4\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ -12 & 16 \end{pmatrix}.$$

Step 3: Subtract B

$$4A - B = \begin{pmatrix} 4 & 8 \\ -12 & 16 \end{pmatrix} - \begin{pmatrix} 0 & 7 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} 4-0 & 8-7 \\ -12-2 & 16-(-4) \end{pmatrix}.$$

Simplifying each entry, $4 - 0 = 4,\ 8 - 7 = 1,\ -12 - 2 = -14,\ 16 + 4 = 20$.

Step 4: Conclusion

Quick Tip to Solve: Multiply every entry of A by 4 first, then subtract the matching entries of B.

Section B: Long Answer Type Questions (any 4 to be answered, 5 marks each)

LA-31. Prove that $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$. [5 marks]

Solution:

Concept:

- Let Δ be the determinant on the left side. The right side is a product of differences, so we aim to use column and row operations that create common factors and zeros.
- We treat x, y, z as nonzero while dividing. The result is a polynomial identity, so it also holds when some variable is zero.

Step 1: Make the third row look like the first two Multiply C_1 by x, C_2 by y and C_3 by z. The determinant gets multiplied by xyz:

$$xyz\,\Delta = \begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ xyz & xyz & xyz \end{vmatrix}$$

Take xyz common from the third row and cancel it from both sides:

$$\Delta = \begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix}$$

Step 2: Bring the row of ones to the top Do $R_3 \leftrightarrow R_2$ and then $R_2 \leftrightarrow R_1$. Two swaps change the sign twice, so the value stays the same:

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$$

Step 3: Create zeros in the first row Apply $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$:

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ x^2-y^2 & y^2-z^2 & z^2 \\ x^3-y^3 & y^3-z^3 & z^3 \end{vmatrix}$$

Step 4: Expand along the first row Only the entry 1 in the third place is nonzero, and its sign is $(-1)^{1+3} = +$:

$$\Delta = (x^2-y^2)(y^3-z^3) - (y^2-z^2)(x^3-y^3)$$

Use $a^2-b^2 = (a-b)(a+b)$ and $a^3-b^3 = (a-b)(a^2+ab+b^2)$. Take $(x-y)(y-z)$ common:

$$\Delta = (x-y)(y-z)\big[(x+y)(y^2+yz+z^2) - (y+z)(x^2+xy+y^2)\big]$$

Step 5: Simplify the bracket Expand each product:

$$(x+y)(y^2+yz+z^2) = xy^2+xyz+xz^2+y^3+y^2z+yz^2$$

$$(y+z)(x^2+xy+y^2) = x^2y+xy^2+y^3+x^2z+xyz+y^2z$$

Subtract. The terms xy^2, xyz, y^3, y^2z cancel and we are left with

$$xz^2+yz^2-x^2y-x^2z = xz(z-x)+y(z-x)(z+x) = (z-x)(xz+yz+xy)$$

Step 6: Put it together

$$\Delta = (x-y)(y-z)(z-x)(xy+yz+zx)$$

This is the required right side.

Step 7: Conclusion The determinant equals $(x-y)(y-z)(z-x)(xy+yz+zx)$, as we had to prove.

Quick Tip to Solve: Multiply the columns by x, y, z and take xyz common from the last row. Then use $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$ to get two zeros in a row.

LA-32. **If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$ then prove that $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$.** [5 marks]

Solution:

Concept:

- Give each inverse sine a name.
- Let $A = \sin^{-1}x$, $B = \sin^{-1}y$, $C = \sin^{-1}z$.
- Then $x = \sin A$, $y = \sin B$, $z = \sin C$, and the given condition is $A + B + C = \pi$.

Step 1: Rewrite each term of the left side The principal value of $\sin^{-1}x$ lies in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, where cosine is not negative. So $\sqrt{1-x^2} = \sqrt{1-\sin^2 A} = \cos A$.
So $x\sqrt{1-x^2} = \sin A \cos A = \frac{1}{2}\sin 2A$. In the same way for y and z:

$$\text{LHS} = \tfrac{1}{2}\big(\sin 2A + \sin 2B + \sin 2C\big)$$

Step 2: Add the first two sines Use $\sin P + \sin Q = 2\sin\dfrac{P+Q}{2}\cos\dfrac{P-Q}{2}$:

$$\sin 2A + \sin 2B = 2\sin(A+B)\cos(A-B)$$

Since $A + B = \pi - C$, we have $\sin(A+B) = \sin C$. So this equals $2\sin C\cos(A-B)$.

Step 3: Handle the third sine

$$\sin 2C = 2\sin C\cos C$$

Because $C = \pi - (A+B)$, we get $\cos C = -\cos(A+B)$. So $\sin 2C = -2\sin C\cos(A+B)$.

Step 4: Add everything

$$\sin 2A + \sin 2B + \sin 2C = 2\sin C\big[\cos(A-B) - \cos(A+B)\big]$$

Now $\cos(A-B) - \cos(A+B) = 2\sin A\sin B$. So the sum is $4\sin A\sin B\sin C$.

Step 5: Return to x, y, z

$$\text{LHS} = \tfrac{1}{2} \cdot 4 \sin A \sin B \sin C = 2xyz$$

This is the right side.

Step 6: Conclusion We have shown that $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$.

Quick Tip to Solve: Put $x = \sin A$, $y = \sin B$, $z = \sin C$ with $A + B + C = \pi$.
The left side becomes $\frac{1}{2}(\sin 2A + \sin 2B + \sin 2C) = 2 \sin A \sin B \sin C$.

LA-33. Solve : $(x^2 - y^2)\dfrac{dy}{dx} = 2xy$. [5 marks]

Solution:

Step 1: Recognise the type Write the equation as $\dfrac{dy}{dx} = \dfrac{2xy}{x^2 - y^2}$. The numerator and the denominator are both of degree 2 in x and y, so the equation is homogeneous.
We use the substitution $y = vx$.

Step 2: Substitute y = vx Then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. Putting this in:

$$v + x\frac{dv}{dx} = \frac{2vx^2}{x^2 - v^2x^2} = \frac{2v}{1 - v^2}$$

Step 3: Separate the variables

$$x\frac{dv}{dx} = \frac{2v}{1 - v^2} - v = \frac{2v - v + v^3}{1 - v^2} = \frac{v(1 + v^2)}{1 - v^2}$$

Move the v terms to one side and the x terms to the other:

$$\frac{1 - v^2}{v(1 + v^2)}\,dv = \frac{dx}{x}$$

Step 4: Split into partial fractions We check that

$$\frac{1 - v^2}{v(1 + v^2)} = \frac{1}{v} - \frac{2v}{1 + v^2}$$

Indeed, the right side is $\dfrac{(1 + v^2) - 2v^2}{v(1 + v^2)} = \dfrac{1 - v^2}{v(1 + v^2)}$.

Step 5: Integrate both sides

$$\int \left(\frac{1}{v} - \frac{2v}{1 + v^2}\right) dv = \int \frac{dx}{x}$$

$$\log|v| - \log(1 + v^2) = \log|x| + \log|C|$$

Combine the logarithms:

$$\frac{v}{1 + v^2} = Cx$$

Step 6: Go back to x and y Replace $v = \frac{y}{x}$. Then $\frac{v}{1+v^2} = \frac{y/x}{(x^2+y^2)/x^2} = \frac{xy}{x^2+y^2}$:

$$\frac{xy}{x^2+y^2} = Cx \quad \Longrightarrow \quad \frac{y}{x^2+y^2} = C$$

Write $c = \frac{1}{C}$. Then

$$x^2 + y^2 = cy$$

The line $y = 0$ also satisfies the given equation, but it is only a limiting case ($c \to \infty$) of this family.

Step 7: Conclusion The general solution of the equation is the family of circles

Quick Tip to Solve: The equation is homogeneous, so put $y = vx$ and separate the variables. Use $\frac{1-v^2}{v(1+v^2)} = \frac{1}{v} - \frac{2v}{1+v^2}$ before integrating.

LA-34. Prove that $\int_0^{\pi/2} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx = \pi\sqrt{2}$. [5 marks]

Solution:

Step 1: Rewrite the integrand Let $I = \int_0^{\pi/2} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$. Write tangent and cotangent in terms of sine and cosine:

$$\sqrt{\tan x} + \sqrt{\cot x} = \frac{\sin x}{\sqrt{\sin x \cos x}} + \frac{\cos x}{\sqrt{\sin x \cos x}} = \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}$$

Step 2: Choose a substitution The numerator $\sin x + \cos x$ is the derivative of $\sin x - \cos x$. So let

$$t = \sin x - \cos x, \qquad dt = (\cos x + \sin x)\, dx$$

Step 3: Express sin x cos x in t Square the substitution: $t^2 = \sin^2 x + \cos^2 x - 2\sin x \cos x = 1 - 2\sin x \cos x$.

$$\sin x \cos x = \frac{1 - t^2}{2}$$

Step 4: Change the limits When $x = 0$: $t = 0 - 1 = -1$.
When $x = \frac{\pi}{2}$: $t = 1 - 0 = 1$.

Step 5: Evaluate the integral

$$I = \int_{-1}^{1} \frac{dt}{\sqrt{\frac{1-t^2}{2}}} = \sqrt{2} \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}}$$

$$I = \sqrt{2}\left[\sin^{-1} t\right]_{-1}^{1} = \sqrt{2}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \pi\sqrt{2}$$

Step 6: Conclusion The value of the integral is $\pi\sqrt{2}$, which proves the result.

Quick Tip to Solve: Write the integrand as $\frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}$ and put $t = \sin x - \cos x$.

Then $\sin x \cos x = \dfrac{1-t^2}{2}$ and the limits become -1 to 1.

LA-35. Find $\dfrac{dy}{dx}$, when $x^y + y^x = 1$. [5 marks]

Solution:

Concept:

- The variable y appears in the exponent in x^y, and x appears in the exponent in y^x. Such powers cannot be differentiated by the ordinary power rule.
- The method for them is logarithmic differentiation: write each term as $u = x^y$ and $v = y^x$, take the log of each separately, and differentiate.
- The equation becomes $u + v = 1$, so $\dfrac{du}{dx} + \dfrac{dv}{dx} = 0$.

Step 1: Differentiate the first term Let $u = x^y$. Taking logs gives $\ln u = y \ln x$.
Differentiate both sides with respect to x. The right side is a product of y and $\ln x$, and y depends on x:

$$\frac{1}{u}\frac{du}{dx} = y \cdot \frac{1}{x} + \ln x \cdot \frac{dy}{dx}$$

Multiply by $u = x^y$:

$$\frac{du}{dx} = x^y\left(\frac{y}{x} + \ln x \frac{dy}{dx}\right) = y\,x^{y-1} + x^y \ln x \frac{dy}{dx}$$

Step 2: Differentiate the second term Let $v = y^x$. Taking logs gives $\ln v = x \ln y$.
Differentiate both sides with respect to x. Again use the product rule, and the chain rule for $\ln y$:

$$\frac{1}{v}\frac{dv}{dx} = \ln y + x \cdot \frac{1}{y}\frac{dy}{dx}$$

Multiply by $v = y^x$:

$$\frac{dv}{dx} = y^x \ln y + x\,y^{x-1}\frac{dy}{dx}$$

Step 3: Add the two derivatives Since $\dfrac{du}{dx} + \dfrac{dv}{dx} = 0$, we get

$$y\,x^{y-1} + x^y \ln x \frac{dy}{dx} + y^x \ln y + x\,y^{x-1}\frac{dy}{dx} = 0$$

Collect the terms that contain $\dfrac{dy}{dx}$ on the left and move the rest to the right:

$$\left(x^y \ln x + x\,y^{x-1}\right)\frac{dy}{dx} = -\left(y\,x^{y-1} + y^x \ln y\right)$$

Step 4: Solve for dy/dx Divide both sides by the bracket on the left:

$$\frac{dy}{dx} = -\frac{y\,x^{y-1} + y^x \ln y}{x^y \ln x + x\,y^{x-1}}$$

A quick check: the numerator and denominator are exactly the two partial derivatives of $x^y + y^x$, which is what the implicit rule gives.

Step 5: Conclusion Differentiating each power by taking logs and adding the results gives the derivative below.

Quick Tip to Solve: Write $u = x^y$ and $v = y^x$, take logs of each and differentiate, then use $u' + v' = 0$.

LA-36. Prove that $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = a^2b^2$. [5 marks]

Solution:

Concept:

- We must show that the square of the magnitude of the cross product plus the square of the dot product equals a^2b^2, where $a = |\vec{a}|$ and $b = |\vec{b}|$.
- Both products are defined through the angle between the two vectors, so the natural plan is to write each in terms of that angle.

Step 1: Key formulae Let θ be the angle between $\vec{a}$ and $\vec{b}$, with $0 \le \theta \le \pi$.
1. Cross product: $\vec{a} \times \vec{b} = ab \sin\theta\, \hat{n}$, where $\hat{n}$ is a unit vector perpendicular to both. So $|\vec{a} \times \vec{b}| = ab \sin\theta$ (the sine is not negative on this range).
2. Dot product: $\vec{a} \cdot \vec{b} = ab \cos\theta$.

Step 2: Square each product

$$|\vec{a} \times \vec{b}|^2 = a^2b^2 \sin^2\theta$$

$$(\vec{a} \cdot \vec{b})^2 = a^2b^2 \cos^2\theta$$

Step 3: Add them Add the two results and take a^2b^2 out as a common factor:

$$|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = a^2b^2\left(\sin^2\theta + \cos^2\theta\right)$$

Since $\sin^2\theta + \cos^2\theta = 1$ for every angle,

$$|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = a^2b^2$$

If one of the vectors is zero, both sides are zero, so the result holds in that case too. Hence the statement is proved.

Step 4: Conclusion Writing both products with the angle θ and using $\sin^2\theta + \cos^2\theta = 1$ proves the identity.

Quick Tip to Solve: Write the cross and dot products in terms of the angle θ between the vectors, square them and add.

LA-37. Maximize $Z = x + y$
subject to $x - y \le -1$
$-x + y \le 0,\ x \ge 0,\ y \ge 0$ [5 marks]

Solution:

Concept:

- We have to maximise $Z = x + y$ over the region given by $x - y \le -1$, $-x + y \le 0$, $x \ge 0$ and $y \ge 0$.
- In the graphical method we first draw the region common to all the inequalities, the feasible region. The maximum can exist only if that region contains at least one point.

Step 1: Rewrite the two main constraints The first constraint is $x - y \le -1$. Moving terms gives $y \ge x + 1$. The feasible points lie on or above the line $y = x + 1$.

The second constraint is $-x + y \leq 0$. This gives $y \leq x$. The feasible points lie on or below the line $y = x$.

Step 2: Compare the two lines The lines $y = x + 1$ and $y = x$ both have slope 1, so they are parallel. The line $y = x + 1$ lies one unit above $y = x$.
A point must be on or above the upper line and also on or below the lower line. Take any x: we need $x + 1 \leq y \leq x$.
This would need $x + 1 \leq x$, that is $1 \leq 0$, which is false for every x.

Step 3: Conclusion about the feasible region The two half-planes do not overlap, so no point can satisfy both. The conditions $x \geq 0$ and $y \geq 0$ only restrict the plane further and cannot bring a point back.
The feasible region is empty. A linear function has no maximum value over an empty set, so the problem has no solution.

Step 4: Conclusion The constraints as printed contradict each other, so the problem has no feasible solution and Z has no maximum.

Quick Tip to Solve: Rewrite the first two constraints as $y \geq x + 1$ and $y \leq x$ and check whether any point can satisfy both.

LA-38. 10 coins are tossed. What is the probability that exactly 5 heads appear ? [5 marks]

Solution:

Concept:

- Each toss of a fair coin has two outcomes, head or tail, each with probability $\frac{1}{2}$. The 10 tosses are independent of one another.
- This is a set of Bernoulli trials, so the number of heads X follows a binomial distribution with $n = 10$ and $p = \frac{1}{2}$.

Step 1: Key formula For a binomial distribution, the chance of exactly r successes is

$$P(X = r) = {}^{n}C_{r}\, p^{r} q^{n-r}$$

where $q = 1 - p$ is the probability of failure.

Step 2: Put in the values Here $n = 10$, $r = 5$, $p = \frac{1}{2}$ and $q = \frac{1}{2}$. So

$$P(X = 5) = {}^{10}C_{5}\left(\frac{1}{2}\right)^{5}\left(\frac{1}{2}\right)^{5} = {}^{10}C_{5}\left(\frac{1}{2}\right)^{10}$$

Step 3: Calculate The number of ways to choose which 5 of the 10 tosses are heads is

$${}^{10}C_{5} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{30240}{120} = 252$$

Also $2^{10} = 1024$. Therefore

$$P(X = 5) = \frac{252}{1024} = \frac{63}{256}$$

This is about 0.246, so a bit less than a quarter of the time we see exactly 5 heads.

Step 4: Conclusion The probability of getting exactly 5 heads in 10 tosses is $\frac{63}{256}$.

Quick Tip to Solve: Number of heads in 10 fair tosses is binomial with $n = 10$, $p = \frac{1}{2}$. Use ${}^{10}C_{5}/2^{10}$.