CUET PG 2026 Statistcs

Question Paper with Solutions

Conducted by National Testing Agency

General Instructions

1. **Duration:** The total duration of the examination is 90 minutes.
2. **Total Marks:** The complete paper carries a maximum of 300 marks.
3. **Structure:** The paper consists of 1 Section:
   - **Section A:** 75 Questions
4. **Compulsory Questions:** All 75 questions are compulsory.
5. Each question has four options. Only **one** option is correct.
6. **Right Answer:** Marks as per question scheme (Total 300 marks).
7. **Incorrect Answer:** No negative marking.
8. **Unanswered/Marked for Review:** 0 marks.

1. The Sum $\sum_{r=1}^{20}(r^2+1)\times r!$ is equal to

(A) $21 \times 21!$

(B) $20 \times 21!$

(C) $21!$

(D) $20 \times 21 \times 20!$

Correct Answer: (B) $20 \times 21!$

Solution:

Concept:

- We use the telescoping sum method (Method of Differences) to evaluate sums involving factorials.

- The goal is to express the general term T_r as a difference of two consecutive terms, i.e., $T_r = V_r - V_{r-1}$.

Step 1: Express the general term T_r in terms of factorials

The general term of the summation is:

$$T_r = (r^2 + 1) \cdot r!$$

We need to manipulate $r^2 + 1$. We can write:

$$r^2 + 1 = (r^2 + r) - (r - 1) = r(r + 1) - (r - 1)$$

Step 2: Simplify the term to create a telescoping structure

Now, substitute the expression back into T_r:

$$T_r = [r(r + 1) - (r - 1)] \cdot r!$$

Multiply the terms:

$$T_r = r(r + 1) \cdot r! - (r - 1) \cdot r!$$

Since $(r + 1) \cdot r! = (r + 1)!$, we have:

$$T_r = r \cdot (r + 1)! - (r - 1) \cdot r!$$

Step 3: Apply the summation to find the total sum

Let $V_r = r \cdot (r + 1)!$. Then $V_{r-1} = (r - 1) \cdot r!$. The sum is:

$$S = \sum_{r=1}^{20} (V_r - V_{r-1})$$

$$S = (V_1 - V_0) + (V_2 - V_1) + \cdots + (V_{20} - V_{19})$$

By telescoping properties, all intermediate terms cancel:

$$S = V_{20} - V_0$$

Step 4: Calculate final values

$$V_{20} = 20 \cdot (20+1)! = 20 \cdot 21!$$

$$V_0 = 0 \cdot (0+1)! = 0$$

Thus, $S = 20 \cdot 21! - 0 = 20 \cdot 21!$.

Quick Tip: To solve series involving factorials, look for ways to split the polynomial into factors that complete the factorial, such as writing $r \cdot r! = (r+1)! - r!$. Always check the lower bound of the summation to correctly identify the remaining V_0 or V_1 term.

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2. Value of $\sum_{n=0}^{\infty} \frac{2}{(2n+1)(2n+3)}$ is

(A) ∞

(B) 1

(C) 1/3

(D) 4/6

Correct Answer: (B) 1

Solution:

Concept:

- Any rational function where the denominator is a product of linear factors in an arithmetic progression can be summed using partial fractions.
- The infinite sum is the limit of the partial sum S_N as $N \to \infty$.

Step 1: Split the general term using partial fractions

Let the general term be $a_n = \frac{2}{(2n+1)(2n+3)}$. Notice that the difference between the factors in the denominator is $(2n+3)-(2n+1) = 2$. Since the numerator is exactly 2, we can split it directly:

$$a_n = \frac{(2n+3)-(2n+1)}{(2n+1)(2n+3)} = \frac{1}{2n+1} - \frac{1}{2n+3}$$

Step 2: Calculate the partial sum S_N

$$S_N = \sum_{n=0}^{N} \left(\frac{1}{2n+1} - \frac{1}{2n+3} \right)$$

Writing out the terms:

$$S_N = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{2N+1} - \frac{1}{2N+3}\right)$$

Step 3: Evaluate the limit as N approaches infinity

After cancellation:

$$S_N = 1 - \frac{1}{2N+3}$$

Taking the limit for the infinite sum:

$$S = \lim_{N \to \infty} \left(1 - \frac{1}{2N+3}\right)$$

Since $\frac{1}{\infty} \to 0$, we get:

$$S = 1 - 0 = 1$$

Quick Tip: For telescoping series of the form $\sum \frac{k}{f(n)f(n+k)}$, if $f(n)$ is linear, the sum is usually the first part of the first term minus the last part of the last term. Always verify the common difference matches the numerator.

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3. If $f(1) = 1$ and $f'(1) = -1$ then the value of $\frac{d}{dx}\left[\frac{f(x^3)}{xf(x^2)}\right]$ at $x = 1$ is equal to

(A) 1

(B) 0

(C) 2

(D) -2

Correct Answer: (D) -2

Solution:

Concept:

- Quotient Rule: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$.

- Chain Rule: $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$.

Step 1: Define the components for differentiation

Let $u(x) = f(x^3)$ and $v(x) = xf(x^2)$. Differentiating $u(x)$ using the chain rule:

$$u'(x) = f'(x^3) \cdot 3x^2$$

Differentiating $v(x)$ using the product rule and chain rule:

$$v'(x) = 1 \cdot f(x^2) + x \cdot f'(x^2) \cdot 2x = f(x^2) + 2x^2 f'(x^2)$$

Step 2: Apply the quotient rule

Let $y = \frac{u(x)}{v(x)}$. Then:

$$\frac{dy}{dx} = \frac{[3x^2 f'(x^3)] \cdot [xf(x^2)] - [f(x^3)] \cdot [f(x^2) + 2x^2 f'(x^2)]}{(xf(x^2))^2}$$

Step 3: Substitute $x = 1$ and given values

Given $f(1) = 1$ and $f'(1) = -1$.

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{[3(1)^2 f'(1)] \cdot [1 \cdot f(1)] - [f(1)] \cdot [f(1) + 2(1)^2 f'(1)]}{(1 \cdot f(1))^2}$$

Substitute the values:

$$= \frac{[3(-1)] \cdot [1] - [1] \cdot [1 + 2(-1)]}{1^2}$$

$$= \frac{-3 - [1 - 2]}{1}$$

$$= \frac{-3 - (-1)}{1} = -3 + 1 = -2$$

Quick Tip: In composite function derivatives, identify the "inner" and "outer" functions clearly. Plugging in numerical values early in the derivative expression can prevent complex algebraic expansion errors.

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4. The function $f(x) = \int_{e^x}^{e^{2x}} t \cdot \log_e t \, dt$ has an absolute minima at $x = 0$ and a local maxima at $x =$

(A) $-\log_e 4$

(B) $\log_e 2$

(C) 1

(D) $-\log_e 2$

Correct Answer: (D) $-\log_e 2$

Solution:

Concept:

- Leibniz Rule for differentiation under the integral sign is used when limits are functions of x.
- Critical points are found where $f'(x) = 0$.

Step 1: Differentiate $f(x)$ using the Leibniz Rule

The formula is $\frac{d}{dx}\int_{g(x)}^{h(x)} \phi(t)dt = \phi(h(x))h'(x) - \phi(g(x))g'(x)$. Here, $\phi(t) = t\log_e t$.

$$f'(x) = [e^{2x}\log_e(e^{2x})]\cdot\frac{d}{dx}(e^{2x}) - [e^x\log_e(e^x)]\cdot\frac{d}{dx}(e^x)$$

$$f'(x) = [e^{2x}\cdot 2x]\cdot 2e^{2x} - [e^x\cdot x]\cdot e^x$$

$$f'(x) = 4xe^{4x} - xe^{2x}$$

Step 2: Solve for critical points $f'(x) = 0$

$$xe^{2x}(4e^{2x} - 1) = 0$$

Since $e^{2x} \neq 0$, we have: 1) $x = 0$ 2) $4e^{2x} - 1 = 0 \implies e^{2x} = 1/4 \implies 2x = \log_e(1/4) = -2\log_e 2 \implies x = -\log_e 2$

Step 3: Apply the first derivative test

Check signs of $f'(x) = xe^{2x}(4e^{2x} - 1)$:

- For $x < -\log_e 2$: x is $(-)$ and $(4e^{2x} - 1)$ is $(-)$. So $f'(x) > 0$.
- For $-\log_e 2 < x < 0$: x is $(-)$ and $(4e^{2x} - 1)$ is $(+)$. So $f'(x) < 0$.
- For $x > 0$: x is $(+)$ and $(4e^{2x} - 1)$ is $(+)$. So $f'(x) > 0$.

At $x = -\log_e 2$, the sign changes from $+$ to $-$, so it is a local maxima. At $x = 0$, the sign changes from $-$ to $+$, so it is a local minima.

Quick Tip: Remember that $\log_e(e^k) = k$. This simplification is key in such problems. Always factorize the derivative completely to easily determine the intervals of increase and decrease.

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5. In the Taylor series expansion of function $f(x) = e^{x^2-x}$, coefficient of x^3 is

(A) $7/3$

(B) -7

(C) $-7/6$

(D) $7/2$

Correct Answer: (C) $-7/6$

Solution:

Concept:

- The series expansion for e^u is $1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots$
- Substitute $u = x^2 - x$ into this expansion and extract the coefficient of the x^3 term.

Step 1: Write out the expansion for $f(x)$

$$f(x) = e^{(x^2-x)} = 1 + (x^2 - x) + \frac{(x^2-x)^2}{2} + \frac{(x^2-x)^3}{6} + \frac{(x^2-x)^4}{24} + \dots$$

Step 2: Expand relevant terms to find x^3

We only need terms where the power of x can be 3.

- $(x^2 - x)^2 = x^4 - 2x^3 + x^2$. The coefficient of x^3 here is -2.
- $(x^2 - x)^3 = (x^2)^3 - 3(x^2)^2(x) + 3(x^2)(x)^2 - x^3 = x^6 - 3x^5 + 3x^4 - x^3$. The coefficient of x^3 here is -1.
- For powers $n \geq 4$, $(x^2 - x)^n$ will have its lowest power as x^n. Thus, $x^4, x^5, \dots$ do not contribute to the x^3 coefficient.

Step 3: Calculate total coefficient

From $\frac{(x^2-x)^2}{2}$, we get: $\frac{-2x^3}{2} = -1x^3$. From $\frac{(x^2-x)^3}{6}$, we get: $\frac{-1x^3}{6} = -\frac{1}{6}x^3$. Summing these coefficients:

$$\text{Coefficient of } x^3 = -1 + \left(-\frac{1}{6}\right) = -\frac{6}{6} - \frac{1}{6} = -\frac{7}{6}$$

Quick Tip: When using series expansion for a polynomial exponent like $x^k + \ldots$, only terms up to $\frac{u^n}{n!}$ where n is the power you are looking for can contribute. Expansion is often much faster than computing the third derivative $f'''(0)/3!$.

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6. Match List - I with List - II.

	List - I		List - II
A.	The value of x where $f(x) = 9x(x-1)^2$, $0 \le x \le 2$ attains its maximum is	I.	e
B.	The maximum value of $f(x) = \frac{1}{x} e^{-\frac{1}{2}(\log_e x - 2)^2}$ attains at $x =$	II.	$\frac{2}{3}$
C.	Function $f(x) = x^2(1-x)^6$; $0 < x < 1$ attains its maximum at $x =$	III.	$\frac{1}{3}$
D.	The maximum value of function $f(x) = x^2 e^{-3x}$ attains at x	IV.	$\frac{1}{4}$

Choose the correct answer from the options given below

(A) A-III, B-I, C-IV, D-II

(B) A-III, B-I, C-II, D-IV

(C) A-IV, B-I, C-III, D-II

(D) A-II, B-III, C-I, D-IV

Correct Answer: (A) A-III, B-I, C-IV, D-II

Solution:

Concept:

- To find the point where a function attains its maximum, we find its critical points by setting the first derivative to zero ($f'(x) = 0$).
- We use the product rule and chain rule for differentiation.

Step 1: Solve for List-I part A

For $f(x) = 9x(x-1)^2$:

$$f'(x) = 9[1\cdot(x-1)^2 + x\cdot 2(x-1)] = 9(x-1)[(x-1)+2x] = 9(x-1)(3x-1)$$

Setting $f'(x) = 0$ gives $x = 1$ and $x = 1/3$. The local maximum occurs at $x = 1/3$. (Note: on the interval $[0,2]$, the absolute max is at $x = 2$, but based on the list, the local max at $1/3$ is intended). So, **A matches III**.

Step 2: Solve for List-I part B

For $f(x) = \frac{1}{x}e^{-\frac{1}{2}(\log_e x-2)^2}$, let $y = \log_e f(x) = -\log_e x - \frac{1}{2}(\log_e x-2)^2$.

$$\frac{dy}{dx} = -\frac{1}{x} - \frac{1}{2}\cdot 2(\log_e x-2)\cdot\frac{1}{x} = \frac{-1-\log_e x+2}{x} = \frac{1-\log_e x}{x}$$

Setting $\frac{dy}{dx} = 0 \implies \log_e x = 1 \implies x = e$. So, **B matches I.**

Step 3: Solve for List-I part C

For $f(x) = x^2(1-x)^6$:

$$f'(x) = 2x(1-x)^6 - 6x^2(1-x)^5 = 2x(1-x)^5[(1-x)-3x] = 2x(1-x)^5(1-4x)$$

Setting $f'(x) = 0$ for $0 < x < 1$ gives $x = 1/4$. So, **C matches IV**.

Step 4: Solve for List-I part D

For $f(x) = x^2e^{-3x}$:

$$f'(x) = 2xe^{-3x} - 3x^2e^{-3x} = xe^{-3x}(2-3x)$$

Setting $f'(x) = 0$ for $x > 0$ gives $x = 2/3$. So, **D matches II.**

Quick Tip: For functions of the form $f(x) = x^m(1-x)^n$, the maximum occurs at $x = \frac{m}{m+n}$. For functions of the form $f(x) = x^me^{-ax}$, the maximum occurs at $x = \frac{m}{a}$.

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7. The function $f(x) = |x^2+x-6|$ is not differentiable at $x = a$ and $x = b$ then $(b-a)^2$ equals

(A) 25

(B) 9

(C) 13

(D) 5

Correct Answer: (A) 25

Solution:

Concept:

- A function $f(x) = |g(x)|$ is not differentiable at the points where $g(x) = 0$, provided $g'(x) \neq 0$ at those points (i.e., the roots are simple).

Step 1: Find the roots of the expression inside the absolute value

We need to find when $x^2 + x - 6 = 0$. Factoring the quadratic:

$$x^2 + 3x - 2x - 6 = 0$$

$$x(x+3) - 2(x+3) = 0$$

$$(x-2)(x+3) = 0$$

Thus, the roots are $x = 2$ and $x = -3$.

Step 2: Identify the points of non-differentiability

The function $f(x) = |(x-2)(x+3)|$ will have "corners" or "kinks" at its roots because the linear factors change sign. Therefore, it is not differentiable at $a = -3$ and $b = 2$.

Step 3: Calculate the required value

We need to find $(b-a)^2$:

$$(b-a)^2 = (2-(-3))^2$$

$$= (2+3)^2$$

$$= 5^2 = 25$$

Quick Tip: The points of non-differentiability for $|ax^2+bx+c|$ correspond to the real and distinct roots of the quadratic. The square of the difference of roots is given by the formula $(\alpha-\beta)^2 = \frac{D}{a^2} = \frac{b^2-4ac}{a^2}$.

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8. The value of $\lim_{x\to 1} \frac{\int_{2\log_e x}^{3\log_e x} e^t dt}{x-1}$ equals

(A) 0

(B) 1

(C) -1

(D) ∞

Correct Answer: (B) 1

Solution:

Concept:

- We use L'Hôpital's rule for limits of the form $0/0$.
- We use the Leibniz Integral Rule to differentiate the numerator: $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f(h(x))h'(x) - f(g(x))g'(x)$.

Step 1: Check the form of the limit

As $x \to 1$, the limits of integration become $2\log_e 1 = 0$ and $3\log_e 1 = 0$. The numerator is $\int_0^0 e^t dt = 0$. The denominator is $1-1=0$. This is a $0/0$ indeterminate form.

Step 2: Apply L'Hôpital's rule

Differentiate the numerator using the Leibniz rule:

$$\frac{d}{dx}\left[\int_{2\log_e x}^{3\log_e x} e^t dt\right] = e^{3\log_e x}\cdot\frac{d}{dx}(3\log_e x) - e^{2\log_e x}\cdot\frac{d}{dx}(2\log_e x)$$

$$= x^3\cdot\frac{3}{x} - x^2\cdot\frac{2}{x}$$

$$= 3x^2 - 2x$$

Differentiate the denominator:

$$\frac{d}{dx}(x-1) = 1$$

Step 3: Evaluate the limit

$$\lim_{x\to 1} \frac{3x^2 - 2x}{1} = 3(1)^2 - 2(1) = 1$$

Quick Tip: Always simplify terms like $e^{n\log_e x}$ to x^n before proceeding with further calculations. Leibniz rule is the most efficient way to handle limits where the variable appears in the boundaries of an integral.

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9. If $f(x) = \begin{cases} \frac{\log_e\left(1+\frac{x}{a}\right) - \log_e\left(1-\frac{x}{b}\right)}{x} & \text{if } x \neq 0 \\ k & \text{if } x = 0 \end{cases}$ **is continuous at** $x = 0$**, then value of** k **is:**

(A) 0

(B) 1

(C) $\frac{1}{a} + \frac{1}{b}$

(D) $\frac{1}{a} - \frac{1}{b}$

Correct Answer: (C) $\frac{1}{a} + \frac{1}{b}$

Solution:

Concept:

- For a function to be continuous at $x = 0$, we must have $k = \lim_{x\to 0} f(x)$.
- Use the standard limit: $\lim_{u\to 0} \frac{\log_e(1+u)}{u} = 1$.

Step 1: Set up the limit for k

$$k = \lim_{x\to 0} \frac{\log_e\left(1+\frac{x}{a}\right) - \log_e\left(1-\frac{x}{b}\right)}{x}$$

Step 2: Separate the limit into two parts

$$k = \lim_{x\to 0} \left[\frac{\log_e\left(1+\frac{x}{a}\right)}{x} - \frac{\log_e\left(1-\frac{x}{b}\right)}{x} \right]$$

Step 3: Adjust the denominators to match the arguments

$$k = \lim_{x\to 0}\left[\frac{\log_e\left(1+\frac{x}{a}\right)}{\frac{x}{a}\cdot a} - \frac{\log_e\left(1-\frac{x}{b}\right)}{-\frac{x}{b}\cdot(-b)}\right]$$

$$k = \frac{1}{a}\lim_{x\to 0}\frac{\log_e\left(1+\frac{x}{a}\right)}{\frac{x}{a}} + \frac{1}{b}\lim_{x\to 0}\frac{\log_e\left(1-\frac{x}{b}\right)}{-\frac{x}{b}}$$

Step 4: Evaluate the limits

Using the standard limit $\lim_{u\to 0}\frac{\log_e(1+u)}{u} = 1$:

$$k = \frac{1}{a}(1) + \frac{1}{b}(1) = \frac{1}{a} + \frac{1}{b}$$

Quick Tip: For continuity problems at $x = 0$, expansion using $\log_e(1+u) \approx u$ for small u often gives the answer instantly: $k \approx \frac{(x/a)-(-x/b)}{x} = \frac{1}{a} + \frac{1}{b}$. Be careful with signs when the term inside the log is $(1-u)$.

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10. The value of integral $\int_0^1 \int_x^1 \frac{1}{1+y^2} dy dx$ is equal to

(A) $1-\ln(2)$

(B) $\frac{1}{2}\ln(2)$

(C) $\frac{\pi}{4}$

(D) $\frac{\pi}{2}-\ln 2$

Correct Answer: (B) $\frac{1}{2}\ln(2)$

Solution:

Concept:

- To evaluate this double integral, it is easier to change the order of integration.
- Identify the region of integration: $0 \le x \le 1$ and $x \le y \le 1$.

Step 1: Sketch and describe the region in reverse order

The region is a triangle with vertices at $(0,0), (0,1)$, and $(1,1)$. In the original order, x goes from 0 to 1, and for a fixed x, y goes from the line $y = x$ to $y = 1$. In the new order, y goes from 0 to 1. For a fixed y, x goes from the line $x = 0$ to $x = y$.

Step 2: Rewrite the integral with the new order

$$I = \int_0^1 \left(\int_0^y \frac{1}{1+y^2} dx \right) dy$$

Step 3: Evaluate the inner integral

$$I = \int_0^1 \left[\frac{x}{1+y^2} \right]_{x=0}^{x=y} dy$$

$$I = \int_0^1 \frac{y}{1+y^2} dy$$

Step 4: Evaluate the outer integral

Use the substitution $u = 1 + y^2$, then $du = 2y dy$.

$$I = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} [\ln(1+y^2)]_0^1$$

$$I = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2} \ln 2$$

Quick Tip: Always try changing the order of integration if the inner integral seems difficult or doesn't have an easy elementary antiderivative. In many cases, the resulting integrand after the first integration becomes a simple derivative form (like $f'(y)/f(y)$).

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11. The area of region in the first quadrant that is bounded by $y = \sqrt{x}$, $y = 2 - x$ and x-axis is

(A) 5/6

(B) 2/3

(C) 1

(D) 7/6

Correct Answer: (D) 7/6

Solution:

Concept:

- Area under a curve $y = f(x)$ from $x = a$ to $x = b$ is given by $\int_a^b f(x)\,dx$.
- When multiple curves define a boundary, identify intersection points and split the integral into sub-regions if necessary.

Step 1: Find the intersection points of the curves

First, find where $y = \sqrt{x}$ and $y = 2 - x$ intersect:

$$\sqrt{x} = 2 - x$$

Squaring both sides:

$$x = (2-x)^2 \implies x = 4 - 4x + x^2$$

$$x^2 - 5x + 4 = 0 \implies (x-4)(x-1) = 0$$

For $x = 4$, $y = \sqrt{4} = 2$ but $y = 2 - 4 = -2$. Since the region is in the first quadrant, $x = 4$ is rejected. For $x = 1$, $y = \sqrt{1} = 1$ and $y = 2 - 1 = 1$. So, the intersection point is $(1, 1)$.

Step 2: Set up the definite integrals for the area

The region is bounded by the x-axis $(y = 0)$ from $x = 0$ to $x = 2$. From $x = 0$ to $x = 1$, the upper boundary is $y = \sqrt{x}$. From $x = 1$ to $x = 2$, the upper boundary is $y = 2 - x$. The total area A is:

$$A = \int_0^1 \sqrt{x}\,dx + \int_1^2 (2-x)\,dx$$

Step 3: Evaluate the integrals

$$A = \left[\frac{x^{3/2}}{3/2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$$

$$A = \frac{2}{3}(1-0) + ((4-2) - (2-1/2))$$

$$A = \frac{2}{3} + (2 - 1.5) = \frac{2}{3} + 0.5$$

$$A = \frac{2}{3} + \frac{1}{2} = \frac{4+3}{6} = \frac{7}{6}$$

Quick Tip: To avoid splitting the integral, you can integrate with respect to y. The boundaries become $x = y^2$ and $x = 2-y$. Area $A = \int_0^1 [(2-y)-y^2]\,dy = [2y - \frac{y^2}{2} - \frac{y^3}{3}]_0^1 = 2-0.5-1/3 = 7/6$.

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12. Integral $\int_0^2 \int_{y^2}^{y+2} f(x,y)\,dx dy$ equals

(A) $\int_0^1 \int_0^{\sqrt{x}} f\,dydx + \int_1^2 \int_{x-2}^{\sqrt{x}} f\,dydx$
(B) $\int_0^2 \int_0^{\sqrt{x}} f\,dydx + \int_2^4 \int_{x-2}^{x} f\,dydx$
(C) $\int_0^2 \int_0^{\sqrt{x}} f\,dydx + \int_2^4 \int_{x-2}^{\sqrt{x}} f\,dydx$
(D) $\int_0^2 \int_0^{\sqrt{x}} f\,dydx + \int_2^4 \int_0^{\sqrt{x}} f\,dydx$

Correct Answer: (C) $\int_0^2 \int_0^{\sqrt{x}} f\,dydx + \int_2^4 \int_{x-2}^{\sqrt{x}} f\,dydx$

Solution:

Concept:

- Changing the order of integration requires sketching the region bounded by the limits.
- Original order is $dxdy$, new order is $dydx$.

Step 1: Identify the boundaries of the integration region

The given limits are: Lower limit of y: $y = 0$ Upper limit of y: $y = 2$ Lower limit of x: $x = y^2 \implies y = \sqrt{x}$ Upper limit of x: $x = y+2 \implies y = x-2$

Step 2: Sketch and split the region for the new order

The region starts at $x = 0$ and ends at $x = 4$ (where $y = 2$). The lower boundary of y is $y = 0$ until it hits the line $y = x-2$ at $x = 2$. For $x \in [0,2]$: y ranges from 0 to $\sqrt{x}$. For $x \in [2,4]$: y ranges from the lower line $x-2$ to the upper curve $\sqrt{x}$.

Step 3: Write the summation of integrals

The integral becomes:

$$I = \int_0^2 \left(\int_0^{\sqrt{x}} f(x,y)\,dy \right) dx + \int_2^4 \left(\int_{x-2}^{\sqrt{x}} f(x,y)\,dy \right) dx$$

Quick Tip: Always draw the region. For double integrals with variable limits, the horizontal/vertical strips must cover the entire area. Splitting is required whenever the "bottom" or "top" boundary function changes form.

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13. The area of bounded region R defined as $R = \{(x,y) : 0 < x < 2 \cap 1 < y < 3 \cap y > x\}$ is

(A) 3/4

(B) 1/2

(C) 7/2

(D) 7/8

Correct Answer: (C) 7/2

Solution:

Concept:

- The area is defined by the intersection of three conditions: a rectangle ($x \in [0,2], y \in [1,3]$) and a half-plane ($y > x$).

Step 1: Identify the intersection of the boundaries

The rectangular region has vertices $(0,1),(2,1),(2,3)$, and $(0,3)$. The line $y = x$ intersects the boundaries of this rectangle: - At $y = 1$, $x = 1$. - At $x = 2$, $y = 2$. So the line $y = x$ passes through $(1,1)$ and $(2,2)$.

Step 2: Set up the area calculation by sub-regions

Since we need $y > x$ within the rectangle, we can divide the region along x: For $x \in [0,1]$: y is bounded only by $[1,3]$ because even at $x = 1$, $y = 1$ is the minimum. Area $A_1 = \int_0^1 \int_1^3 dy\,dx = 1 \times (3-1) = 2$. For $x \in [1,2]$: the condition $y > x$ means the lower bound of y is now x. Area $A_2 = \int_1^2 \int_x^3 dy\,dx = \int_1^2 (3-x)\,dx$.

Step 3: Calculate final value

$$A_2 = \left[3x - \frac{x^2}{2}\right]_1^2 = (6-2)-(3-0.5) = 4-2.5 = 1.5 = 3/2$$

Total Area $A = A_1 + A_2 = 2 + 3/2 = 7/2$.

Quick Tip: Area can often be calculated geometrically. The rectangle area is $2 \times 2 = 4$. The part excluded is the small triangle with vertices $(1,1),(2,1),(2,2)$ which has area $\frac{1}{2} \times 1 \times 1 = 1/2$. Area $= 4 - 1/2 = 7/2$.

---

14. If A and B are symmetric matrices of same order then A. AB is symmetric iff $AB = BA$ B. $AB + BA$ is skew symmetric matrix C. $AB - BA$ is symmetric matrix D. $(A+B)^n$ is symmetric for all $n \in \mathbb{N}$ Choose the correct answer from the options given below

(A) A, B only

(B) B, C only

(C) C, D only

(D) A, D only

Correct Answer: (D) A, D only

Solution:

Concept:

- A matrix M is symmetric if $M^T = M$ and skew-symmetric if $M^T = -M$.
- For any matrices X, Y, $(XY)^T = Y^T X^T$.

Step 1: Analyze statement A

Given $A^T = A$ and $B^T = B$. For AB to be symmetric, $(AB)^T = AB$. $(AB)^T = B^T A^T = BA$. So AB is symmetric if and only if $BA = AB$. **A is correct.**

Step 2: Analyze statements B and C

For $AB + BA$: $(AB+BA)^T = (AB)^T + (BA)^T = BA + AB = AB + BA$. It is symmetric, not skew-symmetric. **B is incorrect.** For $AB - BA$: $(AB - BA)^T = (AB)^T - (BA)^T = BA - AB = -(AB - BA)$. It is skew-symmetric, not symmetric. **C is incorrect.**

Step 3: Analyze statement D

Since A and B are symmetric, $(A+B)^T = A^T + B^T = A + B$. The sum is symmetric. Any power of a symmetric matrix is also symmetric. Thus $((A+B)^n)^T = ((A+B)^T)^n = (A+B)^n$. **D is correct.**

Quick Tip: For symmetric matrices, product symmetry is rare and requires commutativity. Sums of symmetric matrices are always symmetric, and sums of skew-symmetric are always skew-symmetric.

---

15. If A is an invertible symmetric matrix then A. $(A^{-1})^T = A^{-1}$ B. adj $A = (\text{adj } A)^T$ C. A^{-1} is skew-symmetric D. $|A| = 0$ Choose the correct answer from the options given below

(A) A, B only

(B) A, C only

(C) B, C only

(D) C, D only

Correct Answer: (A) A, B only

Solution:

Concept:

- Matrix inversion and transpose operations commute: $(A^T)^{-1} = (A^{-1})^T$.
- Adjugate of a symmetric matrix is symmetric.

Step 1: Verify statement A

Since A is symmetric, $A^T = A$. Using the property $(A^T)^{-1} = (A^{-1})^T$: $(A)^{-1} = (A^{-1})^T$. Thus, the inverse of a symmetric matrix is symmetric. **A is correct.**

Step 2: Verify statement B

We know $A^{-1} = \frac{\text{adj } A}{|A|}$. Since A^{-1} is symmetric (from step 1), $\frac{\text{adj } A}{|A|}$ must be symmetric. Because $|A|$ is a scalar, adj A must be symmetric. **B is correct.**

Step 3: Verify statements C and D

As shown in step 1, A^{-1} is symmetric, so it cannot be skew-symmetric (unless it's the zero matrix, which isn't possible for an invertible matrix). **C is incorrect.** Invertibility means $|A| \neq 0$. **D is incorrect.**

Quick Tip: Inverse and Adjugate operations preserve the symmetry of the original matrix. Symmetry is one of the most robust properties under matrix arithmetic (except multiplication).

---

16. Let $A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$ and if $B = |A|\ \text{adj}(A)$. Then $|B|$ is equal to

(A) 3

(B) 9

(C) 81

(D) 243

Correct Answer: (D) 243

Solution:

Concept:

- Determinant of a scalar multiple: $|kA| = k^n|A|$ for an $n \times n$ matrix.
- Determinant of an adjoint: $|\text{adj}(A)| = |A|^{n-1}$.
- Property combination: $||A|\ \text{adj}(A)| = |A|^n|\text{adj}(A)| = |A|^n|A|^{n-1} = |A|^{2n-1}$.

Step 1: Calculate the determinant of matrix A

Expand along the first row:

$$|A| = 2[(1)(2) - (-1)(0)] - 1[(1)(2) - (-1)(1)] + (-2)[(1)(0) - (1)(1)]$$

$$|A| = 2[2 - 0] - 1[2 + 1] - 2[0 - 1]$$

$$|A| = 4 - 3 + 2 = 3$$

Step 2: Apply determinant properties to find $|B|$

Since A is a 3×3 matrix, $n = 3$. The given relation is $B = |A|\ \text{adj}(A)$. Taking determinant on both sides:

$$|B| = ||A|\ \text{adj}(A)|$$

Using the property $|kA| = k^n|A|$, where $k = |A|$:

$$|B| = (|A|)^3|\text{adj}(A)|$$

Using the property $|\text{adj}(A)| = |A|^{3-1} = |A|^2$:

$$|B| = |A|^3 \cdot |A|^2 = |A|^5$$

Step 3: Compute final value

Substitute $|A| = 3$:

$$|B| = 3^5$$

$$|B| = 3 \times 3 \times 3 \times 3 \times 3 = 243$$

Quick Tip: For any $n \times n$ matrix, the identity $||A|\text{adj}(A)| = |A|^{2n-1}$ is very useful for competitive exams. Always calculate the determinant carefully by choosing the row or column with the most zeros to minimize calculation errors.

---

17. Let $AX = B$ be a system of n-linear equations in n unknowns then

(A) System is consistent and infinitely many solution if $|A| = 0$.

(B) System is inconsistent and finitely many solution if $|A| = 0$

(C) System is consistent and unique solution if $|A| \neq 0$

(D) System is consistent if $|A| = 0$ and $(\text{adj } A)B \neq 0$

Correct Answer: (C) System is consistent and unique solution if $|A| \neq 0$

Solution:

Concept:

- A system of linear equations $AX = B$ is classified based on the determinant of the coefficient matrix A.
- If $|A| \neq 0$, the matrix is non-singular and invertible, leading to a unique solution.
- If $|A| = 0$, the system is either inconsistent or has infinite solutions.

Step 1: Evaluate the case for non-singular matrix

When $|A| \neq 0$, A^{-1} exists. Multiplying $AX = B$ by A^{-1} on the left:

$$A^{-1}(AX) = A^{-1}B \implies IX = A^{-1}B \implies X = A^{-1}B$$

This provides exactly one set of values for X, hence a unique solution.

Step 2: Evaluate the singular case $|A| = 0$

If $|A| = 0$, we examine the term $(\text{adj } A)B$:

1. If $(\text{adj } A)B \neq 0$, the system is inconsistent (no solution).
2. If $(\text{adj } A)B = 0$, the system is either consistent with infinitely many solutions or inconsistent.

Step 3: Match with options

Option (A) is not always true because it requires $(\text{adj } A)B = 0$. Option (B) contains the contradiction "inconsistent and finitely many"; inconsistent means zero solutions. Option (D) is incorrect because that specific condition implies inconsistency. Option (C) correctly states the fundamental property of non-singular systems.

Quick Tip: Always start by checking $|A|$. If it's non-zero, you don't need to check any further conditions for uniqueness. Consistency means at least one solution exists (unique or infinite).

---

18. The eigen vectors of the matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ is

(A) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(B) $\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

(C) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(D) $\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Correct Answer: (C) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Solution:

Concept:

- Characteristic equation: $|A - \lambda I| = 0$.
- Eigenvector equation: $(A - \lambda I)X = 0$.

Step 1: Find the characteristic roots (Eigenvalues)

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(2-\lambda) - 4 = 0$$

$$\lambda^2 - 7\lambda + 10 - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0 \implies (\lambda - 6)(\lambda - 1) = 0$$

Eigenvalues are $\lambda_1 = 6$ and $\lambda_2 = 1$.

Step 2: Determine eigenvector for $\lambda = 6$

$$(A - 6I)X = \begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies -x_1 + 4x_2 = 0 \implies x_1 = 4x_2$$

Choosing $x_2 = 1$, we get $X_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

Step 3: Determine eigenvector for $\lambda = 1$

$$(A-1I)X = \begin{bmatrix} 5-1 & 4 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x_1 + x_2 = 0 \implies x_1 = -x_2$$

Choosing $x_2 = -1$, we get $X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Quick Tip: Once eigenvalues are found, you can verify options by checking if $AX = \lambda X$. Sum of eigenvalues equals the trace of the matrix $(5+2=7)$ and their product equals the determinant $(10-4=6)$.

---

19. Solution of differential equation $(x^2+y^2)dx - 2xy\,dy = 0$**, where c is constant, is**

(A) $x^2+y^2 = cy$

(B) $x^2-y^2 = cy$

(C) $x^2+y^2 = cx$

(D) $x^2-y^2 = cx$

Correct Answer: (D) $x^2-y^2 = cx$

Solution:

Concept:

- This is a homogeneous differential equation because all terms have the same degree (degree 2).
- We use the substitution $y = vx$, which implies $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Step 1: Rearrange the equation into $\frac{dy}{dx}$ form

$$2xy\,dy = (x^2+y^2)dx \implies \frac{dy}{dx} = \frac{x^2+y^2}{2xy}$$

Step 2: Substitute $y = vx$

$$v + x\frac{dv}{dx} = \frac{x^2 + (vx)^2}{2x(vx)}$$

$$v + x\frac{dv}{dx} = \frac{x^2(1+v^2)}{2x^2 v} = \frac{1+v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1+v^2-2v^2}{2v} = \frac{1-v^2}{2v}$$

Step 3: Separate variables and integrate

$$\int \frac{2v}{1-v^2}dv = \int \frac{1}{x}dx$$

Let $1 - v^2 = t \implies -2v\,dv = dt$.

$$-\int \frac{dt}{t} = \int \frac{dx}{x} \implies -\ln|t| = \ln|x| + \ln|k|$$

$$\ln|t|^{-1} = \ln|kx| \implies \frac{1}{1-v^2} = kx$$

Step 4: Substitute back $v = y/x$

$$\frac{1}{1-y^2/x^2} = kx \implies \frac{x^2}{x^2-y^2} = kx$$

$$\frac{x}{x^2-y^2} = k \implies x^2 - y^2 = \frac{1}{k}x$$

Letting $\frac{1}{k} = c$, we get $x^2 - y^2 = cx$.

Quick Tip: Homogeneous equations can often be solved by identifying if the numerator is the derivative of the denominator (with some scaling). Verify by differentiating the answer: $2x - 2yy' = c$; substitute $c = (x^2 - y^2)/x$ to check for consistency.

---

20. The integrating factor for the differential equation $x\log_e x\,dy = (2\log_e x - y)dx$ is

(A) $\log_e x$

(B) $e^{-\log_e x}$

(C) x

(D) $\frac{1}{\log_e x}$

Correct Answer: (A) $\log_e x$

Solution:

Concept:

- A linear differential equation of the first order has the form $\frac{dy}{dx} + P(x)y = Q(x)$.
- The Integrating Factor (I.F.) is given by $e^{\int P(x)dx}$.

Step 1: Convert the equation to standard linear form

Divide both sides by $x \log_e x \, dx$:

$$\frac{dy}{dx} = \frac{2\log_e x - y}{x \log_e x}$$

$$\frac{dy}{dx} = \frac{2}{x} - \frac{y}{x \log_e x}$$

$$\frac{dy}{dx} + \left(\frac{1}{x \log_e x}\right) y = \frac{2}{x}$$

Step 2: Identify the term $P(x)$

Comparing with standard form:

$$P(x) = \frac{1}{x \log_e x}$$

Step 3: Calculate the Integrating Factor

$$\text{I.F.} = e^{\int P(x)dx} = e^{\int \frac{1}{x \log_e x} dx}$$

To integrate $\int \frac{1}{x \log_e x} dx$, let $\log_e x = t \implies \frac{1}{x} dx = dt$.

$$\int \frac{1}{t} dt = \ln|t| = \ln(\log_e x)$$

Thus:

$$\text{I.F.} = e^{\ln(\log_e x)} = \log_e x$$

Quick Tip: Always ensure the coefficient of $\frac{dy}{dx}$ is exactly 1 before identifying $P(x)$. The property $e^{\ln u} = u$ is frequently used to simplify the integrating factor.

21. Which of the following differential equation is satisfied by $y_1(x) = e^x$, $y_2(x) = xe^x$ and $y_3 = e^{2x}$?

(A) $\frac{d^3y}{dx^3} + \frac{4d^2y}{dx^2} + \frac{5dy}{dx} + 2y = 0$
(B) $\frac{d^3y}{dx^3} - \frac{4d^2y}{dx^2} + \frac{5dy}{dx} - 2y = 0$
(C) $\frac{d^3y}{dx^3} + \frac{4d^2y}{dx^2} - \frac{5dy}{dx} - 2y = 0$
(D) $\frac{d^3y}{dx^3} - \frac{4d^2y}{dx^2} + \frac{5dy}{dx} + 2y = 0$

Correct Answer: (B) $\frac{d^3y}{dx^3} - \frac{4d^2y}{dx^2} + \frac{5dy}{dx} - 2y = 0$

Solution:

Concept:

- A linear homogeneous differential equation with constant coefficients has solutions of the form e^{mx}, where m is a root of the auxiliary (characteristic) equation.
- If a root m is repeated k times, the solutions are $e^{mx}, xe^{mx}, \ldots, x^{k-1}e^{mx}$.

Step 1: Identify the roots of the auxiliary equation from the given solutions

The solutions are $y_1 = e^x$, $y_2 = xe^x$, and $y_3 = e^{2x}$.

- The presence of e^x and xe^x indicates that $m = 1$ is a repeated root with multiplicity at least 2.
- The presence of e^{2x} indicates that $m = 2$ is a root.

Thus, the roots of the auxiliary equation are $m = 1, 1, 2$.

Step 2: Formulate the auxiliary equation

Using the roots identified, the auxiliary equation in terms of m is:

$$(m-1)^2(m-2) = 0$$

Expanding the terms:

$$(m^2 - 2m + 1)(m-2) = 0$$

$$m^2(m-2) - 2m(m-2) + 1(m-2) = 0$$

$$m^3 - 2m^2 - 2m^2 + 4m + m - 2 = 0$$

$$m^3 - 4m^2 + 5m - 2 = 0$$

Step 3: Construct the corresponding differential equation

Replacing m^n with $\frac{d^n y}{dx^n}$:

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = 0$$

This matches option (B).

Quick Tip: Always check for multiplicity. If you see terms like $x^n e^{ax}$, it means the root a is repeated $n+1$ times. The sum of the roots in the auxiliary equation $m^3 - Am^2 + Bm - C = 0$ is A, which corresponds to the coefficient of the second-highest derivative with a negative sign.

---

22. If A and B are two non-mutually exclusive events such that $P(A|B) = P(B|A)$ then

(A) $A \subseteq B$ but $A \neq B$

(B) $A = B$

(C) $A \cap B = \phi$

(D) $P(A) = P(B)$

Correct Answer: (D) $P(A) = P(B)$

Solution:

Concept:

- Conditional probability definition: $P(A|B) = \frac{P(A\cap B)}{P(B)}$, provided $P(B) > 0$.
- Similarly, $P(B|A) = \frac{P(B\cap A)}{P(A)}$, provided $P(A) > 0$.

Step 1: Equate the given conditional probability expressions

We are given:

$$P(A|B) = P(B|A)$$

Substitute the standard definitions:

$$\frac{P(A\cap B)}{P(B)} = \frac{P(B\cap A)}{P(A)}$$

Step 2: Simplify the equation using the property of intersections

Since $P(A\cap B) = P(B\cap A)$, the numerator on both sides is the same. The problem states the events are non-mutually exclusive, meaning $P(A\cap B) \neq 0$. We can therefore divide both sides by $P(A\cap B)$:

$$\frac{1}{P(B)} = \frac{1}{P(A)}$$

Step 3: Conclude the final relationship

By taking reciprocals on both sides:

$$P(A) = P(B)$$

This shows that the two events have equal probabilities. Note that this does not necessarily mean the events themselves are identical $(A = B)$.

Quick Tip: If $P(A|B) = P(B|A)$, it always implies $P(A) = P(B)$ as long as the intersection is non-empty. If additionally the events were independent, then $P(A) = P(B)$ would follow immediately from $P(A|B) = P(A)$ and $P(B|A) = P(B)$.

---

23. Let E and F be two events, if $P(E|F) = 0.5$, $P(E|\bar{F}) = 0.6$ and $P(F) = 0.6$ then $P(E)$ equals

(A) 0.56

(B) 0.44

(C) 0.54

(D) 0.46

Correct Answer: (C) 0.54

Solution:

Concept:

- Total Probability Theorem: $P(E) = P(E\cap F) + P(E\cap \bar{F})$.
- This can be written in conditional terms as: $P(E) = P(E|F)P(F) + P(E|\bar{F})P(\bar{F})$.

Step 1: Identify the given values and derived values

Given:

- $P(F) = 0.6$
- $P(E|F) = 0.5$
- $P(E|\bar{F}) = 0.6$

Derived value:

$$P(\bar{F}) = 1 - P(F) = 1 - 0.6 = 0.4$$

Step 2: Apply the Total Probability Theorem

Substitute the values into the formula:

$$P(E) = (P(E|F) \cdot P(F)) + (P(E|\bar{F}) \cdot P(\bar{F}))$$

$$P(E) = (0.5 \cdot 0.6) + (0.6 \cdot 0.4)$$

Step 3: Calculate the final result

$$P(E) = 0.30 + 0.24$$

$$P(E) = 0.54$$

Quick Tip: The Total Probability Theorem essentially partitions the sample space. Think of E as happening either "with F" or "without F". A quick weighted average check: $P(E)$ must lie between the two conditional probabilities (0.5 and 0.6). 0.54 satisfies this.

---

24. If $P(E) = \frac{1}{3}$, $P(F) = \frac{1}{5}$ and $P(E \cup F) = \frac{1}{2}$ then $P(E|\bar{F}) + P(F|\bar{E})$ is equal to

(A) 5/4

(B) 5/8

(C) 3/4

(D) 7/8

Correct Answer: (B) 5/8

Solution:

Concept:

- Addition theorem: $P(E \cap F) = P(E) + P(F) - P(E \cup F)$.
- Probability of intersection with complement: $P(E \cap \bar{F}) = P(E) - P(E \cap F)$.

Step 1: Calculate the intersection probability $P(E \cap F)$

Using the addition theorem:

$$P(E \cap F) = \frac{1}{3} + \frac{1}{5} - \frac{1}{2}$$

Common denominator is 30:

$$P(E \cap F) = \frac{10}{30} + \frac{6}{30} - \frac{15}{30} = \frac{1}{30}$$

Step 2: Calculate the first conditional probability $P(E|\bar{F})$

$$P(E|\bar{F}) = \frac{P(E \cap \bar{F})}{P(\bar{F})} = \frac{P(E) - P(E \cap F)}{1 - P(F)}$$

$$P(E|\bar{F}) = \frac{\frac{1}{3} - \frac{1}{30}}{1 - \frac{1}{5}} = \frac{\frac{10-1}{30}}{\frac{4}{5}} = \frac{\frac{9}{30}}{\frac{4}{5}} = \frac{3}{10} \times \frac{5}{4} = \frac{3}{8}$$

Step 3: Calculate the second conditional probability $P(F|\bar{E})$

$$P(F|\bar{E}) = \frac{P(F \cap \bar{E})}{P(\bar{E})} = \frac{P(F) - P(E \cap F)}{1 - P(E)}$$

$$P(F|\bar{E}) = \frac{\frac{1}{5} - \frac{1}{30}}{1 - \frac{1}{3}} = \frac{\frac{6-1}{30}}{\frac{2}{3}} = \frac{\frac{5}{30}}{\frac{2}{3}} = \frac{1}{6} \times \frac{3}{2} = \frac{1}{4}$$

Step 4: Sum the two values

$$\text{Sum} = \frac{3}{8} + \frac{1}{4} = \frac{3}{8} + \frac{2}{8} = \frac{5}{8}$$

Quick Tip: Keep fractions in their simplest common denominator form to make final summation easier. Venn diagrams can help visualize $P(E \cap \bar{F})$ as the "only E" part and $P(F \cap \bar{E})$ as the "only F" part.

---

25. Which of the following statements are correct?

A. Ogives curves are used to obtain median

B. Histogram are used to obtain mode

C. Boxplots are used to determine mean

D. Pie charts are used to determine quantile

Choose the correct answer from the options given below

(A) A and B only
(B) A and C only
(C) B and C only
(D) A and D only

Correct Answer: (A) A and B only

Solution:

Concept:

- Graphical representations in statistics are used to visualize and estimate different measures of central tendency and dispersion.

Step 1: Analyze statement A

An Ogive is a cumulative frequency graph. By finding the value on the x-axis corresponding to the 50th percentile (N/2) on the y-axis, we can estimate the **median**. **A is correct.**

Step 2: Analyze statement B

In a frequency distribution, the mode can be estimated from a Histogram by identifying the highest rectangle (the modal class) and using interpolating lines between the adjacent rectangles. **B is correct.**

Step 3: Analyze statement C

A Boxplot (box-and-whisker plot) displays the minimum, first quartile, median, third quartile, and maximum. While some software adds a marker for the mean, it is primarily used for **medians and quartiles**, not for determining the mean. **C is generally incorrect in a standard context.**

Step 4: Analyze statement D

Pie charts represent components of a whole as a percentage of the total area. They are not designed to determine **quantiles** (like deciles or percentiles), which require ordered data or cumulative structures. **D is incorrect.**

Quick Tip: Remember: "O" for Ogive and Median (positional), "H" for Histogram and Mode (frequency density). Boxplots are best for comparing variability and detecting outliers across different datasets.

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26. Consider $x_1, x_2, \ldots, x_n$ observations such that $\sum_{i=1}^{n} x_i^2 = 500$ and $\sum_{i=1}^{n} x_i = 50$. Then a minimum number of observations required is

(A) 25

(B) 5

(C) 10

(D) 15

Correct Answer: (B) 5

Solution:

Concept:

- For any set of real observations, the variance σ^2 must be greater than or equal to zero ($\sigma^2 \geq 0$).
- Variance formula: $\sigma^2 = \frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2$.

Step 1: Set up the inequality based on the variance property

Substitute the given sums into the variance inequality:

$$\frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2 \geq 0$$

$$\frac{500}{n} - \left(\frac{50}{n}\right)^2 \geq 0$$

Step 2: Simplify the inequality to solve for n

$$\frac{500}{n} - \frac{2500}{n^2} \geq 0$$

Multiply the entire inequality by n^2 (since n^2 is always positive):

$$500n - 2500 \geq 0$$

$$500n \geq 2500$$

Step 3: Find the minimum value of n

$$n \geq \frac{2500}{500}$$

$$n \geq 5$$

The smallest integer satisfying this condition is 5.

Quick Tip: This problem essentially uses the Cauchy-Schwarz inequality in the form $n\sum x_i^2 \geq (\sum x_i)^2$. Always remember that in statistics, the root-mean-square (RMS) is always greater than or equal to the arithmetic mean (AM).

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27. If $P(E) = \frac{1}{3}$, $P(F) = \frac{2}{5}$ and $P(E \cup F) - P(E \cap F) = \frac{1}{5}$ then $P(E \cup F)$ is equal to

(A) $\frac{11}{15}$
(B) $\frac{4}{15}$
(C) $\frac{8}{15}$
(D) $\frac{7}{15}$

Correct Answer: (D) $\frac{7}{15}$

Solution:

Concept:

- Addition Theorem: $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.
- Symmetric Difference: $P(E \cup F) - P(E \cap F)$ represents the probability that exactly one of the events occurs.

Step 1: Express $P(E \cap F)$ in terms of $P(E \cup F)$

From the addition theorem:

$$P(E \cap F) = P(E) + P(F) - P(E \cup F)$$

Substitute this into the given equation $P(E \cup F) - P(E \cap F) = \frac{1}{5}$:

$$P(E \cup F) - [P(E) + P(F) - P(E \cup F)] = \frac{1}{5}$$

Step 2: Simplify and solve for $P(E \cup F)$

$$2P(E \cup F) - P(E) - P(F) = \frac{1}{5}$$

Substitute the known probabilities $P(E) = \frac{1}{3}$ and $P(F) = \frac{2}{5}$:

$$2P(E \cup F) - \frac{1}{3} - \frac{2}{5} = \frac{1}{5}$$

$$2P(E \cup F) = \frac{1}{5} + \frac{1}{3} + \frac{2}{5} = \frac{3}{5} + \frac{1}{3}$$

Step 3: Calculate final result

Common denominator is 15:

$$2P(E \cup F) = \frac{9+5}{15} = \frac{14}{15}$$

$$P(E \cup F) = \frac{14}{30} = \frac{7}{15}$$

Quick Tip: The expression $P(E\cup F)-P(E\cap F)$ is the same as $P(E\cap\bar{F})+P(\bar{E}\cap F)$. Drawing a Venn diagram helps visualize that we are taking the union and removing the common part.

---

28. Let E, F and G be mutually independent events such that $P(E)=0.4, P(F)=0.6$ and $P(G)=0.8$ then $P(\bar{E}\cup\bar{F}\cup G)$ is

(A) 0.192

(B) 0.048

(C) 0.952

(D) 0.808

Correct Answer: (C) 0.952

Solution:

Concept:

- De Morgan's Law: $\bar{E}\cup\bar{F}\cup G=\overline{E\cap F\cap\bar{G}}$.
- For independent events: $P(A\cap B\cap C)=P(A)P(B)P(C)$.

Step 1: Identify the complementary event probability

We want to calculate $P(\bar{E}\cup\bar{F}\cup G)$. By the complement rule:

$$P(\bar{E}\cup\bar{F}\cup G)=1-P(\overline{\bar{E}\cup\bar{F}\cup G})$$

Using De Morgan's laws:

$$\overline{\bar{E}\cup\bar{F}\cup G}=E\cap F\cap\bar{G}$$

Step 2: Calculate the probability of the intersection

Since events $E, F,$ and G (and thus $\bar{G}$) are mutually independent:

$$P(E\cap F\cap\bar{G})=P(E)\cdot P(F)\cdot P(\bar{G})$$

Given $P(G)=0.8 \implies P(\bar{G})=1-0.8=0.2.$

$$P(E\cap F\cap\bar{G})=0.4\times0.6\times0.2$$

$$P(E \cap F \cap \bar{G}) = 0.24 \times 0.2 = 0.048$$

Step 3: Subtract from 1 to get the final answer

$$P(\bar{E} \cup \bar{F} \cup G) = 1 - 0.048$$

$$P(\bar{E} \cup \bar{F} \cup G) = 0.952$$

Quick Tip: Whenever you see multiple unions of complements, try converting to the complement of an intersection. Independence makes calculating intersections extremely straightforward—just multiply the individual probabilities.

---

29. Let E, F and G be events such that $P(E|G) = 0.05$ and $P(F|G) = 0.05$ which of the following statement must be true?

(A) $P(E \cap F|G) = (0.05)^2$
(B) $P(\bar{E} \cap \bar{F}|G) \geq 0.90$
(C) $P(E \cup F|G) \leq 0.05$
(D) $P(E \cup F|G) \geq 1 - (0.05)^2$

Correct Answer: (B) $P(\bar{E} \cap \bar{F}|G) \geq 0.90$

Solution:

Concept:

- Boole's Inequality (Union Bound): $P(A \cup B) \leq P(A) + P(B)$. This holds for conditional probabilities as well.
- De Morgan's Law for complements: $\bar{E} \cap \bar{F} = \overline{E \cup F}$.

Step 1: Apply the union bound to the conditional events

We know that for any events E and F:

$$P(E \cup F|G) \leq P(E|G) + P(F|G)$$

Substitute the given values:

$$P(E \cup F|G) \leq 0.05 + 0.05$$

$$P(E \cup F|G) \leq 0.10$$

Step 2: Calculate the probability of the complementary event

The event $\bar{E} \cap \bar{F}$ is the complement of $E \cup F$.

$$P(\bar{E} \cap \bar{F}|G) = 1 - P(E \cup F|G)$$

Step 3: Determine the lower bound

Since $P(E \cup F|G)$ is at most 0.10, the probability of its complement must be at least:

$$P(\bar{E} \cap \bar{F}|G) \geq 1 - 0.10$$

$$P(\bar{E} \cap \bar{F}|G) \geq 0.90$$

Quick Tip: This problem tests Boole's Inequality. Option (A) is only true if E and F are conditionally independent given G, which isn't stated. In inequality-based questions, always check the extreme cases where events have no overlap.

---

30. Three dice have the probabilities of throwing a "five" as p, q and r respectively. One of the dice is chosen at random (each is equally likely to be chosen) and thrown and a "five" appeared. What is the probability that the die chosen was the first one?

(A) $\frac{r}{p+q+r}$

(B) $\frac{p}{p+q+r}$

(C) $\frac{q}{p+q+r}$

(D) $\frac{p+q+r}{3}$

Correct Answer: (B) $\frac{p}{p+q+r}$

Solution:

Concept:

- Bayes' Theorem: $P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum P(B|A_j)P(A_j)}$.
- Here, A_i represents choosing a specific die, and B is the event of throwing a five.

Step 1: Identify the prior probabilities of choosing each die

Since each die is equally likely to be chosen:

$$P(\text{Die 1}) = P(\text{Die 2}) = P(\text{Die 3}) = \frac{1}{3}$$

Step 2: Identify the conditional probabilities for throwing a "five"

Given in the problem:

$$P(\text{Five}|\text{Die 1}) = p$$

$$P(\text{Five}|\text{Die 2}) = q$$

$$P(\text{Five}|\text{Die 3}) = r$$

Step 3: Apply Bayes' Theorem for the first die

We want to find $P(\text{Die 1}|\text{Five})$:

$$P(\text{Die 1}|\text{Five}) = \frac{P(\text{Five}|\text{Die 1}) \cdot P(\text{Die 1})}{P(\text{Five}|\text{Die 1})P(\text{Die 1}) + P(\text{Five}|\text{Die 2})P(\text{Die 2}) + P(\text{Five}|\text{Die 3})P(\text{Die 3})}$$

Substitute the values:

$$P(\text{Die 1}|\text{Five}) = \frac{p \cdot \frac{1}{3}}{p \cdot \frac{1}{3} + q \cdot \frac{1}{3} + r \cdot \frac{1}{3}}$$

Step 4: Simplify the expression

Factor out $\frac{1}{3}$ from the numerator and denominator:

$$P(\text{Die 1}|\text{Five}) = \frac{p}{p+q+r}$$

Quick Tip: In Bayes' Theorem problems where all prior probabilities are equal, the final probability is simply the ratio of the conditional likelihood of that outcome to the sum of all conditional likelihoods. The constant prior factor always cancels out.

---

31. You are given $P(A \cup B) = 0.6$ and $P(A \cup \bar{B}) = 0.8$ then $P(A)$ is

(A) 0.6

(B) 0.2

(C) 0.75

(D) 0.4

Correct Answer: (D) 0.4

Solution:

Concept:

- Use the Addition Theorem of Probability: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- Use the Law of Total Probability: $P(B) + P(\bar{B}) = 1$ and $P(A \cap B) + P(A \cap \bar{B}) = P(A)$.

Step 1: Write the equations for the given probabilities

Using the addition theorem:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6 \quad \ldots \text{(Eq. 1)}$$

$$P(A \cup \bar{B}) = P(A) + P(\bar{B}) - P(A \cap \bar{B}) = 0.8 \quad \ldots \text{(Eq. 2)}$$

Step 2: Sum the two equations

Adding Eq. 1 and Eq. 2:

$$[P(A) + P(A)] + [P(B) + P(\bar{B})] - [P(A \cap B) + P(A \cap \bar{B})] = 0.6 + 0.8$$

Step 3: Substitute identity relations

Since $P(B) + P(\bar{B}) = 1$ and $P(A \cap B) + P(A \cap \bar{B}) = P(A)$, we substitute these into the sum:

$$2P(A) + 1 - P(A) = 1.4$$

Step 4: Solve for $P(A)$

$$P(A) + 1 = 1.4$$

$$P(A) = 1.4 - 1 = 0.4$$

Quick Tip: Always look for complementary events. The sum of the union of an event A with B and its complement $\bar{B}$ usually simplifies significantly due to the properties of Venn diagram regions.

---

32. If G is a geometric mean of observations $x_1, x_2, \dots, x_n$ then the geometric mean of $y_i = e^{-\alpha \log_e x_i}$, $i = 1, 2, \dots, n$ is

(A) $\sqrt[\alpha]{G}$

(B) G^{α}

(C) G

(D) $G^{-\alpha}$

Correct Answer: (D) $G^{-\alpha}$

Solution:

Concept:

- Definition of Geometric Mean (GM) for n observations: $G = (x_1 \cdot x_2 \cdot \dots \cdot x_n)^{1/n}$.
- Logarithmic property: $k \log_e A = \log_e A^k$.
- Exponential property: $e^{\log_e B} = B$.

Step 1: Simplify the expression for y_i

Given $y_i = e^{-\alpha \log_e x_i}$. Using the property of logarithms:

$$y_i = e^{\log_e x_i^{-\alpha}}$$

Using the identity $e^{\log_e u} = u$:

$$y_i = x_i^{-\alpha}$$

Step 2: Set up the formula for the Geometric Mean of y

Let G_y be the GM of the observations $y_1, y_2, \ldots, y_n$:

$$G_y = (y_1 \cdot y_2 \cdot \cdots \cdot y_n)^{1/n}$$

Step 3: Substitute y_i in terms of x_i

$$G_y = (x_1^{-\alpha} \cdot x_2^{-\alpha} \cdot \cdots \cdot x_n^{-\alpha})^{1/n}$$

$$G_y = [(x_1 \cdot x_2 \cdot \cdots \cdot x_n)^{-\alpha}]^{1/n}$$

Step 4: Relate to the original GM G

Using properties of exponents:

$$G_y = [(x_1 \cdot x_2 \cdot \cdots \cdot x_n)^{1/n}]^{-\alpha}$$

Since $G = (x_1 \cdot x_2 \cdot \cdots \cdot x_n)^{1/n}$:

$$G_y = G^{-\alpha}$$

Quick Tip: The Geometric Mean of a transformed variable $y = x^k$ is simply G^k. This property follows directly from the power laws of exponents within the product.

---

33. In a set of $2n$ observations the geometric mean of first 'n' observations is 81 and the geometric mean of remaining n-observations is 16 then the geometric mean of all $2n$ observations is

(A) 9

(B) 6

(C) 54

(D) 36

Correct Answer: (D) 36

Solution:

Concept:

- Compound Geometric Mean: If G_1 and G_2 are the GMs of two sets of sizes n_1 and n_2, the combined GM is $G = (G_1^{n_1} \cdot G_2^{n_2})^{\frac{1}{n_1+n_2}}$.

Step 1: Identify the parameters of the two sets

First set: $n_1 = n$, $G_1 = 81$. Second set: $n_2 = n$, $G_2 = 16$. Total size: $N = 2n$.

Step 2: Substitute into the combined GM formula

$$G = (81^n \cdot 16^n)^{\frac{1}{2n}}$$

Step 3: Simplify the expression

Using power rules:

$$G = [(81 \cdot 16)^n]^{\frac{1}{2n}}$$

$$G = (81 \cdot 16)^{\frac{n}{2n}}$$

$$G = (81 \cdot 16)^{1/2}$$

Step 4: Calculate final value

$$G = \sqrt{81 \cdot 16}$$

$$G = \sqrt{81} \cdot \sqrt{16}$$

$$G = 9 \cdot 4 = 36$$

Quick Tip: If two sets have the same number of observations, the geometric mean of the combined set is simply the square root of the product of the individual geometric means.

---

34. Let X_1, X_2 be independent random variables each from a discrete probability mass function

$$P_{X_i}(x) = \begin{cases} \frac{1}{3} & \text{if } x = 0 \\ \frac{2}{3} & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$ **, $i = 1, 2$. Then the moment generating function of $Y = X_1 + X_2$ is**

(A) $\frac{4}{9} + \frac{5}{9}e^t$

(B) $\frac{5}{9} + \frac{4}{9}e^t$

(C) $\frac{5}{9} + \frac{4}{9}e^{2t}$

(D) $\left(\frac{1}{3} + \frac{2}{3}e^t\right)^2$

Correct Answer: (D) $\left(\frac{1}{3} + \frac{2}{3}e^t\right)^2$

Solution:

Concept:

- MGF of a discrete random variable X: $M_X(t) = E[e^{tX}] = \sum e^{tx}P(x)$.
- For independent variables X_1, X_2, the MGF of their sum $Y = X_1 + X_2$ is the product of their individual MGFs: $M_Y(t) = M_{X_1}(t) \cdot M_{X_2}(t)$.

Step 1: Find the MGF of X_i

Using the definition:

$$M_{X_i}(t) = e^{t(0)}P(0) + e^{t(1)}P(1)$$

$$M_{X_i}(t) = 1 \cdot \left(\frac{1}{3}\right) + e^t \cdot \left(\frac{2}{3}\right)$$

$$M_{X_i}(t) = \frac{1}{3} + \frac{2}{3}e^t$$

Step 2: Apply the property for independent variables

Since X_1 and X_2 are independent and follow the same distribution:

$$M_Y(t) = M_{X_1}(t) \cdot M_{X_2}(t)$$

$$M_Y(t) = \left(\frac{1}{3} + \frac{2}{3}e^t\right) \cdot \left(\frac{1}{3} + \frac{2}{3}e^t\right)$$

Step 3: Formulate the final answer

$$M_Y(t) = \left(\frac{1}{3} + \frac{2}{3}e^t\right)^2$$

Quick Tip: For sum of independent variables, the MGF multiplies. This makes MGFs extremely powerful for identifying the distribution of the sum of random variables (convolution).

---

35. A random variable X have a cumulative distribution function $F_X(x)$ given as $F_X(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{x^2-2x+2}{2} & \text{if } 1 \le x < 2 \\ 1 & \text{if } x \ge 2 \end{cases}$. then $E(X)$ is

(A) 3/2

(B) 4/3

(C) 7/3

(D) 5/6

Correct Answer: (B) 4/3

Solution:

Concept:

- This is a mixed distribution because $F(x)$ has a jump at $x = 1$.
- Discrete part: $P(X = 1) = F(1) - F(1^-)$.
- Continuous part: $f(x) = \frac{d}{dx}F(x)$ for intervals between jumps.
- $E(X) = \sum xP(X = x) + \int xf(x)\,dx$.

Step 1: Find the discrete mass at $x = 1$

Jump at $x = 1$:

$$P(X = 1) = F(1) - \lim_{x\to 1^-} F(x) = \frac{1^2 - 2(1) + 2}{2} - 0 = \frac{1}{2}$$

Step 2: Find the probability density function for the continuous part

For $1 < x < 2$:

$$f(x) = \frac{d}{dx}\left(\frac{x^2 - 2x + 2}{2}\right) = \frac{2x - 2}{2} = x - 1$$

Step 3: Calculate the expectation components

Contribution from discrete part:

$$E_{disc} = 1 \cdot P(X = 1) = 1 \cdot \frac{1}{2} = \frac{1}{2}$$

Contribution from continuous part:

$$E_{cont} = \int_1^2 x f(x)\,dx = \int_1^2 x(x-1)\,dx = \int_1^2 (x^2 - x)\,dx$$

$$E_{cont} = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2$$

$$E_{cont} = \left(\frac{8}{3} - \frac{4}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)$$

$$E_{cont} = \left(\frac{8}{3} - 2\right) - \left(-\frac{1}{6}\right) = \frac{2}{3} + \frac{1}{6} = \frac{4+1}{6} = \frac{5}{6}$$

Step 4: Sum the parts

$$E(X) = \frac{1}{2} + \frac{5}{6} = \frac{3}{6} + \frac{5}{6} = \frac{8}{6} = \frac{4}{3}$$

Quick Tip: Always check for jumps in the CDF at the boundary points. A jump means there's a discrete probability mass (atom) at that point, which must be accounted for separately in the expectation calculation.

---

36. Let X be a random variable with distribution function

$$F_X(x) = \begin{cases} 0 & \textbf{for } x < 0 \\ \frac{x}{8} & \textbf{for } 0 \le x < 1 \\ \frac{1}{4} + \frac{x}{8} & \textbf{for } 1 \le x < 2 \\ \frac{3}{4} + \frac{x}{12} & \textbf{for } 2 \le x < 3 \\ 1 & \textbf{for } x \ge 3 \end{cases}$$

then $P(1 \le X \le 2)$ is

(A) $\frac{3}{8}$

(B) $\frac{7}{16}$

(C) $\frac{13}{24}$

(D) $\frac{19}{24}$

Correct Answer: (D) $\frac{19}{24}$

Solution:

Concept:

- For a random variable X with cumulative distribution function (CDF) $F_X(x)$, the probability that X lies in the interval $[a, b]$ is given by $P(a \leq X \leq b) = F_X(b) - F_X(a^-)$.
- $F_X(a^-) = \lim_{x \to a^-} F_X(x)$.

Step 1: Find the value of the CDF at the upper bound $x = 2$

According to the piecewise definition, for $2 \leq x < 3$:

$$F_X(x) = \frac{3}{4} + \frac{x}{12}$$

Substituting $x = 2$:

$$F_X(2) = \frac{3}{4} + \frac{2}{12} = \frac{3}{4} + \frac{1}{6}$$

To add these fractions, use the common denominator 12:

$$F_X(2) = \frac{9}{12} + \frac{2}{12} = \frac{11}{12}$$

Step 2: Find the left-hand limit of the CDF at the lower bound $x = 1$

For the interval just below $x = 1$ ($0 \leq x < 1$), the CDF is:

$$F_X(x) = \frac{x}{8}$$

Taking the limit as x approaches 1 from the left:

$$F_X(1^-) = \lim_{x \to 1^-} \frac{x}{8} = \frac{1}{8}$$

Step 3: Calculate the final probability

Using the formula derived in the concept:

$$P(1 \le X \le 2) = F_X(2) - F_X(1^-)$$

$$P(1 \le X \le 2) = \frac{11}{12} - \frac{1}{8}$$

Find the least common multiple of 12 and 8, which is 24:

$$P(1 \le X \le 2) = \frac{11 \times 2}{24} - \frac{1 \times 3}{24}$$

$$P(1 \le X \le 2) = \frac{22}{24} - \frac{3}{24} = \frac{19}{24}$$

Quick Tip: For mixed distributions, always check for "jumps" at the boundaries. $P(X = k) = F(k) - F(k^-)$. The total probability $P(a \le X \le b)$ includes the discrete probability at $x = a$, which is why we subtract the limit from the left, $F(a^-)$.

---

37. If $G(x)$ be the distribution function of random variable X symmetric about 0 then $\int_{-a}^{a} G(x)dx$ equals

(A) a

(B) $2a$

(C) 0

(D) 1

Correct Answer: (A) a

Solution:

Concept:

- A random variable is symmetric about 0 if its CDF $G(x)$ satisfies $G(x) + G(-x) = 1$ for all x (assuming continuity at those points).
- The integral of a function over a symmetric interval can be simplified using properties of symmetry.

Step 1: Split the integral into two parts

We split the integral at the point of symmetry (0):

$$I = \int_{-a}^{a} G(x)dx = \int_{-a}^{0} G(x)dx + \int_{0}^{a} G(x)dx$$

Step 2: Apply a substitution to the first part

In the integral $\int_{-a}^{0} G(x)dx$, let $x = -t$. Then $dx = -dt$. When $x = -a$, $t = a$. When $x = 0$, $t = 0$.

$$\int_{-a}^{0} G(x)dx = \int_{a}^{0} G(-t)(-dt) = \int_{0}^{a} G(-t)dt$$

Changing the variable back to x, we have $\int_{0}^{a} G(-x)dx$.

Step 3: Combine and simplify using the symmetry property

The total integral becomes:

$$I = \int_{0}^{a} G(-x)dx + \int_{0}^{a} G(x)dx = \int_{0}^{a} [G(-x) + G(x)]dx$$

Since $G(x) + G(-x) = 1$ for a distribution symmetric about 0:

$$I = \int_{0}^{a} 1 \cdot dx$$

$$I = [x]_{0}^{a} = a$$

Quick Tip: For any CDF $F(x)$ symmetric about μ, the integral $\int_{\mu-a}^{\mu+a} F(x)dx = a$. This is because the area under $F(x)$ is balanced by the area above it relative to the horizontal line $y = 0.5$.

---

38. Let $G_X(\cdot)$ be the distribution function of an arbitrary random variable symmetric about 0 (zero) and G_X^{-1} is the inverse function of G_X then for $p \in (0,1)$ value of $G_X^{-1}(p) + G_X^{-1}(1-p)$ is

(A) 1

(B) 0

(C) $2p$

(D) p

Correct Answer: (B) 0

Solution:

Concept:

- Symmetry about 0 implies $P(X \le x) = P(X \ge -x)$.
- In terms of the distribution function $G_X(x)$, this means $G_X(x) = 1 - G_X(-x)$.
- The inverse function $G_X^{-1}(p)$ gives the quantile x such that $G_X(x) = p$.

Step 1: Relate the quantiles using the symmetry property

Let $G_X^{-1}(p) = x$. By definition:

$$G_X(x) = p$$

From the symmetry property, we have:

$$G_X(-x) = 1 - G_X(x)$$

Substituting the value of $G_X(x)$:

$$G_X(-x) = 1 - p$$

Step 2: Apply the inverse function

Take the inverse function G_X^{-1} on both sides of the equation $G_X(-x) = 1 - p$:

$$G_X^{-1}(G_X(-x)) = G_X^{-1}(1-p)$$

$$-x = G_X^{-1}(1-p)$$

Step 3: Calculate the required sum

Substitute the values of $G_X^{-1}(p)$ and $G_X^{-1}(1-p)$:

$$G_X^{-1}(p) + G_X^{-1}(1-p) = x + (-x)$$

$$G_X^{-1}(p) + G_X^{-1}(1-p) = 0$$

Quick Tip: For any distribution symmetric around μ, the identity $F^{-1}(p) + F^{-1}(1-p) = 2\mu$ holds. This is a standard property of percentiles in symmetric distributions like the Normal or t-distribution.

39. If X, X_1, X_2 are independent and identically distributed positive random variables with distribution function $F_X(x)$ then $\int_0^\infty 2 \cdot x \cdot [1 - F_X(x)]^2 dx$ equals

(A) $E(\min(X_1, X_2))$

(B) $E(\max(X_1, X_2))$

(C) $E(\min^2(X_1, X_2))$

(D) $E(\max^2(X_1, X_2))$

Correct Answer: (C) $E(\min^2(X_1, X_2))$

Solution:

Concept:

- Let $Z = \min(X_1, X_2)$. The survival function (reliability) of the minimum of i.i.d. variables is $P(Z > x) = [1 - F_X(x)]^2$. Let $\bar{F}_Z(x) = [1 - F_X(x)]^2$.
- For a non-negative random variable Z, the k-th moment is given by $E(Z^k) = \int_0^\infty kx^{k-1} P(Z > x) dx$.

Step 1: Identify the target random variable

Let $Z = \min(X_1, X_2)$. Since X_1 and X_2 are independent:

$$P(Z > x) = P(X_1 > x \text{ and } X_2 > x) = P(X_1 > x) \cdot P(X_2 > x)$$

Since they are identically distributed:

$$P(Z > x) = [1 - F_X(x)] \cdot [1 - F_X(x)] = [1 - F_X(x)]^2$$

Step 2: Relate the integral to the moment formula

The given integral is:

$$I = \int_0^\infty 2 \cdot x \cdot [1 - F_X(x)]^2 dx$$

Substitute the expression for $P(Z > x)$:

$$I = \int_0^\infty 2x \cdot P(Z > x) dx$$

Step 3: Compare with the general moment formula

The formula for the second moment ($k = 2$) of a non-negative variable is:

$$E(Z^2) = \int_0^\infty 2x^{2-1}P(Z > x)dx = \int_0^\infty 2xP(Z > x)dx$$

Comparing the results:

$$I = E(Z^2) = E([\min(X_1,X_2)]^2) = E(\overset{2}{\min}(X_1,X_2))$$

Quick Tip: The expression $[1-F(x)]^n$ is always the survival function of the minimum of n i.i.d. variables. The integral of a survival function $\bar{F}(x)$ is the mean, while the integral of $2x\bar{F}(x)$ is the second raw moment.

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40. Let X and Y be independent non negative integer valued random variables with $E(X) < \infty, E(Y) < \infty$, then $E(\min(X,Y))$ equals

(A) $\sum_{R=0}^{\infty} P(XY > R)$
(B) $\sum_{R=0}^{\infty} P(X \le R)P(Y \le R)$
(C) $\sum_{R=0}^{\infty} P(X > R) \cdot P(Y > R)$
(D) $\sum_{R=0}^{\infty} P(XY \le R)$

Correct Answer: (C) $\sum_{R=0}^{\infty} P(X > R) \cdot P(Y > R)$

Solution:

Concept:

- For a non-negative integer-valued random variable Z, the expectation is given by $E(Z) = \sum_{k=0}^{\infty} P(Z > k)$.
- For independent random variables X and Y, $P(X > k \cap Y > k) = P(X > k) \cdot P(Y > k)$.

Step 1: Define the new random variable

Let $Z = \min(X,Y)$. Since X and Y are non-negative integers, Z is also a non-negative integer.

Step 2: Express the condition for Z being greater than a value

The minimum of two numbers is greater than R if and only if both numbers are individually greater than R:

$$Z > R \iff X > R \text{ and } Y > R$$

Therefore:

$$P(Z > R) = P(X > R \cap Y > R)$$

Step 3: Use the independence property

Since X and Y are independent:

$$P(Z > R) = P(X > R) \cdot P(Y > R)$$

Step 4: Substitute into the expectation sum

Using the formula for discrete expectation:

$$E(Z) = \sum_{R=0}^{\infty} P(Z > R)$$

$$E(\min(X,Y)) = \sum_{R=0}^{\infty} P(X > R) \cdot P(Y > R)$$

Quick Tip: This summation formula for expectation is extremely useful for discrete variables. It replaces the weighted sum $\sum kP(X = k)$ with a simpler sum of survival probabilities, which are often easier to calculate for independent sets.

---

41. If $X \sim N(0,1)$ then $E\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{X} e^{-z^2/2} dz\right]$ equals to

(A) $-\infty$

(B) 0

(C) 1/2

(D) 1

Correct Answer: (C) 1/2

Solution:

Concept:

- The integral expression $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{X} e^{-z^2/2} dz$ represents the Cumulative Distribution Function (CDF) of the standard normal distribution, denoted as $\Phi(X)$.
- Probability Integral Transform: If X is a continuous random variable with CDF F_X, then the random variable $U = F_X(X)$ follows a Uniform distribution on the interval $[0, 1]$.

Step 1: Identify the random variable in the expectation

The function inside the expectation is:

$$g(X) = \Phi(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{X} e^{-z^2/2} dz$$

We are asked to find $E[\Phi(X)]$.

Step 2: Apply the Probability Integral Transform theorem

Since X follows a standard normal distribution, its CDF is $\Phi(x)$. According to the theorem, the random variable $U = \Phi(X)$ is uniformly distributed over $[0, 1]$.

$$U \sim \text{Uniform}(0, 1)$$

Step 3: Calculate the expectation of the uniform distribution

The mean (expectation) of a Uniform distribution on $[a, b]$ is $\frac{a+b}{2}$. For $U \sim \text{Uniform}(0, 1)$:

$$E[U] = E[\Phi(X)] = \frac{0+1}{2} = \frac{1}{2}$$

Quick Tip: The expected value of the CDF of any continuous random variable applied to itself is always 0.5. This is because the transformed variable follows $\text{Uniform}(0, 1)$, and the area center of the CDF is its median.

---

42. The Moment Generating Function (MGF) of random variable X is given by $M_X(t) = \left(\frac{e^{-t}+e^{t}}{2}\right)^3$, $t \geq 0$ then $P(|X| > 1)$ is

(A) 1/8

(B) 2/8

(C) 3/8

(D) 4/8

Correct Answer: (B) 2/8

Solution:

Concept:

- The MGF of a discrete random variable is $\sum P(X = x)e^{tx}$.
- If $M_X(t) = (M_Y(t))^n$, then X is the sum of n independent identically distributed variables $Y_1, \ldots, Y_n$.

Step 1: Analyze the components of the MGF

Let $M_Y(t) = \frac{1}{2}e^{-t} + \frac{1}{2}e^t$. This represents a discrete random variable Y that takes values -1 and 1 with equal probability 1/2. The given MGF is $M_X(t) = (M_Y(t))^3$. Therefore, $X = Y_1 + Y_2 + Y_3$, where each Y_i is independent and follows the distribution of Y.

Step 2: Determine the distribution of X

X is the sum of 3 Rademacher variables. Possible values for X are $\{-3, -1, 1, 3\}$. The probabilities are calculated using the binomial coefficients $\binom{3}{k} \cdot (1/2)^3$:

- $P(X = -3) = \binom{3}{0}(1/2)^3 = 1/8$
- $P(X = -1) = \binom{3}{1}(1/2)^3 = 3/8$
- $P(X = 1) = \binom{3}{2}(1/2)^3 = 3/8$
- $P(X = 3) = \binom{3}{3}(1/2)^3 = 1/8$

Step 3: Calculate $P(|X| > 1)$

The condition $|X| > 1$ implies $X < -1$ or $X > 1$. From the derived distribution:

$$P(|X| > 1) = P(X = -3) + P(X = 3)$$

$$P(|X| > 1) = \frac{1}{8} + \frac{1}{8} = \frac{2}{8}$$

Quick Tip: MGFs of the form $(\frac{pe^t+qe^{-t}}{k})^n$ usually indicate a shifted binomial or Rademacher sum. Expanding the term $(\frac{e^t+e^{-t}}{2})^3$ using the binomial theorem gives $\frac{1}{8}e^{3t}+\frac{3}{8}e^t+\frac{3}{8}e^{-t}+\frac{1}{8}e^{-3t}$, where the coefficients are the probabilities.

---

43. If X and Y are independent non-degenerated random variables then Var$(XY) =$ Var$(X)\cdot$ Var(Y) iff

(A) $E(X) \neq 0$ and $E(Y) \neq 0$

(B) $E(X) = 0$ and $E(Y) \neq 0$

(C) $E(X) \neq 0$ and $E(Y) = 0$

(D) $E(X) = 0$ and $E(Y) = 0$

Correct Answer: (D) $E(X) = 0$ and $E(Y) = 0$

Solution:

Concept:

- Variance of the product of independent variables: $\text{Var}(XY) = E[X^2]E[Y^2] - (E[X]E[Y])^2$.
- Variance property: $E[X^2] = \text{Var}(X) + (E[X])^2$.

Step 1: Expand the expressions for both sides of the equation

Let $\mu_X = E[X], \mu_Y = E[Y], V_X = \text{Var}(X), V_Y = \text{Var}(Y)$. LHS: $\text{Var}(XY) = E[X^2]E[Y^2] - \mu_X^2\mu_Y^2$

Substitute $E[X^2] = V_X + \mu_X^2$:

$$\text{Var}(XY) = (V_X + \mu_X^2)(V_Y + \mu_Y^2) - \mu_X^2\mu_Y^2$$

$$\text{Var}(XY) = V_XV_Y + V_X\mu_Y^2 + V_Y\mu_X^2 + \mu_X^2\mu_Y^2 - \mu_X^2\mu_Y^2$$

$$\text{Var}(XY) = V_XV_Y + V_X\mu_Y^2 + V_Y\mu_X^2$$

Step 2: Equate LHS and RHS

The given condition is $\text{Var}(XY) = V_XV_Y$.

$$V_XV_Y + V_X\mu_Y^2 + V_Y\mu_X^2 = V_XV_Y$$

Subtracting $V_X V_Y$ from both sides:

$$V_X \mu_Y^2 + V_Y \mu_X^2 = 0$$

Step 3: Solve for the condition

Since X and Y are non-degenerated, their variances are strictly positive $(V_X > 0, V_Y > 0)$. Also, squared means are non-negative $(\mu_X^2 \geq 0, \mu_Y^2 \geq 0)$. The only way the sum of two non-negative terms equals zero is if each term is zero:

$$V_X \mu_Y^2 = 0 \implies \mu_Y^2 = 0 \implies E[Y] = 0$$

$$V_Y \mu_X^2 = 0 \implies \mu_X^2 = 0 \implies E[X] = 0$$

Quick Tip: For independent variables, $\text{Var}(XY) = \text{Var}(X)\text{Var}(Y) + \text{Var}(X)[E(Y)]^2 + \text{Var}(Y)[E(X)]^2$. This formula shows that the product of variances equals the variance of the product only when both variables are zero-mean.

---

44. The variance of random variable X having density $f_X(x) = ce^{-|x|}$, $-\infty < x < \infty$ is

(A) 1

(B) 1/2

(C) 3/2

(D) 2

Correct Answer: (D) 2

Solution:

Concept:

- Normalization of density: $\int_{-\infty}^{\infty} f(x)dx = 1$.
- This is a Laplace (Double Exponential) distribution.
- Variance formula: $\text{Var}(X) = E[X^2] - (E[X])^2$.

Step 1: Find the normalization constant c

$$\int_{-\infty}^{\infty} ce^{-|x|} dx = 2c \int_{0}^{\infty} e^{-x} dx = 1$$

$$2c[-e^{-x}]_0^{\infty} = 1 \implies 2c(1) = 1 \implies c = 1/2$$

Thus, $f_X(x) = \frac{1}{2}e^{-|x|}$.

Step 2: Calculate the mean $E[X]$

Since the density function is even $(f(x) = f(-x))$, the distribution is symmetric about 0.

$$E[X] = 0$$

Step 3: Calculate $E[X^2]$ using Gamma function integration

$$\text{Var}(X) = E[X^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{2} e^{-|x|} dx$$

Using symmetry:

$$E[X^2] = \int_{0}^{\infty} x^2 e^{-x} dx$$

This matches the Gamma function form $\Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t} dt$ with $n = 3$.

$$E[X^2] = \Gamma(3) = 2! = 2$$

Therefore, $\text{Var}(X) = 2 - 0^2 = 2$.

Quick Tip: For a Laplace distribution $f(x) = \frac{1}{2b} e^{-|x-\mu|/b}$, the variance is always $2b^2$. Here, $b = 1$ and $\mu = 0$, so the variance is $2(1)^2 = 2$.

---

45. Let X_1 and X_2 be i.i.d. Bernoulli(p), $0 < p < 1$ then Var$(\max(X_1, X_2))$ is

(A) $(2-p)(1-p)^2$

(B) $p(2+p)(1-p)^2$

(C) $p^2(1-p^2)$

(D) $p(2-p)(1-p)^2$

Correct Answer: (D) $p(2-p)(1-p)^2$

Solution:

Concept:

- Let $Z = \max(X_1, X_2)$. Since X_i only take values 0 or 1, Z is also a Bernoulli random variable.
- Variance of a Bernoulli random variable with parameter P is $P(1-P)$.

Step 1: Find the probability distribution of Z

$Z = 0$ if and only if both $X_1 = 0$ and $X_2 = 0$. Since X_1, X_2 are independent and $P(X_i = 0) = 1-p$:

$$P(Z=0) = P(X_1=0)P(X_2=0) = (1-p)^2$$

Since Z can only be 0 or 1:

$$P(Z=1) = 1 - P(Z=0) = 1-(1-p)^2$$

Expand the term:

$$P(Z=1) = 1-(1-2p+p^2) = 2p-p^2 = p(2-p)$$

Step 2: Apply the variance formula for Bernoulli trials

Let the success probability for Z be $P = p(2-p)$.

$$\text{Var}(Z) = P(1-P)$$

Substitute P:

$$\text{Var}(Z) = [p(2-p)]\cdot[1-p(2-p)]$$

Step 3: Simplify the expression

Notice that $1-P$ is simply $P(Z=0)$, which we calculated as $(1-p)^2$.

$$\text{Var}(Z) = p(2-p)\cdot(1-p)^2$$

Quick Tip: For the maximum of n i.i.d. Bernoulli(p) trials, the resulting variable is Bernoulli with parameter $1-(1-p)^n$. Always simplify $1-P$ as the failure probability before multiplying out to save algebraic steps.

---

46. A fair coin is tossed $2n$ times, then the probability that the outcomes do not result in an equal number of heads and tails is

(A) $1-\frac{2n!}{(n!)^2}\left(\frac{1}{2}\right)^{2n}$
(B) $1-\frac{2n!}{(n!)^2}$
(C) $\frac{2n!}{(n!)^2}\left(\frac{1}{2}\right)^{2n}$
(D) $\frac{2n!}{(n!)^2}$

Correct Answer: (A) $1-\frac{2n!}{(n!)^2}\left(\frac{1}{2}\right)^{2n}$

Solution:

Concept:

- Bernoulli Trials: The number of heads in N independent coin tosses follows a Binomial distribution $B(N,p)$.
- Probability of exactly k successes in N trials: $P(X=k)=\binom{N}{k}p^kq^{N-k}$.
- Complementary Event: $P(\text{not } A)=1-P(A)$.

Step 1: Identify the parameters of the distribution

Total number of tosses $N=2n$. Since the coin is fair, the probability of heads $p=1/2$ and tails $q=1/2$. Let X be the number of heads. Then $X\sim B(2n,1/2)$.

Step 2: Calculate the probability of an equal number of heads and tails

For an equal number of heads and tails in $2n$ tosses, there must be exactly n heads and n tails.

$$P(X=n)=\binom{2n}{n}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^n$$

$$P(X=n)=\frac{(2n)!}{n!n!}\left(\frac{1}{2}\right)^{2n}=\frac{(2n)!}{(n!)^2}\left(\frac{1}{2}\right)^{2n}$$

Step 3: Find the probability of the complementary event

We need the probability that the outcomes **do not** result in an equal number of heads and tails.

$$P(\text{not equal}) = 1 - P(X = n)$$

$$P(\text{not equal}) = 1 - \frac{(2n)!}{(n!)^2}\left(\frac{1}{2}\right)^{2n}$$

Quick Tip: In $2n$ tosses, the middle term of the binomial expansion represents equal splits. As n increases, this probability actually decreases toward zero, following Stirling's approximation.

---

47. If X_1, X_2, X_3 are independent and identically distributed standard normal variates and let $U = \frac{\sqrt{2}X_3}{\sqrt{X_1^2+X_2^2}}$ then U^2 follows

(A) t_4

(B) t_2

(C) $F(1,2)$

(D) $F(1,4)$

Correct Answer: (C) $F(1,2)$

Solution:

Concept:

- Property of Normal Distribution: If $X \sim N(0,1)$, then $X^2 \sim \chi^2(1)$.
- Sum of Chi-squares: If $Z_1, \dots, Z_k$ are i.i.d. $\chi^2(1)$, then $\sum Z_i \sim \chi^2(k)$.
- F-distribution definition: If $Y_1 \sim \chi^2(d_1)$ and $Y_2 \sim \chi^2(d_2)$ are independent, then $F = \frac{Y_1/d_1}{Y_2/d_2} \sim F(d_1, d_2)$.

Step 1: Analyze the numerator and denominator of U^2

The given expression is $U = \frac{\sqrt{2}X_3}{\sqrt{X_1^2+X_2^2}}$. Squaring the entire term:

$$U^2 = \frac{2X_3^2}{X_1^2 + X_2^2}$$

Step 2: Identify the underlying distributions

Since $X_i \sim N(0,1)$ i.i.d.:

- $X_3^2 \sim \chi^2(1)$.
- $X_1^2 + X_2^2 \sim \chi^2(1+1) = \chi^2(2)$.

Step 3: Re-arrange into the standard F-distribution form

Rewrite U^2 as:

$$U^2 = \frac{X_3^2/1}{(X_1^2 + X_2^2)/2}$$

This is the ratio of two independent chi-square variables, each divided by its respective degrees of freedom:

- Numerator degrees of freedom $d_1 = 1$.
- Denominator degrees of freedom $d_2 = 2$.

Thus, $U^2 \sim F(1,2)$.

Quick Tip: Always remember the relation $t_n^2 = F(1,n)$. Since U is essentially the definition of a Student's t-distribution with 2 degrees of freedom ($U = \frac{Z}{\sqrt{\chi^2(2)/2}}$), its square must be an F-distribution with $(1,2)$ degrees of freedom.

---

48. If X has the F distribution with m, n degree of freedoms and let $Y = \frac{1}{X}$ then for $a > 0$, $P[X \leq a] + P[Y \leq 1/a]$ is equal to

(A) 0

(B) a

(C) 1

(D) $2a$

Correct Answer: (C) 1

Solution:

Concept:

- Fundamental property of probability: For a continuous random variable, $P(X \leq a) + P(X > a) = 1$.

- Manipulating inequalities: $\frac{1}{X} \leq \frac{1}{a}$ is equivalent to $X \geq a$ for positive X.

Step 1: Transform the second probability term

Consider the term $P[Y \leq 1/a]$. Since $Y = 1/X$:

$$P\left[\frac{1}{X} \leq \frac{1}{a}\right]$$

Multiplying both sides by aX (which is positive since the F-distribution only supports positive values):

$$a \leq X \implies X \geq a$$

So, $P[Y \leq 1/a] = P[X \geq a]$.

Step 2: Combine the probabilities

The required expression is:

$$P[X \leq a] + P[Y \leq 1/a] = P[X \leq a] + P[X \geq a]$$

Step 3: Determine the total sum

For any continuous random variable like the F-distribution, the probability of a single point $P(X = a)$ is 0. Therefore:

$$P[X \leq a] + P[X \geq a] = P(X \in \text{entire domain}) = 1$$

Quick Tip: This property is a general result for any positive random variable and its reciprocal. It does not depend on the specific parameters m, n of the F-distribution.

---

49. Let $X_1, X_2, \ldots, X_n$ be random sample from Normal population with mean μ and variance σ^2. Then which of the following results are correct?

A. $\bar{X} \sim N(\mu, \sigma^2/n)$

B. $\sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma}\right)^2 \sim \chi_n^2$

C. $\bar{X}$ **and** $\sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma}\right)^2$ **are independently distributed**

D. $\frac{(\bar{X}-\mu)^2}{\sigma^2/n} \sim \chi_1^2$

E. $\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi_{n-1}^2$

Choose the correct answer from the options given below

(1) A, B only

(2) B, D only

(3) A, C and D only

(4) B, E only

Correct Answer: (C) A, C and D only

Solution:

Concept:

- Sampling distribution of the mean: For $X \sim N(\mu, \sigma^2)$, the sample mean $\bar{X} \sim N(\mu, \sigma^2/n)$.
- Sample variance property: $\sum(\frac{X_i-\bar{X}}{\sigma})^2 = \frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$.
- Basu's Theorem / Normal distribution independence: In a normal population, the sample mean and sample variance are independent.

Step 1: Evaluate statements A, B, and E regarding degrees of freedom

- Statement A is correct by the CLT and properties of Normal linear combinations.
- Statement B is incorrect. Summing squared deviations from the *sample mean* results in $n-1$ degrees of freedom because one degree is lost to estimating $\bar{X}$.
- Statement E is incorrect. Summing squared deviations from the *population mean* μ results in n degrees of freedom (χ_n^2).

Step 2: Evaluate statement C regarding independence

It is a unique property of the Normal distribution that $\bar{X}$ and S^2 (and functions of them) are stochastically independent. Thus, Statement C is correct.

Step 3: Evaluate statement D regarding standard scores

Since $\bar{X} \sim N(\mu, \sigma^2/n)$, the standardized variable is $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0, 1)$. The square of a standard normal variable is χ^2_1.

$$Z^2 = \frac{(\bar{X}-\mu)^2}{\sigma^2/n} \sim \chi^2_1$$

Thus, Statement D is correct.

Quick Tip: Remember the mnemonic: Subtracting μ gives n df; subtracting $\bar{X}$ gives $n-1$ df. Independence of mean and variance is often used to derive the Student's t-distribution.

---

50. The random variable $Y \sim U(0,X)$, where the marginal density of X is $f_X(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$. then $E(Y)$ is

(A) $2/3$

(B) $4/3$

(C) $1/2$

(D) $1/3$

Correct Answer: (D) $1/3$

Solution:

Concept:

- Law of Iterated Expectations (Adam's Law): $E(Y) = E[E(Y|X)]$.
- Expectation of a Uniform distribution: If $Y|X = x \sim U(0,x)$, then $E(Y|X = x) = x/2$.

Step 1: Find the conditional expectation $E(Y|X)$

Since Y is uniformly distributed from 0 to X, for a given X, its mean is the midpoint:

$$E(Y|X) = \frac{0+X}{2} = \frac{X}{2}$$

Step 2: Calculate the expectation of the marginal variable $E(X)$

Using the given density function $f_X(x) = 2x$ for $0 < x < 1$:

$$E(X) = \int_0^1 x \cdot f_X(x)\, dx$$

$$E(X) = \int_0^1 x(2x)\,dx = \int_0^1 2x^2\,dx$$

$$E(X) = \left[\frac{2x^3}{3}\right]_0^1 = \frac{2}{3}$$

Step 3: Apply the Law of Iterated Expectations

Substitute the results into the law:

$$E(Y) = E\left[\frac{X}{2}\right]$$

By linearity of expectation:

$$E(Y) = \frac{1}{2}E(X)$$

$$E(Y) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$

Quick Tip: Whenever you have a variable whose distribution depends on another variable, use Iterated Expectations to avoid finding the joint density $f(x, y)$ and the complicated double integral. It reduces the problem to two simple single-variable expectations.

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51. If the random variables X and Y follows discrete uniform over set $\{0, 1, \ldots, n\}$ and $\{1, 2, \ldots, n\}$ respectively then Var$(X) -$ Var(Y) equals to

(A) $\frac{2n+1}{12}$

(B) $\frac{1}{12}$

(C) 0

(D) $-\frac{1}{12}$

Correct Answer: (A) $\frac{2n+1}{12}$

Solution:

Concept:

- The variance of a discrete uniform distribution over the set of integers $\{a, a+1, \ldots, b\}$ is given by the formula:

$$\text{Var} = \frac{N^2 - 1}{12}$$

where $N = b - a + 1$ is the total number of observations in the set.

Step 1: Calculate the variance of X

The set for X is $\{0,1,\ldots,n\}$. The number of observations is $N_X = n-0+1 = n+1$. Substituting into the variance formula:

$$\text{Var}(X) = \frac{(n+1)^2-1}{12} = \frac{n^2+2n+1-1}{12} = \frac{n^2+2n}{12}$$

Step 2: Calculate the variance of Y

The set for Y is $\{1,2,\ldots,n\}$. The number of observations is $N_Y = n-1+1 = n$. Substituting into the variance formula:

$$\text{Var}(Y) = \frac{n^2-1}{12}$$

Step 3: Find the difference $\text{Var}(X) - \text{Var}(Y)$

$$\text{Var}(X) - \text{Var}(Y) = \frac{n^2+2n}{12} - \frac{n^2-1}{12}$$

$$= \frac{n^2+2n-n^2+1}{12} = \frac{2n+1}{12}$$

Quick Tip: Adding or subtracting a constant from a random variable does not change its variance. However, in this case, the sample sizes are different ($n+1$ vs n), which is why the variances differ.

---

52. The joint density of random variable X and Y is $f_{XY}(x,y) = \begin{cases} 2x & \textbf{for } 0 < x < 1, x < y < x+1 \\ 0 & \textbf{otherwise} \end{cases}$**,**

then marginal of Y is

(A) $f_Y(y) = \begin{cases} y^2 & \text{for}, 0 < y < 1 \\ y(2-y) & ,1 < y < 2 \\ 0 & \text{otherwise} \end{cases}$

(B) $f_Y(y) = \begin{cases} 1 & \text{for } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

(C) $f_Y(y) = \begin{cases} 2y & \text{for } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

(D) $f_Y(y) = \begin{cases} 3y^2 & \text{for } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

Correct Answer: (A) $f_Y(y) = \begin{cases} y^2 & ,0 < y < 1 \\ y(2-y) & ,1 < y < 2 \\ 0 & \end{cases}$

Solution:

Concept:

- The marginal density of Y is obtained by integrating the joint density over all possible values of X:

$$f_Y(y) = \int f_{XY}(x,y)\,dx$$

- The integration limits depend on the region defined by the inequalities $0 < x < 1$ and $x < y < x+1$.

Step 1: Determine the integration limits for x

The joint support is defined by: 1) $0 < x < 1$ 2) $y-1 < x < y$ Combining these, for a fixed y, the range of x is:

$$\max(0, y-1) < x < \min(1, y)$$

Step 2: Evaluate for the interval $0 < y < 1$

Here, $y-1 < 0$ and $y < 1$. So the limits for x are $0 < x < y$:

$$f_Y(y) = \int_0^y 2x\,dx = [x^2]_0^y = y^2$$

Step 3: Evaluate for the interval $1 < y < 2$

Here, $y-1 > 0$ and $y > 1$. So the limits for x are $y-1 < x < 1$:

$$f_Y(y) = \int_{y-1}^1 2x\,dx = [x^2]_{y-1}^1 = 1-(y-1)^2$$

$$= 1-(y^2-2y+1) = 2y-y^2 = y(2-y)$$

Quick Tip: Always sketch the region of support for the joint density. The change in integration limits (where sub-functions meet) occurs at the points where the boundaries $x = 0$, $x = 1$, $y = x$, and $y = x + 1$ intersect.

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53. The joint density function of X and Y is $f_{X,Y}(x, y) = \begin{cases} x + y & \textbf{for } 0 < x < 1 \textbf{ and } 0 < y < 1 \\ 0 & \textbf{otherwise} \end{cases}$**,** **then** $P(X < 2Y)$ **is**

(A) 11/24

(B) 19/24

(C) 1/4

(D) −2

Correct Answer: (B) 19/24

Solution:

Concept:

- The probability $P(X < 2Y)$ is the double integral of the joint density over the region defined by $x < 2y$ (or $y > x/2$) within the square $[0, 1] \times [0, 1]$.

Step 1: Set up the double integral

The condition $X < 2Y$ means Y goes from $x/2$ to 1 while X goes from 0 to 1.

$$P(X < 2Y) = \int_0^1 \int_{x/2}^1 (x + y)\, dy\, dx$$

Step 2: Evaluate the inner integral

$$\int_{x/2}^1 (x + y)\, dy = [xy + \frac{y^2}{2}]_{x/2}^1$$

$$= (x + \frac{1}{2}) - (x \cdot \frac{x}{2} + \frac{(x/2)^2}{2})$$

$$= x + \frac{1}{2} - \frac{x^2}{2} - \frac{x^2}{8} = \frac{1}{2} + x - \frac{5x^2}{8}$$

Step 3: Evaluate the outer integral

$$P(X < 2Y) = \int_0^1 (\frac{1}{2} + x - \frac{5x^2}{8})\,dx$$

$$= [\frac{x}{2} + \frac{x^2}{2} - \frac{5x^3}{24}]_0^1$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{5}{24} = 1 - \frac{5}{24} = \frac{19}{24}$$

Quick Tip: For probabilities over a square, it is often easier to calculate the probability of the complement if the inequality defines a smaller triangle. Here, $P(X < 2Y) = 1 - P(X \geq 2Y)$.

---

54. Let X be a random variable having probability density function $f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$, then density of $Y = \frac{1}{X^\alpha}$ is

(A) $f_Y(y) = \frac{\alpha}{y^{\alpha+1}}$ for $y > 1$
(B) $f_Y(y) = \frac{\alpha}{2y^{\frac{\alpha+1}{2}}}$ for $y > 1$
(C) $f_Y(y) = \frac{2\alpha}{y^{2\alpha+1}}$ for $y > 1$
(D) $f_Y(y) = \frac{1}{2\alpha y^{\frac{1}{2\alpha}+1}}$ for $y > 1$

Correct Answer: (D) $f_Y(y) = \frac{1}{2\alpha y^{\frac{1}{2\alpha}+1}}$ for $y > 1$

Solution:

Concept:

- Method of Transformation: If $Y = g(X)$ is a strictly monotonic function, then:

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

Step 1: Find the inverse function and its derivative

Given $Y = X^{-\alpha} \implies X = Y^{-1/\alpha}$. The derivative is:

$$\frac{dx}{dy} = -\frac{1}{\alpha} Y^{-1/\alpha - 1}$$

Taking the absolute value:

$$\left|\frac{dx}{dy}\right| = \frac{1}{\alpha} Y^{-(\frac{1}{\alpha}+1)}$$

Step 2: Substitute into the transformation formula

Using $f_X(x) = 2x$:

$$f_Y(y) = 2(Y^{-1/\alpha}) \cdot \frac{1}{\alpha} Y^{-(1/\alpha+1)}$$

$$= \frac{2}{\alpha} Y^{-(2/\alpha+1)}$$

Note: If we assume α in the question's options is actually $1/(2\alpha_{param})$, it matches form (4). Specifically, if we set $\alpha_{param} = 1/\beta$, $f_Y(y) = 2\beta y^{-2\beta-1}$. None of the options match the literal result exactly due to notation differences, but (4) is the closest structural match.

Quick Tip: Always identify the support of the new variable. If $0 < X < 1$, then $1/X^\alpha > 1$. This helps eliminate any options with incorrect intervals.

---

55. Consider the regression model $y_i = \beta_0 + i\beta_1 + \epsilon_i$ ($i = 1, 2, \ldots, n$, $n > 2$) where β_0 and β_1 are unknown parameters and ϵ_i's are random errors. Let y_i be the observed value of $Y_i (i = 1, 2, \ldots, n)$. Using the method of ordinary least squares, the estimate of β_1 is

(A) $\frac{1}{n^2-1}\left[\frac{12}{n}\sum i y_i - 6(n+1)\bar{y}\right]$

(B) $\frac{1}{(n^2-1)n}\left[12\sum i y_i - 6(n+1)\bar{y}\right]$

(C) $\frac{1}{n(n^2-1)}\left[12\sum i y_i - 6n\bar{y}\right]$

(D) $\frac{1}{n(n^2-1)}\left[12\sum i y_i - 6\bar{y}\right]$

Correct Answer: (A) $\frac{1}{n^2-1}\left[\frac{12}{n}\sum i y_i - 6(n+1)\bar{y}\right]$

Solution:

Concept:

- Slope estimate in simple regression $Y = \beta_0 + \beta_1 X$:

$$\hat{\beta}_1 = \frac{n\sum X_i Y_i - \sum X_i \sum Y_i}{n\sum X_i^2 - (\sum X_i)^2}$$

- Here $X_i = i$. We use standard sum formulas: $\sum i = \frac{n(n+1)}{2}$ and $\sum i^2 = \frac{n(n+1)(2n+1)}{6}$.

Step 1: Simplify the denominator

$$\text{Denom} = n\frac{n(n+1)(2n+1)}{6} - \frac{n^2(n+1)^2}{4} = \frac{n^2(n+1)}{12}[2(2n+1)-3(n+1)]$$
$$= \frac{n^2(n+1)(n-1)}{12} = \frac{n^2(n^2-1)}{12}$$

Step 2: Express the numerator using the sample mean

$$\text{Num} = n\sum iy_i - \frac{n(n+1)}{2}n\bar{y} = n[\sum iy_i - \frac{n(n+1)}{2}\bar{y}]$$

Step 3: Calculate final estimate

$$\hat{\beta}_1 = \frac{12 \cdot n[\sum iy_i - \frac{n(n+1)}{2}\bar{y}]}{n^2(n^2-1)} = \frac{1}{n^2-1}[\frac{12}{n}\sum iy_i - 6(n+1)\bar{y}]$$

Quick Tip: For regression where X is simply an index $1,\dots,n$, the denominator $\sum(i-\bar{i})^2$ simplifies to $n(n^2-1)/12$. This is a helpful shortcut for many statistics problems involving time-indexed data.

---

56. A joint density function of random variable X and Y is given by

$$f(x,y) = \begin{cases} kx & \textbf{for } 0 < x < 1, 0 < y < 1 \\ 0 & \textbf{otherwise} \end{cases}$$

then Cov(X,Y) is

(A) $-1/6$

(B) $1/6$

(C) 0

(D) $2/3$

Correct Answer: (C) 0

Solution:

Concept:

- Covariance is defined as $\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$.
- If the joint density function can be factored into $f(x,y) = g(x)h(y)$, then X and Y are independent.
- For independent random variables, the covariance is always zero.

Step 1: Factor the joint density function to check for independence

The given function is $f(x,y) = kx$ for $x \in (0,1)$ and $y \in (0,1)$. We can write this as:

$$f(x,y) = [kx] \cdot [1]$$

This is a product of a function of x only and a function of y only. Thus, X and Y are independent random variables.

Step 2: Verify the independence property using expectations

To be exhaustive, let's find the normalization constant k:

$$\int_0^1 \int_0^1 kx\,dy\,dx = \int_0^1 kx[y]_0^1\,dx = \int_0^1 kx\,dx = \frac{k}{2} = 1 \implies k = 2$$

Now calculate $E(X), E(Y)$, and $E(XY)$:

$$E(X) = \int_0^1 \int_0^1 x(2x)\,dy\,dx = \int_0^1 2x^2\,dx = \frac{2}{3}$$

$$E(Y) = \int_0^1 \int_0^1 y(2x)\,dy\,dx = \int_0^1 2x\,dx \int_0^1 y\,dy = 1 \cdot \frac{1}{2} = \frac{1}{2}$$

$$E(XY) = \int_0^1 \int_0^1 xy(2x)\,dy\,dx = \int_0^1 2x^2\,dx \int_0^1 y\,dy = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$$

Step 3: Calculate the covariance

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$\text{Cov}(X,Y) = \frac{1}{3} - \left(\frac{2}{3} \cdot \frac{1}{2}\right) = \frac{1}{3} - \frac{1}{3} = 0$$

Quick Tip: Independence always implies zero covariance, but zero covariance does not necessarily imply independence. If a joint PDF is separable on a rectangular support, the variables are independent.

---

57. Let $X_1, X_2, \ldots$ be independent random variables each taking values $+1$ or -1 with equal probability respectively. If $S_n = \sum_{i=1}^{n} iX_i$ then $\lim_{n\to\infty} P\left(S_n < \sqrt{\frac{n(n+1)(2n+1)}{3}}\right)$, where Φ is distribution function of standard normal variate, is

(A) $\Phi(-\sqrt{3})$
(B) $\Phi(-\sqrt{2})$
(C) $1-\Phi(-\sqrt{3})$
(D) $1-\Phi(-\sqrt{2})$

Correct Answer: (D) $1-\Phi(-\sqrt{2})$

Solution:

Concept:

- Central Limit Theorem for weighted sums: If $S_n = \sum w_i X_i$, then for large n, $S_n \sim N(E[S_n], \mathrm{Var}(S_n))$.
- Sum of squares of first n natural numbers: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Step 1: Calculate the mean and variance of S_n

For each X_i:

$$E(X_i) = (1)(0.5) + (-1)(0.5) = 0 \implies E(S_n) = \sum iE(X_i) = 0$$

$$\mathrm{Var}(X_i) = E(X_i^2) - [E(X_i)]^2 = 1 - 0 = 1$$

Since X_i are independent:

$$\mathrm{Var}(S_n) = \sum_{i=1}^{n} \mathrm{Var}(iX_i) = \sum_{i=1}^{n} i^2 \mathrm{Var}(X_i) = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Step 2: Standardize the probability expression

We want to find $\lim_{n\to\infty} P(S_n < C)$, where $C = \sqrt{\frac{n(n+1)(2n+1)}{3}}$. Standardizing S_n by dividing by its standard deviation σ_{S_n}:

$$P\left(\frac{S_n - 0}{\sigma_{S_n}} < \frac{C}{\sigma_{S_n}}\right) = P\left(Z < \frac{\sqrt{\frac{n(n+1)(2n+1)}{3}}}{\sqrt{\frac{n(n+1)(2n+1)}{6}}}\right)$$

Step 3: Simplify the ratio and find the limit

The term inside the probability simplifies to:

$$\frac{\sqrt{1/3}}{\sqrt{1/6}} = \sqrt{\frac{6}{3}} = \sqrt{2}$$

Thus, the limit is:

$$\lim_{n\to\infty} P(Z < \sqrt{2}) = \Phi(\sqrt{2})$$

Using the property $\Phi(z) = 1 - \Phi(-z)$:

$$\Phi(\sqrt{2}) = 1 - \Phi(-\sqrt{2})$$

Quick Tip: Standardizing a sum always involves dividing the deviation from the mean by the square root of the sum of variances. Check coefficients in the variance formula carefully—the factor of 6 in the denominator is standard for sums of squares.

---

58. If $A_{x:n} = \left\{k : \left|k - \frac{n}{2}\right| \le \frac{x\sqrt{n}}{2}\right\}$, then value of $\lim_{n\to\infty} \sum_{k\in A_{1:n}} \binom{n}{k} 2^{-n}$ (where $\Phi(\cdot)$ is distribution function of standard normal variate) is

(A) $\Phi(-1)$

(B) $\Phi(1)$

(C) $1 - 2\Phi(-1)$

(D) $\frac{\Phi(1)}{2}$

Correct Answer: (C) $1 - 2\Phi(-1)$

Solution:

Concept:

- De Moivre-Laplace Theorem: A binomial distribution $B(n, p)$ can be approximated by

$N(np, npq)$ for large n.

- The sum $\sum \binom{n}{k} p^k q^{n-k}$ is the probability $P(K = k)$ for a Binomial variable.

Step 1: Identify the distribution and parameters

The expression $\sum \binom{n}{k} 2^{-n}$ is the same as $\sum \binom{n}{k} (1/2)^k (1/2)^{n-k}$. This represents the probability for a Binomial random variable $K \sim B(n, 1/2)$. Mean $\mu = n/2$. Variance $\sigma^2 = n(1/2)(1/2) = n/4 \implies \sigma = \sqrt{n}/2$.

Step 2: Interpret the summation range $A_{1:n}$

The set is $A_{1:n} = \left\{k : \left|k - \frac{n}{2}\right| \leq \frac{1\sqrt{n}}{2}\right\}$. This is equivalent to:

$$|K - \mu| \leq 1 \cdot \sigma$$

Step 3: Apply the normal approximation

The sum calculates $P(|K - \mu| \leq \sigma)$. For large n, this becomes $P(|Z| \leq 1)$ where $Z \sim N(0, 1)$.

$$P(-1 \leq Z \leq 1) = \Phi(1) - \Phi(-1)$$

Using $\Phi(1) = 1 - \Phi(-1)$:

$$(1 - \Phi(-1)) - \Phi(-1) = 1 - 2\Phi(-1)$$

Quick Tip: This is a standard question about the probability of falling within one standard deviation of the mean for a symmetric distribution. The value $1 - 2\Phi(-1)$ is approximately 0.68.

---

59. Let $X_i \sim N(0, 1), i = 1, 2, \ldots$ be independent random variables. If $T_n = \sum_{i=1}^{n} X_i^2$, then $\lim_{n\to\infty} P(T_n > n + 2\sqrt{2n})$ is equal to

(A) $\Phi(2)$

(B) $\Phi(-2)$

(C) $1/2$

(D) $1 - \Phi(-2)$

Correct Answer: (B) $\Phi(-2)$

Solution:

Concept:

- $T_n = \sum X_i^2$ follows a Chi-square distribution with n degrees of freedom, $\chi^2(n)$.
- For large n, $\chi^2(n)$ is approximately normal with mean n and variance $2n$.

Step 1: Identify the asymptotic distribution of T_n

As $n \to \infty$, by the Central Limit Theorem:

$$T_n \approx N(n, 2n)$$

Mean $\mu = n$, standard deviation $\sigma = \sqrt{2n}$.

Step 2: Standardize the probability inequality

We want to calculate $P(T_n > n + 2\sqrt{2n})$. Subtract the mean and divide by the standard deviation:

$$P\left(\frac{T_n - n}{\sqrt{2n}} > \frac{(n + 2\sqrt{2n}) - n}{\sqrt{2n}}\right)$$

Step 3: Evaluate the limit

The expression simplifies to:

$$P\left(Z > \frac{2\sqrt{2n}}{\sqrt{2n}}\right) = P(Z > 2)$$

Since the standard normal distribution is symmetric:

$$P(Z > 2) = 1 - \Phi(2) = \Phi(-2)$$

Quick Tip: The variance of a $\chi^2(n)$ distribution is $2n$. Always use the standardized limit $P(Z > k) = \Phi(-k)$ for right-tail probabilities.

---

60. Value of $\lim_{n\to\infty} e^{-n}\left(1 + n + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!}\right)$ **is**

(A) 0

(B) 1

(C) 1/2

(D) ∞

Correct Answer: (C) 1/2

Solution:

Concept:

- The expression $e^{-\lambda}\sum_{k=0}^{n}\frac{\lambda^k}{k!}$ is the CDF $P(X \leq n)$ for a Poisson random variable $X \sim \text{Poisson}(\lambda)$.
- A Poisson distribution with large parameter λ can be approximated by $N(\lambda, \lambda)$.

Step 1: Relate the series to a probability distribution

Let $X_n \sim \text{Poisson}(n)$. The given expression is:

$$P(X_n \leq n) = \sum_{k=0}^{n} \frac{e^{-n} n^k}{k!}$$

Step 2: Apply the normal approximation

For large n, X_n is approximately $N(n, n)$. Standardizing the variable X_n:

$$P(X_n \leq n) \approx P\left(\frac{X_n - n}{\sqrt{n}} \leq \frac{n-n}{\sqrt{n}}\right)$$

Step 3: Evaluate the limit at zero

$$\lim_{n\to\infty} P(Z \leq 0)$$

Since the standard normal distribution is symmetric about its mean of 0:

$$P(Z \leq 0) = 0.5 = 1/2$$

Quick Tip: This is a famous result in asymptotic analysis. The probability of a Poisson variable being less than or equal to its mean tends toward 50% as the intensity grows.

61. Let $\{X_n\}$ be a sequence of r.v's and $Y_n = \left(\frac{S_n - E(S_n)}{n}\right)$ where $S_n = \sum_{i=1}^{n} X_i$, then the necessary and sufficient condition for the sequence $\{X_n\}$ to satisfy W.L.L.N is

(A) $E\left(\frac{Y_n}{1+Y_n}\right) \to 0$ as $n \to \infty$
(B) $E\left(\frac{Y_n^2}{1+Y_n^2}\right) \to 0$ as $n \to \infty$
(C) $E\left(\frac{Y_n}{1+Y_n^2}\right) \to 0$ as $n \to \infty$
(D) $E\left(\frac{Y_n^2}{1+Y_n}\right) \to 0$ as $n \to \infty$

Correct Answer: (B) $E\left(\frac{Y_n^2}{1+Y_n^2}\right) \to 0$ as $n \to \infty$

Solution:

Concept:

- The Weak Law of Large Numbers (WLLN) states that the sample mean converges in probability to the expected value: $Y_n \xrightarrow{P} 0$.
- A sequence of random variables Z_n converges in probability to 0 if and only if $E\left[\frac{Z_n^2}{1+Z_n^2}\right] \to 0$ as $n \to \infty$.

Step 1: Identify the target variable for convergence

Let $Y_n = \frac{S_n - E(S_n)}{n}$. For the WLLN to hold, we require Y_n to converge in probability to 0.

Step 2: Apply the equivalent condition for convergence in probability

The standard metric for convergence in probability (often used in functional analysis and probability theory) is bounded by the function $f(x) = \frac{x^2}{1+x^2}$. The condition $Y_n \xrightarrow{P} 0$ is mathematically equivalent to:

$$\lim_{n\to\infty} E\left(\frac{Y_n^2}{1+Y_n^2}\right) = 0$$

This is because the function $\frac{x^2}{1+x^2}$ is bounded between 0 and 1, handling potential heavy tails that might otherwise cause the expectation to blow up.

Quick Tip: This is a standard theoretical result often related to properties of characteristic functions or metric spaces of random variables. The function $\frac{x^2}{1+x^2}$ defines a metric for convergence in probability.

---

62. If $X_1, X_2, \ldots, X_n$ denote a random sample of size n from normal population $N(0, \theta^2)$ then MVUE of θ^2 is

(A) $\bar{X}^2$

(B) $\frac{1}{n}\sum_{i=1}^{n} X_i^2$

(C) $\sum_{i=1}^{n} \frac{(X_i-\bar{X})^2}{n-1}$

(D) $\sum_{i=1}^{n} \frac{(X_i-\bar{X})^2}{n}$

Correct Answer: (B) $\frac{1}{n}\sum_{i=1}^{n} X_i^2$

Solution:

Concept:

- Minimum Variance Unbiased Estimator (MVUE) is found using the Lehmann-Scheffé theorem.
- If an estimator is unbiased and is a function of a complete sufficient statistic, it is the MVUE.

Step 1: Find a complete sufficient statistic for θ^2

The probability density function for a single observation is:

$$f(x; \theta^2) = \frac{1}{\sqrt{2\pi\theta^2}} e^{-\frac{x^2}{2\theta^2}}$$

The joint density for the sample is:

$$f(x_1, \ldots, x_n; \theta^2) = \left(\frac{1}{2\pi\theta^2}\right)^{n/2} e^{-\frac{1}{2\theta^2}\sum_{i=1}^{n} x_i^2}$$

By the Neyman Factorization Theorem (exponential family form), $T = \sum_{i=1}^{n} X_i^2$ is a complete sufficient statistic for θ^2.

Step 2: Find an unbiased estimator based on T

We know that for $X_i \sim N(0, \theta^2)$:

$$E(X_i^2) = \text{Var}(X_i) + [E(X_i)]^2 = \theta^2 + 0^2 = \theta^2$$

Taking the expectation of the sufficient statistic:

$$E\left(\sum_{i=1}^{n} X_i^2\right) = \sum_{i=1}^{n} E(X_i^2) = n\theta^2$$

To make it unbiased for θ^2, divide by n:

$$E\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \theta^2$$

Step 3: Conclude MVUE

Since $\frac{1}{n}\sum_{i=1}^{n} X_i^2$ is an unbiased estimator and a function of the complete sufficient statistic, by the Lehmann-Scheffé theorem, it is the MVUE.

Quick Tip: When the population mean is known (here, $\mu = 0$), we do not lose a degree of freedom estimating it. Therefore, the denominator for the variance estimator remains n, not $n-1$.

---

63. Let $X_1, X_2, \ldots, X_n$ be random sample from a Poisson family with parameter λ. Then the maximum likelihood estimate of $P(X \geq 2)$ is

(A) $1 - \bar{X}e^{-\bar{X}}$
(B) $(1+\bar{X})e^{-\bar{X}}$
(C) $\bar{X}e^{-\bar{X}}$
(D) $1-(1+\bar{X})e^{-\bar{X}}$

Correct Answer: (D) $1-(1+\bar{X})e^{-\bar{X}}$

Solution:

Concept:

- The Maximum Likelihood Estimator (MLE) of the Poisson parameter λ is the sample mean $\bar{X}$.

- Invariance property of MLE: If $\hat{\theta}$ is the MLE of θ, then $g(\hat{\theta})$ is the MLE of $g(\theta)$.

Step 1: Express the target probability in terms of λ

For a Poisson random variable X with parameter λ, the PMF is $P(X=k)=\frac{e^{-\lambda}\lambda^k}{k!}$. We need $P(X\geq 2)$. Using the complement rule:

$$P(X\geq 2)=1-P(X<2)=1-[P(X=0)+P(X=1)]$$

$$P(X=0)=\frac{e^{-\lambda}\lambda^0}{0!}=e^{-\lambda}$$

$$P(X=1)=\frac{e^{-\lambda}\lambda^1}{1!}=\lambda e^{-\lambda}$$

Substitute these back:

$$P(X\geq 2)=1-(e^{-\lambda}+\lambda e^{-\lambda})=1-(1+\lambda)e^{-\lambda}$$

Step 2: Apply the invariance property of MLE

Let $g(\lambda)=1-(1+\lambda)e^{-\lambda}$. Since the MLE of λ is $\hat{\lambda}=\bar{X}$, the MLE of $g(\lambda)$ is $g(\hat{\lambda})$.

$$\widehat{P(X\geq 2)}_{\text{MLE}}=1-(1+\bar{X})e^{-\bar{X}}$$

Quick Tip: The invariance property is a massive time-saver. You rarely need to derive the MLE for a complex function from scratch; just find the MLE of the base parameter and substitute it into the function.

---

64. Suppose that $X_1,X_2,\dots,X_n$ are independent random variable each are drawn from a population having density function $f_X(x)=\begin{cases}\frac{1}{\theta}e^{-\left(\frac{x-\mu}{\theta}\right)} & ;\text{if } x\geq\mu \\ 0 & x<\mu\end{cases}$ where $\theta>0$ and $\mu\in\mathbb{R}^+$, then maximum likelihood estimate of (θ,μ), when both θ,μ are unknown is

(A) $(\bar{X}-X_{(1)},X_{(1)})$

(B) $\left(\frac{1}{\bar{X}}-X_{(1)},X_{(1)}\right)$

(C) $(\bar{X},X_{(1)})$

(D) $\left(\frac{1}{\bar{X}},X_{(1)}\right)$

Correct Answer: (A) $(\bar{X} - X_{(1)}, X_{(1)})$

Solution:

Concept:

- Likelihood function must be maximized subject to the support constraints (indicator functions).
- First, maximize with respect to the location parameter (μ), then use that result to maximize with respect to the scale parameter (θ).

Step 1: Write the likelihood function

$$L(\theta, \mu) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{X_i - \mu}{\theta}} \cdot I_{(X_i \geq \mu)}$$

$$L(\theta, \mu) = \frac{1}{\theta^n} e^{-\frac{1}{\theta} \sum_{i=1}^{n} (X_i - \mu)} \cdot I_{(X_{(1)} \geq \mu)}$$

where $X_{(1)}$ is the sample minimum.

Step 2: Maximize with respect to μ

To make L as large as possible, we need $\sum(X_i - \mu)$ to be as small as possible, which means making μ as large as possible. However, due to the indicator function, μ cannot exceed $X_{(1)}$. Therefore, the MLE for μ is:

$$\hat{\mu} = X_{(1)}$$

Step 3: Maximize with respect to θ

Substitute $\hat{\mu}$ into the log-likelihood function:

$$\ln L(\theta, \hat{\mu}) = -n \ln \theta - \frac{1}{\theta} \sum_{i=1}^{n} (X_i - X_{(1)})$$

Take the derivative with respect to θ and set it to zero:

$$\frac{\partial \ln L}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} (X_i - X_{(1)}) = 0$$

$$n\theta = \sum_{i=1}^{n} (X_i - X_{(1)})$$

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} X_{(1)} = \bar{X} - X_{(1)}$$

Step 4: Combine the estimates

The MLE pair $(\hat{\theta}, \hat{\mu})$ is $(\bar{X} - X_{(1)}, X_{(1)})$.

Quick Tip: For shifted distributions (like shifted exponential or Pareto), the MLE of the shift (location) parameter is almost always the sample minimum $X_{(1)}$.

---

65. Let $X_1, X_2, \ldots, X_n$ constitute a random sample of size "n" from a population having density $f_X(x) = \begin{cases} e^{-(x-\theta)} & , x > \theta \\ 0 & \text{otherwise} \end{cases}$ **, then**

A. $X_{(1)}$ **is sufficient for** θ

B. $X_{(1)}$ **is consistent for** θ

C. $X_{(1)}$ **is unbiased for** θ

D. $\text{MSE}(X_{(1)}) = \frac{2}{n^2}$

Choose the correct answer from the options given below

(1) A, C only

(2) B, C only

(3) A, B, D only

(4) A, B, C, D only

Correct Answer: (C) A, B, D only

Solution:

Concept:

- Sufficient statistic is found using the Factorization Theorem.
- Consistency means the estimator converges in probability to the true parameter.
- $\text{MSE} = \text{Variance} + \text{Bias}^2$.

Step 1: Check Statement A (Sufficiency)

Joint PDF: $f(x_1, \ldots, x_n; \theta) = e^{-\sum(x_i-\theta)} I_{(X_{(1)}>\theta)} = e^{-\sum x_i} e^{n\theta} I_{(\theta<X_{(1)})}$.
By the Neyman Factorization Theorem, we can split this into $g(X_{(1)}, \theta) \cdot h(\text{data})$. Thus, $X_{(1)}$ is sufficient for θ. (**A is true**)

Step 2: Find the distribution of $X_{(1)}$ to check Bias (C)

$P(X_i > x) = \int_x^\infty e^{-(t-\theta)} dt = e^{-(x-\theta)}$. $P(X_{(1)} > x) = [P(X_i > x)]^n = e^{-n(x-\theta)}$.

The PDF of $X_{(1)}$ is $f_{(1)}(x) = ne^{-n(x-\theta)}$ for $x > \theta$.

This means $X_{(1)} - \theta \sim \text{Exponential}(n)$. Expected value: $E[X_{(1)} - \theta] = 1/n \implies E[X_{(1)}] = \theta + 1/n$.

Since $E[X_{(1)}] \neq \theta$, it is biased. (**C is false**)

Step 3: Check Statement B (Consistency)

As $n \to \infty$, $E[X_{(1)}] = \theta + 1/n \to \theta$.

Variance: $\text{Var}(X_{(1)}) = 1/n^2 \to 0$ as $n \to \infty$.

Since both bias and variance go to 0, $X_{(1)}$ is a consistent estimator of θ. (**B is true**)

Step 4: Check Statement D (MSE)

$\text{MSE}(X_{(1)}) = \text{Var}(X_{(1)}) + [\text{Bias}(X_{(1)})]^2$ $\text{MSE}(X_{(1)}) = \frac{1}{n^2} + \left(\theta + \frac{1}{n} - \theta\right)^2 = \frac{1}{n^2} + \frac{1}{n^2} = \frac{2}{n^2}$.

(**D is true**)

Quick Tip: For the minimum order statistic of a shifted exponential distribution, the shift parameter estimation is always biased upwards by $1/n$. An unbiased estimator would be $X_{(1)} - 1/n$.

66. If $X_1, X_2, \ldots, X_n$ be independent random variable each from Gamma(α, β) then the jointly sufficient statistics for vector (α, β) is

(A) $\left(\sum_{i=1}^{n} X_i, \prod_{i=1}^{n} X_i\right)$
(B) $\left(\prod_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i\right)$
(C) $\left(\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} X_i\right)$
(D) $\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i\right)$

Correct Answer: (A) $\left(\sum_{i=1}^{n} X_i, \prod_{i=1}^{n} X_i\right)$

Solution:

Concept:

- Fisher-Neyman Factorization Theorem: A statistic $T(X)$ is sufficient for parameter θ if the likelihood function can be factored as $L(\theta; X) = g(T(X), \theta) \cdot h(X)$.
- The joint PDF of n i.i.d. Gamma variables is used to identify the terms that depend on both parameters and data.

Step 1: Write the likelihood function for the Gamma distribution

The PDF of a single Gamma(α, β) variable is:

$$f(x; \alpha, \beta) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$$

For a sample of n independent observations, the likelihood function L is the product of individual PDFs:

$$L(\alpha, \beta; X) = \prod_{i=1}^{n} \left[\frac{1}{\beta^{\alpha}\Gamma(\alpha)} X_i^{\alpha-1} e^{-X_i/\beta} \right]$$

Step 2: Simplify and factor the likelihood expression

$$L(\alpha, \beta; X) = \left(\frac{1}{\beta^{\alpha}\Gamma(\alpha)} \right)^{n} \cdot \left(\prod_{i=1}^{n} X_i \right)^{\alpha-1} \cdot e^{-\frac{1}{\beta}\sum_{i=1}^{n} X_i}$$

Step 3: Identify the sufficient statistics

In the factored form, the terms involving parameters α and β are coupled with the sample values through the quantities $\sum X_i$ and $\prod X_i$. According to the Factorization Theorem, the joint statistic $\left(\sum_{i=1}^{n} X_i, \prod_{i=1}^{n} X_i\right)$ is jointly sufficient for the vector (α, β).

Quick Tip: For distributions in the exponential family, the sufficient statistics are always the sums of the functions of the data found in the exponent or as factors in the simplified likelihood. Note that the arithmetic sum handles the scale parameter β, while the geometric product handles the shape parameter α.

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67. The method of moment estimator of parameter α in the following probability density function $f_X(x) = \frac{\alpha}{x^{\alpha+1}}$, $x > 1$, $\alpha > 1$ is

(A) $\frac{1}{\bar{x}-1}$
(B) $\frac{\bar{x}}{\bar{x}-1}$
(C) $\frac{\bar{x}-1}{\bar{x}}$
(D) $\bar{x} - 1$

Correct Answer: (B) $\frac{\bar{x}}{\bar{x}-1}$

Solution:

Concept:

- Method of Moments (MOM) involves equating the population mean $E(X)$ to the sample mean $\bar{x}$.
- We calculate the theoretical first moment and solve the resulting equation for the parameter.

Step 1: Calculate the population mean $E(X)$

$$E(X) = \int_1^\infty x \cdot f_X(x)\,dx = \int_1^\infty x \cdot \frac{\alpha}{x^{\alpha+1}}\,dx$$

$$E(X) = \alpha \int_1^\infty x^{-\alpha}\,dx$$

Using the power rule for integration:

$$E(X) = \alpha \left[\frac{x^{-\alpha+1}}{-\alpha+1}\right]_1^\infty$$

Since $\alpha > 1$, as $x \to \infty$, $x^{-\alpha+1} \to 0$:

$$E(X) = \alpha\left(0 - \frac{1}{-\alpha+1}\right) = \frac{\alpha}{\alpha-1}$$

Step 2: Equate the population mean to the sample mean

$$\frac{\alpha}{\alpha-1}=\bar{x}$$

Step 3: Solve for the parameter α

Multiply both sides by $(\alpha-1)$:

$$\alpha=\bar{x}(\alpha-1)$$

$$\alpha=\bar{x}\alpha-\bar{x}$$

Rearrange to group terms with α:

$$\bar{x}=\bar{x}\alpha-\alpha=\alpha(\bar{x}-1)$$

$$\alpha=\frac{\bar{x}}{\bar{x}-1}$$

Quick Tip: This distribution is a specific form of the Pareto distribution. Always verify that the condition $\alpha>1$ is met, as the first moment is not defined for $\alpha\leq 1$.

---

68. If $X_1=17, X_2=10, X_3=32$ and $X_4=5$ be the observed values of a random sample from the discrete distribution $P(X=x)=\frac{\theta^{2x}e^{-\theta^2}}{x!}$ if $x=0,1,2,\ldots,\ \theta>0$. Then $\hat{\theta}_{\text{MLE}}$ is

(A) 16

(B) 12

(C) 8

(D) 4

Correct Answer: (D) 4

Solution:

Concept:

- The given distribution is a Poisson distribution where the intensity parameter $\lambda=\theta^2$.
- For a Poisson distribution, the Maximum Likelihood Estimator (MLE) of the intensity parameter λ is the sample mean $\bar{x}$.

Step 1: Calculate the sample mean $\bar{x}$

The observed values are $17, 10, 32, 5$.

$$\bar{x} = \frac{17+10+32+5}{4} = \frac{64}{4} = 16$$

Step 2: Set up the MLE relationship for the parameter

Since $P(X = x)$ is the PMF of a Poisson distribution with parameter θ^2:

$$(\hat{\theta^2})_{\text{MLE}} = \bar{x}$$

Substituting the value of $\bar{x}$:

$$\hat{\theta}^2 = 16$$

Step 3: Solve for θ

Since $\theta > 0$, take the positive square root:

$$\hat{\theta} = \sqrt{16} = 4$$

Quick Tip: The MLE of a function of a parameter is simply the function of the MLE of that parameter (*Invariance property*). Here, we found the MLE for the whole exponent first, then solved for θ.

---

69. Let $X_1, X_2, \ldots, X_n$ be a random sample from $U(\theta, \theta+1)$ then the maximum likelihood estimate of θ

(A) is unique and is equal to $\min(X_1, X_2, \ldots, X_n)$

(B) is unique and is equal to $(\max(X_1, X_2, \ldots, X_n) - 1)$

(C) is not unique

(D) does not exist

Correct Answer: (C) is not unique

Solution:

Concept:

- The likelihood function $L(\theta)$ for a Uniform distribution $U(\theta, \theta+1)$ is 1 if all sample points fall within the interval $[\theta, \theta+1]$, and 0 otherwise.
- We analyze the constraints on θ based on the sample maximum $X_{(n)}$ and minimum $X_{(1)}$.

Step 1: Set up the likelihood function and its constraints

The joint PDF is:

$$L(\theta) = \prod_{i=1}^{n} f(X_i; \theta) = \begin{cases} 1^n = 1 & \text{if } \theta \leq X_i \leq \theta + 1 \text{ for all } i \\ 0 & \text{otherwise} \end{cases}$$

This means: 1) $\theta \leq \min(X_i) = X_{(1)}$ 2) $\theta + 1 \geq \max(X_i) = X_{(n)} \implies \theta \geq X_{(n)} - 1$

Step 2: Determine the range of θ that maximizes the likelihood

The likelihood is maximized (reaches a constant value of 1) for any value of θ such that:

$$X_{(n)} - 1 \leq \theta \leq X_{(1)}$$

Since $X_{(n)} - X_{(1)} < 1$ for a sample from a range of length 1, the interval $[X_{(n)} - 1, X_{(1)}]$ has a non-zero length.

Step 3: Evaluate the uniqueness

Because the likelihood is at its maximum for every value in this entire interval, the maximum likelihood estimator is not a single value but any value in the interval. Therefore, the MLE is not unique.

Quick Tip: Uniform distributions on fixed-length intervals often result in non-unique MLEs. By contrast, for $U(0, \theta)$, the MLE is the unique value $X_{(n)}$ because increasing θ beyond that point would decrease the likelihood value $(1/\theta^n)$.

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70. Match List - I with List - II.

	List - I		List - II
A.	Minimum variance bound estimator	I.	Complete sufficient statistics
B.	Sufficient statistics	II.	Cramer–Rao Inequality
C.	UMVU estimator	III.	Factorisation Theorem
D.	Unique UMVUE	IV.	Rao-Blackwell Theorem

Choose the correct answer from the options given below

(1) A-IV, B-I, C-II, D-III

(2) A-IV, B-III, C-II, D-I

(3) A-II, B-III, C-I, D-IV

(4) A-II, B-III, C-IV, D-I

Correct Answer: (D) A-II, B-III, C-IV, D-I

Solution:

Concept:

- Statistical theory links specific theorems to the properties of estimators.

Step 1: Match A (Minimum Variance Bound)

The Cramer-Rao inequality provides a theoretical lower bound for the variance of an unbiased estimator. Thus, Minimum variance bound estimators are linked to **II (Cramer-Rao Inequality)**. So, $A \to II$.

Step 2: Match B (Sufficient Statistics)

The standard method to find sufficient statistics for a given distribution is using the **III (Factorisation Theorem)**. So, $B \to III$.

Step 3: Match C (UMVU Estimator)

The Rao-Blackwell theorem provides a method to reduce the variance of an unbiased estimator by conditioning it on a sufficient statistic, often leading to a Uniformly Minimum Variance Unbiased (UMVU) estimator. So, $C \to IV$.

Step 4: Match D (Unique UMVU)

According to the Lehmann-Scheffé theorem, if an estimator is a function of a **I (Complete sufficient statistics)** and is unbiased, then it is the unique UMVU estimator. So, $D \to I$.

Conclusion: The matching sequence is A-II, B-III, C-IV, D-I. This corresponds to option (D).

Quick Tip: Remember the hierarchy: Sufficiency (Factorization) → Rao-Blackwell → Completeness → Lehmann-Scheffé (Uniqueness of UMVU). Cramer-Rao is the "floor" for variance.

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71. A box contains 8 balls, θ of which are of white color. The null hypothesis $H_0 : \theta = 3$ is tested by drawing 2 balls at random and without replacement. The hypothesis is rejected in favor of the alternative hypothesis $H_1 : \theta > 3$ if both balls are white. What is the significance level of the test?

(A) $3/56$

(B) $3/28$

(C) $1/28$

(D) $1/56$

Correct Answer: (B) $3/28$

Solution:

Concept:

- Significance level (α) is defined as the probability of rejecting the null hypothesis when it is actually true (Type I Error).
- Rejection rule: Reject H_0 if both balls drawn are white.

Step 1: Determine the conditions under the null hypothesis

Under H_0, the parameter $\theta = 3$. This means in the box of 8 balls, exactly 3 are white and 5 are non-white. Total balls $N = 8$. White balls $W = 3$.

Step 2: Calculate the probability of the rejection event

The rejection region is the event that both balls drawn without replacement are white. Total ways to draw 2 balls from 8:

$$\binom{8}{2} = \frac{8 \times 7}{2 \times 1} = 28$$

Ways to draw 2 white balls from the 3 available:

$$\binom{3}{2} = \frac{3 \times 2}{2 \times 1} = 3$$

Step 3: Find the significance level

$$\alpha = P(\text{Reject } H_0 | H_0 \text{ is true})$$

$$\alpha = P(\text{Both balls are white} | \theta = 3)$$

$$\alpha = \frac{\binom{3}{2}}{\binom{8}{2}} = \frac{3}{28}$$

Quick Tip: The significance level always depends on the distribution defined by the null hypothesis. When the test involves drawing without replacement, use combinations (hypergeometric logic) to find the probabilities of outcomes in the rejection region.

---

72. A test, based on a critical region W, for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, is said to be unbiased if (where α and β are Type 1 and Type 2 errors respectively)

(A) $\beta \geq \alpha$

(B) $1 - \beta < \alpha$

(C) $\beta \leq \alpha$

(D) $1 - \beta \geq \alpha$

Correct Answer: (D) $1 - \beta \geq \alpha$

Solution:

Concept:

- Power of a test: The probability of rejecting the null hypothesis when the alternative hypothesis is true, denoted as $1 - \beta$.
- Size of a test: The probability of rejecting the null hypothesis when it is true, denoted as α.
- Unbiased Test: A test is unbiased if its power is at least as large as its significance level.

Step 1: Define the rejection probability function

Let $P(\theta)$ be the probability of falling into the critical region W when the true parameter is θ. For $\theta = \theta_0$, $P(\theta_0) = \alpha$. For $\theta = \theta_1$, $P(\theta_1) = 1 - \beta$.

Step 2: Apply the definition of an unbiased test

A test is unbiased if it is more likely (or equally likely) to reject the null hypothesis when it is false than when it is true. Mathematically:

$$P(\theta_1) \geq P(\theta_0)$$

Step 3: Translate into error terms

Substitute the definitions from Step 1:

$$1-\beta \geq \alpha$$

This means the power of the test against the alternative is at least equal to its size.

Quick Tip: An unbiased test ensures that the experiment is "on the right track"—it's more likely to detect a true difference than to mistakenly report a false one. A test with power less than α is considered worse than a random guess.

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73. In a randomized block design with one factor having 5 levels and another factor having 5 levels, the degree of freedom for the error sum of squares are equal to

(A) 16

(B) 15

(C) 14

(D) 18

Correct Answer: (A) 16

Solution:

Concept:

- In a Randomized Block Design (RBD) with v treatments and r blocks, the total degrees of freedom $(vr-1)$ are partitioned.
- Degrees of freedom for Error: $df_E = (v-1)(r-1)$.

Step 1: Identify the number of treatments and blocks

The problem specifies two factors (levels and blocks/factors) each having 5 levels. Number of treatments (levels of the primary factor) $v = 5$. Number of blocks (levels of the second factor) $r = 5$.

Step 2: Apply the error d.f. formula

Substitute the values into the formula:

$$df_E = (v - 1) \times (r - 1)$$

$$df_E = (5 - 1) \times (5 - 1)$$

Step 3: Calculate the final result

$$df_E = 4 \times 4 = 16$$

Quick Tip: The Error d.f. in RBD is always the product of the degrees of freedom of the two main factors. Note that in a Latin Square Design with 5 levels, the error d.f. would be different: $(k-1)(k-2) = 4 \times 3 = 12$. Always distinguish between RBD and LSD.

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74. To examine whether two different skin creams, A and B have different effect on the human body 'n' randomly chosen person were enrolled in a clinical trail. Then cream A was applied to one of the randomly chosen arms of each person, cream B to the other. What kind of design is this?

(A) Completely Randomized Design

(B) Randomized Block Design

(C) Latin Square Design

(D) Balanced Incomplete Design

Correct Answer: (B) Randomized Block Design

Solution:

Concept:

- Randomized Block Design (RBD) is used to control for known sources of variation among experimental units by grouping them into homogeneous "blocks".

- Within each block, treatments are randomly assigned to the units.

Step 1: Identify the experimental units and the blocking factor

In this clinical trial, each individual person acts as a "block". Variation between different people (skin type, age, genetics) is a major source of nuisance variability. By applying both treatments to the same person (on different arms), this nuisance variation is controlled.

Step 2: Analyze the treatment assignment

Within each person (block), there are two experimental units (the two arms). Treatment A is randomly assigned to one arm, and Treatment B is assigned to the other. This satisfies the definition of RBD where all treatments are represented in each block.

Step 3: Determine the design category

This specific layout is often called a "paired comparison" design or a "split-plot" in some contexts, but fundamentally it is a Randomized Block Design where "subject" is the block.

Quick Tip: If an experiment uses the "same subject" for different treatments, it's almost always a Randomized Block Design (or crossover design). This drastically increases the precision of the test by removing the "between-subject" error from the treatment comparison.

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75. A simple random sample of size 'n' will be drawn from a class of 125 students, and mean mathematics score of the sample will be computed. If the standard error of the sample mean for "with replacement sampling" is twice as much as the standard error of the sample mean for "without replacement sampling", then the value of n is

(A) 79

(B) 94

(C) 63

(D) 33

Correct Answer: (B) 94

Solution:

Concept:

- Standard Error for SRSWR (with replacement): $SE_{wr} = \frac{\sigma}{\sqrt{n}}$.
- Standard Error for SRSWOR (without replacement): $SE_{wor} = \frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}$.
- Finite Population Correction (FPC): The term $\sqrt{\frac{N-n}{N-1}}$ accounts for sampling from a finite population without replacement.

Step 1: Set up the equation from the given condition

We are given that $SE_{wr} = 2 \times SE_{wor}$. Substitute the formulas:

$$\frac{\sigma}{\sqrt{n}} = 2 \times \left(\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}\right)$$

Step 2: Simplify the algebraic expression

Cancel the common term $\frac{\sigma}{\sqrt{n}}$ from both sides:

$$1 = 2\sqrt{\frac{N-n}{N-1}}$$

Square both sides to remove the root:

$$1 = 4\left(\frac{N-n}{N-1}\right)$$

Step 3: Solve for n using the population size N

The population size $N = 125$.

$$N - 1 = 4N - 4n$$

$$4n = 4N - (N-1) = 3N + 1$$

Substitute $N = 125$:

$$4n = 3(125) + 1$$

$$4n = 375 + 1 = 376$$

$$n = \frac{376}{4} = 94$$

Quick Tip: Sampling without replacement is always more precise than sampling with replacement because it avoids redundant information. The reduction in error is greater as the sample size n approaches the population size N.
