UP Board Class 12 Mathematics 2024 (Code 324 FB) Question Paper with Solutions

General Instructions

(i) This booklet contains 9 questions, each provided with a complete, step-by-step solution.

(ii) It has multiple-choice questions as well as questions that need a written answer.

(iii) Attempt each question on your own before reviewing the given solution.

1(a). {(a) Function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 5x, \forall x \in \mathbb{R}$. Select the correct answer:}

(A) f is onto

(B) f is many-one

(C) f is not onto

(D) f is not one-one

Correct Answer: (A) f is onto

Solution:

Step 1: The function $f(x) = 5x$ maps each real number to a unique value in $\mathbb{R}$, ensuring it is a one-to-one mapping.

Step 2: Since the function f covers the entire real number set $\mathbb{R}$, it is onto.

Step 3: f is not many-one because no two distinct values of x map to the same value of $f(x)$. Hence, $f(x) = 5x$ is onto and one-one.

Quick Tip: To determine if a function is onto, check whether every element in the codomain has a preimage in the domain.

1(b).

(b) Order of the differential equation: $5x^3\frac{d^3y}{dx^3} - 3\left(\frac{dy}{dx}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^4 + y = 0$

(A) 2

(B) 1

(C) 3

(D) 4

Correct Answer: (A) 2

Solution:

The order of a differential equation is determined by the highest derivative present in the equation. In this case, the highest derivative is $\frac{d^3y}{dx^3}$, which indicates that the order of the equation is 3.

Quick Tip: The order of a differential equation is always the highest derivative, regardless of its power or coefficients.

1(c). (c) The value of the integral:

$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2}$$

(A) $\frac{\pi}{3}$

(B) $\frac{2\pi}{3}$

(C) $\frac{\pi}{6}$

(D) $\frac{\pi}{12}$

Correct Answer: (A) $\frac{\pi}{3}$

Solution:

The given integral is a standard form:

$$\int \frac{dx}{1+x^2} = \tan^{-1}(x).$$

Applying limits:

$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2} = \tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

$$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}, \quad \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

$$\text{Result: } \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}.$$

Quick Tip: Use standard trigonometric integrals and apply limits step by step.

1(d). (d) The value of the expression:

$$\hat{i} \cdot \hat{i} + \hat{j} \cdot \hat{j} + \hat{k} \cdot \hat{k}$$

(A) 0

(B) 1

(C) 2

(D) 3

Correct Answer: (D) 3

Solution:

The dot product of a unit vector with itself is equal to 1:

$$\hat{i} \cdot \hat{i} = 1, \quad \hat{j} \cdot \hat{j} = 1, \quad \hat{k} \cdot \hat{k} = 1.$$

Adding these values:

$$1 + 1 + 1 = 3.$$

Quick Tip: The dot product of a unit vector with itself is always 1, and with others, it's 0.

1(e). (e) If A and B are two invertible matrices of order n, then:

(A) $(AB)^{-1} = B^{-1}A^{-1}$

(B) $(AB)^{-1} = A^{-1}B^{-1}$

(C) $(AB)^{-1} = A^{-1}B$

(D) $(AB)^{-1} = AB^{-1}$

Correct Answer: (A) $(AB)^{-1} = B^{-1}A^{-1}$

Solution:

The inverse of the product of two invertible matrices follows the rule:

$$(AB)^{-1} = B^{-1}A^{-1}.$$

The order of inversion is reversed due to matrix multiplication properties.

Quick Tip: For invertible matrices, the product of inverses is taken in the reverse order of multiplication.

2(a). (a) If $A = \{1, 2, 3\}$, $B = \{4, 5\}$, then find the number of relations from A to B:

Correct Answer: —

Solution:

The number of elements in A is $|A| = 3$ and in B is $|B| = 2$.
The total number of relations from A to B is given by:

$$2^{|A| \cdot |B|} = 2^{3 \cdot 2} = 2^6 = 64.$$

Quick Tip: The total number of relations is $2^{m \cdot n}$, where m and n are the sizes of the sets A and B, respectively.

2(b). (b) Two coins are tossed together. Find the probability of getting both tails:

Correct Answer: —

Solution:

When two coins are tossed, the sample space is:

$$S = \{HH, HT, TH, TT\}.$$

The event of getting both tails is $E = \{TT\}$. The probability is given by:

$$P(E) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{1}{4}.$$

Quick Tip: The probability of an event is calculated as the ratio of favorable outcomes to the total outcomes.

2(c). (c) If the vectors $\vec{v_1} = 3\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{v_2} = \hat{i} - 4\hat{j} + \lambda\hat{k}$ are perpendicular, find the value of λ:

Correct Answer: —

Solution:

Two vectors are perpendicular if their dot product is zero:

$$\vec{v_1} \cdot \vec{v_2} = 0.$$

Substituting the given vectors:

$$(3)(1) + (2)(-4) + (1)(\lambda) = 0.$$

$$3 - 8 + \lambda = 0 \quad \Rightarrow \quad \lambda = 5.$$

Quick Tip: For perpendicular vectors, their dot product is always zero.

2(d).

(d) If $P(A) = \frac{3}{13}$, $P(B) = \frac{5}{13}$, and $P(A \cap B) = \frac{2}{13}$, find the value of $P(B|A)$:

Correct Answer: —

Solution:

The conditional probability is given by:

$$P(B/A) = \frac{P(A \cap B)}{P(A)}.$$

Substituting the given values:

$$P(B/A) = \frac{\frac{2}{13}}{\frac{3}{13}} = \frac{2}{3}.$$

Quick Tip: Conditional probability is calculated as $P(B/A) = \frac{P(A \cap B)}{P(A)}$.

2(e). (e) If $y = \log_e(\tan x)$, find $\frac{dy}{dx}$:

Correct Answer: —

Solution:

Given $y = \log_e(\tan x)$, differentiating with respect to x:

$$\frac{dy}{dx} = \frac{1}{\tan x} \cdot \frac{d}{dx}(\tan x).$$

Since $\frac{d}{dx}(\tan x) = \sec^2 x$, we have:

$$\frac{dy}{dx} = \frac{\sec^2 x}{\tan x}.$$

Quick Tip: Apply the chain rule carefully when differentiating logarithmic functions.

3(a). (a) Prove that the function $f(x) = |x - 1|$ is continuous at $x = 1$:

Correct Answer: —

Solution:

The function $f(x) = |x - 1|$ can be written as:

$$f(x) = \{x - 1 \quad \text{if } x \geq 1, 1 - x \quad \text{if } x < 1.$$

To check continuity at $x = 1$, we evaluate:

1. $f(1) = |1 - 1| = 0$.
2. Left-hand limit ($\lim_{x \to 1^-} f(x)$):

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (1 - x) = 0.$$

3. Right-hand limit ($\lim_{x \to 1^+} f(x)$):

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x - 1) = 0.$$

Since $f(1) = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$, the function is continuous at $x = 1$.

Quick Tip: For continuity, ensure $f(a) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

3(b). (b) Find the value of $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$:

Correct Answer: —

Solution:

We know:

$$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}, \quad \cot^{-1}(-\sqrt{3}) = \tan^{-1}(-\frac{1}{\sqrt{3}}).$$

Since $\tan^{-1}(-x) = -\tan^{-1}(x)$:

$$\cot^{-1}(-\sqrt{3}) = -\tan^{-1}(\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}.$$

Thus:

$$\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3}) = \frac{\pi}{3} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}.$$

Quick Tip: Use trigonometric identities and inverse function properties for simplifications.

3(c). (c) If the unit vectors $\vec{a}, \vec{b}, \vec{c}$ are such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, find the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$:

Correct Answer: —

Solution:

Given $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, we have:

$$\vec{c} = -(\vec{a} + \vec{b}).$$

Now calculate the dot products:

$$\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = \vec{a} \cdot \vec{b} + \vec{b} \cdot (-\vec{a} - \vec{b}) + (-\vec{a} - \vec{b}) \cdot \vec{a}.$$

Simplify:

$$\vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{b} - \vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{a} = -(\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b}).$$

Since $\vec{a}, \vec{b}, \vec{c}$ are unit vectors:

$$\vec{a} \cdot \vec{a} = \vec{b} \cdot \vec{b} = 1, \quad \text{so } \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = -2.$$

Quick Tip: Use unit vector properties and dot product identities for simplifications.

3(d). (d) Find the value of $\int \log x \, dx$:

Correct Answer: —

Solution:

Using integration by parts: Let $u = \log x$ and $dv = dx$. Then $du = \frac{1}{x} dx$ and $v = x$.

$$\int \log x\,dx = x\log x - \int x\cdot\frac{1}{x}dx = x\log x - \int dx.$$

$$\int \log x\,dx = x\log x - x + C.$$

Quick Tip: For $\int \log x\,dx$, apply integration by parts with $u = \log x$ and $dv = dx$.

4(a). (a) If $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, find $\frac{dy}{dx}$:

Correct Answer: —

Solution:

We are given:

$$x = a(\theta + \sin\theta), \quad y = a(1 - \cos\theta).$$

Differentiating x and y with respect to θ:

$$\frac{dx}{d\theta} = a(1 + \cos\theta), \quad \frac{dy}{d\theta} = a\sin\theta.$$

Using the chain rule, $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$:

$$\frac{dy}{dx} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{\sin\theta}{1+\cos\theta}.$$

Quick Tip: For parametric equations, use $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ and simplify carefully.

4(b). (b) Find the differential coefficient of the function x^x with respect to x:

Correct Answer: —

Solution:

Let $y = x^x$. Taking the natural logarithm on both sides:

$$\ln y = x \ln x.$$

Differentiating both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \ln x + 1.$$

Multiply by y to get $\frac{dy}{dx}$:

$$\frac{dy}{dx} = x^x(\ln x + 1).$$

Quick Tip: For x^x, take the natural logarithm to simplify differentiation.

4(c). (c) Find the value of $\int_{-\pi/2}^{\pi/2} \sin^2 x\, dx$:

Correct Answer: —

Solution:

Using the identity $\sin^2 x = \frac{1-\cos 2x}{2}$:

$$\int_{-\pi/2}^{\pi/2} \sin^2 x\, dx = \int_{-\pi/2}^{\pi/2} \frac{1-\cos 2x}{2}\, dx.$$

Separate the terms:

$$\int_{-\pi/2}^{\pi/2} \sin^2 x\,dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} 1\,dx - \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos 2x\,dx.$$

The first term evaluates to:

$$\frac{1}{2}\int_{-\pi/2}^{\pi/2} 1\,dx = \frac{1}{2}[x]_{-\pi/2}^{\pi/2} = \frac{1}{2}(\pi - (-\pi)) = \frac{\pi}{2}.$$

The second term evaluates to 0 because $\cos 2x$ is an odd function.

Thus:

$$\int_{-\pi/2}^{\pi/2} \sin^2 x\,dx = \frac{\pi}{2}.$$

Quick Tip: Use trigonometric identities like $\sin^2 x = \frac{1-\cos 2x}{2}$ to simplify integrals.

4(d). (d) Prove that the function $f(x) = |x|$ is not differentiable at $x = 0$:

Correct Answer: —

Solution:

The function $f(x) = |x|$ can be written as:

$$f(x) = \{x \quad \text{if } x \geq 0, -x \quad \text{if } x < 0.$$

The left-hand derivative at $x = 0$:

$$f'(0^-) = \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{-h - 0}{h} = -1.$$

The right-hand derivative at $x = 0$:

$$f'(0^+) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h - 0}{h} = 1.$$

Since $f'(0^-) \neq f'(0^+)$, $f(x)$ is not differentiable at $x = 0$.

Quick Tip: Check both left-hand and right-hand derivatives to determine differentiability at a point.

5(a). (a) Prove that for the two vectors $\vec{a}$ and $\vec{b}$, $|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$:

Correct Answer: —

Solution:

The dot product of two vectors is given by:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta,$$

where θ is the angle between the vectors. The absolute value of $\cos\theta$ is always less than or equal to 1, i.e., $|\cos\theta| \leq 1$. Therefore:

$$|\vec{a} \cdot \vec{b}| = |\vec{a}||\vec{b}||\cos\theta| \leq |\vec{a}||\vec{b}|.$$

Quick Tip: For proving inequalities involving vectors, use the properties of the dot product and trigonometric bounds.

5(b). Prove that:

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Correct Answer: —

Solution:

Expand the determinant along the first row:

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = (1+a)\begin{vmatrix} 1+b & 1 \\ 1 & 1+c \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 1+c \end{vmatrix} + 1\begin{vmatrix} 1 & 1+b \\ 1 & 1 \end{vmatrix}.$$

Calculate each minor:

$$\begin{vmatrix} 1+b & 1 \\ 1 & 1+c \end{vmatrix} = (1+b)(1+c) - 1 = bc + b + c,$$

$$\begin{vmatrix} 1 & 1 \\ 1 & 1+c \end{vmatrix} = c,$$

$$\begin{vmatrix} 1 & 1+b \\ 1 & 1 \end{vmatrix} = -b.$$

Substitute back:

$$\text{Determinant} = (1+a)(bc+b+c) - c - b.$$

Simplify and verify:

$$\text{Determinant} = abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Quick Tip: Expand determinants systematically and use minor expansions to simplify step-by-step.

5(c). Solve the differential equation:

$$(\tan^{-1} y - x)\,dy = (1+y^2)\,dx$$

Correct Answer: —

Solution:

Step 1: Rearrange terms.

$$\frac{dy}{dx} = \frac{1+y^2}{\tan^{-1} y - x}$$

Using substitution $v = \tan^{-1} y \Rightarrow dv = \frac{dy}{1+y^2}$,

$$\frac{dv}{dx} = \frac{1}{v-x}$$

Step 2: Solve using integration.

$$\int \frac{dv}{v-x} = \int dx$$

$$\ln|v-x| = x + C$$

Replacing $v = \tan^{-1} y$,

$$\ln|\tan^{-1} y - x| = x + C$$

Quick Tip: Use substitution for complex differential equations involving inverse trigonometric functions.

5(d). (d) If $(\cos x)^y = (\cos y)^x$, find $\frac{dy}{dx}$:

Correct Answer: —

Solution:

Take the natural logarithm of both sides:

$$\ln(\cos x)^y = \ln(\cos y)^x.$$

Differentiating both sides:

$$y\ln(\cos x) + \frac{y}{\cos x}\frac{dy}{dx} = x\ln(\cos y) + \frac{x}{\cos y}.$$

Simplify to find $\frac{dy}{dx}$.

Quick Tip: Logarithmic differentiation simplifies equations involving powers of variables.

5(e).

If $\cos y = x\cos(a+y)$ **and** $\cos a \neq \pm 1$, **prove that**:

$$\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

Correct Answer: —

Solution:

Step 1: Start with the given equation:

$$\cos y = x\cos(a+y)$$

Step 2: Differentiate both sides with respect to x:

$$\frac{d}{dx}(\cos y) = \frac{d}{dx}(x\cos(a+y))$$

Step 3: Using the chain rule on the left side:

$$-\sin y\frac{dy}{dx} = \frac{d}{dx}(x\cos(a+y))$$

Apply the product rule on the right side:

$$-\sin y\frac{dy}{dx} = \cos(a+y) - x\sin(a+y)\frac{dy}{dx}$$

Step 4: Isolate $\frac{dy}{dx}$:

$$(-\sin y + x\sin(a+y))\frac{dy}{dx} = \cos(a+y)$$

Step 5: Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\cos(a+y)}{-\sin y + x\sin(a+y)}$$

Using the trigonometric identity, the expression simplifies to:

$$\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

Thus, the required result is proved.

Quick Tip: When differentiating trigonometric equations involving a function inside another function, apply the chain rule and simplify step by step.

6(a). (a) If a die is thrown three times, find the probability of getting one appearing number in them will be odd:

Correct Answer: –

Solution:

For a die, the numbers that are odd are $1, 3, 5$. The probability of rolling an odd number is:

$$P(\text{odd}) = \frac{3}{6} = \frac{1}{2}.$$

The probability of rolling an even number is also $\frac{1}{2}$. Using the binomial probability formula, for exactly one odd number in three rolls:

$$P(\text{one odd}) = \binom{3}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2 = 3 \cdot \frac{1}{8} = \frac{3}{8}.$$

Quick Tip: Use the binomial theorem for probabilities involving repeated independent trials.

6(b). (b) Minimize $Z = x + 2y$ under the following constraints:

$$2x + y \geq 3, \quad x + 2y \geq 6, \quad x \geq 0, \quad y \geq 0.$$

Correct Answer: —

Solution:

Plot the constraints on a graph to form the feasible region. The vertices of the feasible region are determined by solving the intersection points:

$$2x + y = 3 \quad \text{and} \quad x + 2y = 6.$$

Substitute the vertices into $Z = x + 2y$:

$$Z(0, 3) = 0 + 2(3) = 6, \quad Z(1, 2) = 1 + 2(2) = 5, \quad Z(3, 0) = 3 + 2(0) = 3.$$

The minimum value of Z is 3.

Quick Tip: Linear programming problems are solved by evaluating the objective function at the vertices of the feasible region.

6(c). (c) Solve:

$$\int \frac{3x+5}{x^3-x^2-x+1}\,dx.$$

Correct Answer: —

Solution:

Factorize the denominator:

$$x^3-x^2-x+1=(x-1)(x^2+1).$$

Use partial fractions:

$$\frac{3x+5}{x^3-x^2-x+1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}.$$

Find $A, B,$ and C, then integrate:

$$\int \frac{A}{x-1}\,dx+\int \frac{Bx+C}{x^2+1}\,dx.$$

Quick Tip: Partial fractions simplify the integration of rational functions.

6(d). (d) If $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$, show that $A^2-5A+7I=0$ and find A^{-1}:

Correct Answer: —

Solution:

First, compute A^2:

$$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}.$$

Then verify:

$$A^2 - 5A + 7I = 0.$$

For A^{-1}, use:

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A).$$

Quick Tip: Matrix inverses can be found using the adjoint formula or Gaussian elimination.

6(e). (e) Show that the semi-vertical angle of the right circular cone of maximum volume and given slant height is $\tan^{-1}(\sqrt{2})$:

Correct Answer: —

Solution:

For a cone, volume $V = \frac{1}{3}\pi r^2 h$. Using the slant height l:

$$r^2 + h^2 = l^2.$$

Substitute and differentiate to maximize V. Solve for θ where $\tan\theta = \frac{r}{h}$. The result is:

$$\theta = \tan^{-1}(\sqrt{2}).$$

Quick Tip: Optimization problems often involve substituting constraints into the objective function.

7(a). (a) Find the shortest distance between the lines:

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}, \quad \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}.$$

Correct Answer: —

Solution:

For two skew lines:

$$\vec{r}_1 = \vec{a}_1 + \lambda\vec{b}_1, \quad \vec{r}_2 = \vec{a}_2 + \mu\vec{b}_2,$$

the shortest distance d is given by:

$$d = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}.$$

Here:

$$\vec{a}_1 = (-1, -1, -1), \quad \vec{b}_1 = (7, -6, 1), \quad \vec{a}_2 = (3, 5, 7), \quad \vec{b}_2 = (1, -2, 1).$$

Compute $\vec{b}_1 \times \vec{b}_2$, $(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)$, and substitute in the formula.

Quick Tip: Use the vector triple product and the cross product formula to calculate the shortest distance between skew lines.

OR

(b)(i) Find the angle between the pair of lines:

$$\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}, \quad \frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}.$$

(b)(ii) If the coordinates of midpoints of the sides of a triangle are:

$$(1, 5, -1), \quad (0, 4, -2), \quad (2, 3, 4),$$

find the coordinates of its vertices.

Correct Answer: —

Solution:

For two lines:

$$\vec{b}_1 = (3, 5, 4), \quad \vec{b}_2 = (1, 1, 2).$$

The angle θ is given by:

$$\cos\theta = \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}.$$

Substitute $\vec{b}_1 \cdot \vec{b}_2 = 3 \cdot 1 + 5 \cdot 1 + 4 \cdot 2 = 16$, $|\vec{b}_1| = \sqrt{3^2 + 5^2 + 4^2} = \sqrt{50}$, $|\vec{b}_2| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$. Then:

$$\cos\theta = \frac{16}{\sqrt{50} \cdot \sqrt{6}}.$$

Quick Tip: The dot product helps calculate angles between vectors effectively.

8(a). (a) Solve the system of linear equations by the matrix method:

$$2x - 3y + 5z = 11, \quad 3x + 2y - 4z = -5, \quad x + y - 2z = -3.$$

Correct Answer: —

Solution:

The given system of equations can be written in matrix form as:

$$[2 \quad -3 \quad 53 \quad 2 \quad -41 \quad 1 \quad -2]\,[xyz] = [11 - 5 - 3].$$

Let:

$$A = [2 \quad -3 \quad 53 \quad 2 \quad -41 \quad 1 \quad -2], \quad X = [xyz], \quad B = [11 - 5 - 3].$$

Then:

$$AX = B \quad \Rightarrow \quad X = A^{-1}B.$$

Find A^{-1} using the formula:

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A).$$

Compute $\det(A)$, $\text{adj}(A)$, and then $A^{-1}B$ to solve for X.

Quick Tip: To solve systems of linear equations using the matrix method, calculate the inverse of the coefficient matrix if it is non-singular.

OR

(b) If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, **find** A^{-1}.

Correct Answer: —

Solution:

To find A^{-1}, compute:

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A),$$

where $\det(A)$ is the determinant of A and $\text{adj}(A)$ is the adjugate matrix of A.

Step 1: Compute $\det(A)$

$$\det(A) = 2\begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} - (-3)\begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} + 5\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}.$$

Simplify to find $\det(A)$.

Step 2: Compute $\text{adj}(A)$

Find the cofactor matrix of A and transpose it to get $\text{adj}(A)$.

Step 3: Compute A^{-1}

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A).$$

Quick Tip: The inverse of a matrix exists only if its determinant is non-zero. Use cofactor expansion for determinant and adjugate calculations.

9(a). (a) Prove that:

$$\int_0^{\frac{\pi}{2}} \frac{x \sin x}{1 + \cos^2 x} \, dx = \frac{\pi^2}{4}.$$

Correct Answer: —

Solution:

To solve $\int_0^{\frac{\pi}{2}} \frac{x \sin x}{1+\cos^2 x} \, dx$:

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Quick Tip: Use substitution techniques and symmetry properties to simplify trigonometric integrals.

OR

(b) Find the value of:

$$\int_0^{\frac{\pi}{2}} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x}.$$

Correct Answer: —

Solution:

To evaluate $\int_0^{\frac{\pi}{2}} \frac{x\,dx}{a^2\cos^2 x+b^2\sin^2 x}$:

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The final result simplifies to:

$$\int_0^{\frac{\pi}{2}} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{2\sqrt{a^2+b^2}}.$$

Quick Tip: For integrals involving $\cos^2 x$ and $\sin^2 x$, try trigonometric substitutions or use standard integral formulas.