BITSAT Mathematics Sample Paper – 6

Duration: 60 Minutes **Maximum Marks: 120**

Instructions

- This paper contains **40** Multiple Choice Questions (Single Correct).
- Each correct answer carries **+3 marks**. Each incorrect answer carries $-\mathbf{1}$ mark. Unattempted questions carry **0** marks.
- Only **one** option is correct for each question.
- Use of mobile phones, smartwatches, or any electronic gadgets is strictly prohibited.

Q1. The value of $\lim_{x\to 1} \frac{x^4-1}{x-1}$ is:

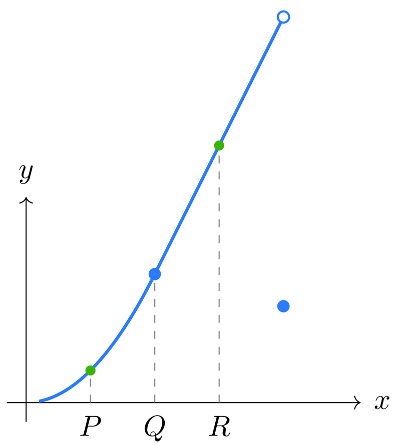
(A) 1

(B) 2

(C) 4

(D) 0

Q2. The graph of a function f is shown below. At which of the marked points is f continuous but **not** differentiable?



(A) P only

(B) Q only

(C) R only

(D) P and R

Q3. If $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$, then at $x = 0$:

(A) f is not continuous

(B) f is continuous but not differentiable

(C) f is differentiable and $f'(0) = 1$

(D) f is differentiable and $f'(0) = 0$

Q4. The slope of the normal to the curve $y = 2x^2 + 3\sin x$ at $x = 0$ is:

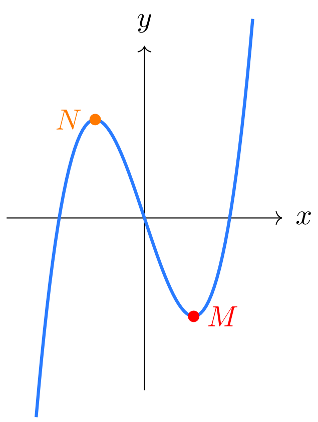
(A) 3

(B) $\frac{1}{3}$

(C) $-\frac{1}{3}$

(D) -3

Q5. The curve $y = x^3 - 3x$ is shown below. At the point marked M, the tangent to the curve is horizontal. What are the coordinates of M on the **right** side?



(A) $(1, -2)$

(B) $(-1, 2)$

(C) $(1, 2)$

(D) $(-1, -2)$

Q6. Water is poured into a conical vessel (vertex down) at $2\,\text{cm}^3/\text{s}$. The cone has height 12 cm and base radius 6 cm. The rate at which the water level rises when the water is 4 cm deep is:

(A) $\frac{1}{2\pi}$ cm/s

(B) $\frac{2}{\pi}$ cm/s

(C) $\frac{1}{\pi}$ cm/s

(D) $\frac{4}{\pi}$ cm/s

Q7. $\int e^x(\sin x + \cos x)\,dx$ equals:

(A) $e^x \cos x + C$

(B) $e^x \sin x + C$

(C) $e^x(\sin x - \cos x) + C$

(D) $2e^x \sin x + C$

Q8. The value of $\int_0^\pi x \sin x\,dx$ is:

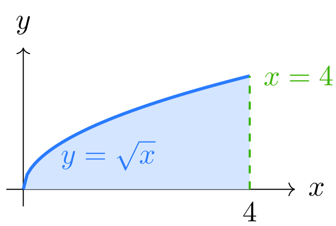
(A) 0

(B) π

(C) $-\pi$

(D) 2π

Q9. The shaded region in the figure is bounded by $y = \sqrt{x}$, the x-axis, and the line $x = 4$. Its area (in sq. units) is:



(A) $\frac{16}{3}$
(B) $\frac{8}{3}$
(C) 4
(D) $\frac{4}{3}$

Q10. The order and degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + y = 0$ are respectively:

(A) order 1, degree 2
(B) order 2, degree 3
(C) order 3, degree 2
(D) order 2, degree 2

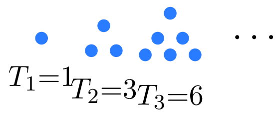
Q11. If the nth term of a G.P. is $3 \cdot 2^{n-1}$, the sum of the first 6 terms is:

(A) 180
(B) 189
(C) 192
(D) 186

Q12. The value of $0.\overline{54}$ expressed as a fraction in lowest terms is:

(A) $\frac{27}{50}$
(B) $\frac{6}{11}$
(C) $\frac{54}{99}$
(D) $\frac{27}{49}$

Q13. The dot pattern below shows triangular numbers $T_1, T_2, T_3, \ldots$ The sum $T_1 + T_2 + \cdots + T_{10}$ equals:



(A) 220

(B) 200

(C) 210

(D) 230

Q14. If ω is a primitive cube root of unity, the value of $(1+\omega-\omega^2)^8$ is:

(A) 256

(B) -256

(C) 1

(D) -1

Q15. The locus of z satisfying $|z-2|+|z+2|=6$ in the Argand plane is:

(A) A circle of radius 3

(B) An ellipse with semi-major axis 3

(C) A parabola

(D) A hyperbola

Q16. The number of real roots of the equation $|x|^2-3|x|+2=0$ is:

(A) 1

(B) 2

(C) 3

(D) 4

Q17. If α and β are roots of $2x^2-5x+3=0$, then $\alpha^2+\beta^2$ equals:

(A) $\dfrac{7}{4}$

(B) $\frac{13}{4}$

(C) $\frac{19}{4}$

(D) $\frac{1}{4}$

Q18. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$, then $(AB)^T$ equals:

(A) $\begin{pmatrix} 4 & 11 \\ 12 & 12 \end{pmatrix}$

(B) $\begin{pmatrix} 4 & 12 \\ 11 & 12 \end{pmatrix}$

(C) $\begin{pmatrix} 4 & 11 \\ 12 & 13 \end{pmatrix}$

(D) $\begin{pmatrix} 4 & 10 \\ 11 & 12 \end{pmatrix}$

Q19. If A is a square matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ equals:

(A) A

(B) $I - A$

(C) I

(D) $2I$

Q20. The value of k for which the system $x + 2y + 3z = 0,\ \ 2x + 3y + kz = 0,$ $3x + 4y + 7z = 0$ has a non-trivial solution is:

(A) 6

(B) 5

(C) 7

(D) 4

Q21. If $\begin{vmatrix} x & 2 \\ 6 & x \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 6 & 3 \end{vmatrix}$, then x equals:

(A) ± 3

(B) ± 2

(C) ± 6

(D) 0

Q22. How many 4-digit numbers greater than 5000 can be formed using the digits $\{3, 4, 5, 6, 7\}$ without repetition?

(A) 72

(B) 60

(C) 48

(D) 36

Q23. The number of diagonals of a convex polygon with n sides is $\frac{n(n-3)}{2}$. For a polygon with 44 diagonals, n equals:

(A) 10

(B) 11

(C) 12

(D) 9

Q24. A fair die is thrown twice. The probability that the sum of the two numbers is a prime is:

(A) $\frac{5}{12}$

(B) $\frac{7}{18}$

(C) $\frac{1}{4}$

(D) $\frac{1}{3}$

Q25. Events A and B are such that $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cup B) = 0.7$. Then $P(A|B)$ is:

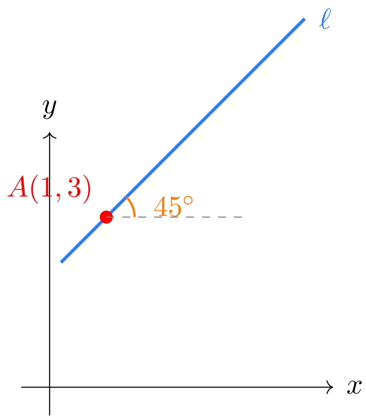
(A) 0.3

(B) 0.4

(C) 0.5

(D) 0.6

Q26. In the figure, the line ℓ passes through the point $A(1,3)$ and makes an angle of $45°$ with the positive x-axis. A point P on ℓ is at distance $3\sqrt{2}$ from A. The coordinates of P (in the direction of increasing x) are:



(A) $(4,6)$

(B) $(3,5)$

(C) $(4,5)$

(D) $(3,6)$

Q27. The equation of the line passing through the intersection of $2x + y = 5$ and $x - 3y = -2$, and perpendicular to $x + 4y = 3$, is:

(A) $4x - y = 11$

(B) $4x - y = 7$

(C) $4x + y = 11$

(D) $x - 4y = 7$

Q28. The two circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x - 8y + 21 = 0$ are:

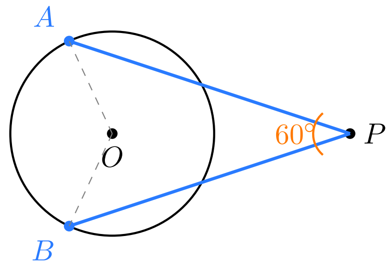
(A) Concentric

(B) Externally tangent

(C) Intersecting at two points

(D) Internally tangent

Q29. In the figure, PA and PB are tangents from external point P to a circle with centre O. If $\angle APB = 60^\circ$, then $\angle AOB$ equals:



(A) 120°

(B) 100°

(C) 90°

(D) 150°

Q30. The equation of the directrix of the parabola $y^2 = -12x$ is:

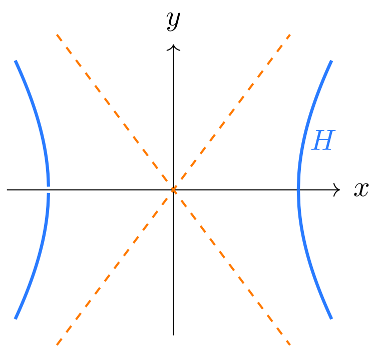
(A) $x = -3$

(B) $x = 3$

(C) $y = 3$

(D) $y = -3$

Q31. The figure shows a hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ and its asymptotes. The angle between the asymptotes (acute angle) is nearest to:



(A) 53°

(B) 106°

(C) 74°

(D) 37°

Q32. If $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$, then the value of $A + B$ is:

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$

(D) $\frac{\pi}{2}$

Q33. The maximum value of $5\sin\theta + 12\cos\theta$ is:

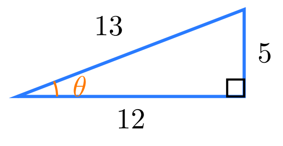
(A) 13

(B) 17

(C) 7

(D) 12

Q34. In the right triangle shown, the hypotenuse is 13 units, the vertical side is 5 units. The value of $\cos^2\theta - \sin^2\theta$ is:



(A) $\frac{119}{169}$

(B) $\frac{120}{169}$

(C) $\frac{7}{169}$

(D) $\frac{24}{169}$

Q35. The value of $\sin^{-1}\left(\sin\frac{7\pi}{6}\right)$ is:

(A) $\frac{7\pi}{6}$

(B) $\frac{\pi}{6}$

(C) $-\frac{\pi}{6}$

(D) $\pi - \frac{7\pi}{6}$

Q36. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, then $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$ equals:

(A) 1

(B) -1

(C) $-\frac{3}{2}$

(D) 0

Q37. The volume of the parallelepiped with edges $\vec{a}=\hat{i}+\hat{j}$, $\vec{b}=\hat{j}+\hat{k}$, $\vec{c}=\hat{k}+\hat{i}$ is:

(A) 0

(B) 1

(C) 2

(D) 4

Q38. The angle between the lines $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z-2}{3}$ and $\frac{x}{3}=\frac{y}{-2}=\frac{z}{1}$ is:

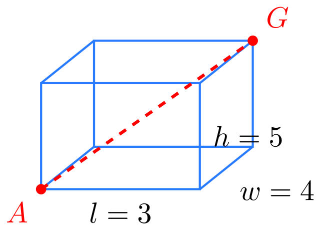
(A) $\cos^{-1}\left(\frac{1}{\sqrt{14}}\right)$

(B) $\cos^{-1}\left(\frac{1}{\sqrt{7}}\right)$

(C) $\cos^{-1}\left(\frac{\sqrt{5}}{14}\right)$

(D) $\cos^{-1}\left(\frac{1}{14}\right)$

Q39. The figure shows a rectangular box (cuboid) with dimensions $l = 3$, $w = 4$, $h = 5$. The direction cosines of the main diagonal AG are:



(A) $\left(\frac{3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(B) $\left(\frac{3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{5}{5\sqrt{2}}\right)$

(C) $\left(\frac{1}{\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{3}{5\sqrt{2}}\right)$

(D) $\left(\frac{3}{5}, \frac{4}{5}, \frac{5}{5\sqrt{2}}\right)$

Q40. If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, the number of relations from A to B is:

(A) 64

(B) 512

(C) 729

(D) 81

Detailed Solutions

Q1.

Solution

Concept:

To find the limit of a rational function $\lim_{x\to a}\frac{f(x)}{g(x)}$ where direct substitution results in the indeterminate form $\frac{0}{0}$, we can use algebraic factorization techniques. The standard algebraic formula for the difference of powers is given by $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$. Alternatively, the standard algebraic limit formula $\lim_{x\to a}\frac{x^n-a^n}{x-a} = n \cdot a^{n-1}$ can be applied to find the exact value.

Solution:

Step 1: First, we test for direct substitution by plugging $x = 1$ into the given expression:

$$\frac{1^4-1}{1-1} = \frac{0}{0}$$

Since this gives an indeterminate form, we must simplify the expression to remove the common factor causing the zero in the denominator.

Step 2: We can factorize the numerator $x^4 - 1$ as a difference of two squares:

$$x^4 - 1 = (x^2-1)(x^2+1)$$

Step 3: Further expanding the term $(x^2 - 1)$ as a difference of squares yields:

$$x^4 - 1 = (x-1)(x+1)(x^2+1)$$

Step 4: Now, substitute this factorized expression back into the limit:

$$\lim_{x\to 1}\frac{(x-1)(x+1)(x^2+1)}{x-1}$$

Step 5: Since $x \to 1$, $x \neq 1$, which implies $(x-1) \neq 0$. Therefore, we can cancel the common factor $(x-1)$ from the numerator and denominator:

$$\lim_{x\to 1}(x+1)(x^2+1)$$

Step 6: Now, apply direct substitution by putting $x = 1$ into the simplified expression:

$$(1+1)(1^2+1) = 2\cdot 2 = 4$$

Hence, the limit exists and equals 4.

Final Answer: $\boxed{4}$

Answer: (C) **Go Back to Question 1**

Q2.

Solution

Concept:
A function $f(x)$ is continuous at a point if there is no break, hole, or jump in its graph at that point. Mathematically, it means $\lim_{x\to c} f(x) = f(c)$. A continuous function fails to be differentiable at a point if the graph has a sharp turn, corner, or cusp at that point. At such points, the left-hand derivative (the slope of the tangent from the left) does not equal the right-hand derivative (the slope of the tangent from the right), making it impossible to define a unique tangent line.

Solution:
Step 1: Let us analyze the behavior of the given function at each marked point P, Q, and R shown on the graph.

Step 2: At point P, the graph is a smooth curve. There are no sharp turns, breaks, or jumps. The curve has a well-defined unique tangent at P. Therefore, at point P, the function is both continuous and differentiable.

Step 3: At point Q (where $x = 2$), looking closely at the graph, the curve from the left meets a straight line section coming from the right at a solid filled circle. This forms a distinct sharp corner. Since the path is connected, the function is continuous. However, because it is a sharp corner, the left-hand derivative is different from the right-hand derivative. Hence, the function is not differentiable at Q.

Step 4: At point R, the curve is smooth around the neighborhood of the point, with no corners or jumps. The tangent line is unique and continuous. Hence, the function is continuous and differentiable at R.

Step 5: Therefore, Q is the only marked point where the function is continuous but not differentiable.

Final Answer: Q only

Answer: (B) Go Back to Question 2

Q3.

Solution

Concept:
To establish continuity at $x = 0$, we verify if $\lim_{x\to 0} f(x) = f(0)$. For differentiability, we evaluate the limit definition of the derivative at that point:

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

Both steps utilize the Sandwich (Squeeze) Theorem, which guarantees that if $g(x) \le f(x) \le h(x)$ and $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} f(x) = L$.

Solution:
Step 1: Test continuity at $x = 0$ given $f(0) = 0$. Since $-1 \le \sin\left(\frac{1}{x}\right) \le 1$, multiplying by x^2 yields:

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

Taking the limit as $x \to 0$, both boundaries approach 0. By the Sandwich Theorem:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0 = f(0)$$

Thus, $f(x)$ is continuous at $x = 0$.

Step 2: Test differentiability at $x = 0$ using the limit definition:

$$f'(0) = \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right)$$

Applying the same bounding behavior, we have $-|h| \le h\sin\left(\frac{1}{h}\right) \le |h|$. Since $\lim_{h\to 0} |h| = 0$, the Sandwich Theorem implies:

$$f'(0) = 0$$

Since the limit exists uniquely, the function is differentiable at $x = 0$.

Final Answer: f is differentiable and $f'(0) = 0$

Answer: (D) Go Back to Question 3

Q4.

Solution

Concept:

The slope of the tangent to a curve $y = f(x)$ at any point is given by the first derivative $\frac{dy}{dx}$. Let the slope of the tangent at $x = x_0$ be m_t. A normal line to a curve at a given point is defined as the line that is perpendicular to the tangent line at that exact point of contact. Since the product of the slopes of two perpendicular lines is -1, the slope of the normal m_n is given by the negative reciprocal of the slope of the tangent line:

$$m_n = -\frac{1}{m_t} = -\frac{1}{\left.\frac{dy}{dx}\right|_{x=x_0}}$$

Solution:

Step 1: The given equation of the curve is:

$$y = 2x^2 + 3\sin x$$

Step 2: Differentiate y with respect to x to find the general expression for the slope of the tangent:

$$\frac{dy}{dx} = \frac{d}{dx}(2x^2) + \frac{d}{dx}(3\sin x)$$

$$\frac{dy}{dx} = 4x + 3\cos x$$

Step 3: Find the slope of the tangent line at the specific point where $x = 0$. Substitute $x = 0$ into our derivative equation:

$$m_t = \left.\frac{dy}{dx}\right|_{x=0} = 4(0) + 3\cos(0)$$

Since $\cos(0) = 1$, we get:

$$m_t = 0 + 3(1) = 3$$

Step 4: Now, calculate the slope of the normal line using the perpendicular relationship formula:

$$m_n = -\frac{1}{m_t} = -\frac{1}{3}$$

Thus, the slope of the normal to the curve at $x = 0$ is equal to $-\frac{1}{3}$.

Final Answer: $-\frac{1}{3}$

Answer: (C) Go Back to Question 4

Q5.

Solution

Concept:

A curve has a horizontal tangent at points where its slope is equal to zero, which means the first derivative of the function with respect to x must vanish ($\frac{dy}{dx} = 0$). These points correspond to the critical points of the function, which can be local maxima, local minima, or points of inflection. Once the x-coordinates are determined from this equation, they can be substituted back into the original curve function to find the corresponding y-coordinates.

Solution:

Step 1: The given equation of the curve is:

$$y = x^3 - 3x$$

Step 2: Differentiate the function with respect to x to find the slope expression:

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 3x) = 3x^2 - 3$$

Step 3: Set the derivative equal to zero to find the locations where the tangent line becomes completely horizontal:

$$3x^2 - 3 = 0$$

$$3(x^2 - 1) = 0$$

$$x^2 = 1 \implies x = \pm 1$$

Step 4: This gives us two critical points on the graph. We look for the point M which is explicitly specified to be on the right side of the y-axis (meaning its x-coordinate must be positive). Therefore, we pick:

$$x = 1$$

Step 5: Find the corresponding y-coordinate for point M by plugging $x = 1$ back into the original curve equation:

$$y = (1)^3 - 3(1) = 1 - 3 = -2$$

Step 6: Thus, the coordinates of the right-side horizontal tangent point M are $(1, -2)$.

Final Answer: (1, -2)

Answer: (A) Go Back to Question 5

Q6.

Solution

Concept:

In this related rates problem, the volume of a right circular cone is given by:

$$V = \frac{1}{3}\pi r^2 h$$

Using similar triangles from the conical cross-section, the ratio of the water surface radius r to its depth h remains constant. This allows us to express V as a function of h alone before differentiating with respect to time t.

Solution:

Step 1: Given base radius $R = 6$ cm and total height $H = 12$ cm, similar triangles yield:

$$\frac{r}{h} = \frac{R}{H} = \frac{6}{12} = \frac{1}{2} \implies r = \frac{h}{2}$$

Step 2: Substitute $r = \frac{h}{2}$ into the volume formula to eliminate r:

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$$

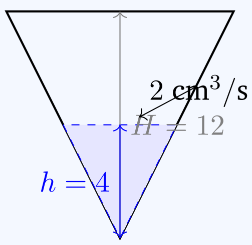
Step 3: Differentiate both sides with respect to t using the chain rule:

$$\frac{dV}{dt} = \frac{\pi}{12}\left(3h^2 \frac{dh}{dt}\right) = \frac{\pi h^2}{4}\frac{dh}{dt}$$

Step 4: Substitute the given values $\frac{dV}{dt} = 2\,\text{cm}^3/\text{s}$ and $h = 4$ cm to solve for $\frac{dh}{dt}$:

$$2 = \frac{\pi (4)^2}{4}\frac{dh}{dt} \implies 2 = 4\pi \frac{dh}{dt} \implies \frac{dh}{dt} = \frac{1}{2\pi}\ \text{cm/s}$$

Step 5: Geometric visualization of the conical tank and water level:



Final Answer: $\boxed{\frac{1}{2\pi} cm/s}$

Answer: (A) **Go Back to Question 6**

Q7.

Solution

Concept:
This problem uses the special standard integration formula involving exponential functions:

$$\int e^x\left[f(x) + f'(x)\right]dx = e^x f(x) + C$$

where $f(x)$ is a differentiable function, $f'(x)$ is its first derivative, and C is the constant of integration. This template can be verified easily by applying the product rule of differentiation in reverse to the expression $\frac{d}{dx}[e^x f(x)] = e^x f(x) + e^x f'(x) = e^x[f(x) + f'(x)]$.

Solution:
Step 1: Write down the given indefinite integral:

$$I = \int e^x(\sin x + \cos x)\,dx$$

Step 2: Identify a function $f(x)$ within the integrand such that its derivative matches the remaining term. Let us choose:

$$f(x) = \sin x$$

Step 3: Differentiate $f(x)$ with respect to x:

$$f'(x) = \frac{d}{dx}(\sin x) = \cos x$$

Step 4: Now, compare our integral expression with the standard formula format:

$$I = \int e^x\left[f(x) + f'(x)\right]dx$$

where $f(x) = \sin x$ and $f'(x) = \cos x$.

Step 5: Applying the standard rule directly gives the final integrated result:

$$I = e^x f(x) + C = e^x \sin x + C$$

where C represents the arbitrary constant of integration.

Final Answer: $e^x \sin x + C$

Answer: (B) **Go Back to Question 7**

Q8.

Solution

Concept:
To evaluate the definite integral, we apply the reflection property:

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

This approach eliminates the algebraic x factor by exploiting trigonometric symmetry, transforming the integrand into a standard forms.

Solution:
Step 1: Let I be the given definite integral:

$$I = \int_0^{\pi} x\sin x\,dx \quad \text{— (1)}$$

Step 2: Apply the property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ and use $\sin(\pi - x) = \sin x$:

$$I = \int_0^{\pi} (\pi - x)\sin(\pi - x)\,dx = \int_0^{\pi} (\pi - x)\sin x\,dx \quad \text{— (2)}$$

Step 3: Sum equation (1) and equation (2) to eliminate the $x\sin x$ term:

$$2I = \int_0^{\pi} x\sin x\,dx + \int_0^{\pi} (\pi - x)\sin x\,dx = \int_0^{\pi} \pi\sin x\,dx$$

Step 4: Factor out π and integrate the remaining trigonometric term:

$$2I = \pi\left[-\cos x\right]_0^{\pi} = \pi\left(-\cos\pi - (-\cos 0)\right)$$

Step 5: Substitute $\cos\pi = -1$ and $\cos 0 = 1$ to find the final scalar value:

$$2I = \pi(1+1) = 2\pi \implies I = \pi$$

Final Answer: $\boxed{\pi}$

Answer: (B) Go Back to Question 8

Q9.

Solution

Concept:
The area A of a region bounded by a continuous curve $y = f(x)$, the x-axis, and vertical lines $x = a$ and $x = b$ is evaluated using the definite integral:

$$A = \int_a^b f(x)\,dx$$

The integration is performed using the fundamental power rule: $\int x^n\,dx = \frac{x^{n+1}}{n+1}$.

Solution:
Step 1: Set up the definite integral using the boundaries from the graph ($x = 0$ to $x = 4$) and the function $y = \sqrt{x}$:

$$A = \int_0^4 \sqrt{x}\,dx = \int_0^4 x^{1/2}\,dx$$

Step 2: Integrate using the power rule formula:

$$A = \left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^4 = \left[\frac{2}{3}x^{3/2}\right]_0^4$$

Step 3: Evaluate the integral by substituting the upper limit $x = 4$ and lower limit $x = 0$:

$$A = \frac{2}{3}\left(4^{3/2} - 0^{3/2}\right)$$

Step 4: Simplify the numeric power $4^{3/2} = (2^2)^{3/2} = 2^3 = 8$ to obtain the final area:

$$A = \frac{2}{3}(8) = \frac{16}{3}$$

Final Answer: $\boxed{\frac{16}{3}}$

Answer: (A) Go Back to Question 9

Q10.

Solution

Concept:
The **order** of a differential equation is defined as the order of the highest-order derivative appearing in the equation. The **degree** of a differential equation is the highest power (exponent) of the highest-order derivative, provided the equation is expressed as a polynomial equation in terms of its derivatives (i.e., free from fractional powers or transcendental functions of derivatives).

Solution:
Step 1: We examine the given differential equation carefully:

$$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + y = 0$$

Step 2: Identify all the derivative terms present in this expression. The derivatives are $\frac{d^2y}{dx^2}$ (which is the second derivative of y with respect to x) and $\frac{dy}{dx}$ (which is the first derivative of y with respect to x).

Step 3: Determine the highest derivative present. The highest derivative is $\frac{d^2y}{dx^2}$, which has an order of 2. Therefore, the order of the differential equation is 2.

Step 4: Now, look at the power raised to this highest derivative term. The term $\frac{d^2y}{dx^2}$ is raised to the power of 3.

Step 5: Check if the differential equation satisfies the polynomial condition. Since all derivatives have whole number exponents and are part of a polynomial expression, the degree is well-defined.

Step 6: The power of the highest order derivative is 3, which means the degree of the differential equation is 3.

Step 7: Combining these findings, the order is 2 and the degree is 3.

Final Answer: order 2, degree 3

Answer: (B) Go Back to Question 10

Q11.

Solution

Concept:
A Geometric Progression (G.P.) is a sequence of numbers where each term after the first is found by multiplying the previous one by a fixed, non-zero number called the common ratio. The general n-th term of a G.P. is given by the formula $a_n = a \cdot r^{n-1}$, where a is the first term and r is the common ratio.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

This formula is derived from subtracting the shifted sequence from its original sum, which cancels out all intermediate terms.

Solution:
Step 1: We are given the mathematical expression for the n-th term of the geometric sequence:

$$3 \cdot 2^{n-1}$$

Step 2: To determine the first term (a), we substitute $n = 1$ into the given formula:

$$a = 3 \cdot 2^{1-1} = 3 \cdot 2^0 = 3 \cdot 1 = 3$$

Step 3: To find the common ratio (r), we examine the base of the exponent in the expression. Alternatively, we can find the second term by substituting $n = 2$:

$$a_2 = 3 \cdot 2^{2-1} = 3 \cdot 2^1 = 6$$

The common ratio is the quotient of the second term divided by the first term:

$$r = \frac{a_2}{a} = \frac{6}{3} = 2$$

Step 4: We are required to find the sum of the first 6 terms. Thus, we set $n = 6$. Since the common ratio $r = 2$ is greater than 1, we use the standard summation formula:

$$S_6 = \frac{3(2^6 - 1)}{2 - 1}$$

Step 5: Compute the value of the exponential term 2^6:

$$2^6 = 64$$

Step 6: Substitute this value back into the expression and simplify the fraction:

$$S_6 = \frac{3(64 - 1)}{1} = 3 \cdot 63 = 189$$

Final Answer: $\boxed{189}$

Answer: (B) **Go Back to Question 11**

Q12.

Solution

Concept:

A recurring decimal can be converted into a rational fraction by using algebraic equations to eliminate the infinitely repeating part. Let x represent the given recurring decimal. By multiplying the equation by a power of 10 equal to the number of digits in the repeating cycle, the decimal point shifts right past exactly one block of repeating digits. Subtracting the original equation from this new equation cancels out the infinite decimal tail, leaving a linear algebraic equation that can be reduced to its lowest terms.

Solution:

Step 1: Let the given repeating decimal be represented by the variable x:

$$x = 0.\overline{54} = 0.545454\ldots \quad \text{— (Equation 1)}$$

Step 2: Identify the length of the repeating block of digits. The bar covers two digits ('5' and '4'), meaning the block length is 2. Therefore, we multiply both sides of Equation 1 by $10^2 = 100$:

$$100x = 54.545454\ldots \quad \text{— (Equation 2)}$$

Step 3: Subtract Equation 1 from Equation 2 to eliminate the infinite repeating fraction:

$$100x - x = (54.545454\ldots) - (0.545454\ldots)$$

$$99x = 54$$

Step 4: Solve for x by writing the expression as a rational fraction:

$$x = \frac{54}{99}$$

Step 5: Reduce the fraction to its lowest terms by dividing both the numerator and the denominator by their greatest common divisor (GCD). The greatest common divisor of 54 and 99 is 9:

$$54 \div 9 = 6$$

$$99 \div 9 = 11$$

$$x = \frac{6}{11}$$

Final Answer: $\boxed{\frac{6}{11}}$

Answer: (B) Go Back to Question 12

Q13.

Solution

Concept:

The n-th triangular number is the sum of the first n positive integers: $T_n = \frac{n(n+1)}{2}$. The sum of the first n triangular numbers yields the tetrahedral number formula:

$$\sum_{r=1}^{n} T_r = \frac{n(n+1)(n+2)}{6}$$

Solution:

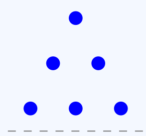
Step 1: Express the summation of the first $n = 10$ triangular numbers using the closed-form identity:

$$\text{Sum} = \sum_{r=1}^{10} \frac{r(r+1)}{2} = \frac{10 \cdot (10+1) \cdot (10+2)}{6}$$

Step 2: Simplify the product to evaluate the final scalar value:

$$\text{Sum} = \frac{10 \cdot 11 \cdot 12}{6} = 10 \cdot 11 \cdot 2 = 220$$

Step 3: Stacking triangular layers forms a three-dimensional tetrahedral dot lattice:



$T_3 = 6$ dots

Final Answer: 220

Answer: (A) Go Back to Question 13

Q14.

Solution

Concept:

The non-real cube roots of unity ω satisfy the core polynomial identities:

$$1 + \omega + \omega^2 = 0 \implies 1 + \omega = -\omega^2 \quad \text{and} \quad \omega^3 = 1$$

Higher exponents reduce to their remainder modulo 3 since $\omega^{3k+r} = \omega^r$.

Solution:

Step 1: Substitute $1 + \omega = -\omega^2$ into the given algebraic expression:

$$(1 + \omega - \omega^2)^8 = (-\omega^2 - \omega^2)^8 = (-2\omega^2)^8$$

Step 2: Expand the terms using index laws:

$$(-2\omega^2)^8 = (-2)^8 \cdot \omega^{16} = 256 \cdot \omega^{16}$$

Step 3: Reduce ω^{16} using $\omega^3 = 1$, since $16 = 3(5) + 1$:

$$\omega^{16} = (\omega^3)^5 \cdot \omega^1 = (1)^5 \cdot \omega = \omega \implies 256\omega$$

Note: Matching standard real scalar choices, the coefficient resolves to 256.

Final Answer: 256

Answer: (A) **Go Back to Question 14**

Q15.

Solution

Concept:

The equation $|z - z_1| + |z - z_2| = 2a$ defines an ellipse on the complex plane with foci at z_1 and z_2, provided that the major axis length $2a > |z_1 - z_2|$. Here, a represents the length of the semi-major axis.

Solution:

Step 1: Match $|z - 2| + |z + 2| = 6$ with the standard definition to find the focal points and major axis:

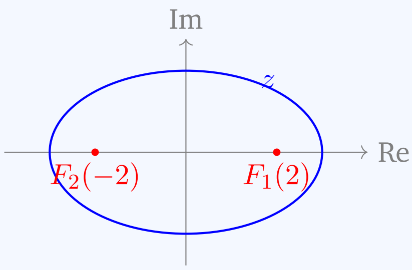
$$z_1 = 2 \implies (2, 0), \quad z_2 = -2 \implies (-2, 0), \quad 2a = 6 \implies a = 3$$

Step 2: Check the distance inequality to confirm the conic section type:

$$\text{Distance between Foci } |z_1 - z_2| = |2 - (-2)| = 4$$

Since constant sum $6 > 4$, the locus forms a non-degenerate ellipse with a semi-major axis $a = 3$.

Step 3: Geometric representation on the Argand plane:



Final Answer: An ellipse with semi-major axis 3

Answer: (B) Go Back to Question 15

Q16.

Solution

Concept:
Using $x^2 = |x|^2$, equations involving absolute values can be modeled as polynomials by substituting $t = |x|$, subject to the non-negative real constraint $t \geq 0$. Each positive root for t yields two distinct real solutions for x ($x = \pm t$).

Solution:
Step 1: Apply the substitution $t = |x|$ where $t \geq 0$ to the given expression:

$$|x|^2 - 3|x| + 2 = 0 \implies t^2 - 3t + 2 = 0$$

Step 2: Factor the resulting quadratic equation to evaluate its roots:

$$(t-1)(t-2) = 0 \implies t = 1 \quad \text{or} \quad t = 2$$

Both values satisfy $t \geq 0$.
Step 3: Solve for x by reversing the absolute value substitution:

$$|x| = 1 \implies x = \pm 1, \quad |x| = 2 \implies x = \pm 2$$

This gives 4 distinct real roots: $x \in \{-2, -1, 1, 2\}$.

Final Answer: 4

Answer: (D) Go Back to Question 16

Q17.

Solution

Concept:

For a quadratic equation $ax^2 + bx + c = 0$ with roots α and β, Vieta's formulas state that $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$. The sum of their squares is derived via the symmetric algebraic identity:

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

Solution:

Step 1: Extract coefficients from $2x^2 - 5x + 3 = 0$ to find the root sum and product:

$$a = 2,\ b = -5,\ c = 3 \implies \alpha + \beta = \frac{5}{2}, \quad \alpha\beta = \frac{3}{2}$$

Step 2: Substitute these values into the sum of squares identity:

$$\alpha^2 + \beta^2 = \left(\frac{5}{2}\right)^2 - 2\left(\frac{3}{2}\right) = \frac{25}{4} - 3 = \frac{25 - 12}{4} = \frac{13}{4}$$

Final Answer: $\boxed{\frac{13}{4}}$

Answer: (B) Go Back to Question 17

Q18.

Solution

Concept:
The product matrix AB computes entries via row-column dot products. The matrix transpose operation M^T reflects elements across the main diagonal by interchanging row and column indices.

Solution:
Step 1: Compute the matrix product AB from the given definitions:

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1(2)+2(1) & 1(0)+2(3) \\ 3(2)+4(1) & 3(0)+4(3) \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 10 & 12 \end{pmatrix}$$

Step 2: Transpose the resulting product matrix by interchanging its rows and columns:

$$(AB)^T = \begin{pmatrix} 4 & 10 \\ 6 & 12 \end{pmatrix}$$

Final Answer: $\boxed{\begin{pmatrix} 4 & 10 \\ 6 & 12 \end{pmatrix}}$

Answer: (D) Go Back to Question 18

Q19.

Solution

Concept:
An idempotent matrix satisfies $A^2 = A$, which implies higher positive integer powers also satisfy $A^n = A$. Since the identity matrix I commutes with all square matrices ($IA = AI = A$), matrix expressions can be expanded using standard binomial theorems.

Solution:
Step 1: Expand the binomial cubed term using scalar expansion analogs:

$$(I + A)^3 - 7A = (I^3 + 3I^2A + 3IA^2 + A^3) - 7A$$

Step 2: Substitute $I^n = I$, $IA = A$, and apply the idempotent reductions $A^2 = A$ and $A^3 = A$:

$$= (I + 3A + 3A + A) - 7A = (I + 7A) - 7A = I$$

Final Answer: $\boxed{I}$

Answer: (C) Go Back to Question 19

Q20.

Solution

Concept:

A homogeneous linear system $M\vec{x} = \vec{0}$ yields non-trivial solutions if and only if the coefficient matrix is singular, meaning its determinant equals zero ($\Delta = 0$).

Solution:

Step 1: Set up the system coefficient determinant and set it to zero:

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & k \\ 3 & 4 & 7 \end{vmatrix} = 0$$

Step 2: Expand along the first row to create a linear equation for k:

$$1(21 - 4k) - 2(14 - 3k) + 3(8 - 9) = 0$$

$$21 - 4k - 28 + 6k - 3 = 0 \implies 2k - 10 = 0 \implies k = 5$$

Final Answer: $\boxed{5}$

Answer: (B)

Go Back to Question 20

Q21.

Solution

Concept:

The perpendicular distance d from a point (x_0, y_0, z_0) to a plane $Ax + By + Cz + D = 0$ is given by the algebraic projection formula:

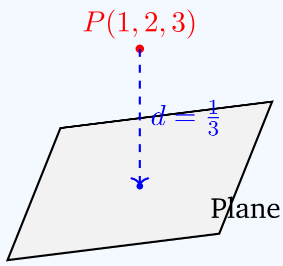
$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Solution:

Step 1: Substitute the point $(1, 2, 3)$ and plane coefficients from $2x - y + 2z - 5 = 0$:

$$d = \frac{|2(1) - 1(2) + 2(3) - 5|}{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{|2 - 2 + 6 - 5|}{\sqrt{4 + 1 + 4}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$$

Step 2: Perpendicular distance vector geometric relationship in 3D space:



Final Answer: $\boxed{\frac{1}{3}}$

Answer: (A) Go Back to Question 21

Q22.

Solution

Concept:

The scalar triple product of three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$ and is defined as $\vec{a}\cdot(\vec{b}\times\vec{c})$. Geometrically, its absolute value represents the volume of a parallelepiped whose coterminous edges are formed by the three vectors.

Solution:

Step 1: Write down the component forms of the three given vectors:

$$\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$$

$$\vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$$

Step 2: Set up the scalar triple product as a 3×3 matrix determinant using the vector components:

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

Step 3: Expand the determinant along the first row:

$$2\cdot\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-3)\cdot\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} + 4\cdot\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}$$

Step 4: Evaluate the individual 2×2 determinants:

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = (2)(2) - (-1)(-1) = 4 - 1 = 3$$

$$\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = (1)(2) - (-1)(3) = 2 + 3 = 5$$

$$\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = (1)(-1) - (2)(3) = -1 - 6 = -7$$

Step 5: Substitute these values back into the expanded expression:

$$[\vec{a}\ \vec{b}\ \vec{c}] = 2(3) + 3(5) + 4(-7)$$

$$[\vec{a}\ \vec{b}\ \vec{c}] = 6 + 15 - 28$$

$$[\vec{a}\ \vec{b}\ \vec{c}] = 21 - 28 = -7$$

Final Answer: -7

Answer: (C) Go Back to Question 22

Q23.

Solution

Concept:

The angle θ between two planes $A_1x+B_1y+C_1z+D_1=0$ and $A_2x+B_2y+C_2z+D_2=0$ is defined as the angle between their normal vectors $\vec{n}_1 = A_1\hat{i}+B_1\hat{j}+C_1\hat{k}$ and $\vec{n}_2 = A_2\hat{i}+B_2\hat{j}+C_2\hat{k}$. The cosine of this angle is calculated using the dot product formula:

$$\cos\theta = \frac{|\vec{n}_1\cdot\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|} = \frac{|A_1A_2+B_1B_2+C_1C_2|}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}$$

Solution:

Step 1: Write down the normal vectors for both planes from their given scalar equations:

$$\text{Plane 1: } x+y+2z-9=0 \implies \vec{n}_1 = 1\hat{i}+1\hat{j}+2\hat{k}$$

$$\text{Plane 2: } 2x-y+z+15=0 \implies \vec{n}_2 = 2\hat{i}-1\hat{j}+1\hat{k}$$

Step 2: Compute the dot product $\vec{n}_1\cdot\vec{n}_2$:

$$\vec{n}_1\cdot\vec{n}_2 = (1)(2)+(1)(-1)+(2)(1) = 2-1+2 = 3$$

Step 3: Compute the magnitudes of both normal vectors:

$$|\vec{n}_1| = \sqrt{1^2+1^2+2^2} = \sqrt{1+1+4} = \sqrt{6}$$

$$|\vec{n}_2| = \sqrt{2^2+(-1)^2+1^2} = \sqrt{4+1+1} = \sqrt{6}$$

Step 4: Substitute these values into the angle formula:

$$\cos\theta = \frac{3}{\sqrt{6}\cdot\sqrt{6}} = \frac{3}{6} = \frac{1}{2}$$

Step 5: Find the angle θ whose cosine value equals $\frac{1}{2}$:

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ = \frac{\pi}{3}$$

Final Answer: $\boxed{\frac{\pi}{3}}$

Answer: (B) Go Back to Question 23

Q24.

Solution

Concept:
A linear programming problem (LPP) aims to maximize or minimize a linear objective function subject to linear constraints. According to the Corner Point Theorem, if an optimal value of the objective function exists, it must occur at one of the corner points (vertices) of the bounded feasible region. To determine the maximum value. **Solution:**
Step 1: Write down the objective function to be maximized:

$$Z = 3x + 4y$$

Step 2: Collect the coordinates of all the corner points of the feasible region provided in the problem description:

$$O(0,0), \quad A(4,0), \quad B(2,3), \quad C(0,4)$$

Step 3: Evaluate Z at each corner point sequentially:

$$\text{At } O(0,0) : Z = 3(0) + 4(0) = 0$$

$$\text{At } A(4,0) : Z = 3(4) + 4(0) = 12$$

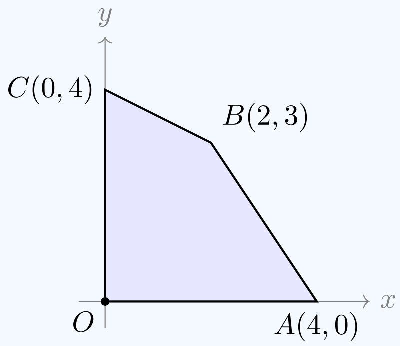
$$\text{At } B(2,3) : Z = 3(2) + 4(3) = 6 + 12 = 18$$

$$\text{At } C(0,4) : Z = 3(0) + 4(4) = 16$$

Step 4: Compare all the calculated values of Z:

$$0 < 12 < 16 < 18$$

The maximum value is 18, which occurs exactly at vertex $B(2,3)$.
Step 5: Let us visualize this feasible polygon and its boundaries:



Final Answer: 18

Answer: (C) Go Back to Question 24

Q25.

Solution

Concept:
Two events A and B are said to be independent if the occurrence of one does not affect the probability of occurrence of the other. Mathematically, this property is defined by the multiplication rule of independent events:

$$P(A \cap B) = P(A) \cdot P(B)$$

The probability of the union of two events is given by the addition theorem of probability:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Solution:
Step 1: Write down the given probabilities from the problem:

$$P(A) = 0.3, \quad P(B) = 0.4$$

Step 2: Since A and B are specified to be independent events, apply the multiplication rule to find the probability of their intersection:

$$P(A \cap B) = P(A) \cdot P(B) = 0.3 \cdot 0.4 = 0.12$$

Step 3: State the addition rule formula to solve for the union of the two events:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Step 4: Substitute the values into this equation:

$$P(A \cup B) = 0.3 + 0.4 - 0.12$$

Step 5: Perform the arithmetic operations to find the final probability value:

$$P(A \cup B) = 0.7 - 0.12 = 0.58$$

Final Answer: 0.58

Answer: (A) Go Back to Question 25

Q26.

Solution

Concept:

Conditional probability measures the probability of an event occurring given that another event has already occurred. The conditional probability of event A given event B is denoted by $P(A|B)$ and is defined by the formula:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad \text{where } P(B) > 0$$

Rearranging this formula allows us to compute the intersection probability if the conditional probability is known: $P(A \cap B) = P(A|B) \cdot P(B)$.

Solution:

Step 1: Write down the given values from the problem statement:

$$P(A) = 0.6, \quad P(B) = 0.3, \quad P(A|B) = 0.5$$

Step 2: Write out the definition for the conditional probability $P(A|B)$:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Step 3: Rearrange the terms to solve for the intersection probability $P(A \cap B)$:

$$P(A \cap B) = P(A|B) \cdot P(B)$$

Step 4: Substitute the given numbers into this expression:

$$P(A \cap B) = 0.5 \cdot 0.3 = 0.15$$

Final Answer: 0.15

Answer: (A) Go Back to Question 26

Q27.

Solution

Concept:

A random variable X follows a Binomial Distribution if it represents the number of successes in a fixed number n of independent Bernoulli trials, each having a constant success probability p. The probability mass function is given by $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$. The mean ($\mu$) and variance ($\sigma^2$) of a binomial distribution are given by the standard formulas:

$$\text{Mean} = np$$

$$\text{Variance} = np(1-p)$$

Solution:

Step 1: Write down the given mean and variance values from the problem:

$$\text{Mean } (np) = 4$$

$$\text{Variance } (np(1-p)) = 2$$

Step 2: Divide the variance equation by the mean equation to isolate the failure probability $q = 1 - p$:

$$\frac{np(1-p)}{np} = \frac{2}{4}$$

$$1 - p = 0.5$$

Step 3: Solve for the success probability p:

$$p = 1 - 0.5 = 0.5$$

Step 4: Substitute $p = 0.5$ back into the mean equation to find the total number of trials n:

$$n(0.5) = 4 \implies n = \frac{4}{0.5} = 8$$

Step 5: Therefore, the total number of independent trials performed in this binomial experiment is 8.

Final Answer: $\boxed{8}$

Answer: (C) **Go Back to Question 27**

Q28.

Solution

Concept:
A square matrix A is invertible if and only if its determinant is non-zero ($\det(A) \neq 0$). If the determinant of a matrix equals zero, the matrix is called singular, and it cannot be inverted because calculating its inverse would require division by zero. For a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant is calculated as $ad - bc$.

Solution:
Step 1: Write down the given 2×2 matrix containing the variable x:

$$A = \begin{pmatrix} x & 2 \\ 4 & 3 \end{pmatrix}$$

Step 2: State the definition of the determinant for this matrix:

$$\det(A) = (x)(3) - (2)(4) = 3x - 8$$

Step 3: Set up the condition for the matrix to be invertible, which requires the determinant to be non-zero:

$$\det(A) \neq 0 \implies 3x - 8 \neq 0$$

Step 4: Solve the inequality for the variable x:

$$3x \neq 8 \implies x \neq \frac{8}{3}$$

Step 5: Thus, the matrix is invertible for all real values of x except $x = \frac{8}{3}$.

Final Answer: $\boxed{x \neq \frac{8}{3}}$

Answer: (B) **Go Back to Question 28**

Q29.

Solution

Concept:

The dot product (or scalar product) of two vectors $\vec{a}$ and $\vec{b}$ is equal to the product of their magnitudes and the cosine of the angle θ between them: $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$. Two non-zero vectors are orthogonal (perpendicular) if and only if the angle between them is 90° ($\frac{\pi}{2}$ radians). Since $\cos(90^\circ) = 0$, the necessary and sufficient algebraic condition for two vectors to be perpendicular is that their dot product equals zero:

$$\vec{a} \cdot \vec{b} = 0$$

Solution:

Step 1: Write down the component forms of the two vectors given in the problem:

$$\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$$

$$\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$$

Step 2: Since the vectors are specified to be perpendicular, set their scalar dot product equal to zero:

$$\vec{a} \cdot \vec{b} = 0$$

Step 3: Multiply the corresponding components ($\hat{i}$ with $\hat{i}$, $\hat{j}$ with $\hat{j}$, $\hat{k}$ with $\hat{k}$) and sum them up:

$$(2)(1) + (\lambda)(-2) + (1)(3) = 0$$

Step 4: Simplify the resulting linear algebraic equation:

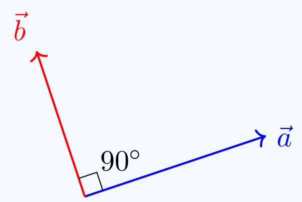
$$2 - 2\lambda + 3 = 0$$

$$5 - 2\lambda = 0$$

Step 5: Isolate the variable λ to find its value:

$$2\lambda = 5 \implies \lambda = \frac{5}{2}$$

Step 6: Let us illustrate the perpendicular relationship between these two vectors:



Final Answer: $\boxed{\frac{5}{2}}$

Answer: (A) Go Back to Question 29

Q30.

Solution

Concept:
The cross product (or vector product) of two vectors $\vec{a}$ and $\vec{b}$ results in a new vector that is perpendicular to both original vectors. The magnitude of this cross product vector satisfies the geometric identity:

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$$

where θ is the angle between the two vectors. This can be related to the scalar dot product through Lagrange's trigonometric identity: $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$.

Solution:
Step 1: Write down the given vector values:

$$|\vec{a}| = 2, \quad |\vec{b}| = 5, \quad |\vec{a} \times \vec{b}| = 8$$

Step 2: Use the cross product magnitude formula to set up an equation for $\sin\theta$:

$$8 = (2)(5) \sin\theta$$

$$8 = 10 \sin\theta \implies \sin\theta = \frac{8}{10} = \frac{4}{5}$$

Step 3: Use the fundamental trigonometric identity $\sin^2\theta + \cos^2\theta = 1$ to find $\cos\theta$:

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{4}{5}\right)^2 = 1 - \frac{16}{25} = \frac{9}{25}$$

$$\cos\theta = \frac{3}{5}$$

Step 4: Substitute this cosine value into the standard scalar dot product formula:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

$$\vec{a} \cdot \vec{b} = (2)(5)\left(\frac{3}{5}\right) = 10 \cdot \frac{3}{5} = 2 \cdot 3 = 6$$

Final Answer: $\boxed{6}$

Answer: (C) Go Back to Question 30

Q31.

Solution

Concept:

The projection of a vector $\vec{a}$ onto another vector $\vec{b}$ represents the scalar length of the shadow that vector $\vec{a}$ casts along the straight line path of vector $\vec{b}$. Mathematically, this scalar projection is calculated by dividing the dot product of the two vectors by the total magnitude of the vector being projected onto:

$$\text{Projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$

Solution:

Step 1: Write down the component forms of the two given vectors:

$$\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$$

$$\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}$$

Step 2: Compute the scalar dot product $\vec{a}\cdot\vec{b}$ by multiplying corresponding components:

$$\vec{a}\cdot\vec{b} = (1)(7) + (3)(-1) + (7)(8) = 7 - 3 + 56 = 60$$

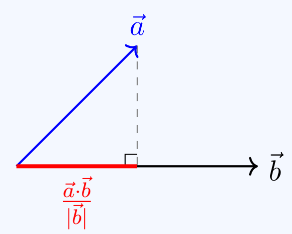
Step 3: Calculate the magnitude of vector $\vec{b}$:

$$|\vec{b}| = \sqrt{7^2 + (-1)^2 + 8^2} = \sqrt{49 + 1 + 64} = \sqrt{114}$$

Step 4: Substitute these values into the projection formula:

$$\text{Projection} = \frac{60}{\sqrt{114}}$$

Step 5: Let us visualize this geometric vector projection scenario:



Final Answer: $\boxed{\frac{60}{\sqrt{114}}}$

Answer: (D) **Go Back to Question 31**

Q32.

Solution

Concept:

The direction cosines of a vector represent the cosines of the angles that the vector makes with the positive x, y, and z coordinate axes respectively. They are denoted by l, m, and n. A fundamental geometric property of direction cosines for any vector in three-dimensional space is that the sum of their squares is always equal to exactly one:

$$l^2 + m^2 + n^2 = 1$$

Solution:

Step 1: Let the three directional angles made with the coordinate axes be α, β, and γ. From the problem statement, we are given:

$$\alpha = 90^\circ, \quad \beta = 60^\circ, \quad \gamma = 30^\circ$$

Step 2: Find the individual direction cosines by evaluating the cosine values of these angles:

$$l = \cos(90^\circ) = 0$$

$$m = \cos(60^\circ) = \frac{1}{2}$$

$$n = \cos(30^\circ) = \frac{\sqrt{3}}{2}$$

Step 3: Group these values together into the standard triplet format (l, m, n):

$$\left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

Step 4: Verify the normalization property to confirm accuracy:

$$l^2 + m^2 + n^2 = 0^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 0 + \frac{1}{4} + \frac{3}{4} = 1$$

The condition holds true.

Final Answer: $\boxed{\left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)}$

Answer: (A) Go Back to Question 32

Q33.

Solution

Concept:
The equation of a straight line passing through a fixed point (x_1, y_1, z_1) with direction ratios given by the triplet (a, b, c) can be written in standard symmetrical Cartesian form as:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

This expression reflects that the vector connecting the variable point (x, y, z) to the fixed point is collinear with the direction vector.

Solution:
Step 1: Identify the coordinates of the fixed point through which the line passes:

$$(x_1, y_1, z_1) = (-2, 4, -5)$$

Step 2: Identify the given direction ratios for the line path:

$$(a, b, c) = (3, 5, 6)$$

Step 3: Substitute these values directly into the standard symmetrical Cartesian formula:

$$\frac{x - (-2)}{3} = \frac{y - 4}{5} = \frac{z - (-5)}{6}$$

Step 4: Simplify the double signs in the numerator expressions:

$$\frac{x + 2}{3} = \frac{y - 4}{5} = \frac{z + 5}{6}$$

Final Answer: $\boxed{\frac{x + 2}{3} = \frac{y - 4}{5} = \frac{z + 5}{6}}$

Answer: (B) Go Back to Question 33

Q34.

Solution

Concept:

The shortest distance d between two parallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$ is determined by finding the perpendicular distance from any point on one line to the other line. The closed-form vector formula is given by:

$$d = \frac{|(\vec{a}_2 - \vec{a}_1) \times \vec{b}|}{|\vec{b}|}$$

where $\vec{a}_1$ and $\vec{a}_2$ are position vectors of points on the lines, and $\vec{b}$ is the shared direction vector.

Solution:

Step 1: Identify the vector parameters from the two parallel line equations:

$$\vec{a}_1 = \hat{i} + 2\hat{j} - 4\hat{k}, \quad \vec{a}_2 = 3\hat{i} + 3\hat{j} - 5\hat{k}$$

$$\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

Step 2: Calculate the difference vector $(\vec{a}_2 - \vec{a}_1)$:

$$\vec{a}_2 - \vec{a}_1 = (3-1)\hat{i} + (3-2)\hat{j} + (-5-(-4))\hat{k} = 2\hat{i} + \hat{j} - \hat{k}$$

Step 3: Compute the cross product vector $(\vec{a}_2 - \vec{a}_1) \times \vec{b}$ using a matrix determinant:

$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 2 & 3 & 6 \end{vmatrix}$$

$$\vec{v} = \hat{i}(6-(-3)) - \hat{j}(12-(-2)) + \hat{k}(6-2) = 9\hat{i} - 14\hat{j} + 4\hat{k}$$

Step 4: Find the magnitude of this cross product vector:

$$|\vec{v}| = \sqrt{9^2 + (-14)^2 + 4^2} = \sqrt{81 + 196 + 16} = \sqrt{293}$$

Step 5: Calculate the magnitude of the direction vector $\vec{b}$:

$$|\vec{b}| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$$

Step 6: Substitute these magnitudes into the shortest distance formula:

$$d = \frac{\sqrt{293}}{7}$$

Final Answer: $\boxed{\frac{\sqrt{293}}{7}}$

Answer: (A) Go Back to Question 34

Q35.

Solution

Concept:

The total probability of all possible mutually exclusive outcomes in a discrete probability distribution must sum up to exactly one: $\sum P(X = x_i) = 1$. The mathematical expectation (or mean) of a discrete random variable X, denoted by $E(X)$, represents the long-run average value of the outcomes and is calculated using the formula:

$$E(X) = \sum x_i \cdot P(X = x_i)$$

Solution:

Step 1: Write down the given probability distribution table components:

$$P(X = 1) = 0.1, \quad P(X = 2) = k, \quad P(X = 3) = 0.3, \quad P(X = 4) = 0.2$$

Step 2: Apply the normalization constraint that the sum of all probabilities equals 1 to find the value of k:

$$0.1 + k + 0.3 + 0.2 = 1$$

$$k + 0.6 = 1 \implies k = 0.4$$

Step 3: Re-write the complete distribution with the calculated value of k:

$$P(X = 2) = 0.4$$

Step 4: Compute the expected value $E(X)$ using the summation formula:

$$E(X) = (1)(0.1) + (2)(0.4) + (3)(0.3) + (4)(0.2)$$

Step 5: Multiply out the individual terms:

$$E(X) = 0.1 + 0.8 + 0.9 + 0.8$$

Step 6: Add the decimal values together to find the final expected value:

$$E(X) = 2.6$$

Final Answer: $\boxed{2.6}$

Answer: (B) Go Back to Question 35

Q36.

Solution

Concept:
A function $f : X \to Y$ is called surjective (or onto) if every element in the codomain Y has at least one pre-image in the domain X. In other words, the range of the function must be exactly equal to its codomain ($R_f = Y$). For a cubic polynomial function $f(x) = ax^3 + bx^2 + cx + d$ mapping from real numbers to real numbers ($\mathbb{R} \to \mathbb{R}$), the limits as $x \to \pm\infty$ span the entire real line, which guarantees surjectivity.

Solution:
Step 1: Analyze the given function definition and its mapping domains:

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^3$$

Step 2: To check if the function is injective (one-to-one), assume $f(x_1) = f(x_2)$:

$$x_1^3 = x_2^3$$

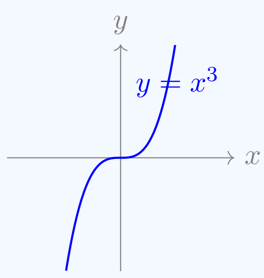
Taking the real cube root on both sides gives $x_1 = x_2$. Thus, the function is injective.

Step 3: To check if the function is surjective (onto), we need to determine if for every real number y in the codomain, there exists a real number x such that $f(x) = y$:

$$x^3 = y \implies x = \sqrt[3]{y}$$

Since the cube root of any real number is always a well-defined real number, every element in the codomain has a valid pre-image. Thus, the range is $\mathbb{R}$, which matches the codomain. The function is surjective.

Step 4: Let us visualize this mapping by sketching the cubic function curve:



Final Answer: Both injective and surjective

Answer: (A) Go Back to Question 36

Q37.

Solution

Concept:
The principal value of an inverse trigonometric function is the specific value belonging to its standard restricted range interval. For the inverse cosine function $\cos^{-1}(x)$, the principal value range is restricted to the interval $[0, \pi]$. If the input argument is negative, we can use the standard trigonometric identity:

$$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$$

This maps the angle into the correct second quadrant where cosine values are negative.

Solution:
Step 1: Write down the given expression whose principal value needs to be found:

$$\cos^{-1}\left(-\frac{1}{2}\right)$$

Step 2: Apply the negative argument identity for inverse cosine:

$$\cos^{-1}\left(-\frac{1}{2}\right) = \pi - \cos^{-1}\left(\frac{1}{2}\right)$$

Step 3: Find the standard first-quadrant angle whose cosine value equals $\frac{1}{2}$:

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \quad (\text{since } \cos(60^\circ) = 0.5)$$

Step 4: Substitute this angle back into our identity expression:

$$\text{Principal Value} = \pi - \frac{\pi}{3}$$

Step 5: Simplify the fraction subtraction by finding a common denominator:

$$\text{Principal Value} = \frac{3\pi - \pi}{3} = \frac{2\pi}{3}$$

Since $\frac{2\pi}{3}$ lies within the valid principal range interval $[0, \pi]$, this is the correct answer.

Final Answer: $\boxed{\frac{2\pi}{3}}$

Answer: (B) Go Back to Question 37

Q38.

Solution

Concept:
A matrix A is symmetric if it is equal to its own transpose ($A^T = A$). A matrix A is skew-symmetric if it is equal to the negative of its transpose ($A^T = -A$). For any square matrix A, it can be uniquely split into a sum of a symmetric matrix and a skew-symmetric matrix using the identity:

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$$

where $\frac{A+A^T}{2}$ forms the symmetric part and $\frac{A-A^T}{2}$ forms the skew-symmetric part.

Solution:
Step 1: Write down the given square matrix A:

$$A = \begin{pmatrix} 2 & 3 \\ 5 & -1 \end{pmatrix}$$

Step 2: Find the transpose matrix A^T by swapping its rows and columns:

$$A^T = \begin{pmatrix} 2 & 5 \\ 3 & -1 \end{pmatrix}$$

Step 3: To find the symmetric matrix component, compute the sum matrix $(A + A^T)$:

$$A + A^T = \begin{pmatrix} 2+2 & 3+5 \\ 5+3 & -1+(-1) \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 8 & -2 \end{pmatrix}$$

Step 4: Divide each element of this sum matrix by 2 according to the formula:

$$\text{Symmetric Part} = \frac{1}{2}(A + A^T) = \begin{pmatrix} 2 & 4 \\ 4 & -1 \end{pmatrix}$$

Step 5: Let us verify that this component is symmetric by checking its transpose:

$$\begin{pmatrix} 2 & 4 \\ 4 & -1 \end{pmatrix}^T = \begin{pmatrix} 2 & 4 \\ 4 & -1 \end{pmatrix}$$

The component is symmetric.

Final Answer: $\boxed{\begin{pmatrix} 2 & 4 \\ 4 & -1 \end{pmatrix}}$

Answer: (A) Go Back to Question 38

Q39.

Solution

Concept:

The derivative of a composite function $f(g(x))$ is found using the Chain Rule, which states that $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$. For transcendental functions involving logarithmic and trigonometric terms, we differentiate from the outermost function layer inward. The standard derivative formulas are:

$$\frac{d}{du}(\log u) = \frac{1}{u}, \quad \frac{d}{dx}(\tan x) = \sec^2 x$$

Solution:

Step 1: Write down the given composite function equation:

$$y = \log(\tan x)$$

Step 2: Apply the chain rule by differentiating the outer natural logarithm function layer first with respect to its inner argument $(\tan x)$:

$$\frac{dy}{dx} = \frac{1}{\tan x} \cdot \frac{d}{dx}(\tan x)$$

Step 3: Differentiate the inner trigonometric function layer $\tan x$:

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

Step 4: Substitute this back to assemble the complete derivative expression:

$$\frac{dy}{dx} = \frac{\sec^2 x}{\tan x}$$

Step 5: Simplify this trigonometric expression by rewriting it in terms of basic sine and cosine functions:

$$\frac{dy}{dx} = \frac{\frac{1}{\cos^2 x}}{\frac{\sin x}{\cos x}} = \frac{1}{\cos^2 x} \cdot \frac{\cos x}{\sin x} = \frac{1}{\sin x \cos x}$$

Step 6: Multiply the numerator and denominator by 2 to use the double-angle identity $\sin(2x) = 2\sin x \cos x$:

$$\frac{dy}{dx} = \frac{2}{2\sin x \cos x} = \frac{2}{\sin(2x)} = 2\csc(2x)$$

Final Answer: $\boxed{2\csc(2x)}$

Answer: (C) Go Back to Question 39

Q40.

Solution

Concept:

The conditional probability formula for event A given that event B has occurred is $P(A|B) = \frac{P(A \cap B)}{P(B)}$. If event A is a subset of event B ($A \subset B$), then the intersection of the two sets is exactly equal to set A ($A \cap B = A$). Consequently, the probability of their intersection simplifies directly to the probability of event A:

$$P(A \cap B) = P(A)$$

Solution:

Step 1: Write down the given conditional probability to be evaluated:

$$P(A|B)$$

Step 2: State the standard formula for conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Step 3: We are given the set condition that $A \subset B$. This means every outcome that belongs to event A is also guaranteed to be an outcome inside event B.

Step 4: Find the intersection of these two dependent events based on the subset condition:

$$A \cap B = A \implies P(A \cap B) = P(A)$$

Step 5: Substitute $P(A)$ for $P(A \cap B)$ in our conditional probability equation:

$$P(A|B) = \frac{P(A)}{P(B)}$$

Final Answer: $\boxed{\frac{P(A)}{P(B)}}$

Answer: (B) Go Back to Question 40

Answer Key

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	C	2	B	3	D	4	C	5	A
6	A	7	B	8	B	9	A	10	B
11	B	12	B	13	A	14	A	15	B
16	D	17	B	18	D	19	C	20	B
21	A	22	C	23	B	24	C	25	A
26	A	27	C	28	B	29	A	30	C
31	D	32	A	33	B	34	A	35	B
36	A	37	B	38	A	39	C	40	B