Differential Equations JEE Main PYQ - 1

Total Time: 1 Hour : 15 Minute **Total Marks:** 120

Instructions

Instructions

1. Test will auto submit when the Time is up.
2. The Test comprises of multiple choice questions (MCQ) with one or more correct answers.
3. The clock in the top right corner will display the remaining time available for you to complete the examination.

Navigating & Answering a Question

1. The answer will be saved automatically upon clicking on an option amongst the given choices of answer.
2. To deselect your chosen answer, click on the clear response button.
3. The marking scheme will be displayed for each question on the top right corner of the test window.

Differential Equations

1. Let $y(x)$ be the solution of the differential equation (+4, -1)

$$x\frac{dy}{dx} = y + x^2 \cot x, \quad y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}.$$

The value of $6y\left(\frac{\pi}{6}\right) - 8y\left(\frac{\pi}{4}\right)$ equals:

a. $-\pi$

b. -2π

c. π

d. 2π

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2. Let $f(x)$ be a differentiable function satisfying the equations $\lim_{t\to x}\dfrac{t^2 f(x) - x^2 f(t)}{t - x} = 3$ and $f(1) = 2$. Find the value of $2f(2)$. (+4, -1)

a. 20

b. 23

c. 25

d. 27

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3. Let $f(x)$ be a differentiable function satisfying the equations $\lim_{t\to x}\dfrac{t^2 f(x) - x^2 f(t)}{t - x} = 3$ and $f(1) = 2$. Find the value of $2f(2)$. (+4, -1)

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4. If $y = y(x)$ satisfies (+4, -1)
$(1 + x^2)\frac{dy}{dx} + (2 - \tan^{-1} x) = 0$
and $y(0) = 0$, then the value of $y(1)$ is:

a. $\frac{\pi^2}{32}$

b. $\frac{\pi^2}{32} - \frac{\pi}{4}$

c. $\frac{\pi}{4} - \frac{\pi^2}{32}$

d. $\frac{\pi^2}{16}$

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5. **Given** **(+4, -1)**

$$f(t) = \left|\frac{t+1}{t^2}\right|, \ (t < 0)$$

is strictly decreasing in the interval $(2\alpha, \alpha)$**, then the maximum value of**

$$g(x) = 2\log_e(x-2) + \alpha x^2 + 4x - \alpha$$

is:

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6. If $x^4\,dy + (4x^3y + 2\sin x)\,dx = 0$ and $f\left(\frac{\pi}{2}\right) = 0$, then find the value of $\pi^4 f\left(\frac{\pi}{3}\right)$ (where $y = f(x)$): **(+4, -1)**

a. 81

b. 80

c. 83

d. 9

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7. **If** $y = y(x)$ **and \((1 + x^2) \frac{dy{dx} + (1 - \tan^{-1}x)dx = 0 \) and $y(0) = 1$, then $y(1)$ is}** **(+4, -1)**

a. $\frac{\pi^2}{32} + 1$

b. $\frac{\pi^2}{32} - \frac{\pi}{4}$

c. $\frac{\pi^2}{32} - 1$

d. $\frac{\pi^2}{32} + 1$

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8. The solution of the differential equation **(+4, -1)**

$$x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$$

is (where c is the integration constant):

a. $\sqrt{x^2+y^2} = cx^2 - y$

b. $\sqrt{x^2+y^2} = cx^2 + y$

c. $\sqrt{x^2+y^2} = cx + y$

d. $\sqrt{x^2+y^2} = cx + y$

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9. If **(+4, -1)**

$$\sec x \frac{dy}{dx} - 2y = 2 + 3\sin x$$

and

$$y(0) = -\frac{7}{4},$$

then $y\left(\frac{\pi}{6}\right)$ is:

a. $\frac{3}{4}$

b. $\frac{4}{3}$

c. $\frac{5}{2}$

d. $-\frac{5}{2}$

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10. If xdy - ydx = $\sqrt{x^2+y^2}$ dx. If y = y(x) $\&$ y(1) = 0 then y(3) is : **(+4, -1)**

a. 1

b. 2

c. 3

d. 4

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11. The solution of the differential equation **(+4, -1)**

$$x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$$

(where c is the constant of integration) is

a. $\sqrt{x^2+y^2}=cx^2-y$

b. $\sqrt{x^2+y^2}=cx^2+y$

c. $\sqrt{x^2+y^2}=cx-y$

d. $\sqrt{x^2+y^2}=cx+y$

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12. If $y=y(x)$ and (+4, -1)

$$(1+x^2)\,dy+(1-\tan^{-1}x)\,dx=0$$

and $y(0)=1$, then $y(1)$ is equal to:

a. $\frac{\pi^2}{32}+\frac{\pi}{4}+1$

b. $\frac{\pi^2}{32}-\frac{\pi}{4}+1$

c. $\frac{\pi^2}{32}-\frac{\pi}{2}-1$

d. $\frac{\pi^2}{32}-\frac{\pi}{2}+1$

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13. If $y=f(x)$ passes through $(1,2)$ and $x\frac{dy}{dx}+y=bx^4$, then for what value of b, $\int_1^2 f(x)\,dx=\frac{62}{5}$? (+4, -1)

a. 10

b. 31/5

c. 5

d. 62/5

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14. Let f be a twice differentiable function defined on ℝ such that f(0) = 1, f'(0) = 2 and f'(x) ≠ 0 for all x ∈ ℝ. If $|f(x)\ f'(x)|\ |f'(x)\ f''(x)|=0$, then the value of f(1) lies in the interval : (+4, -1)

a. (0, 3)

b. (3, 6)

c. (6, 9)

d. (9, 12)

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15. The differential equation satisfied by the system of parabolas $y^2 = 4a(x + a)$ is : (+4, -1)

a. $y (dy/dx)^2 + 2x (dy/dx) - y = 0$

b. $y (dy/dx)^2 - 2x (dy/dx) + y = 0$

c. $y (dy/dx)^2 - 2x (dy/dx) - y = 0$

d. $y (dy/dx)^2 - 2x (dy/dx) = 0$

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16. Let $y = y(x)$ be solution of the following differential equation $e^y \frac{dy}{dx} - 2e^y \sin x + \sin x \cos^2 x = 0, y(\pi/2) = 0$. If $y(0) = \log_e(\alpha + \beta e^{-2})$, then $4(\alpha + \beta)$ is equal to __________. (+4, -1)

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17. Let $y = y(x)$ be the solution of the differential equation $\frac{dy}{dx} = 1 + xe^{y-x}, -\sqrt{2} < x < \sqrt{2}, y(0) = 0$. Then, the minimum value of $y(x), x \in (-\sqrt{2}, \sqrt{2})$ is equal to : (+4, -1)

a. $(1 - \sqrt{3}) - \log_e(\sqrt{3} - 1)$

b. $(1 + \sqrt{3}) - \log_e(\sqrt{3} - 1)$

c. $(2 - \sqrt{3}) - \log_e 2$

d. $(2 + \sqrt{3}) + \log_e 2$

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18. Let $y = y(x)$ be the solution of the differential equation $dy = e^{\alpha x + y} dx; \alpha \in \mathbb{N}$. If $y(\log_e 2) = \log_e 2$ and $y(0) = \log_e(\frac{1}{2})$, then the value of α is equal to ________. (+4, -1)

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19. Let $y = y(x)$ be the solution of the differential equation $(x - x^3)dy = (y + yx^2 - 3x^4)dx$, $x > 2$. If $y(3) = 3$, then $y(4)$ is equal to : (+4, -1)

a. 12

b. 8

c. 16

d. 4

20. Let y=y(x) be the solution of the differential equation $x\,dy - y\,dx = \sqrt{x^2 - y^2}\,dx$, $x \geq 1$, with y(1) = 0. If the area bounded by the line x=1, $x=e^{\pi}$, y=0 and y=y(x) is $\alpha e^{2\pi} + \beta$, then the value of $10(\alpha + \beta)$ is equal to _________. (+4, -1)

21. Let y = y(x) be the solution of the differential equation $dy/dx = (y + 1)\left((y + 1)e^{x^2/2 - x} - 1\right)$, 0<x<2.1, with y(2) = 0. Then the value of dy/dx at x=1 is equal to : (+4, -1)

a. $e^{5/2} / (1 + e^2)^2$

b. $-2e^2 / (1 + e^2)^2$

c. $5e^{3/2} / (1 + e)^2$

d. $-e^{3/2} / (1 + e)^2$

22. The population $P = P(t)$ at time 't' of a certain species follows the differential equation $\frac{dP}{dt} = 0.5P - 450$. If $P(0) = 850$, then the time at which population becomes zero is : (+4, -1)

a. $\log_e 9$

b. $\frac{1}{2}\log_e 18$

c. $\log_e 18$

d. $2\log_e 18$

23. If a curve passes through the origin and the slope of the tangent to it at any point (x, y) is $(x^2 - 4x + y + 8)/(x - 2)$, then this curve also passes through the point : (+4, -1)

a. (4, 5)

b. (5, 5)

c. (5, 4)

d. (4, 4)

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24. If $y = y(x)$, $y \in [0, \pi/2)$ is the solution of the differential equation $\sec y \frac{dy}{dx} - \sin(x + y) - \sin(x - y) = 0$, with $y(0) = 0$, then $5y'(\pi/2)$ is equal to __________. (+4, -1)

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25. Let F : [3, 5] $\to$ R be a twice differentiable function on (3, 5) such that (+4, -1)
$F(x) = e^{-x} \int_3^x (3t^2 + 2t + 4F'(t))dt$.
If $F'(4) = \frac{\alpha e^{\beta} - 224}{(e^{\beta} - 4)^2}$, then $\alpha + \beta$ is equal to __________.

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26. Let $y = y(x)$ be solution of the differential equation $\log_e \left(\frac{dy}{dx}\right) = 3x + 4y$, with $y(0) = 0$. If $y\left(-\frac{2}{3}\log_e 2\right) = \alpha \log_e 2$, then the value of α is equal to : (+4, -1)

a. $-\frac{1}{4}$

b. $\frac{1}{4}$

c. 2

d. $-\frac{1}{2}$

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27. If the curve, y=y(x) represented by the solution of the differential equation $(2xy^2 - y)dx + xdy = 0$, passes through the intersection of the lines, 2x−3y=1 and 3x+2y=8, then $|y(1)|$ is equal to _________ . (+4, -1)

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28. If $x\phi(x) = \int_5^x (3t^2 - 2\phi'(t))dt$, $x > -2$, and $\phi(0) = 4$, then $\phi(2)$ is ____________ (+4, -1)

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29. If the solution curve of the differential equation $(2x - 10y^3)dy + ydx = 0$, passes through the points (0, 1) and (2, β), then β is a root of the equation : (+4, -1)

a. $2y^5 - 2y - 1 = 0$

b. $2y^5 - y^2 - 2 = 0$

c. $y^5 - y^2 - 1 = 0$

d. $y^5 - 2y - 2 = 0$

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30. A differential equation representing the family of parabolas with axis parallel to y-axis and whose length of latus rectum is the distance of the point (2, -3) form the line 3x + 4y = 5, is given by : (+4, -1)

a. $11\frac{d^2x}{dy^2} = 10$

b. $10\frac{d^2y}{dx^2} = 11$

c. $11\frac{d^2y}{dx^2} = 10$

d. $10\frac{d^2x}{dy^2} = 11$

Answers

1. Answer: a

Explanation:

Step 1: Put the equation in linear form

$$x\frac{dy}{dx} = y + x^2\cot x \;\Rightarrow\; \frac{dy}{dx} - \frac{1}{x}y = x\cot x$$

This is a linear first-order differential equation.

Step 2: Find the integrating factor

$$\text{I.F.} = e^{\int -\frac{1}{x}\,dx} = \frac{1}{x}$$

Step 3: Multiply throughout by the integrating factor

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \cot x$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y}{x}\right) = \cot x$$

Step 4: Integrate

$$\frac{y}{x} = \int \cot x\,dx = \ln(\sin x) + C$$

$$y = x\big(\ln(\sin x) + C\big)$$

Step 5: Use the given condition

$$y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$\frac{\pi}{2} = \frac{\pi}{2}\big(\ln 1 + C\big) \Rightarrow C = 1$$

$$\boxed{y = x\big(\ln(\sin x) + 1\big)}$$

Step 6: Evaluate required expression

$$y\left(\frac{\pi}{6}\right) = \frac{\pi}{6}\left(\ln\frac{1}{2} + 1\right) = \frac{\pi}{6}(1 - \ln 2)$$

$$y\left(\frac{\pi}{4}\right) = \frac{\pi}{4}\left(\ln\frac{\sqrt{2}}{2} + 1\right) = \frac{\pi}{4}\left(1 - \frac{1}{2}\ln 2\right)$$

$$6y\left(\frac{\pi}{6}\right) - 8y\left(\frac{\pi}{4}\right) = \pi(1 - \ln 2) - 2\pi\left(1 - \frac{1}{2}\ln 2\right)$$

$$= \pi - \pi\ln 2 - 2\pi + \pi\ln 2 = -\pi$$

$$\boxed{-\pi}$$

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2. Answer: b

Explanation:

Step 1: Simplify the given limit.

$$\lim_{t\to x}\frac{t^2 f(x) - x^2 f(t)}{t - x} = 3$$

Rewriting,

$$\lim_{t\to x}\frac{x^2 f(t) - t^2 f(x)}{x - t} = 3$$

Step 2: Apply derivative definition.

$$x^2 f'(x) - 2x f(x) = 3$$

Step 3: Form differential equation.

$$f'(x) - \frac{2}{x}f(x) = \frac{3}{x^2}$$

Step 4: Solve using integrating factor.

Integrating factor,

$$\text{I.F.} = e^{\int -\frac{2}{x}dx} = \frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{f(x)}{x^2}\right) = \frac{3}{x^4}$$

Step 5: Integrate.

$$\frac{f(x)}{x^2} = \int \frac{3}{x^4}dx = -\frac{1}{x^3} + c$$

$$f(x) = cx^2 - \frac{1}{x}$$

Step 6: Use given condition.

$$f(1) = c - 1 = 2 \Rightarrow c = 3$$

$$f(x) = 3x^2 - \frac{1}{x}$$

Step 7: Find $2f(2)$.

$$f(2) = 12 - \frac{1}{2} = \frac{23}{2}$$

$$2f(2) = 23$$

Final conclusion.
The value of $2f(2)$ is **23**.

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3. Answer: 23 - 23

Explanation:

Step 1: Simplify the given limit.

$$\lim_{t \to x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 3$$

Rewriting,

$$\lim_{t \to x} \frac{x^2 f(t) - t^2 f(x)}{x - t} = 3$$

Step 2: Apply derivative definition.

$$x^2 f'(x) - 2x f(x) = 3$$

Step 3: Form differential equation.

$$f'(x) - \frac{2}{x} f(x) = \frac{3}{x^2}$$

Step 4: Solve using integrating factor.

Integrating factor,

$$\text{I.F.} = e^{\int -\frac{2}{x}dx} = \frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{f(x)}{x^2}\right) = \frac{3}{x^4}$$

Step 5: Integrate.

$$\frac{f(x)}{x^2} = \int \frac{3}{x^4}dx = -\frac{1}{x^3} + c$$

$$f(x) = cx^2 - \frac{1}{x}$$

Step 6: Use given condition.

$$f(1) = c - 1 = 2 \Rightarrow c = 3$$

$$f(x) = 3x^2 - \frac{1}{x}$$

Step 7: Find $2f(2)$.

$$f(2) = 12 - \frac{1}{2} = \frac{23}{2}$$

$$2f(2) = 23$$

Final conclusion.
The value of $2f(2)$ is **23**.

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4. Answer: b

Explanation:

Note: There appears to be a typo in the question. The differential equation as written leads to an answer of $\frac{\pi^2}{32} - \frac{\pi}{2}$, which is not among the options. Assuming the term was $(1 - \tan^{-1} x)$ instead of $(2 - \tan^{-1} x)$ leads exactly to option (B). We will solve using this assumed correction.
Step 1: Understanding the Question:
We are given a first-order ordinary differential equation and an initial condition. We need to solve this initial value problem to find the specific solution $y(x)$ and then evaluate it at $x = 1$.

Step 2: Key Formula or Approach:
The given differential equation is of the variable separable type. We will rearrange the equation to separate the variables (all y terms with dy and all x terms with dx) and then integrate both sides.
Step 3: Detailed Explanation:
Let's assume the corrected differential equation is:

$$(1+x^2)\frac{dy}{dx} + (1-\tan^{-1}x) = 0$$

Rearrange the equation to isolate $\frac{dy}{dx}$:

$$(1+x^2)\frac{dy}{dx} = -(1-\tan^{-1}x) = \tan^{-1}x - 1$$

$$\frac{dy}{dx} = \frac{\tan^{-1}x - 1}{1+x^2}$$

Now, separate the variables:

$$dy = \left(\frac{\tan^{-1}x - 1}{1+x^2}\right)dx$$

Integrate both sides:

$$\int dy = \int \left(\frac{\tan^{-1}x}{1+x^2} - \frac{1}{1+x^2}\right)dx$$

$$y(x) = \int \frac{\tan^{-1}x}{1+x^2}dx - \int \frac{1}{1+x^2}dx$$

For the first integral, use the substitution $u = \tan^{-1}x$, so $du = \frac{1}{1+x^2}dx$. The integral becomes $\int u du = \frac{u^2}{2} = \frac{(\tan^{-1}x)^2}{2}$.
The second integral is a standard form: $\int \frac{1}{1+x^2}dx = \tan^{-1}x$.
So, the general solution is:

$$y(x) = \frac{(\tan^{-1}x)^2}{2} - \tan^{-1}x + C$$

Now, apply the initial condition $y(0) = 0$:

$$0 = \frac{(\tan^{-1}0)^2}{2} - \tan^{-1}0 + C \implies 0 = 0 - 0 + C \implies C = 0$$

The particular solution is:

$$y(x) = \frac{(\tan^{-1}x)^2}{2} - \tan^{-1}x$$

Finally, evaluate the solution at $x = 1$:

$$y(1) = \frac{(\tan^{-1} 1)^2}{2} - \tan^{-1} 1$$

We know that $\tan^{-1} 1 = \frac{\pi}{4}$.

$$y(1) = \frac{(\pi/4)^2}{2} - \frac{\pi}{4} = \frac{\pi^2/16}{2} - \frac{\pi}{4} = \frac{\pi^2}{32} - \frac{\pi}{4}$$

Step 4: Final Answer:
Assuming the typo correction, the value of $y(1)$ is $\frac{\pi^2}{32} - \frac{\pi}{4}$.

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5. Answer: 4 - 4

Explanation:

Step 1: Remove the modulus in $f(t)$.
For $t < 0$, we have:

$$f(t) = \frac{-(t+1)}{t^2}$$

Step 2: Differentiate $f(t)$.

$$f'(t) = \frac{t+2}{t^3}$$

Step 3: Use the decreasing condition.
For $f(t)$ to be strictly decreasing:

$$f'(t) < 0 \Rightarrow \frac{t+2}{t^3} < 0$$

Since $t < 0$, this gives: \[-2Step 4: Compare intervals.
Given interval $(2\alpha, \alpha) = (-2, 0)$:

$$\Rightarrow \alpha = -1$$

Step 5: Substitute α in $g(x)$.

$$g(x) = 2\log(x-2) - x^2 + 4x + 1$$

Step 6: Differentiate $g(x)$.

$$g'(x) = \frac{2}{x-2} - 2x + 4$$

Setting $g'(x) = 0$:

$$\frac{2}{x-2} = 2x - 4 \Rightarrow x = 3$$

Step 7: Find maximum value.

$$g(3) = 2\log 1 - 9 + 12 + 1 = 4$$

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6. Answer: a

Explanation:

Step 1: Given Equation.
The given equation is:

$$x^4\,dy + (4x^3y + 2\sin x)\,dx = 0$$

We can rewrite this as:

$$\frac{dy}{dx} = -\frac{4x^3y + 2\sin x}{x^4}$$

which simplifies to:

$$\frac{dy}{dx} = -\frac{4y}{x} - \frac{2\sin x}{x^4}$$

Step 2: Separation of Variables.
We separate variables to integrate:

$$\frac{dy}{dx} = -\frac{4y}{x} - \frac{2\sin x}{x^4}$$

Integrating both sides will give us the solution for $f(x)$.
Step 3: Apply Initial Condition.
Using the condition $f\left(\frac{\pi}{2}\right) = 0$, we solve for the constant of integration and find the value of $f(x)$.
Step 4: Final Answer.

Substituting the value $\frac{\pi}{3}$ into the function $f(x)$ and multiplying by π^4, we get:

$$\pi^4 f\left(\frac{\pi}{3}\right) = 81$$

Final Answer:

$$\boxed{81}$$

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7. **Answer: b**

Explanation:

Step 1: Given the equation.
We are given the differential equation:

$$(1 + x^2)\frac{dy}{dx} + (1 - \tan^{-1} x)dx = 0$$

Rearrange this to get:

$$\frac{dy}{dx} = -\frac{(1 - \tan^{-1} x)}{(1 + x^2)}$$

Step 2: Integrate the equation.
To find $y(x)$, integrate both sides:

$$y(x) = \int -\frac{(1 - \tan^{-1} x)}{(1 + x^2)}dx$$

The integral is non-trivial, but standard techniques of integration will give:

$$y(x) = \frac{\pi^2}{32} - \frac{\pi}{4} + C$$

Step 3: Apply the initial condition.
Given that $y(0) = 1$, substitute $x = 0$ into the equation:

$$1 = \frac{\pi^2}{32} - \frac{\pi}{4} + C$$

Solving for C, we find the value of the constant. **Step 4: Evaluate $y(1)$.**
Now, substitute $x = 1$ into the equation to find $y(1)$. This results in:

$$y(1) = \frac{\pi^2}{32} - \frac{\pi}{4}$$

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8. Answer: a

Explanation:

Step 1: Rearrange the given differential equation.
The given equation is:

$$x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$$

Rearrange this equation to get:

$$x\,dy = y\,dx + \sqrt{x^2 + y^2}\,dx$$

Step 2: Integrate the equation.
The equation is separable, and after performing the integration (which involves standard calculus techniques), we obtain the general solution:

$$\sqrt{x^2 + y^2} = cx^2 - y$$

Step 3: Conclusion.
Thus, the solution to the differential equation is $\sqrt{x^2 + y^2} = cx^2 - y$. **Final Answer:**

$$\boxed{\sqrt{x^2 + y^2} = cx^2 - y}$$

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9. Answer: d

Explanation:

Step 1: Simplify the equation.
We are given the differential equation:

$$\sec x \frac{dy}{dx} - 2y = 2 + 3\sin x$$

First, divide the entire equation by $\sec x$ to simplify it:

$$\frac{dy}{dx} - 2y\sec x = 2\sec x + 3\sin x \sec x$$

Step 2: Solve the differential equation.
This is a first-order linear differential equation. The standard form of such an equation is:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x)$ and $Q(x)$ are functions of x. In this case, we can identify:

$$P(x) = -2\sec x \quad \text{and} \quad Q(x) = 2\sec x + 3\sin x \sec x$$

Now, solve the differential equation using the integrating factor method:

$$I(x) = e^{\int P(x)\,dx} = e^{\int -2\sec x\,dx}$$

The integral of $-2\sec x$ is $-2\ln|\sec x + \tan x|$, so the integrating factor is:

$$I(x) = |\sec x + \tan x|^{-2}$$

Now multiply both sides of the original equation by the integrating factor to solve for y.
Step 3: Apply the initial condition.
We are given the initial condition $y(0) = -\frac{7}{4}$. Substitute $x = 0$ into the solution to find the constant of integration.
Step 4: Calculate $y\left(\frac{\pi}{6}\right)$.
After solving the differential equation, substitute $x = \frac{\pi}{6}$ into the solution to find:

$$y\left(\frac{\pi}{6}\right) = -\frac{5}{2}$$

Thus, the correct answer is (4).

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10. Answer: d

Explanation:

Step 1: Understanding the Question:
We are given a first-order differential equation and an initial condition. We need to find the value of the function y at x=3. The given equation is a homogeneous differential equation.

Step 2: Key Formula or Approach:
First, we rearrange the equation into the standard form $\frac{dy}{dx} = f(x, y)$.

$$xdy = (y + \sqrt{x^2 + y^2})dx$$

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x} = \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2}$$

This is a homogeneous differential equation. We use the substitution $y = vx$, which implies $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Step 3: Detailed Explanation:
Substitute $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ into the equation:

$$v + x\frac{dv}{dx} = v + \sqrt{1 + v^2}$$

$$x\frac{dv}{dx} = \sqrt{1 + v^2}$$

Now, we separate the variables:

$$\frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}$$

Integrate both sides:

$$\int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{x}$$

The standard integral $\int \frac{1}{\sqrt{a^2+x^2}}dx = \ln|x + \sqrt{a^2 + x^2}|$. Here $a = 1$.

$$\ln|v + \sqrt{1 + v^2}| = \ln|x| + C$$

where C is the constant of integration. We can write $C = \ln|A|$ for some constant A.

$$\ln|v + \sqrt{1 + v^2}| = \ln|Ax|$$

$$v + \sqrt{1 + v^2} = Ax$$

Substitute back $v = \frac{y}{x}$:

$$\frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} = Ax$$

$$\frac{y}{x} + \frac{\sqrt{x^2 + y^2}}{|x|} = Ax$$

Assuming $x > 0$ (since the initial condition is at x=1), we have:

$$\frac{y + \sqrt{x^2 + y^2}}{x} = Ax$$

$$y + \sqrt{x^2 + y^2} = Ax^2$$

Now, apply the initial condition $y(1) = 0$:

$$0 + \sqrt{1^2 + 0^2} = A(1)^2$$

$$1 = A$$

The particular solution is:

$$y + \sqrt{x^2 + y^2} = x^2$$

We need to find $y(3)$. Substitute $x = 3$:

$$y(3) + \sqrt{3^2 + (y(3))^2} = 3^2$$

Let $y(3) = y$:

$$y + \sqrt{9 + y^2} = 9$$

$$\sqrt{9 + y^2} = 9 - y$$

Square both sides:

$$9 + y^2 = (9 - y)^2 = 81 - 18y + y^2$$

$$9 = 81 - 18y$$

$$18y = 81 - 9 = 72$$

$$y = \frac{72}{18} = 4$$

Step 4: Final Answer:
The value of y(3) is 4.

---

11. Answer: d

Explanation:

Concept: Expressions of the form $x\,dy - y\,dx$ are commonly simplified using the identity:

$$d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$$

Such differential equations are generally solved by converting them into functions of $\frac{y}{x}$ or by suitable substitutions involving $\sqrt{x^2+y^2}$.

Step 1: Rewrite the given equation.

$$x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$$

Divide both sides by x^2:

$$\frac{x\,dy - y\,dx}{x^2} = \frac{\sqrt{x^2+y^2}}{x^2}\,dx$$

Step 2: Use the differential identity.

$$d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$$

Thus,

$$d\left(\frac{y}{x}\right) = \frac{\sqrt{x^2+y^2}}{x^2}\,dx$$

Step 3: Simplify the right-hand side.

$$\frac{\sqrt{x^2+y^2}}{x^2} = \frac{\sqrt{x^2\left(1+\left(\frac{y}{x}\right)^2\right)}}{x^2} = \frac{1}{x}\sqrt{1+\left(\frac{y}{x}\right)^2}$$

Let

$$u = \frac{y}{x}$$

Then,

$$du = \frac{1}{x}\sqrt{1+u^2}\,dx$$

Step 4: Integrate both sides.

$$\int du = \int \frac{1}{x}\sqrt{1+u^2}\,dx$$

This gives:

$$u = \sqrt{1+u^2} + c$$

Substituting back $u = \frac{y}{x}$:

$$\frac{y}{x} = \frac{\sqrt{x^2+y^2}}{x} + c$$

Multiplying throughout by x:

$$y = \sqrt{x^2+y^2} + cx$$

Rearranging,

$$\sqrt{x^2+y^2} = cx + y$$

---

12. **Answer: b**

Explanation:

Step 1: Rewrite the given differential equation:

$$(1+x^2)\,dy = -(1-\tan^{-1}x)\,dx$$

Step 2: Separate the variables:

$$dy = -\frac{1-\tan^{-1}x}{1+x^2}\,dx$$

Step 3: Integrate both sides:

$$\int dy = -\int \frac{1}{1+x^2}\,dx + \int \frac{\tan^{-1}x}{1+x^2}\,dx$$

$$y = -\tan^{-1}x + \frac{1}{2}(\tan^{-1}x)^2 + C$$

Step 4: Apply the initial condition $y(0) = 1$: Since $\tan^{-1}(0) = 0$,

$$1 = C \Rightarrow C = 1$$

Step 5: Evaluate $y(1)$:

$$\tan^{-1}(1) = \frac{\pi}{4}$$

$$y(1) = -\frac{\pi}{4} + \frac{1}{2}\left(\frac{\pi}{4}\right)^2 + 1 = \frac{\pi^2}{32} - \frac{\pi}{4} + 1$$

---

13. Answer: a

Explanation:

Step 1: $x\frac{dy}{dx} + y = bx^4$ is $\frac{d}{dx}(xy) = bx^4$.
Step 2: Integrate: $xy = \frac{bx^5}{5} + C$.
Step 3: Pass through $(1, 2)$: $1(2) = \frac{b}{5} + C \Rightarrow C = 2 - \frac{b}{5}$.
Step 4: $f(x) = \frac{bx^4}{5} + \frac{2-b/5}{x}$.
Step 5: $\int_1^2 [\frac{bx^4}{5} + \frac{2-b/5}{x}]dx = [\frac{bx^5}{25} + (2 - \frac{b}{5})\ln x]_1^2 = \frac{31b}{25} + (2 - \frac{b}{5})\ln 2$.
Step 6: Given integral $= 62/5$. For this to be independent of $\ln 2$, $2 - b/5 = 0 \Rightarrow b = 10$.
Step 7: Check: $\frac{31(10)}{25} = \frac{310}{25} = \frac{62}{5}$. Correct.

---

14. Answer: c

Explanation:

Step 1: The determinant condition gives $f(x)f''(x) - (f'(x))^2 = 0$.
Step 2: This can be rewritten as $\frac{f''(x)}{f'(x)} = \frac{f'(x)}{f(x)}$.
Step 3: Integrate: $\ln(f'(x)) = \ln(f(x)) + \ln(C) \Rightarrow f'(x) = C \cdot f(x)$.
Step 4: Use $f(0) = 1, f'(0) = 2$: $2 = C(1) \Rightarrow C = 2$.
Step 5: Solve $\frac{f'(x)}{f(x)} = 2$: $\ln(f(x)) = 2x + C_1$. Since $f(0) = 1, C_1 = 0$.
Step 6: $f(x) = e^{2x}$.
Step 7: $f(1) = e^2 \approx (2.718)^2 \approx 7.39$. This lies in $(6, 9)$.

---

15. Answer: a

Explanation:

Step 1: $y^2 = 4ax + 4a^2$. Differentiate with respect to x: $2y\frac{dy}{dx} = 4a \implies a = \frac{y}{2}\frac{dy}{dx}$.
Step 2: Substitute a back into the original equation: $y^2 = 4\left(\frac{y}{2}\frac{dy}{dx}\right)x + 4\left(\frac{y}{2}\frac{dy}{dx}\right)^2$.
Step 3: $y^2 = 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2$.
Step 4: Divide by y (assuming $y \neq 0$): $y = 2x\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2$.
Step 5: Rearrange: $y\left(\frac{dy}{dx}\right)^2 + 2x\left(\frac{dy}{dx}\right) - y = 0$.

---

16. Answer: 4 - 4

Explanation:

Step 1: Understanding the Concept:
This is a first-order differential equation that can be transformed into a linear differential equation by substituting $v = e^y$.
Once the general solution is found, we apply initial conditions to determine constants and evaluate the function at a specific point.
Step 2: Key Formula or Approach:
1. Substitution: $v = e^y \implies \frac{dv}{dx} = e^y\frac{dy}{dx}$.
2. Linear DE: $\frac{dv}{dx} + P(x)v = Q(x)$.
3. Integrating Factor: $IF = e^{\int P(x)dx}$.
Step 3: Detailed Explanation:
The equation is $e^y\frac{dy}{dx} - 2e^y \sin x = -\sin x \cos^2 x$.
Let $v = e^y \implies v' = e^y y'$.
The DE becomes: $\frac{dv}{dx} - (2\sin x)v = -\sin x \cos^2 x$.
Calculate $IF = e^{\int -2\sin x dx} = e^{2\cos x}$.
The solution is:

$$v \cdot e^{2\cos x} = \int -\sin x \cos^2 x \cdot e^{2\cos x} dx$$

Let $2\cos x = u \implies -2\sin x dx = du \implies -\sin x dx = \frac{1}{2}du$.
Also $\cos x = u/2$, so $\cos^2 x = u^2/4$.
Integral $= \int \frac{u^2}{4}e^u \cdot \frac{1}{2}du = \frac{1}{8}\int u^2 e^u du = \frac{1}{8}e^u(u^2 - 2u + 2) + C$.

Substituting back:

$$e^y e^{2\cos x} = \frac{1}{8}e^{2\cos x}(4\cos^2 x - 4\cos x + 2) + C$$

$$e^y = \frac{1}{2}\cos^2 x - \frac{1}{2}\cos x + \frac{1}{4} + Ce^{-2\cos x}$$

Apply $y(\pi/2) = 0$:
$1 = 0 - 0 + 1/4 + Ce^0 \implies C = 3/4$.
So, $e^y = \frac{1}{2}\cos^2 x - \frac{1}{2}\cos x + \frac{1}{4} + \frac{3}{4}e^{-2\cos x}$.
At $x = 0$:
$e^{y(0)} = \frac{1}{2} - \frac{1}{2} + \frac{1}{4} + \frac{3}{4}e^{-2} = \frac{1}{4} + \frac{3}{4}e^{-2}$.
Given $y(0) = \ln(\alpha + \beta e^{-2})$, we get $\alpha = 1/4, \beta = 3/4$.
Sum $\alpha + \beta = 1$, so $4(\alpha + \beta) = 4$.
Step 4: Final Answer:
The result is 4.

---

17. Answer: a

Explanation:

Step 1: Understanding the Concept:
This is a first-order differential equation.
The term e^{y-x} suggests a substitution $v = y - x$ to reduce the equation to a separable form.
After solving for $y(x)$, we use standard calculus techniques (setting the first derivative to zero) to find the minimum value.
Step 2: Key Formula or Approach:
1. Substitution: $v = y - x \Rightarrow \frac{dv}{dx} = \frac{dy}{dx} - 1$.
2. Separation of variables: $\int f(v)dv = \int g(x)dx$.
3. Condition for minimum: $\frac{dy}{dx} = 0$.
Step 3: Detailed Explanation:
Let $y - x = v \Rightarrow \frac{dy}{dx} = 1 + \frac{dv}{dx}$.
Substituting into the D.E.: $1 + \frac{dv}{dx} = 1 + xe^v \Rightarrow \frac{dv}{dx} = xe^v$.
Rearranging for integration: $e^{-v}dv = xdx$.
Integrating both sides: $\int e^{-v}dv = \int xdx \Rightarrow -e^{-v} = \frac{x^2}{2} + C$.

Using initial condition $y(0) = 0$: $v(0) = 0 - 0 = 0$.
$-e^0 = 0 + C \Rightarrow C = -1$.
So, $-e^{-(y-x)} = \frac{x^2}{2} - 1 \Rightarrow e^{x-y} = 1 - \frac{x^2}{2}$.
Taking natural logarithm: $x - y = \ln(1 - x^2/2) \Rightarrow y(x) = x - \ln(1 - x^2/2)$.
For minimum value, $\frac{dy}{dx} = 0 \Rightarrow 1 + xe^{y-x} = 0 \Rightarrow xe^{y-x} = -1 \Rightarrow e^{x-y} = -x$.
Substituting this into our equation: $-x = 1 - \frac{x^2}{2} \Rightarrow x^2 - 2x - 2 = 0$.
Solving the quadratic: $x = \frac{2 \pm \sqrt{4+8}}{2} = 1 \pm \sqrt{3}$.
Given domain $x \in (-\sqrt{2}, \sqrt{2})$, only $x = 1 - \sqrt{3}$ is valid.
Substitute $x = 1 - \sqrt{3}$ into $y(x)$:
$y_{min} = (1 - \sqrt{3}) - \ln(1 - \frac{(1-\sqrt{3})^2}{2})$.
$y_{min} = (1 - \sqrt{3}) - \ln(1 - \frac{4-2\sqrt{3}}{2}) = (1 - \sqrt{3}) - \ln(\sqrt{3} - 1)$.
Step 4: Final Answer:
The minimum value is $(1 - \sqrt{3}) - \log_e(\sqrt{3} - 1)$.

---

18. **Answer: 2 - 2**

Explanation:

Given differential equation:

$$\frac{dy}{dx} = e^{\alpha x + y}$$

Rewrite:

$$\frac{dy}{dx} = e^{\alpha x} e^{y}$$

Separating variables,

$$e^{-y} dy = e^{\alpha x} dx$$

Integrating both sides,

$$\int e^{-y} dy = \int e^{\alpha x} dx$$

$$-e^{-y} = \frac{1}{\alpha} e^{\alpha x} + C$$

Using the condition $y(0) = \ln \frac{1}{2} = -\ln 2$:

$$-e^{-(-\ln 2)} = \frac{1}{\alpha} + C \Rightarrow -2 = \frac{1}{\alpha} + C$$

$$C = -2 - \frac{1}{\alpha}$$

Using the condition $y(\ln 2) = \ln 2$**:**

$$-e^{-\ln 2} = \frac{1}{\alpha}e^{\alpha \ln 2} + C \Rightarrow -\frac{1}{2} = \frac{2^\alpha}{\alpha} + C$$

Substitute C:

$$-\frac{1}{2} = \frac{2^\alpha}{\alpha} - 2 - \frac{1}{\alpha}$$

$$\frac{3}{2} = \frac{2^\alpha - 1}{\alpha} \Rightarrow 3\alpha = 2(2^\alpha - 1)$$

Since $\alpha \in \mathbb{N}$, test values:

$$\alpha = 2: \quad 3(2) = 2(4-1) = 6 \quad \text{(satisfies)}$$

$$\boxed{\alpha = 2}$$

---

19. Answer: a

Explanation:

The given differential equation is $(x - x^3)dy = (y + yx^2 - 3x^4)dx$.
Let's rearrange it to see if it fits a standard form.
$x(1 - x^2)dy = (y(1 + x^2) - 3x^4)dx$.
$\frac{dy}{dx} = \frac{y(1+x^2)-3x^4}{x(1-x^2)}$.
$\frac{dy}{dx} = \frac{y(1+x^2)}{x(1-x^2)} - \frac{3x^4}{x(1-x^2)}$.
$\frac{dy}{dx} - \frac{y(1+x^2)}{x(1-x^2)} = -\frac{3x^3}{1-x^2}$.
This is a linear differential equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$.
Here, $P(x) = -\frac{1+x^2}{x(1-x^2)} = \frac{1+x^2}{x(x^2-1)}$.
And $Q(x) = -\frac{3x^3}{1-x^2} = \frac{3x^3}{x^2-1}$.
Let's find the integrating factor (I.F.).
I.F. = $e^{\int P(x)dx} = e^{\int \frac{1+x^2}{x(x^2-1)}dx}$.
Using partial fractions for the integrand: $\frac{1+x^2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$.
$A(x^2 - 1) + Bx(x + 1) + Cx(x - 1) = 1 + x^2$.
If $x = 0, -A = 1 \Rightarrow A = -1$.
If $x = 1, 2B = 2 \Rightarrow B = 1$.
If $x = -1, 2C = 2 \Rightarrow C = 1$.

So, $\int P(x)dx = \int(-\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1})dx = -\ln x + \ln(x-1) + \ln(x+1) = \ln\left(\frac{x^2-1}{x}\right)$.

I.F. = $e^{\ln\left(\frac{x^2-1}{x}\right)} = \frac{x^2-1}{x}$.

The solution is $y \cdot (\text{I.F.}) = \int Q(x) \cdot (\text{I.F.})dx + C$.

$y\left(\frac{x^2-1}{x}\right) = \int \frac{3x^3}{x^2-1} \cdot \frac{x^2-1}{x}dx + C$.

$y\left(\frac{x^2-1}{x}\right) = \int 3x^2dx + C = x^3 + C$.

We are given the condition $y(3) = 3$.

$3\left(\frac{3^2-1}{3}\right) = 3^3 + C$.

$3\left(\frac{8}{3}\right) = 27 + C \Rightarrow 8 = 27 + C \Rightarrow C = -19$.

The solution is $y\left(\frac{x^2-1}{x}\right) = x^3 - 19$.

Now we need to find $y(4)$.

$y(4)\left(\frac{4^2-1}{4}\right) = 4^3 - 19$.

$y(4)\left(\frac{15}{4}\right) = 64 - 19 = 45$.

$y(4) = 45 \times \frac{4}{15} = 3 \times 4 = 12$.

So, $y(4) = 12$.

---

20. Answer: 4 - 4

Explanation:

Step 1: Rearrange: $\frac{xdy-ydx}{x^2} = \frac{\sqrt{x^2-y^2}}{x^2}dx = \frac{\sqrt{1-(y/x)^2}}{x}dx$. Let $y/x = \sin\theta \implies d(y/x) = \frac{\sqrt{1-(y/x)^2}}{x}dx \implies \frac{d(y/x)}{\sqrt{1-(y/x)^2}} = \frac{dx}{x}$.

Step 2: Integrate: $\sin^{-1}(y/x) = \ln x + C$. At $x = 1, y = 0 \implies C = 0$. So $y = x\sin(\ln x)$.

Step 3: Area $= \int_1^{e^\pi} x\sin(\ln x)dx$. Let $\ln x = t \implies x = e^t, dx = e^tdt$. Area $= \int_0^\pi e^{2t}\sin tdt$.

Step 4: Using $\int e^{at}\sin btdt = \frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt)$: $= [\frac{e^{2t}}{5}(2\sin t - \cos t)]_0^\pi = \frac{e^{2\pi}}{5}(0-(-1)) - \frac{1}{5}(0-1) = \frac{1}{5}e^{2\pi} + \frac{1}{5}$.

Step 5: $\alpha = 1/5, \beta = 1/5$. $10(\alpha+\beta) = 10(2/5) = 4$.

---

21. Answer: d

Explanation:

Step 1: Let $v = y + 1 \implies \frac{dv}{dx} = \frac{dy}{dx}$. $\frac{dv}{dx} = v(ve^{x^2/2-x} - 1) = v^2e^{x^2/2-x} - v \implies \frac{dv}{dx} + v = v^2e^{x^2/2-x}$.

Step 2: This is a **Bernoulli's Equation**. Divide by v^2: $v^{-2}\frac{dv}{dx} + v^{-1} = e^{x^2/2-x}$. Let $z = v^{-1} \implies \frac{dz}{dx} = -v^{-2}\frac{dv}{dx}$. $-\frac{dz}{dx} + z = e^{x^2/2-x} \implies \frac{dz}{dx} - z = -e^{x^2/2-x}$.

Step 3: Solving the linear differential equation with initial condition $y(2) = 0$ leads to the derivative value at $x = 1$. After finding $z(x)$, we find $y(x)$ and then $y'(1) = -\frac{e^{3/2}}{(1+e)^2}$.

---

22. Answer: d

Explanation:

Step 1: Rearrange the equation: $\frac{dP}{dt} = \frac{P-900}{2} \Rightarrow \frac{dP}{P-900} = \frac{1}{2}dt$.
Step 2: Integrate both sides: $\ln|P - 900| = \frac{t}{2} + C$.
Step 3: Use $P(0) = 850$: $\ln|850 - 900| = 0 + C \Rightarrow C = \ln 50$.
Step 4: For $P(t) = 0$: $\ln|0 - 900| = \frac{t}{2} + \ln 50$.
Step 5: $\frac{t}{2} = \ln 900 - \ln 50 = \ln\left(\frac{900}{50}\right) = \ln 18$.
Step 6: $t = 2\ln 18$.

---

23. Answer: b

Explanation:

Step 1: $\frac{dy}{dx} = \frac{(x-2)^2+y+4}{x-2} = (x-2) + \frac{y+4}{x-2}$.
Step 2: Let $Y = y + 4$ and $X = x - 2$. $\frac{dY}{dX} = X + \frac{Y}{X} \Rightarrow \frac{dY}{dX} - \frac{Y}{X} = X$.
Step 3: Linear D.E.: $I.F. = e^{\int -1/X dX} = 1/X$.
Step 4: $Y(1/X) = \int X(1/X)dX = X + C$.
Step 5: $\frac{y+4}{x-2} = (x-2) + C$. Origin $(0, 0)$ passes through: $\frac{4}{-2} = -2 + C \Rightarrow C = 0$.
Step 6: $y + 4 = (x-2)^2 = x^2 - 4x + 4 \Rightarrow y = x^2 - 4x$.
Step 7: At $x = 5, y = 25 - 20 = 5$. So point $(5, 5)$ lies on it.

---

24. Answer: 2 - 2

Explanation:

Step 1: Simplify the differential equation Using the identity

$$\sin(x+y) + \sin(x-y) = 2\sin x \cos y$$

the given equation becomes

$$\sec y \frac{dy}{dx} - 2\sin x \cos y = 0$$

$$\frac{1}{\cos y}\frac{dy}{dx} = 2\sin x \cos y$$

$$\frac{dy}{dx} = 2\sin x \cos^2 y$$

Step 2: Separate the variables

$$\frac{1}{\cos^2 y}\, dy = 2\sin x\, dx$$

$$\sec^2 y\, dy = 2\sin x\, dx$$

Step 3: Integrate both sides

$$\int \sec^2 y\, dy = \int 2\sin x\, dx$$

$$\tan y = -2\cos x + C$$

Step 4: Use the initial condition Given $y(0) = 0$,

$$\tan 0 = -2\cos 0 + C$$

$$0 = -2 + C \Rightarrow C = 2$$

Hence,

$$\tan y = 2 - 2\cos x$$

Step 5: Find $y(\pi/2)$

$$\tan y\Big|_{x=\pi/2} = 2 - 2\cos\frac{\pi}{2} = 2$$

$$\tan y(\pi/2) = 2$$

Using

$$1 + \tan^2 y = \sec^2 y$$

$$\sec^2 y = 1 + 4 = 5 \Rightarrow \cos^2 y = \frac{1}{5}$$

Step 6: Evaluate $y'(\pi/2)$ From

$$\frac{dy}{dx} = 2\sin x \cos^2 y$$

$$y'(\pi/2) = 2\sin\frac{\pi}{2} \cdot \frac{1}{5} = \frac{2}{5}$$

Step 7: Final value

$$5y'(\pi/2) = 5 \times \frac{2}{5} = 2$$

Answer: $\boxed{2}$

---

25. Answer: 16 - 16

Explanation:

Given:

$$F(x) = e^{-x}\int_3^x (3t^2 + 2t + 4F'(t))\,dt$$

Multiply both sides by e^x:

$$e^x F(x) = \int_3^x (3t^2 + 2t + 4F'(t))\,dt$$

Step 1: Evaluate the integral

Split the integral:

$$\int_3^x (3t^2 + 2t)\,dt + \int_3^x 4F'(t)\,dt$$

$$= \left[t^3 + t^2\right]_3^x + 4\left[F(t)\right]_3^x$$

$$= (x^3 + x^2) - (27 + 9) + 4(F(x) - F(3))$$

From the given definition:

$$F(3) = e^{-3}\int_3^3 (\cdots)\,dt = 0$$

Hence,

$$e^x F(x) = x^3 + x^2 - 36 + 4F(x)$$

Step 2: Solve for $F(x)$

$$(e^x - 4)F(x) = x^3 + x^2 - 36$$

$$\boxed{F(x) = \frac{x^3 + x^2 - 36}{e^x - 4}}$$

Step 3: Differentiate $F(x)$
Using the quotient rule,

$$F'(x) = \frac{(3x^2 + 2x)(e^x - 4) - (x^3 + x^2 - 36)e^x}{(e^x - 4)^2}$$

Step 4: Evaluate $F'(4)$

$$F'(4) = \frac{(3 \cdot 16 + 2 \cdot 4)(e^4 - 4) - (64 + 16 - 36)e^4}{(e^4 - 4)^2}$$

$$= \frac{56(e^4 - 4) - 44e^4}{(e^4 - 4)^2}$$

$$= \frac{12e^4 - 224}{(e^4 - 4)^2}$$

--- **Step 5: Compare with given form**
Given:

$$F'(4) = \frac{\alpha e^\beta - 224}{(e^\beta - 4)^2}$$

Thus,

$$\alpha = 12, \quad \beta = 4$$

$$\boxed{\alpha + \beta = 16}$$

---

26. Answer: a

Explanation:

The given differential equation is $\ln\left(\frac{dy}{dx}\right) = 3x + 4y$.
Exponentiating both sides gives:
$\frac{dy}{dx} = e^{3x+4y} = e^{3x} \cdot e^{4y}$.
This is a variable separable differential equation.

$\frac{dy}{e^{4y}} = e^{3x}dx.$
$e^{-4y}dy = e^{3x}dx.$
Integrate both sides:
$\int e^{-4y}dy = \int e^{3x}dx.$
$\frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + C$, where C is the integration constant.
We are given the initial condition $y(0) = 0$. Substitute $x = 0, y = 0$ to find C.
$\frac{e^{-4(0)}}{-4} = \frac{e^{3(0)}}{3} + C.$
$\frac{1}{-4} = \frac{1}{3} + C \implies C = -\frac{1}{4} - \frac{1}{3} = -\frac{3+4}{12} = -\frac{7}{12}.$
The particular solution is:
$-\frac{e^{-4y}}{4} = \frac{e^{3x}}{3} - \frac{7}{12}.$
Multiply by -12: $3e^{-4y} = -4e^{3x} + 7.$
Now we need to find the value of y when $x = -\frac{2}{3}\ln 2.$
Let $x_0 = -\frac{2}{3}\ln 2.$
$e^{3x_0} = e^{3(-\frac{2}{3}\ln 2)} = e^{-2\ln 2} = e^{\ln(2^{-2})} = 2^{-2} = \frac{1}{4}.$
Substitute this into the solution:
$3e^{-4y} = -4\left(\frac{1}{4}\right) + 7.$
$3e^{-4y} = -1 + 7 = 6.$
$e^{-4y} = \frac{6}{3} = 2.$
Take the natural logarithm of both sides:
$\ln(e^{-4y}) = \ln 2.$
$-4y = \ln 2 \implies y = -\frac{1}{4}\ln 2.$
We are given that $y\left(-\frac{2}{3}\log_e 2\right) = \alpha \log_e 2.$
Comparing the two expressions for y, we get:
$\alpha = -\frac{1}{4}.$

---

27. Answer: 1 - 1

Explanation:

82. Given differential equation:

$$(2xy^2 - y)\,dx + x\,dy = 0$$

Rewrite:

$$\frac{dy}{dx} = \frac{y}{x} - 2y^2$$

This is a Bernoulli equation. Divide by y^2:

$$y^{-2}\frac{dy}{dx} - \frac{1}{x}y^{-1} = -2$$

Let $v = y^{-1}$, then:

$$\frac{dv}{dx} + \frac{1}{x}v = 2$$

Integrating factor:

$$\text{IF} = x$$

$$\frac{d}{dx}(vx) = 2x \Rightarrow vx = x^2 + C$$

$$\frac{x}{y} = x^2 + C$$

Intersection of $2x - 3y = 1$ and $3x + 2y = 8$ gives $(2, 1)$.

$$\frac{2}{1} = 4 + C \Rightarrow C = -2$$

$$\frac{x}{y} = x^2 - 2$$

At $x = 1$:

$$y = -1 \Rightarrow |y(1)| = 1$$

$$\boxed{1}$$

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28. Answer: 4 - 4

Explanation:

Step 1: Understanding the Concept:
We differentiate both sides with respect to x using the Newton-Leibniz formula to convert the integral equation into a linear differential equation.
Step 2: Detailed Explanation:
Differentiate both sides of $x\phi(x) = \int_5^x (3t^2 - 2\phi'(t))dt$ w.r.t x:

$$\phi(x) + x\phi'(x) = 3x^2 - 2\phi'(x)$$

Rearrange the terms:

$$(x+2)\phi'(x) + \phi(x) = 3x^2$$

This is a first-order linear differential equation. Divide by $(x+2)$:

$$\frac{d\phi}{dx} + \frac{1}{x+2}\phi = \frac{3x^2}{x+2}$$

Integrating factor (I.F.) = $e^{\int \frac{1}{x+2}dx} = e^{\ln(x+2)} = x+2$.
Multiplying the equation by I.F.:

$$\frac{d}{dx}[\phi(x)\cdot(x+2)] = \frac{3x^2}{x+2}\cdot(x+2) = 3x^2$$

Integrate both sides:

$$\phi(x)\cdot(x+2) = \int 3x^2 dx = x^3 + C$$

Using initial condition $\phi(0) = 4$:

$$4(0+2) = 0^3 + C \implies C = 8$$

So, $\phi(x) = \frac{x^3+8}{x+2}$.
Using the identity $x^3 + 8 = (x+2)(x^2 - 2x + 4)$:

$$\phi(x) = x^2 - 2x + 4$$

Now, $\phi(2) = 2^2 - 2(2) + 4 = 4 - 4 + 4 = 4$.
Step 3: Final Answer:
The value of $\phi(2)$ is 4.

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29. Answer: c

Explanation:

Step 1: Rearrange the differential equation.

$$(2x - 10y^3)dy + ydx = 0$$

$$y\frac{dx}{dy} + 2x - 10y^3 = 0$$

$$\frac{dx}{dy} + \frac{2}{y}x = 10y^2$$

This is a linear differential equation of the form $\frac{dx}{dy} + P(y)x = Q(y)$, where $P(y) = 2/y$ and $Q(y) = 10y^2$. **Step 2: Find the integrating factor (I.F.).**

$$I.F. = e^{\int P(y)dy} = e^{\int \frac{2}{y}dy} = e^{2\ln y} = e^{\ln y^2} = y^2$$

Step 3: Find the general solution.
The solution is given by $x \cdot (I.F.) = \int Q(y) \cdot (I.F.)dy + C$.

$$x \cdot y^2 = \int (10y^2) \cdot y^2 dy + C$$

$$xy^2 = \int 10y^4 dy + C$$

$$xy^2 = 10\frac{y^5}{5} + C = 2y^5 + C$$

Step 4: Use the initial condition to find C.
The curve passes through (0, 1). So, x=0, y=1.

$$(0)(1)^2 = 2(1)^5 + C \implies 0 = 2 + C \implies C = -2$$

The particular solution is $xy^2 = 2y^5 - 2$. **Step 5: Use the second point to find the equation for β.**
The curve also passes through (2, β). So, x=2, y=β.

$$(2)(\beta)^2 = 2(\beta)^5 - 2$$

Divide by 2:

$$\beta^2 = \beta^5 - 1$$

Rearranging the terms gives the equation that β must satisfy:

$$\beta^5 - \beta^2 - 1 = 0$$

This means β is a root of the equation $y^5 - y^2 - 1 = 0$.

30. Answer: c

Explanation:

Step 1: Find the length of the latus rectum.

The length of the latus rectum (LLR) is the perpendicular distance from the point (2, -3) to the line 3x + 4y - 5 = 0. Using the distance formula $d = \frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$:

$$LLR = \frac{|3(2)+4(-3)-5|}{\sqrt{3^2+4^2}} = \frac{|6-12-5|}{\sqrt{9+16}} = \frac{|-11|}{\sqrt{25}} = \frac{11}{5}$$

Step 2: Write the general equation of the family of parabolas.

The parabolas have their axis parallel to the y-axis. The general equation for such a parabola is:

$$(x-h)^2 = 4a(y-k)$$

where (h, k) is the vertex and $4a$ is the length of the latus rectum. We found that $LLR = 4a = 11/5$. So, the equation of the family is:

$$(x-h)^2 = \frac{11}{5}(y-k)$$

Here, h and k are arbitrary constants (parameters). **Step 3: Form the differential equation.**

Since there are two parameters (h and k), we need to differentiate the equation twice to eliminate them. Differentiate with respect to x:

$$2(x-h) = \frac{11}{5}\frac{dy}{dx}$$

This eliminates k. Now we differentiate again to eliminate h. Differentiate with respect to x again:

$$2(1) = \frac{11}{5}\frac{d^2y}{dx^2}$$

$$2 = \frac{11}{5}\frac{d^2y}{dx^2}$$

Rearranging the terms:

$$10 = 11\frac{d^2y}{dx^2}$$

This is the required differential equation.