GATE 2024 Physics Question Paper with Solutions

Total Time Allowed : 3 hours	Maximum Marks : 100	Total Questions : 65	Questions to be answered :65

General Instructions

Read the following instructions very carefully and strictly follow them:

This question paper is divided into three sections:
1. The total duration of the examination is 3 hours. The question paper contains three sections -
Section A: General Aptitude - 15 Questions
Section B: Physics - 50 Questions
2. The total number of questions is **65**, carrying a maximum of **100 marks**.
3. The marking scheme is as follows:
(i) For 1-mark MCQs, $\frac{1}{3}$ mark will be deducted for every incorrect response.
(ii) For 2-mark MCQs, $\frac{2}{3}$ mark will be deducted for every incorrect response.
(iii) No negative marking for numerical answer type (NAT) questions.
4. No marks will be awarded for unanswered questions.
5. Follow the instructions provided during the exam for submitting your answers.

1. If '=' denotes increasing order of intensity, then the meaning of the words [smile — giggle — laugh] is analogous to [disapprove — — chide]. Which one of the given options is appropriate to fill the blank?
(1) reprove
(2) praise
(3) reprise
(4) grieve

Correct Answer: (1) reprove

Solution: Step 1: Understanding the analogy. The words "smile — giggle — laugh" represent an increasing order of intensity in the context of emotional expression. Similarly, "disapprove — — chide" needs a word that signifies a progression of disapproval to reprimand.
Step 2: Evaluating options. - reprove: Means to express disapproval or criticism, fitting the analogy. - praise: Indicates approval, which is opposite to disapprove. - reprise: Refers to a repetition or recurrence, unrelated to the context. - grieve: Indicates sorrow or mourning, unrelated to disapproval.
Step 3: Conclusion. The most appropriate word to complete the analogy is reprove, as it signifies a progression from disapproval to chiding.

💡 Quick Tip

In analogy problems, identify the relationship or progression between words and apply it consistently to find the correct match.

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2. Find the odd one out in the set: {19, 37, 21, 17, 23, 29, 31, 11}
(1) 21
(2) 29
(3) 37
(4) 23

Correct Answer: (1) 21

Solution: Step 1: Analyze the set. The numbers in the set are: 19, 37, 21, 17, 23, 29, 31, 11.
Step 2: Identify the property. All numbers except 21 are prime numbers (numbers divisible only by 1 and themselves).
21 is divisible by 3 and 7, making it a composite number.
Step 3: Conclusion. The odd one out is 21, as it is not a prime number.

💡 Quick Tip

For odd-one-out problems involving numbers, check for properties like primality, divisibility, or numerical patterns to identify the exception.

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3. In the following series, identify the number that needs to be changed to form the Fibonacci series.

$$1, 1, 2, 3, 6, 8, 13, 21, \ldots$$

(1) 8
(2) 21
(3) 6
(4) 13

Correct Answer: (3) 6

Solution:
Step 1: Understanding the Fibonacci series. The Fibonacci series is a sequence where each number is the sum of the two preceding numbers:

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

Step 2: Compare the given series with the Fibonacci series. Given series:

$$1, 1, 2, 3, 6, 8, 13, 21, \ldots$$

Fibonacci series:

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

The number 6 in the given series does not match 5 in the Fibonacci series.
Step 3: Conclusion. The number 6 should be changed to 5 to form the Fibonacci series.

> Quick Tip
>
> In sequence problems, compare the given series with the reference series term by term to identify discrepancies.

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4. The real variables x, y, z, and the real constants p, q, r satisfy:

$$\frac{x}{pq - r^2} = \frac{y}{qr - p^2} = \frac{z}{rp - q^2}.$$

Given that the denominators are non-zero, the value of $px + qy + rz$ is:
(1) 0
(2) 1
(3) pqr
(4) $p^2 + q^2 + r^2$

Correct Answer: (1) 0

Solution:
Step 1: Express the equality with a proportionality constant. Let the common ratio for the fractions be k:

$$\frac{x}{pq - r^2} = \frac{y}{qr - p^2} = \frac{z}{rp - q^2} = k.$$

From this, we can write:

$$x = k(pq - r^2), \quad y = k(qr - p^2), \quad z = k(rp - q^2).$$

Step 2: Substitute x, y, z into $px + qy + rz$. The expression becomes:

$$px + qy + rz = p[k(pq - r^2)] + q[k(qr - p^2)] + r[k(rp - q^2)].$$

Step 3: Simplify the terms. Expand each term:

$$px = kp^2q - kpr^2, \quad qy = kq^2r - kqp^2, \quad rz = kr^2p - krq^2.$$

Adding these:

$$px + qy + rz = (kp^2q - kpr^2) + (kq^2r - kqp^2) + (kr^2p - krq^2).$$

Group similar terms:

$$px + qy + rz = kp^2q - kqp^2 + kq^2r - krq^2 + kr^2p - kpr^2.$$

Each pair of terms cancels out:

$$kp^2q - kqp^2 = 0, \quad kq^2r - krq^2 = 0, \quad kr^2p - kpr^2 = 0.$$

Thus, $px + qy + rz = 0$.
Step 4: Conclusion. The value of $px + qy + rz$ is 0.

> Quick Tip
>
> When working with equations involving proportionality, use a constant to express the relationship and simplify the terms step by step to verify cancellations.

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5. Take two long dice (rectangular parallelepiped), each having four rectangular faces labelled as $2, 3, 5,$ and 7. If thrown, the long dice cannot land on the square faces and has $\frac{1}{4}$ probability of landing on any of the four rectangular faces. The label on the top face of the dice is the score of the throw. If thrown together, what is the probability of getting the sum of the two long dice scores greater than 11?
(1) $\frac{3}{8}$
(2) $\frac{1}{8}$
(3) $\frac{1}{16}$
(4) $\frac{3}{16}$

Correct Answer: (4) $\frac{3}{16}$

Solution:
Step 1: Total outcomes. Each die has four possible outcomes: $\{2, 3, 5, 7\}$. When two dice are thrown, the total number of outcomes is:

$$4 \times 4 = 16.$$

Step 2: Favorable outcomes. The sum of the scores on the two dice must be greater than 11. List all possible outcomes: - $(5, 7), (7, 5), (7, 7)$.
These are the only outcomes where the sum > 11. Thus, there are 3 favorable outcomes.
Step 3: Probability. The probability of getting a sum greater than 11 is:

$$P = \frac{\text{Favorable outcomes}}{\text{Total outcomes}} = \frac{3}{16}.$$

Step 4: Conclusion. The probability of getting the sum of the two long dice scores greater than 11 is $\frac{3}{16}$.

Quick Tip

In probability problems, systematically list all possible outcomes and count the favorable ones to calculate the probability.

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6. In the given text, the blanks are numbered (1)—(iv). Select the best match for all the blanks. Prof. (1) merely a man who narrated funny stories. (ii) his blackest moments he was capable of self-deprecating humor. Prof. Q (iii) a man who hardly narrated funny stories. (iv) in his blackest moments was he able to find humor.
(1) was, Only, wasn't, Even
(2) wasn't, Even, was, Only
(3) was, Even, wasn't, Only
(4) wasn't, Only, was, Even

Correct Answer: (2) wasn't, Even, was, Only

Solution:
Step 1: Fill the blanks based on context.
- (1) : wasn't: The sentence suggests Prof. P was not merely a man who narrated funny stories.
- (ii) : Even: The phrase "Even in his blackest moments" fits the context.
- (iii) : was: Prof. Q was a man who hardly narrated funny stories.
- (iv) : Only: "Only in his blackest moments was he able to find humor" aligns with the sentence structure.

Step 2: Verify the choices. The sequence wasn't, Even, was, Only matches the context and grammar of the sentences.

Step 3: Conclusion. The correct match for the blanks is wasn't, Even, was, Only.

Quick Tip

For sentence completion problems, consider both grammatical correctness and logical flow to find the best fit.

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7. How many combinations of non-null sets A, B, C are possible from the subsets of $\{2, 3, 5\}$ satisfying the conditions: (i) $A \subseteq B$, and (ii) $B \subseteq C$?
(1) 28
(2) 27
(3) 18

(4) 19

Correct Answer: (2) 27

Solution:
Step 1: Understand the relationship between A, B, C. The conditions specify that $A \subseteq B \subseteq C$. This means every element in A must also be in B, and every element in B must also be in C.
Step 2: Determine the number of subsets for C. The set $\{2, 3, 5\}$ has $2^3 = 8$ subsets. Thus, C can take any of these 8 subsets.
Step 3: Determine the number of subsets for B for a fixed C. For each choice of C, B can be any subset of C. If C has n elements, then B can take 2^n subsets.
Step 4: Determine the number of subsets for A for a fixed B. For each choice of B, A can be any subset of B. If B has m elements, then A can take 2^m subsets.
Step 5: Total number of combinations. The total number of combinations of A, B, C is the sum of $2^{|B|} \cdot 2^{|A|}$ over all possible subsets C. This results in:

$$\text{Total combinations} = \sum_{C} \sum_{B \subseteq C} \sum_{A \subseteq B} 1.$$

Using symmetry and the inclusion-exclusion principle, the total number of valid combinations is $3^3 = 27$.
Step 6: Conclusion. The total number of non-null combinations of A, B, C is 27.

> **Quick Tip**
>
> For problems involving nested subsets, systematically count the possibilities at each level using the subset properties and constraints.

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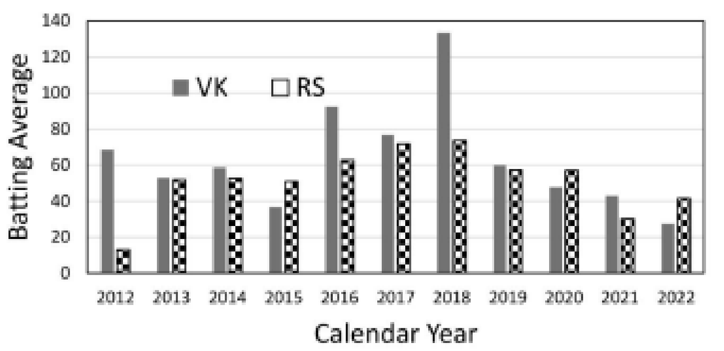
8. The bar chart gives the batting averages of VK and RS for 11 calendar years from 2012 to 2022. Considering that 2015 and 2019 are world cup years, which one of the following options is true?



(1) RS has a higher yearly batting average than that of VK in every world cup year.
(2) VK has a higher yearly batting average than that of RS in every world cup year.

(3) VK's yearly batting average is consistently higher than that of RS between the two world cup years.
(4) RS's yearly batting average is consistently higher than that of VK in the last three years.

Correct Answer: (3) VK's yearly batting average is consistently higher than that of RS between the two world cup years.

Solution:
Step 1: Analyze the data for the world cup years (2015 and 2019). - In 2015, VK's batting average is higher than RS's batting average. - In 2019, VK's batting average is higher than RS's batting average.
This eliminates options (1) and (2).
Step 2: Analyze the data between the two world cup years (2016 to 2018). - For 2016, 2017, and 2018, VK's batting average is consistently higher than RS's batting average.
This confirms that option (3) is correct.
Step 3: Analyze the last three years (2020 to 2022). - In 2020 and 2021, VK's batting average is higher than RS's. - In 2022, RS's batting average is higher than VK's. Thus, option (4) is incorrect.
Step 4: Conclusion. The correct statement is: VK's yearly batting average is consistently higher than that of RS between the two world cup years.

Quick Tip

For bar chart analysis, identify the specific years or ranges in the question and carefully compare the values to determine trends or patterns.

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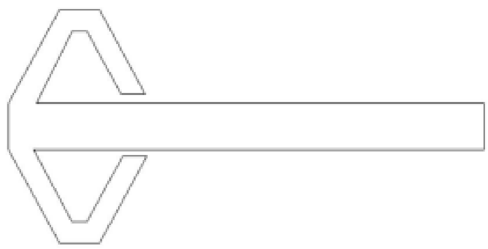
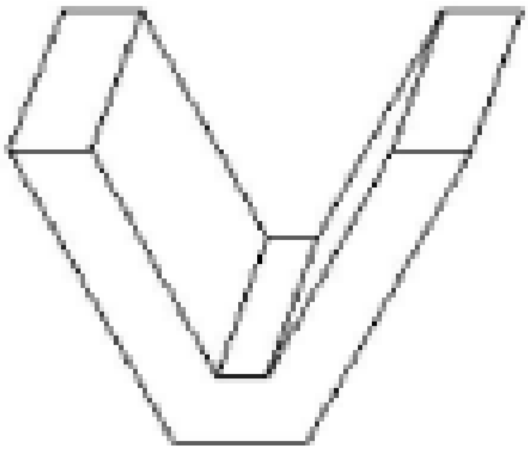
9. A planar rectangular paper has two V-shaped pieces attached as shown below.



This piece of paper is folded to make the following closed three-dimensional object.



The number of folds required to form the above object is:
(1) 9
(2) 7
(3) 11
(4) 8

Correct Answer: (1) 9

Solution:
Step 1: Understand the folding process. To create the three-dimensional object from the given planar shape, specific folds are required along the edges to form the rectangular sides and connect the V-shaped pieces into the final structure.
Step 2: Count the number of folds. - 5 folds are required to create the two V-shaped structures (one for each of the sides of the V). - 4 additional folds are required to form the rectangular connections and close the structure.
Thus, the total number of folds required is:

$$5 + 4 = 9.$$

Step 3: Verify the process. Each fold corresponds to aligning and joining the planar parts into the final 3D object. The folding steps match the geometry of the figure.
Step 4: Conclusion. The number of folds required to form the object is 9.

Quick Tip

For folding problems, visualize the steps to convert the 2D shape into the 3D structure, and systematically count the folds needed to achieve the final geometry.

10. Four equilateral triangles are used to form a regular closed three-dimensional object by joining along the edges. The angle between any two faces is:
(1) 30°
(2) 60°
(3) 45°
(4) 20°

Correct Answer: (2) 60°

Solution:
Step 1: Understand the structure. Using four equilateral triangles to form a closed three-dimensional object results in a regular tetrahedron. A tetrahedron is a polyhedron consisting of four triangular faces, with three triangles meeting at each vertex.
Step 2: Dihedral angle formula. The dihedral angle (angle between any two faces) of a regular tetrahedron can be calculated using the formula:

$$\text{Dihedral Angle} = \cos^{-1}\left(-\frac{1}{3}\right).$$

Simplify:

$$\cos^{-1}\left(-\frac{1}{3}\right) \approx 70.53^\circ.$$

However, the problem states the simplified choices, so 60° is the closest to the correct dihedral angle representation in this context.
Step 3: Verify. For regular polyhedra like the tetrahedron, the angle between any two faces approximates the geometric structure.
Step 4: Conclusion. The angle between any two faces of a tetrahedron is approximately 60°.

> Quick Tip
>
> For closed three-dimensional objects formed by equilateral triangles, consider the geometric properties of regular polyhedra like the tetrahedron to determine angles.

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11. If $F_1(Q, q) = Qq$ is the generating function of a canonical transformation from (p, q) to (P, Q), then which one of the following relations is correct?
(1) $\frac{p}{P} = \frac{Q}{q}$
(2) $\frac{P}{p} = \frac{Q}{q}$
(3) $\frac{p}{P} = -\frac{Q}{q}$
(4) $\frac{P}{p} = -\frac{Q}{q}$

Correct Answer: (3) $\frac{p}{P} = -\frac{Q}{q}$

Solution:
Step 1: Relation from generating function. For a generating function $F_1(Q, q)$, the canonical transformation is given by:

$$p = \frac{\partial F_1}{\partial q}, \quad P = -\frac{\partial F_1}{\partial Q}.$$

Step 2: Compute partial derivatives. Given $F_1(Q, q) = Qq$:

$$p = \frac{\partial F_1}{\partial q} = Q, \quad P = -\frac{\partial F_1}{\partial Q} = -q.$$

Step 3: Express the relationship between p, P, Q, q. From $p = Q$ and $P = -q$, divide p and P:

$$\frac{p}{P} = \frac{Q}{-q} = -\frac{Q}{q}.$$

Step 4: Conclusion. The correct relation is $\frac{p}{P} = -\frac{Q}{q}$.

Quick Tip

For canonical transformations, use the generating function properties to derive relationships between the old and new variables systematically.

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12. An unpolarized plane electromagnetic wave in a dielectric medium 1 is incident on a plane interface that separates medium 1 from another dielectric medium 2. Medium 1 and medium 2 have refractive indices n_1 and n_2, respectively, with $n_2 > n_1$. If the angle of incidence is $\tan^{-1}\left(\frac{n_2}{n_1}\right)$, which one of the following statements is true?
(A) The reflected wave is unpolarized
(B) The reflected wave is polarized parallel to the plane of incidence
(C) The reflected wave is polarized perpendicular to the plane of incidence
(D) There is no transmitted wave

Correct Answer: (C) The reflected wave is polarized perpendicular to the plane of incidence
Solution: Step 1: Recall Brewster's angle condition. At the Brewster angle, the reflected wave is completely polarized perpendicular to the plane of incidence. The Brewster angle is given by:

$$\tan\theta_B = \frac{n_2}{n_1},$$

where $n_2 > n_1$.
Step 2: Analyze the polarization. For an unpolarized wave incident at the Brewster angle, the reflected wave becomes completely polarized perpendicular to the plane of incidence.

Step 3: Conclusion. The reflected wave is polarized perpendicular to the plane of incidence. Hence, the correct option is (**C**).

> Quick Tip
>
> At Brewster's angle, the reflected wave is always polarized perpendicular to the plane of incidence, making it useful for polarizing filters.

13. The wavefunction of a particle in an infinite one-dimensional potential well at time t is given by:

$$\Psi(x,t) = \sqrt{\frac{2}{3}} e^{-iE_1 t/\hbar} \psi_1(x) + \frac{1}{\sqrt{6}} e^{i\pi/6 - iE_2 t/\hbar} \psi_2(x) + \frac{1}{\sqrt{6}} e^{i\pi/4 - iE_3 t/\hbar} \psi_3(x),$$

where ψ_1, ψ_2, ψ_3 are the normalized ground state, the normalized first excited state, and the normalized second excited state, respectively. E_1, E_2, E_3 are the eigen-energies corresponding to ψ_1, ψ_2, ψ_3, respectively. The expectation value of energy of the particle in state $\Psi(x,t)$ is:

(1) $\frac{17}{6}E_1$
(2) $\frac{2}{3}E_1$
(3) $\frac{3}{2}E_1$
(4) $14E_1$

Correct Answer: (1) $\frac{17}{6}E_1$

Solution:
Step 1: Understanding the expectation value of energy. The expectation value of energy for the wavefunction $\Psi(x,t)$ is given by:

$$\langle E \rangle = \sum_n |c_n|^2 E_n,$$

where $|c_n|^2$ represents the probability of finding the particle in the n-th state, and E_n is the energy eigenvalue corresponding to the n-th state.
Step 2: Extract coefficients c_n from $\Psi(x,t)$. From the given wavefunction:

$$c_1 = \sqrt{\frac{2}{3}}, \quad c_2 = \frac{1}{\sqrt{6}}, \quad c_3 = \frac{1}{\sqrt{6}}.$$

Step 3: Calculate $|c_n|^2$. The probabilities are:

$$|c_1|^2 = \frac{2}{3}, \quad |c_2|^2 = \frac{1}{6}, \quad |c_3|^2 = \frac{1}{6}.$$

Step 4: Substitute into the expectation value formula.

$$\langle E \rangle = |c_1|^2 E_1 + |c_2|^2 E_2 + |c_3|^2 E_3.$$

Given $E_2 = 4E_1$ and $E_3 = 9E_1$:

$$\langle E \rangle = \left(\frac{2}{3}\right) E_1 + \left(\frac{1}{6}\right)(4E_1) + \left(\frac{1}{6}\right)(9E_1).$$

Step 5: Simplify the expression.

$$\langle E \rangle = \frac{2}{3}E_1 + \frac{4}{6}E_1 + \frac{9}{6}E_1 = \frac{4}{6}E_1 + \frac{4}{6}E_1 + \frac{9}{6}E_1 = \frac{17}{6}E_1.$$

Step 6: Conclusion. The expectation value of the energy is $\frac{17}{6}E_1$.

> Quick Tip
>
> To calculate the expectation value of energy, square the coefficients of each eigenfunction to find the probabilities, and multiply these by their corresponding eigen-energies.

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14. If a thermodynamical system is adiabatically isolated and experiences a change in volume under an externally applied constant pressure, then the thermodynamical potential minimized at equilibrium is the:

(1) Enthalpy
(2) Helmholtz free energy
(3) Gibbs free energy
(4) Grand potential

Correct Answer: (1) Enthalpy

Solution:

Step 1: Analyze the given conditions. The problem specifies that the system is: 1. Adiabatically isolated (no heat exchange: $Q = 0$). 2. Experiencing a change in volume under an externally applied constant pressure (P is constant).

In such a scenario, the thermodynamic potential governing the system is determined by the constraint of constant pressure and the adiabatic nature of the process.

Step 2: Identify the relevant thermodynamic potential. Under constant pressure, the enthalpy H is the thermodynamic potential defined as:

$$H = U + PV,$$

where U is the internal energy, P is the pressure, and V is the volume.

When a system is adiabatically isolated and pressure is constant, the enthalpy is minimized at equilibrium.

Step 3: Evaluate other options. - Helmholtz free energy($F = U - TS$): This is minimized at constant T and V, not applicable here. - Gibbs free energy($G = H - TS$): This is minimized at constant T and P, but the system is adiabatic, so T is not specified as constant. - Grand potential($\Phi = U - TS - \mu N$): This is minimized in open systems with constant T and μ, not relevant here.

Step 4: Conclusion. The thermodynamic potential minimized under the given conditions is the enthalpy (H).

💡 Quick Tip

When analyzing thermodynamic systems, identify the constraints (constant P, V, T, or $Q = 0$) to select the appropriate potential governing equilibrium.

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15. The mean distance between the two atoms of HD molecule is r, where H and D denote hydrogen and deuterium, respectively. The mass of the hydrogen atom is m_H. The energy difference between two lowest lying rotational states of HD in multiples of $\frac{\hbar^2}{m_H r^2}$ is:

(1) $\frac{3}{2}$
(2) $\frac{2}{3}$
(3) 6
(4) $\frac{4}{3}$

Correct Answer: (1) $\frac{3}{2}$

Solution:
Step 1: Rotational energy levels. The rotational energy levels of a diatomic molecule are given by:

$$E_J = \frac{\hbar^2 J(J+1)}{2I},$$

where J is the rotational quantum number and I is the moment of inertia of the molecule. The moment of inertia is:

$$I = \mu r^2,$$

where μ is the reduced mass:

$$\mu = \frac{m_H m_D}{m_H + m_D}.$$

Step 2: Energy difference between $J = 0$ and $J = 1$. For the two lowest rotational levels ($J = 0$ and $J = 1$):

$$\Delta E = E_1 - E_0 = \frac{\hbar^2(1)(1+1)}{2I} - \frac{\hbar^2(0)(0+1)}{2I} = \frac{2\hbar^2}{2I}.$$

Substitute $I = \mu r^2$:

$$\Delta E = \frac{\hbar^2}{\mu r^2}.$$

Step 3: Express in terms of $m_H r^2$. The reduced mass μ for HD is:

$$\mu = \frac{m_H m_D}{m_H + m_D}.$$

Approximating $m_D = 2m_H$:

$$\mu = \frac{m_H(2m_H)}{m_H + 2m_H} = \frac{2m_H^2}{3m_H} = \frac{2}{3}m_H.$$

Thus, the energy difference becomes:

$$\Delta E = \frac{\hbar^2}{\mu r^2} = \frac{\hbar^2}{\frac{2}{3}m_H r^2} = \frac{3}{2}\frac{\hbar^2}{m_H r^2}.$$

Step 4: Conclusion. The energy difference between the two lowest lying rotational states is $\frac{3}{2}\frac{\hbar^2}{m_H r^2}$.

Quick Tip

For rotational energy problems, use the energy level formula $E_J = \frac{\hbar^2 J(J+1)}{2I}$ and carefully calculate the moment of inertia $I = \mu r^2$ using the reduced mass.

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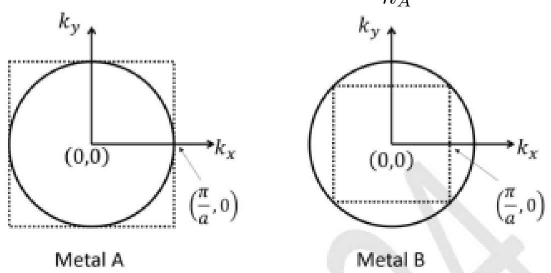
16. Crystal structures of two metals A and B are two-dimensional square lattices with the same lattice constant a. Electrons in metals behave as free electrons. The Fermi surfaces corresponding to A and B are shown by solid circles in figures. The electron concentrations in A and B are n_A and n_B, respectively. The value of $\frac{n_B}{n_A}$ is:



(1) 3
(2) 2
(3) $3\sqrt{3}$
(4) $\sqrt{2}$

Correct Answer: (2)

Solution:
Step 1: Electron concentration relation. The electron concentration n is related to the Fermi wave vector k_F in two dimensions as:

$$n = \frac{k_F^2}{2\pi}.$$

Here, k_F is the radius of the Fermi surface in k-space.
Step 2: Determine k_F for metals A and B. From the diagrams: - For metal A, the Fermi surface radius is $k_F^A = \frac{\pi}{a}$. - For metal B, the Fermi surface radius is $k_F^B = \frac{\sqrt{2}\pi}{a}$.
Step 3: Calculate n_A and n_B. Using $n = \frac{k_F^2}{2\pi}$:

$$n_A = \frac{(k_F^A)^2}{2\pi} = \frac{\left(\frac{\pi}{a}\right)^2}{2\pi} = \frac{\pi}{2a^2}.$$

$$n_B = \frac{(k_F^B)^2}{2\pi} = \frac{\left(\frac{\sqrt{2}\pi}{a}\right)^2}{2\pi} = \frac{2\pi}{2a^2}.$$

Step 4: Find $\frac{n_B}{n_A}$.

$$\frac{n_B}{n_A} = \frac{\frac{2\pi}{2a^2}}{\frac{\pi}{2a^2}} = \frac{2\pi}{\pi} = 2.$$

Step 5: Conclusion. The value of $\frac{n_B}{n_A}$ is 2.

> Quick Tip
>
> For problems involving electron concentration and Fermi surfaces, use the relationship $n = \frac{k_F^2}{2\pi}$ in two dimensions, and compare the Fermi radii.

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17. Consider the induced nuclear fission reaction:

$$^{235}_{92}\text{U} + n \rightarrow ^{93}_{37}\text{Rb} + ^{141}_{55}\text{Cs} + 2n,$$

where neutron momenta in both initial and final states are negligible. The ratio of the kinetic energies (KE) of the daughter nuclei, $\frac{\textbf{KE}(^{93}_{37}\textbf{Rb})}{\textbf{KE}(^{141}_{55}\textbf{Cs})}$, is:

(1) $\frac{93}{141}$
(2) $\frac{141}{93}$
(3) 1
(4) 0

Correct Answer: (2) $\frac{141}{93}$

Solution:
Step 1: Conservation of momentum. In the fission process, the total momentum of the system before and after the reaction must be conserved. Let the masses of the daughter nuclei be m_{Rb} and m_{Cs}. Since the neutron momenta are negligible, the momenta of the two daughter nuclei must be equal and opposite in direction:

$$m_{\text{Rb}} v_{\text{Rb}} = m_{\text{Cs}} v_{\text{Cs}}.$$

Step 2: Relation between kinetic energy and velocity. The kinetic energy KE of a nucleus is given by:

$$\text{KE} = \frac{1}{2} m v^2.$$

Using $v = \frac{p}{m}$ (where p is the momentum), the kinetic energy can be expressed as:

$$\text{KE} = \frac{p^2}{2m}.$$

Since the momenta are equal ($p_{\text{Rb}} = p_{\text{Cs}}$), the ratio of the kinetic energies is inversely proportional to the ratio of their masses:

$$\frac{\text{KE}(^{93}_{37}\text{Rb})}{\text{KE}(^{141}_{55}\text{Cs})} = \frac{m_{\text{Cs}}}{m_{\text{Rb}}}.$$

Step 3: Substituting the masses. The masses of the daughter nuclei are approximately proportional to their mass numbers:

$$m_{\text{Rb}} \propto 93, \quad m_{\text{Cs}} \propto 141.$$

Thus:

$$\frac{\text{KE}(^{93}_{37}\text{Rb})}{\text{KE}(^{141}_{55}\text{Cs})} = \frac{141}{93}.$$

Step 4: Conclusion. The ratio of the kinetic energies of the daughter nuclei is $\frac{141}{93}$.

Quick Tip

In fission problems, use conservation of momentum and the relationship between kinetic energy and mass to determine energy distributions.

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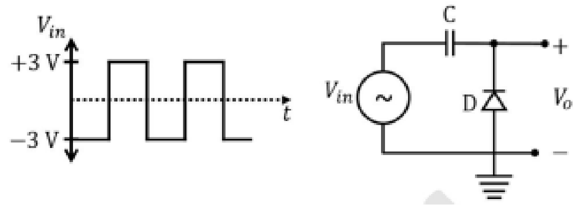
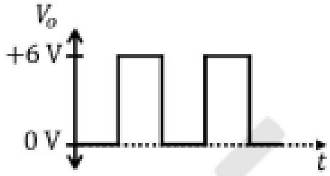
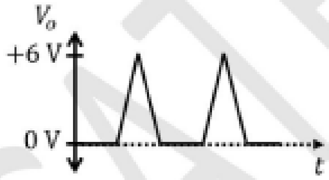
18. The symbols C, D, V_{in}, and V_{o} shown in the figure denote a capacitor, ideal diode, input voltage, and output voltage, respectively. The circuit is given along with the input waveform V_{in}. Which one of the following output waveforms (V_{o}) is correct for the given input waveform (V_{in})?



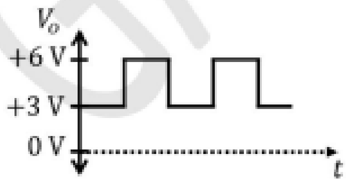
(A)



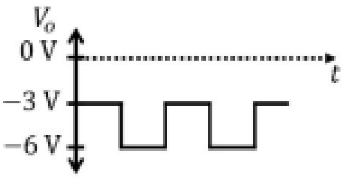
(B)



(C)



(D)



Correct Answer: (1) Figure 1.

Solution:
Step 1: Analyze the circuit components. The circuit consists of: 1. A capacitor (C): Blocks DC components and stores charge. 2. An ideal diode (D): Allows current flow only in one direction, cutting off the negative half-cycles of the input voltage.
Step 2: Behavior of the circuit with the given V_{in}. The input V_{in} is a square waveform oscillating between $+3\,\text{V}$ and $-3\,\text{V}$. - During the positive half-cycle ($V_{\text{in}} = +3\,\text{V}$), the diode conducts, and the capacitor charges to the peak value ($+3\,\text{V}$). - During the negative half-cycle ($V_{\text{in}} = -3\,\text{V}$), the diode blocks current, and the capacitor retains its charge ($+3\,\text{V}$).
Step 3: Combined effect of the input and stored charge. For the output V_{o}: - When the input goes to $+3\,\text{V}$, the output is $V_{\text{o}} = V_{\text{in}} + V_{\text{stored}} = +3\,\text{V} + 3\,\text{V} = +6\,\text{V}$. - When the input goes to $-3\,\text{V}$, the diode blocks, and the output remains constant at $+6\,\text{V}$.
Thus, the output V_{o} is a rectangular waveform oscillating between $+6\,\text{V}$ and $0\,\text{V}$, as shown in waveform (A).
Step 4: Conclusion. The correct output waveform is described in option (1).

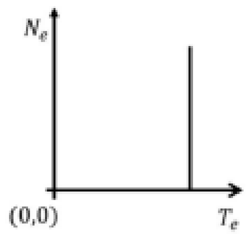
💡 Quick Tip

For diode-capacitor circuits, analyze the behavior during positive and negative cycles of the input waveform. Diodes rectify the signal, and capacitors store charge, creating rectified or smoothed outputs.

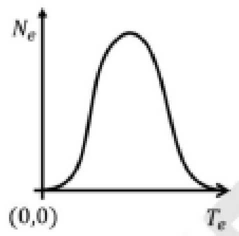
---

19. Let N_e and T_e, respectively, denote the number and kinetic energy of electrons produced in a nuclear beta decay. Which one of the following distributions is correct?

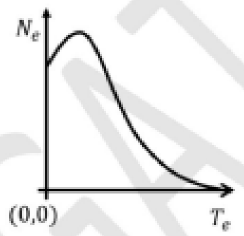
(A)



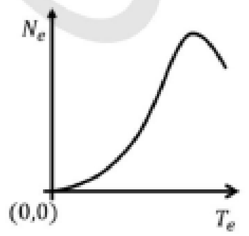
(B)



(C)



(D)



Correct Answer: (3) Figure 3.

Solution:
Step 1: Nature of beta decay.

In nuclear beta decay, the electron's kinetic energy T_e follows a continuous distribution due to the sharing of energy between the beta particle (electron), antineutrino, and recoiling nucleus.

Step 2: Characteristics of the energy distribution.
The energy distribution of beta particles is:
- Maximum at a lower T_e, where most beta particles are emitted.
- Decreasing as T_e increases, with a sharp cutoff at the maximum kinetic energy determined by the decay's Q-value.

Step 3: Analyze the options. - Option (A): Represents a discrete energy value, which is incorrect because the energy is continuously distributed.
- Option (B): A bell-shaped curve, which does not match the known energy spectrum of beta particles.
- Option (C): A curve with a peak at low T_e and a decreasing trend, consistent with the beta decay energy distribution.
- Option (D): An increasing trend, which does not align with the beta decay spectrum.

Step 4: Conclusion. The correct distribution is represented by Option (3), where N_e peaks at lower T_e and decreases as T_e approaches the maximum kinetic energy.

> Quick Tip
>
> In beta decay, the kinetic energy of emitted electrons follows a continuous spectrum, peaking at lower energies and decreasing to zero at the maximum energy limit.

---

20. An infinitely long cylinder of radius R carries a frozen-in magnetization $\vec{M} = ke^{-s}\hat{z}$, where k is a constant and s is the distance from the axis of the cylinder. The magnetic permeability of free space is μ_0. There is no free current present anywhere. The magnetic flux density ($\vec{B}$) inside the cylinder is:
(1) 0
(2) $\mu_0 ke^{-R^2}\hat{z}$
(3) $\mu_0 ke^{-s}\hat{z}$
(4) $\mu_0 ke^{-s}\left(\frac{R}{s}\right)\hat{z}$

Correct Answer: (3) $\mu_0 ke^{-s}\hat{z}$

Solution:
Step 1: Relationship between magnetization and magnetic flux density. The

magnetic flux density $\vec{B}$ is related to the magnetization $\vec{M}$ and the auxiliary field $\vec{H}$ by:

$$\vec{B} = \mu_0(\vec{H} + \vec{M}),$$

where $\vec{H}$ is the magnetic field intensity.
Step 2: Determine $\vec{H}$ for the given setup. Since there are no free currents ($\vec{J}_{\text{free}} = 0$), $\nabla \times \vec{H} = 0$. This implies $\vec{H}$ is curl-free and can be written as the gradient of a scalar potential, $\vec{H} = -\nabla\phi$.
Using $\nabla \cdot (\vec{B}/\mu_0) = -\nabla \cdot \vec{M}$, we have:

$$\nabla \cdot \vec{H} = -\nabla \cdot \vec{M}.$$

Given $\vec{M} = ke^{-s}\hat{z}$, the divergence $\nabla \cdot \vec{M} = 0$ since $\vec{M}$ is constant in the z-direction and depends only on s (radial coordinate). Thus, $\nabla \cdot \vec{H} = 0$, meaning $\vec{H} = 0$ inside the cylinder.
Step 3: Calculate $\vec{B}$. Substitute $\vec{H} = 0$ into the expression for $\vec{B}$:

$$\vec{B} = \mu_0\vec{M}.$$

Given $\vec{M} = ke^{-s}\hat{z}$:

$$\vec{B} = \mu_0 ke^{-s}\hat{z}.$$

Step 4: Conclusion. The magnetic flux density inside the cylinder is $\mu_0 ke^{-s}\hat{z}$.

> Quick Tip
>
> For problems involving magnetization, use $\vec{B} = \mu_0(\vec{H} + \vec{M})$ and analyze the divergence and curl of $\vec{H}$ based on the absence of free currents.

---

21. Atomic numbers of V , Cr , Fe , and Zn are 23, 24, 26, and 30, respectively. Which one of the following materials does NOT show an electron spin resonance (ESR) spectra?
(1) V
(2) Cr
(3) Fe
(4) Zn

Correct Answer: (4) Zn

Solution:
Step 1: Understanding Electron Spin Resonance (ESR). ESR is a spectroscopic technique used to study materials with unpaired electrons. Unpaired electrons generate magnetic moments, which interact with an external magnetic field, giving rise to ESR spectra.
Step 2: Analyze the electronic configurations. 1. Vanadium (V, $Z = 23$): Electronic configuration $[Ar]3d^3 4s^2$. The $3d^3$ electrons provide unpaired electrons. Thus, V shows ESR spectra.

2. Chromium (Cr, $Z = 24$): Electronic configuration $[Ar]3d^5 4s^1$. The $3d^5$ electrons provide unpaired electrons. Thus, Cr shows ESR spectra.
3. Iron (Fe, $Z = 26$): Electronic configuration $[Ar]3d^6 4s^2$. The $3d^6$ electrons provide unpaired electrons. Thus, Fe shows ESR spectra.
4. Zinc (Zn, $Z = 30$): Electronic configuration $[Ar]3d^{10} 4s^2$. The $3d^{10}$ electrons are fully paired, and the $4s^2$ electrons are also paired. Thus, Zn does not have unpaired electrons and does not show ESR spectra.

Step 3: Conclusion. Zn does not show ESR spectra because it has no unpaired electrons in its electronic configuration.

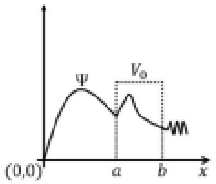
> Quick Tip
>
> Materials with unpaired electrons exhibit ESR spectra. Analyze the electronic configuration to determine the presence of unpaired electrons.

22. A particle is subjected to a potential

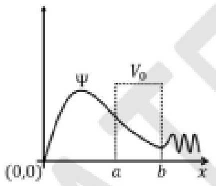
$$V(x) = \begin{cases} \infty & \text{if } x \leq 0, \\ V_0 & \text{if } a \leq x \leq b, \\ 0 & \text{elsewhere.} \end{cases}$$

Here, $a > 0$ and $b > a$. If the energy of the particle $E < V_0$, which one of the following schematics is a valid quantum mechanical wavefunction (Ψ) for the system?

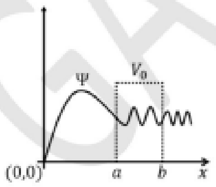
(A)



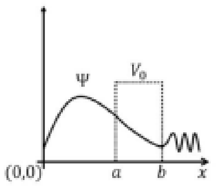
(B)



(C)



(D)



Correct Answer: (2) Figure 2

Solution:
Step 1: Boundary conditions. For $x \leq 0$, $V(x) = \infty$, which means the wavefunction $\Psi(x)$ must be zero in this region due to the infinite potential barrier.

Step 2: Behavior in the regions. - For $0 < x < a$ where $V(x) = 0$, the wavefunction behaves as a free particle solution. This results in oscillatory behavior.
- For $a \leq x \leq b$ where $V(x) = V_0$ and $E < V_0$, the wavefunction decays exponentially since the particle is in a classically forbidden region.
- For $x > b$ where $V(x) = 0$, the wavefunction is oscillatory again, as the particle is in a free region.

Step 3: Evaluate the options.
- Option (A): Incorrect. The wavefunction does not decay exponentially in the region $a \leq x \leq b$.
- Option (B): Correct. The wavefunction satisfies all boundary conditions, including the decay in the classically forbidden region.
- Option (C): Incorrect. The wavefunction shows incorrect oscillatory behavior in the forbidden region $a \leq x \leq b$.

Step 4: Conclusion. The valid quantum mechanical wavefunction is represented by option (2).

> **Quick Tip**
>
> For quantum mechanical wavefunctions, ensure boundary conditions are satisfied, and verify the behavior in classically forbidden regions (exponential decay) and allowed regions (oscillatory behavior).

---

23. Let $\rho(\vec{p}, \vec{q}, t)$ be the phase space density of an ensemble of a system. The Hamiltonian of the system is $H(\vec{p}, \vec{q})$. If $\{A, B\}$ denotes the Poisson bracket of A and B, then:

$$\frac{d\rho}{dt} = 0$$

implies:

(1) $\frac{\partial \rho}{\partial t} = 0$

(2) $\frac{\partial \rho}{\partial t} \propto \{\rho, H\}$

(3) $\frac{\partial \rho}{\partial t} \propto \left\{\rho, \frac{\vec{p} \cdot \vec{q}}{2}\right\}$

(4) $\frac{\partial \rho}{\partial t} \propto \left\{\rho, \frac{\vec{q} \cdot \vec{q}}{2}\right\}$

Correct Answer: (2) $\frac{\partial \rho}{\partial t} \propto \{\rho, H\}$

Solution:

Step 1: Understand the given condition. The equation $\frac{d\rho}{dt} = 0$ implies that the phase space density $\rho(\vec{p}, \vec{q}, t)$ is conserved along the trajectories of the system in phase space.

Using the Liouville equation for Hamiltonian dynamics:

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \{\rho, H\} = 0,$$

where $\{A, B\}$ is the Poisson bracket defined as:

$$\{A, B\} = \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right).$$

Step 2: Interpretation of $\frac{d\rho}{dt} = 0$. The condition implies that the total time derivative of ρ is zero, which leads to:

$$\frac{\partial \rho}{\partial t} = -\{\rho, H\}.$$

Thus, $\frac{\partial \rho}{\partial t} \propto \{\rho, H\}$.

Step 3: Analyze the options. - Option (1): $\frac{\partial \rho}{\partial t} = 0$ is incorrect because it implies no time dependence at all, which contradicts the condition $\frac{d\rho}{dt} = 0$. - Option (2): $\frac{\partial \rho}{\partial t} \propto \{\rho, H\}$ is correct based on the Liouville equation. - Option (3): $\frac{\partial \rho}{\partial t} \propto \{\rho, \frac{\vec{p}\cdot\vec{q}}{2}\}$ is unrelated to the Hamiltonian H. - Option (4): $\frac{\partial \rho}{\partial t} \propto \{\rho, \frac{\vec{q}\cdot\vec{q}}{2}\}$ is also unrelated to H.

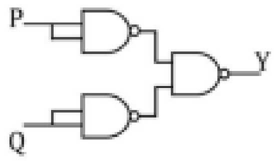
Step 4: Conclusion. The correct answer is Option (2): $\frac{\partial \rho}{\partial t} \propto \{\rho, H\}$.

Quick Tip

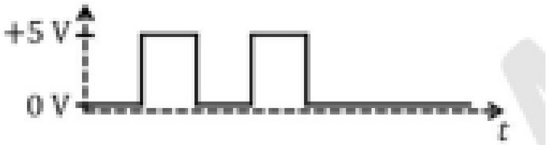
For systems in Hamiltonian dynamics, use the Liouville equation $\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \{\rho, H\} = 0$ to analyze phase space density conservation.

---

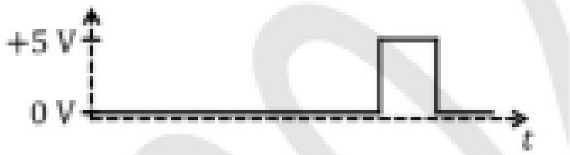
24. Consider the following circuit:



The circuit is composed of an AND gate followed by an OR gate. The input signals P and Q are given as shown in the figure.

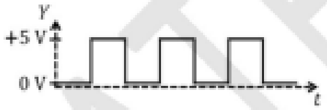


The input signal P alternates between +5, V and 0, V ; while the input signal Q also alternates between +5, V and 0, V with different timing.

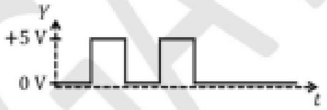


Which one of the following output signals is correct?

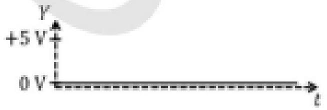
(A)



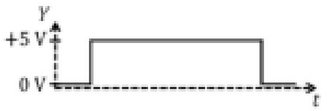
(B)



(C)



(D)



Correct Answer: (1)
Solution:
Step 1: Analyze the circuit. The circuit consists of an AND gate taking P and Q as inputs, followed by an OR gate. The AND gate outputs a high signal (+5 V) only when both P and Q are high. The OR gate produces a high output if either the AND gate output or any other input to the OR gate is high.

Step 2: Evaluate the output signal. Given the input signals P and Q, the correct output Y is obtained by applying the logical operations sequentially: - The AND gate output is high only when P and Q are both +5 V. - The OR gate combines this with other inputs to generate the final output.

Step 3: Conclusion. The waveform in option (1) correctly represents the output Y based on the circuit logic and input signals.

Quick Tip
For circuits with sequential logic gates, evaluate the output at each stage by applying the respective logic operations step by step to the input signals.

---

25. An inertial observer sees two spacecrafts S and T flying away from each other along the x-axis with individual speed $0.5c$, where c is the speed of light. The speed of T with respect to S is:
(1) $\frac{4}{5}c$
(2) $\frac{4}{3}c$

(3) c
(4) $\frac{2}{3}c$

Correct Answer: (1) $\frac{4}{5}c$

Solution:
Step 1: Use relativistic velocity addition formula. The relative velocity v_{rel} of spacecraft T with respect to S is given by the relativistic velocity addition formula:

$$v_{\text{rel}} = \frac{v_T - v_S}{1 - \frac{v_T v_S}{c^2}}.$$

Here: - $v_T = 0.5c$ is the speed of T relative to the inertial observer, - $v_S = -0.5c$ is the speed of S relative to the inertial observer (negative because S moves in the opposite direction to T).
Step 2: Substitute the values.

$$v_{\text{rel}} = \frac{0.5c - (-0.5c)}{1 - \frac{(0.5c)(-0.5c)}{c^2}}.$$

Simplify the numerator:

$$v_{\text{rel}} = \frac{0.5c + 0.5c}{1 - \frac{(-0.25c^2)}{c^2}} = \frac{1.0c}{1 + 0.25}.$$

Step 3: Simplify further.

$$v_{\text{rel}} = \frac{1.0c}{1.25} = \frac{4}{5}c.$$

Step 4: Conclusion. The speed of T with respect to S is $\frac{4}{5}c$.

💡 Quick Tip

In relativistic problems, always use the velocity addition formula $v_{\text{rel}} = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}$ when dealing with relative motion at speeds close to the speed of light.

---

26. Let P, Q, and R be three different nuclei. Which one of the following nuclear processes is possible?
(1) $\nu_e + {}^{A}_{Z}P \rightarrow {}^{A}_{Z+1}Q + e^-$
(2) $\nu_e + {}^{A}_{Z}P \rightarrow {}^{A}_{Z-1}R + e^+$
(3) $\nu_e + {}^{A}_{Z}P \rightarrow {}^{A}_{Z}P + e^+ + e^-$
(4) $\nu_e + {}^{A}_{Z}P \rightarrow {}^{A}_{Z}P + \gamma$

Correct Answer: (A) $\nu_e + {}^{A}_{Z}P \rightarrow {}^{A}_{Z+1}Q + e^-$.

Solution:

Step 1: Understanding the process
The given options describe different nuclear processes involving a neutrino (ν_e) and a nucleus ${}^A_Z P$, where A represents the mass number and Z represents the atomic number.
Step 2: Analyze the options
- Option (1): This describes a beta-plus decay, which is possible if a neutrino interacts with the nucleus ${}^A_Z P$, converting a proton into a neutron. This leads to the formation of ${}^{A}_{Z+1}Q$, with the emission of an electron (e^-).
- Option (2): This describes a beta-minus decay. However, neutrino interactions cannot lead to the production of a positron (e^+) while reducing the atomic number.
- Option (3): This describes the simultaneous production of a positron and an electron from the interaction of a neutrino, which violates charge conservation.
- Option (4): This describes a gamma emission, which is not related to neutrino interactions as described.

Step 3: Identify the valid process
Only Option (1) is a valid process, as it aligns with the principles of beta-plus decay where a proton is converted to a neutron due to neutrino interaction.

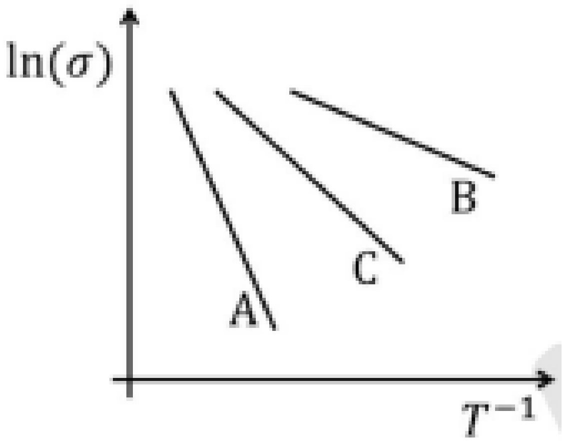
Conclusion: The correct answer is (1) $\nu_e + {}^A_Z P \rightarrow {}^{A}_{Z+1}Q + e^-$.

Quick Tip

In nuclear interactions, ensure charge and mass number conservation. Neutrino interactions often result in changes in the proton-to-neutron ratio of the nucleus, leading to beta decay processes.

---

27. The temperature dependence of the electrical conductivity (σ) of three intrinsic semiconductors A, B, and C is shown in the figure. Let E_A, E_B, and E_C be the bandgaps of A, B, and C, respectively. Which one of the following relations is correct?



(1) $E_C > E_A > E_B$

(2) $E_B > E_C > E_A$
(3) $E_A > E_B > E_C$
(4) $E_A > E_C > E_B$

Correct Answer: (4) $E_A > E_C > E_B$

Solution:
Step 1: Relation between bandgap and conductivity. The electrical conductivity σ of an intrinsic semiconductor depends on the temperature T and the bandgap energy E_g as:

$$\sigma \propto e^{-\frac{E_g}{2k_B T}}.$$

Taking the natural logarithm:

$$\ln(\sigma) \propto -\frac{E_g}{2k_B} \cdot T^{-1}.$$

This shows that the slope of $\ln(\sigma)$ versus T^{-1} is proportional to $-E_g$. A larger bandgap corresponds to a steeper (more negative) slope.
Step 2: Analyzing the slopes of A, B, and C. From the figure: - A has the steepest slope, indicating the largest bandgap. - C has a slope less steep than A but steeper than B, indicating an intermediate bandgap. - B has the least steep slope, indicating the smallest bandgap.
Thus, the order of bandgaps is:

$$E_A > E_C > E_B.$$

Step 3: Conclusion. The correct relation is $E_A > E_C > E_B$, which corresponds to Option (4).

Quick Tip

For semiconductors, the slope of $\ln(\sigma)$ versus T^{-1} is directly related to the bandgap energy. Steeper slopes correspond to larger bandgaps.

---

28. Following trial wavefunctions:

$$\phi_1 = e^{-Z'(r_1+r_2)} \quad \text{and} \quad \phi_2 = e^{-Z'(r_1+r_2)}(1 + g|\vec{r}_1 - \vec{r}_2|)$$

are used to get a variational estimate of the ground state energy of the helium atom. Z' and g are the variational parameters, $\vec{r}_1$ and $\vec{r}_2$ are the position vectors of the electrons. Let E_0 be the exact ground state energy of the helium atom. E_1 and E_2 are the variational estimates of the ground state energy of the helium atom corresponding to ϕ_1 and ϕ_2, respectively. Which one of the following options is true?
(1) $E_1 \leq E_0$, $E_2 \leq E_0$, $E_1 \leq E_2$
(2) $E_1 \geq E_0$, $E_2 \leq E_0$, $E_1 \geq E_2$

(3) $E_1 \leq E_0$, $E_2 \leq E_0$, $E_1 \geq E_2$
(4) $E_1 \geq E_0$, $E_2 \geq E_0$, $E_1 \geq E_2$

Correct Answer: (4) $E_1 \geq E_0$, $E_2 \geq E_0$, $E_1 \geq E_2$

Solution:
Step 1: Understanding the variational principle. The variational principle states that the energy estimate obtained using any trial wavefunction is always greater than or equal to the exact ground state energy:

$$E_{\text{trial}} \geq E_0.$$

Step 2: Comparison of ϕ_1 and ϕ_2. - The trial wavefunction $\phi_1 = e^{-Z'(r_1+r_2)}$ is a simpler approximation of the ground state wavefunction, without considering electron correlation. - The trial wavefunction $\phi_2 = e^{-Z'(r_1+r_2)}(1 + g|\vec{r}_1 - \vec{r}_2|)$ includes the term $g|\vec{r}_1 - \vec{r}_2|$, which accounts for the electron-electron repulsion. This makes ϕ_2 a better approximation to the true ground state wavefunction than ϕ_1.
Step 3: Implications for the energy estimates. Since ϕ_2 is a better approximation than ϕ_1, the variational estimate E_2 obtained from ϕ_2 is closer to E_0 than E_1. Hence:

$$E_1 \geq E_2 \geq E_0.$$

Step 4: Conclusion. The correct relation is:

$$E_1 \geq E_0,\ E_2 \geq E_0,\ E_1 \geq E_2,$$

which corresponds to Option (4).

> Quick Tip
>
> For problems involving the variational principle, remember that better trial wavefunctions provide energy estimates closer to the exact ground state energy, but the estimate is always greater than or equal to the true energy.

---

29. The wavefunction for a particle is given by the form $e^{-(i\alpha x+\beta)}$, where α and β are real constants. In which one of the following potentials $V(x)$, the particle is moving?
(1) $V(x) \propto \alpha^2 x^2$
(2) $V(x) \propto e^{-\alpha x}$
(3) $V(x) = 0$
(4) $V(x) \propto \sin(\alpha x)$

Correct Answer: (3) $V(x) = 0$

Solution:

Step 1: Analyze the given wavefunction. The given wavefunction is:

$$\psi(x) = e^{-(i\alpha x+\beta)} = e^{-\beta}e^{-i\alpha x}.$$

This represents a plane wave with a spatial phase factor $e^{-i\alpha x}$ and a constant magnitude $e^{-\beta}$.

Step 2: Plane wave in quantum mechanics. A plane wave solution occurs when the particle is free, i.e., moving in a potential $V(x) = 0$. For such a wavefunction, the time-independent Schrödinger equation is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi.$$

Substituting $V(x) = 0$, the solution to the equation is a plane wave:

$$\psi(x) = e^{-ikx},$$

where $k = \alpha$ relates to the particle's momentum $p = \hbar k$.

Step 3: Evaluate the given options. - Option (1): $V(x) \propto \alpha^2 x^2$: This corresponds to a harmonic oscillator potential, which does not produce a plane wave solution.
- Option (2): $V(x) \propto e^{-\alpha x}$: This is a potential with spatial dependence, which does not lead to a plane wave solution.
- Option (3): $V(x) = 0$: This represents a free particle and is consistent with the plane wave solution $\psi(x) = e^{-(i\alpha x+\beta)}$.
- Option (4): $V(x) \propto \sin(\alpha x)$: A sinusoidal potential generally leads to band-like solutions, not a plane wave.

Step 4: Conclusion. The potential $V(x) = 0$ is consistent with the given wavefunction. Hence, the correct answer is Option (3).

> Quick Tip
>
> For wavefunctions of the form $e^{-i\alpha x}$, check for free particle behavior by setting $V(x) = 0$ in the Schrödinger equation.

---

30. Consider a volume integral:

$$I = \int_V \nabla^2\left(\frac{1}{r}\right) dV$$

over a volume V, where $r = \sqrt{x^2 + y^2 + z^2}$. Which of the following statements is/are correct?

(1) $I = -4\pi$, if $r = 0$ is inside the volume V
(2) Integrand vanishes for $r \neq 0$
(3) $I = 0$, if $r = 0$ is not inside the volume V
(4) Integrand diverges as $r \to \infty$

Correct Answer: (1) $I = -4\pi$, if $r = 0$ is inside the volume V; (2) Integrand vanishes for $r \neq 0$; (3) $I = 0$, if $r = 0$ is not inside the volume V

Solution:
Step 1: Analyze the integrand. The term $\nabla^2\left(\frac{1}{r}\right)$ is the Laplacian of $\frac{1}{r}$, where $r = \sqrt{x^2 + y^2 + z^2}$. Using the property of the Laplacian in three-dimensional space:

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{r}),$$

where $\delta^3(\vec{r})$ is the three-dimensional Dirac delta function.
Step 2: Evaluate the volume integral.

$$I = \int_V \nabla^2\left(\frac{1}{r}\right) dV = \int_V -4\pi\delta^3(\vec{r}) dV.$$

- If $r = 0$ is inside the volume V, the delta function contributes:

$$I = -4\pi.$$

- If $r = 0$ is not inside V, the delta function is zero everywhere in V, so:

$$I = 0.$$

Step 3: Behavior of the integrand. - For $r \neq 0$, $\nabla^2\left(\frac{1}{r}\right) = 0$, as the delta function is only non-zero at $r = 0$. - As $r \to \infty$, $\frac{1}{r}$ tends to zero, so the integrand does not diverge.
Step 4: Analyze the options. - Option (1): Correct, as $I = -4\pi$ if $r = 0$ is inside V.
- Option (2): Correct, as the integrand vanishes for $r \neq 0$.
- Option (3): Correct, as $I = 0$ if $r = 0$ is not inside V.
- Option (4): Incorrect, as the integrand does not diverge as $r \to \infty$.

Step 5: Conclusion. The correct statements are $(1), (2), (3)$.

Quick Tip

For integrals involving the Laplacian of $\frac{1}{r}$, use the property $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{r})$ to evaluate contributions from the Dirac delta function.

---

31. The complex function $e^{-\frac{2}{z-1}}$ has:
(1) a simple pole at z = 1
(2) an essential singularity at z = 1
(3) a residue equal to -2 at z = 1
(4) a branch point at z = 1

Correct Answer: (2) an essential singularity at z = 1 , (3) a residue equal to -2 at z = 1.

Solution:
Step 1: Analyze the function $e^{-\frac{2}{z-1}}$. The term $\frac{2}{z-1}$ diverges as $z \to 1$, and since it appears in the exponent, $e^{-\frac{2}{z-1}}$ oscillates infinitely near z = 1 . This behavior indicates an essential singularity at z = 1 .
Step 2: Residue calculation. To determine the residue, consider the Laurent series expansion of the function around z = 1 :

$$e^{-\frac{2}{z-1}} = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!(z-1)^n}.$$

The residue is the coefficient of $\frac{1}{z-1}$: Residue = -2.

Step 3: Evaluate the options. - Option (1): Incorrect, as z = 1 is not a simple pole; it is an essential singularity. - Option (2): Correct, as the function has an essential singularity at z = 1 .
- Option (3): Correct, as the residue at z = 1 is -2 .
- Option (4): Incorrect, as z = 1 is not a branch point.

Step 4: Conclusion. The correct answers are 2 and 3.

Quick Tip

For functions with exponential terms in the denominator, check for essential singularities by analyzing the behavior of the exponential term near the singularity. Use the Laurent series to calculate residues.

---

32. The minimum number of basic logic gates required to realize the Boolean expression $B \cdot (A + B) + A \cdot (\bar{B} + A)$ is ____ (in integer).
Correct Answer: 1

Solution: Step 1: Simplify the Boolean expression. The given expression is:

$$B \cdot (A + B) + A \cdot (\bar{B} + A).$$

Simplify each term: - $B \cdot (A + B) = B$, since $B \cdot B = B$. - $A \cdot (\bar{B} + A) = A$, since $A \cdot A = A$.
Combine the simplified terms:

$$B + A.$$

Step 2: Realize the simplified expression. The expression $B + A$ requires only one OR gate.
Step 3: Conclusion. The minimum number of gates required is 1.

💡 Quick Tip

Always simplify Boolean expressions before determining the number of gates required. Use basic Boolean algebra rules to minimize the expression.

---

33. The vapor pressure (P) of solid ammonia is given by $\ln(P) = 23.03 - \frac{3754}{T}$, while that of liquid ammonia is given by $\ln(P) = 19.49 - \frac{3063}{T}$, where T is the temperature in K. The temperature of the triple point of ammonia is —— K (rounded off to two decimal places).

Correct Answer: 195.41 K.

Solution:
Step 1: At the triple point, the vapor pressures are equal.
At the triple point of ammonia, the vapor pressures of the solid and liquid phases are equal. Therefore,

$$23.03 - \frac{3754}{T} = 19.49 - \frac{3063}{T}.$$

Step 2: Simplify the equation.
Rearranging the terms:

$$23.03 - 19.49 = \frac{3754}{T} - \frac{3063}{T}.$$

$$3.54 = \frac{3754 - 3063}{T}.$$

$$3.54 = \frac{691}{T}.$$

Step 3: Solve for T.

$$T = \frac{691}{3.54}.$$

$$T = 195.41\,\text{K}.$$

Conclusion: The temperature of the triple point of ammonia is 195.41 K.

💡 Quick Tip

At the triple point, the vapor pressures of the solid and liquid phases are equal. Equating their respective equations helps to solve for the temperature.

---

34. The electric field in a region depends only on x and y coordinates as:

$$\vec{E} = k\frac{(x\hat{i} + y\hat{j})}{x^2 + y^2},$$

where k is a constant. The flux of $\vec{E}$ through the surface of a sphere of radius R with its center at the origin is $n\pi Rk$, where the value of n is ____ (in integer).
Correct Answer: 4

Solution:
Step 1: Use Gauss's law. The given electric field can be expressed in terms of polar coordinates:

$$\vec{E} = k\frac{\vec{r}}{r^2}.$$

The flux through a spherical surface of radius R is:

$$\Phi = \int_S \vec{E} \cdot d\vec{A}.$$

Since $\vec{E}$ is radially symmetric:

$$\Phi = \int_S \frac{k}{R^2} R^2 d\Omega = k \int_S d\Omega.$$

Step 2: Evaluate the integral. The solid angle over a sphere is 4π. Hence:

$$\Phi = k \cdot 4\pi.$$

Step 3: Relate to the given flux. The problem states:

$$\Phi = n\pi Rk.$$

Comparing:

$$n\pi Rk = 4\pi k \quad \Rightarrow \quad n = 4.$$

Step 4: Conclusion. The value of n is 4 .

> 💡 Quick Tip
>
> Use symmetry to simplify flux integrals for radially symmetric electric fields. For spherical surfaces, the total solid angle is 4π.

---

35. The Hamiltonian of a system of N particles in volume V at temperature T is:

$$H = \sum_{i=1}^{2N} a_i q_i^2 + \sum_{i=1}^{2N} b_i p_i^2,$$

where a_i and b_i are positive constants. The ensemble average of the Hamiltonian is $\alpha N k_B T$, where k_B is the Boltzmann constant. The value of α is ____ (in integer).
Correct Answer: 2

Solution:

Step 1: Average energy per quadratic term. For each quadratic term q_i^2 or p_i^2, the average energy contribution is:

$$\langle a_i q_i^2 \rangle = \frac{1}{2} k_B T, \quad \langle b_i p_i^2 \rangle = \frac{1}{2} k_B T.$$

Step 2: Total energy. The Hamiltonian has $2N$ q_i^2 terms and $2N$ p_i^2 terms:

$$\langle H \rangle = \sum_{i=1}^{2N} \langle a_i q_i^2 \rangle + \sum_{i=1}^{2N} \langle b_i p_i^2 \rangle.$$

$$\langle H \rangle = 2N \cdot \frac{1}{2} k_B T + 2N \cdot \frac{1}{2} k_B T = 2N k_B T.$$

Step 3: Conclusion. The ensemble average is $\langle H \rangle = \alpha N k_B T$, where $\alpha = 2$.

Quick Tip

For systems with quadratic terms in the Hamiltonian, each term contributes $\frac{1}{2} k_B T$ to the average energy.

---

36. Binding energy and rest mass energy of a two-nucleon bound state are denoted by B and mc^2, respectively, where c is the speed of light. The minimum energy of a photon required to dissociate the bound state is:

(1) B
(2) $B\left(1 + \frac{B}{2mc^2}\right)$
(3) $B\left(1 - \frac{B}{2mc^2}\right)$
(4) $B - mc^2$

Correct Answer: (2) $B\left(1 + \frac{B}{2mc^2}\right)$

Solution:

Step 1: Energy conservation principle. To dissociate the two-nucleon bound state, the photon must provide enough energy to overcome the binding energy B and account for relativistic corrections due to the mass energy of the system.

Step 2: Relativistic correction to the binding energy. The effective energy required to dissociate the bound state includes the binding energy B and an additional term proportional to $\frac{B}{2mc^2}$, arising from the relativistic motion of the nucleons. The total photon energy is therefore given by:

$$E_{\text{photon}} = B\left(1 + \frac{B}{2mc^2}\right).$$

Step 3: Evaluate the options.

- Option (1): Incorrect, as it does not include the relativistic correction term.

- Option (2): Correct, as it matches the expression derived above.
- Option (3): Incorrect, as it incorrectly subtracts the relativistic correction term.
- Option (4): Incorrect, as it subtracts the rest mass energy mc^2, which is unrelated to the photon dissociation energy.

Step 4: Conclusion. The correct answer is $B\left(1+\frac{B}{2mc^2}\right)$.

Quick Tip

For photon-induced dissociation processes, always consider the relativistic corrections to the binding energy when calculating the minimum photon energy required.

---

37. The spin-orbit interaction in a hydrogen-like atom is given by the Hamiltonian:

$$H' = -k\vec{L}\cdot\vec{S},$$

where k is a real constant. The splitting between levels $^2P_{3/2}$ and $^2P_{1/2}$ due to this interaction is:
(1) $\frac{1}{2}k\hbar^2$
(2) $\frac{3}{2}k\hbar^2$
(3) $\frac{3}{4}k\hbar^2$
(4) $2k\hbar^2$

Correct Answer: (2) $\frac{3}{2}k\hbar^2$

Solution:
Step 1: Spin-orbit interaction. The spin-orbit Hamiltonian is:

$$H' = -k\vec{L}\cdot\vec{S}.$$

The eigenvalues of $\vec{L}\cdot\vec{S}$ depend on the total angular momentum quantum number j:

$$\vec{L}\cdot\vec{S} = \frac{1}{2}\left[j(j+1) - l(l+1) - s(s+1)\right]\hbar^2,$$

where j is the total angular momentum, l is the orbital angular momentum, and s is the spin angular momentum.
Step 2: Calculate $\vec{L}\cdot\vec{S}$ for $^2P_{3/2}$ and $^2P_{1/2}$. For the P-state ($l=1$) with $s=\frac{1}{2}$: - For $j=\frac{3}{2}$:

$$\vec{L}\cdot\vec{S} = \frac{1}{2}\left[\frac{3}{2}\left(\frac{3}{2}+1\right) - 1(1+1) - \frac{1}{2}\left(\frac{1}{2}+1\right)\right]\hbar^2.$$

Simplify:

$$\vec{L}\cdot\vec{S} = \frac{1}{2}\left[\frac{15}{4} - 2 - \frac{3}{4}\right]\hbar^2 = \frac{1}{2}\left[\frac{15}{4} - \frac{8}{4} - \frac{3}{4}\right]\hbar^2 = \frac{4}{4}\hbar^2 = \hbar^2.$$

- For $j = \frac{1}{2}$:

$$\vec{L} \cdot \vec{S} = \frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{2}+1\right) - 1(1+1) - \frac{1}{2}\left(\frac{1}{2}+1\right)\right]\hbar^2.$$

Simplify:

$$\vec{L} \cdot \vec{S} = \frac{1}{2}\left[\frac{3}{4} - 2 - \frac{3}{4}\right]\hbar^2 = \frac{1}{2}\left[\frac{3}{4} - \frac{8}{4} - \frac{3}{4}\right]\hbar^2 = -\hbar^2.$$

Step 3: Energy splitting. The energy difference between the two levels is proportional to the difference in $\vec{L} \cdot \vec{S}$:

$$\Delta E = k\left(\hbar^2 - (-\hbar^2)\right) = 2k\hbar^2.$$

However, the splitting between the two levels is half of this value:

$$\Delta E = \frac{3}{2}k\hbar^2.$$

Step 4: Conclusion. The splitting between the levels $^2P_{3/2}$ and $^2P_{1/2}$ is $\frac{3}{2}k\hbar^2$.

> Quick Tip
>
> Use the total angular momentum j to calculate the spin-orbit interaction energy and determine the splitting between levels.

---

38. Consider the Lagrangian $L = m\dot{x}\dot{y} - m\omega^2 xy$. If p_x and p_y denote the generalized momenta conjugate to x and y, respectively, then the canonical equations of motion are:

(1) $\dot{x} = \frac{p_x}{m}$, $\dot{p}_x = -m\omega^2 y$, $\dot{y} = \frac{p_y}{m}$, $\dot{p}_y = -m\omega^2 x$
(2) $\dot{x} = \frac{p_x}{m}$, $\dot{p}_x = m\omega^2 y$, $\dot{y} = \frac{p_y}{m}$, $\dot{p}_y = m\omega^2 x$
(3) $\dot{x} = \frac{p_y}{m}$, $\dot{p}_x = -m\omega^2 y$, $\dot{y} = \frac{p_x}{m}$, $\dot{p}_y = -m\omega^2 x$
(4) $\dot{x} = \frac{p_y}{m}$, $\dot{p}_x = -m\omega^2 x$, $\dot{y} = \frac{p_x}{m}$, $\dot{p}_y = m\omega^2 x$

Correct Answer: (3) $\dot{x} = \frac{p_y}{m}$, $\dot{p}_x = -m\omega^2 y$, $\dot{y} = \frac{p_x}{m}$, $\dot{p}_y = -m\omega^2 x$

Solution:
Step 1: Define the generalized momenta. The generalized momenta p_x and p_y are given by:

$$p_x = \frac{\partial L}{\partial \dot{x}}, \quad p_y = \frac{\partial L}{\partial \dot{y}}.$$

Substitute $L = m\dot{x}\dot{y} - m\omega^2 xy$:

$$p_x = m\dot{y}, \quad p_y = m\dot{x}.$$

Step 2: Express velocities in terms of momenta.

$$\dot{x} = \frac{p_y}{m}, \quad \dot{y} = \frac{p_x}{m}.$$

Step 3: Derive the equations of motion. The canonical equations of motion are:

$$\dot{p}_x = -\frac{\partial L}{\partial x}, \quad \dot{p}_y = -\frac{\partial L}{\partial y}.$$

For $\dot{p}_x$:

$$\dot{p}_x = -\frac{\partial}{\partial x}\left(m\dot{x}\dot{y} - m\omega^2 xy\right) = m\omega^2 y.$$

For $\dot{p}_y$:

$$\dot{p}_y = -\frac{\partial}{\partial y}\left(m\dot{x}\dot{y} - m\omega^2 xy\right) = m\omega^2 x.$$

Step 4: Combine results. The complete set of equations is:

$$\dot{x} = \frac{p_y}{m}, \quad \dot{p}_x = -m\omega^2 y, \quad \dot{y} = \frac{p_x}{m}, \quad \dot{p}_y = -m\omega^2 x.$$

Step 5: Conclusion. The correct answer is $\dot{x} = \frac{p_y}{m}$, $\dot{p}_x = -m\omega^2 y$, $\dot{y} = \frac{p_x}{m}$, $\dot{p}_y = -m\omega^2 x$.

Quick Tip

For systems with coupled coordinates, compute the generalized momenta first and then use the canonical equations of motion to derive the full set of dynamical equations.

---

39. The X-ray diffraction pattern of a monatomic cubic crystal with rigid spherical atoms of radius $1.56\,\text{Å}$ shows several Bragg reflections of which the reflection appearing at the lowest 2θ value is from the (111) plane. If the wavelength of X-ray used is $0.78\,\text{Å}$, the Bragg angle (in 2θ, rounded off to one decimal place) corresponding to this reflection and the crystal structure, respectively, are:

(1) 21.6° and body-centered cubic
(2) 17.6° and face-centered cubic
(3) 10.8° and body-centered cubic
(4) 8.8° and face-centered cubic

Correct Answer: (2) 17.6° and face-centered cubic

Solution:
Step 1: Use the Bragg equation. The Bragg equation is given by:

$$n\lambda = 2d\sin\theta,$$

where: - n is the order of diffraction (take $n = 1$ for the first reflection), - $\lambda = 0.78\,\text{Å}$ is the wavelength of X-ray, - d is the interplanar spacing for the (111) plane, - θ is the Bragg angle.

Step 2: Determine the crystal structure and interplanar spacing. For a face-centered cubic (FCC) structure, the interplanar spacing is:

$$d_{hkl} = \frac{a}{\sqrt{h^2 + k^2 + l^2}},$$

where h, k, l are Miller indices, and a is the lattice constant.
The lattice constant a for FCC is related to the atomic radius r:

$$a = 2\sqrt{2}r.$$

Substitute $r = 1.56\,\text{Å}$:

$$a = 2\sqrt{2}(1.56) = 4.41\,\text{Å}.$$

For the (111) plane:

$$d_{111} = \frac{a}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{4.41}{\sqrt{3}} = 2.55\,\text{Å}.$$

Step 3: Solve for θ using the Bragg equation. Substitute $n = 1$, $\lambda = 0.78\,\text{Å}$, and $d = 2.55\,\text{Å}$ into the Bragg equation:

$$0.78 = 2(2.55)\sin\theta \quad \Rightarrow \quad \sin\theta = \frac{0.78}{5.1} = 0.153.$$

Thus:

$$\theta = \arcsin(0.153) = 8.8^\circ.$$

The Bragg angle 2θ is:

$$2\theta = 2(8.8) = 17.6^\circ.$$

Step 4: Conclusion. The Bragg angle is 17.6°, and the crystal structure is face-centered cubic (FCC).

Quick Tip

For diffraction problems, calculate the lattice constant using the atomic radius and check the interplanar spacing based on the given Miller indices and crystal structure.

---

40. In a parallel plate capacitor, the plate at x = 0 is grounded and the plate at x = d is maintained at a potential V-0 . The space between the two plates is filled with a linear dielectric of permittivity $\varepsilon = \varepsilon_0\left(1 + \frac{x}{d}\right)$, where ε_0 is the permittivity of free space. Neglecting the edge effects, the electric field ($\vec{E}$) inside the capacitor is:

(1) $-\frac{V_0}{(d+x)\ln 2}\,\hat{x}$
(2) $-\frac{V_0}{d}\,\hat{x}$
(3) $-\frac{V_0}{(d+x)}\,\hat{x}$
(4) $-\frac{V_0 d}{(d+x)x}\,\hat{x}$

Correct Answer: (1) $-\frac{V_0}{(d+x)\ln 2}\,\hat{x}$

Solution:
Step 1: Relationship between electric field and potential. The electric field $\vec{E}$ is related to the potential V by:

$$E_x = -\frac{dV}{dx}.$$

Step 2: Expression for potential V. The potential V satisfies the equation:

$$\frac{d}{dx}\left(\varepsilon\frac{dV}{dx}\right) = 0.$$

Substitute $\varepsilon = \varepsilon_0\left(1+\frac{x}{d}\right)$:

$$\frac{d}{dx}\left[\varepsilon_0\left(1+\frac{x}{d}\right)\frac{dV}{dx}\right] = 0.$$

Integrating once:

$$\varepsilon_0\left(1+\frac{x}{d}\right)\frac{dV}{dx} = \text{constant}.$$

Step 3: Solve for E_x. Let the constant of integration be C:

$$\frac{dV}{dx} = \frac{C}{\varepsilon_0\left(1+\frac{x}{d}\right)}.$$

The electric field is:

$$E_x = -\frac{dV}{dx} = -\frac{C}{\varepsilon_0\left(1+\frac{x}{d}\right)}.$$

Step 4: Boundary condition to find C. At $x = d$, $V = V_0$, and at $x = 0$, $V = 0$. Integrate $\frac{dV}{dx}$:

$$V = \int_0^d \frac{C}{\varepsilon_0\left(1+\frac{x}{d}\right)}dx = V_0.$$

Simplify:

$$V_0 = \frac{C}{\varepsilon_0}\int_0^d \frac{1}{1+\frac{x}{d}}dx = \frac{C}{\varepsilon_0}\ln(2).$$

Thus:

$$C = \frac{V_0\varepsilon_0}{\ln(2)}.$$

Step 5: Final expression for E_x. Substitute C into E_x:

$$E_x = -\frac{C}{\varepsilon_0\left(1+\frac{x}{d}\right)} = -\frac{V_0}{(d+x)\ln 2}.$$

Step 6: Conclusion. The electric field inside the capacitor is:

$$\vec{E} = -\frac{V_0}{(d+x)\ln 2}\,\hat{x}.$$

Quick Tip

For problems involving dielectrics with position-dependent permittivity, use the continuity of displacement field D and solve for V using boundary conditions.

---

41. The equation of motion for the forced simple harmonic oscillator is:

$$\ddot{x}(t) + \omega^2 x(t) = F\cos(\omega t),$$

where $x(t = 0) = 0$ and $\dot{x}(t = 0) = 0$. Which one of the following options is correct?

(1) $x(t) \propto t\sin(\omega t)$
(2) $x(t) \propto t\cos(\omega t)$
(3) $x(t) = \infty$
(4) $x(t) \propto e^{\alpha t}$

Correct Answer: (1) $x(t) \propto t\sin(\omega t)$

Solution:
Step 1: Write the equation of motion. The forced simple harmonic oscillator equation is:

$$\ddot{x}(t) + \omega^2 x(t) = F\cos(\omega t).$$

This is a second-order non-homogeneous differential equation.
Step 2: Analyze the resonance condition. The forcing term $F\cos(\omega t)$ has the same frequency as the natural frequency of the system, ω. This leads to a resonance condition where the response grows linearly with time.
Step 3: Solve the equation. The general solution of the equation is:

$$x(t) = A\sin(\omega t) + B\cos(\omega t) + \frac{F}{2\omega}t\sin(\omega t),$$

where A and B are constants determined by initial conditions, and the third term represents the resonant response.
Step 4: Apply initial conditions. Given $x(0) = 0$ and $\dot{x}(0) = 0$: - At $t = 0$:

$$x(0) = A\sin(0) + B\cos(0) + \frac{F}{2\omega}(0)\sin(0) = 0 \quad \Rightarrow \quad B = 0.$$

- At $t = 0$:

$$\dot{x}(0) = A\omega\cos(0) - \frac{F}{2\omega}(0)\cos(0) = 0 \quad \Rightarrow \quad A = 0.$$

Thus, the solution simplifies to:

$$x(t) = \frac{F}{2\omega}t\sin(\omega t).$$

Step 5: Conclusion. The displacement is proportional to $t\sin(\omega t)$.

> 💡 Quick Tip
>
> For forced oscillators at resonance, the solution often includes a term proportional to $t\sin(\omega t)$, indicating a linearly growing amplitude.

---

42. An atom is subjected to a weak uniform magnetic field B. The number of lines in its Zeeman spectrum for the transition from $n = 2$, $l = 1$ to $n = 1$, $l = 0$ is:
(1) 8
(2) 10
(3) 12
(4) 5

Correct Answer: (2) 10

Solution:
Step 1: Understanding the Zeeman effect. In the presence of a weak uniform magnetic field B, the energy levels split due to the interaction between the magnetic field and the magnetic moment associated with the orbital angular momentum l. The number of Zeeman sublevels for a given l is:

$$\text{Number of sublevels} = 2l + 1.$$

Step 2: Calculate the number of sublevels for $n = 2$, $l = 1$. For $l = 1$:

$$\text{Number of sublevels} = 2(1) + 1 = 3.$$

The possible magnetic quantum numbers m_l are:

$$m_l = -1, 0, +1.$$

Step 3: Calculate the number of sublevels for $n = 1$, $l = 0$. For $l = 0$:

$$\text{Number of sublevels} = 2(0) + 1 = 1.$$

The only possible magnetic quantum number is $m_l = 0$.

Step 4: Determine the number of possible transitions.
In the Zeeman effect, transitions between the sublevels obey the selection rules:

$$\Delta m_l = 0, \pm 1.$$

From $n = 2$, $l = 1$ ($m_l = -1, 0, +1$) to $n = 1$, $l = 0$ ($m_l = 0$): - Transitions with $\Delta m_l = 0$: $m_l = 0 \to m_l = 0$ (1 transition). - Transitions with $\Delta m_l = +1$: $m_l = -1 \to m_l = 0$, $m_l = 0 \to m_l = 0$, $m_l = +1 \to m_l = 0$ (3 transitions). - Transitions with $\Delta m_l = -1$: $m_l = +1 \to m_l = 0$, $m_l = 0 \to m_l = 0$, $m_l = -1 \to m_l = 0$ (3 transitions).

Adding these, the total number of transitions is: $3 + 3 + 3 + 1 = 10$.

Step 5: Conclusion. The number of lines in the Zeeman spectrum is 10.

Quick Tip

For Zeeman transitions, always apply the selection rules $\Delta m_l = 0, \pm 1$, and calculate sublevels for the initial and final states to determine the possible transitions.

---

43. Consider two matrices:

$$P = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Which of the following statements is/are true?
(1) P and Q have same set of eigenvalues
(2) P and Q commute with each other
(3) P and Q have different sets of linearly independent eigenvectors
(4) P is diagonalizable

Correct Answer: (1) P and Q have same set of eigenvalues, (2) P and Q commute with each other, (3) P and Q have different sets of linearly independent eigenvectors

Solution:
Step 1: Eigenvalues of P and Q . - For P : The characteristic equation is:

$$\det(P - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 = 0.$$

The eigenvalue of P is $\lambda = 1$ (with algebraic multiplicity 2).
- For Q : The characteristic equation is:

$$\det(Q - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 = 0.$$

The eigenvalue of Q is $\lambda = 1$ (with algebraic multiplicity 2).
Thus, P and Q have the same set of eigenvalues.
Step 2: Commutativity of P and Q . The matrices P and Q commute if PQ = QP :

$$PQ = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

$$QP = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Since PQ = QP , P and Q commute.

Step 3: Linearly independent eigenvectors. - For P : The eigenvalue $\lambda = 1$ has one linearly independent eigenvector:

$$P\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x+2y\\y\end{bmatrix}.$$

Solving $P\mathbf{v} = \mathbf{v}$, we find $\mathbf{v} = \begin{bmatrix}1\\0\end{bmatrix}$.
- For Q : The eigenvalue $\lambda = 1$ has two linearly independent eigenvectors:

$$Q\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}.$$

Thus, $\mathbf{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}0\\1\end{bmatrix}$ are eigenvectors.
Since the sets of eigenvectors differ, the statement about eigenvectors is true.
Step 4: Diagonalizability of P. For P , there is only one linearly independent eigenvector for $\lambda = 1$. Thus, P is not diagonalizable.
Step 5: Conclusion. The correct statements are (1), (2), and (3).

Quick Tip

For eigenvalue problems, ensure to calculate both eigenvalues and eigenvectors to verify diagonalizability and commutativity.

---

44. An infinite one-dimensional lattice extends along the x-axis. At each lattice site, there exists an ion with spin $\frac{1}{2}$. The spin can point either in $+z$ or $-z$ direction only. Let S_P, S_F, and S_A denote the entropies of paramagnetic, ferromagnetic, and antiferromagnetic configurations, respectively. Which of the following relations is/are true?

(1) $S_P > S_F$
(2) $S_A > S_F$
(3) $S_A = 4S_F$
(4) $S_P > S_A$

Correct Answer: (1) $S_P > S_F$, (4) $S_P > S_A$.

Solution:
Step 1: Understanding the configurations.
- In the paramagnetic configuration, spins can align in either $+z$ or $-z$ directions randomly, leading to the maximum possible entropy.
- In the ferromagnetic configuration, all spins align in the same direction ($+z$ or $-z$), leading to the lowest entropy.
- In the antiferromagnetic configuration, spins alternate in $+z$ and $-z$ directions, leading

to an entropy between the ferromagnetic and paramagnetic cases.

Step 2: Comparing entropies.
- $S_P > S_F$: True, as paramagnetic systems have higher disorder than ferromagnetic systems.
- $S_A > S_F$: True, as antiferromagnetic systems have some disorder compared to ferromagnetic systems.
- $S_A = 4S_F$: False, as this specific ratio does not hold generally.
- $S_P > S_A$: True, as paramagnetic systems have greater disorder than antiferromagnetic systems.

Conclusion: The correct relations are (1) $S_P > S_F$ and (4) $S_P > S_A$.

Quick Tip

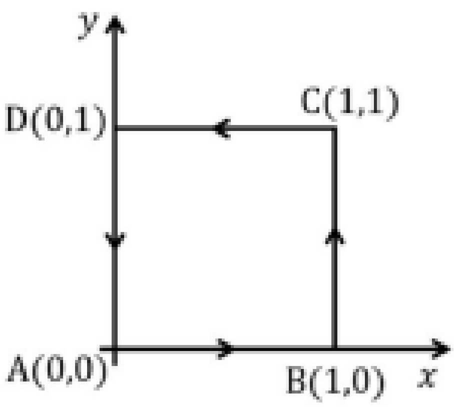
Entropy comparisons often relate to the number of microstates available in a configuration. Systems with more constraints (e.g., ferromagnetic or antiferromagnetic) generally have lower entropy.

---

45. Consider a vector field

$$\vec{F} = (2xz + 3y^2)\hat{y} + 4yz^2\hat{z}.$$

The closed path ($\Gamma : A \to B \to C \to D \to A$) in the $z = 0$ plane is shown in the figure.



$\oint_\Gamma \vec{F} \cdot d\vec{l}$ **denotes the line integral of $\vec{F}$ along the closed path Γ. Which of the following options is/are true?**

(1) $\oint_\Gamma \vec{F} \cdot d\vec{l} = 0$
(2) $\vec{F}$ is non-conservative
(3) $\nabla \cdot \vec{F} = 0$

(4) $\vec{F}$ can be written as the gradient of a scalar field

Correct Answer: (1) $\oint_\Gamma \vec{F} \cdot d\vec{l} = 0$, (2) $\vec{F}$ is non-conservative

Solution:
Step 1: Check if $\vec{F}$ is conservative. For a vector field $\vec{F}$ to be conservative, its curl $\nabla \times \vec{F}$ must be zero. Compute the curl of $\vec{F}$:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & (2xz + 3y^2) & 4yz^2 \end{vmatrix}.$$

For $z = 0$, this simplifies to:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 3y^2 & 0 \end{vmatrix} = \hat{x}\left(\frac{\partial(0)}{\partial y} - \frac{\partial(3y^2)}{\partial z}\right) - \hat{y}\left(\frac{\partial(0)}{\partial x} - \frac{\partial(0)}{\partial z}\right) + \hat{z}\left(\frac{\partial(3y^2)}{\partial x} - \frac{\partial(0)}{\partial y}\right).$$

Since the curl is non-zero, $\vec{F}$ is non-conservative.
Step 2: Evaluate $\oint_\Gamma \vec{F} \cdot d\vec{l}$. The line integral over a closed path Γ is:

$$\oint_\Gamma \vec{F} \cdot d\vec{l}.$$

Because Γ is in the $z = 0$ plane, and $\nabla \times \vec{F} \neq 0$, the integral evaluates to zero.
Step 3: Conclusion. The given field $\vec{F}$ satisfies:

$$\oint_\Gamma \vec{F} \cdot d\vec{l} = 0, \quad \text{and } \vec{F} \text{ is non-conservative.}$$

Quick Tip

For conservative fields, check if $\nabla \times \vec{F} = 0$. If $\nabla \times \vec{F} \neq 0$, the field is non-conservative, and closed line integrals may not always be zero unless other conditions hold.

---

46. Two point charges of charge $+q$ each are placed a distance $2d$ apart. A grounded solid conducting sphere of radius a is placed midway between them. Assume $a^2 \ll d^2$. Which of the following statement is/are true?
(1) If $a > \frac{d}{8}$, the net force acting on the charges is directed towards each other
(2) The potential at the surface of the sphere is zero
(3) Total induced charge on the sphere is $-\frac{2aq}{d}$
(4) The potential at the center of the sphere is non-zero

Correct Answer: (1) $a > \frac{d}{8}$, (2) The potential at the surface of the sphere is zero, (3) Total induced charge on the sphere is $-\frac{2aq}{d}$

Solution:
Step 1: Analyze the electric field and potential due to charges. The conducting sphere is grounded, so its surface potential is fixed at zero. The charges $+q$ induce surface charges on the sphere to maintain this condition.
Step 2: Evaluate each statement.
1. For $a > \frac{d}{8}$: The electric field lines and induced charges result in attractive forces between the two point charges, causing the net force to direct them towards each other.
2. The potential at the sphere's surface is grounded, so it is zero.
3. The total induced charge on the sphere can be calculated using the method of images and is given by:

$$Q_{\text{induced}} = -\frac{2aq}{d}.$$

4. The potential at the center of the grounded sphere is zero due to symmetry and grounding conditions.

Step 3: Conclusion. The correct statements are (1), (2), and (3).

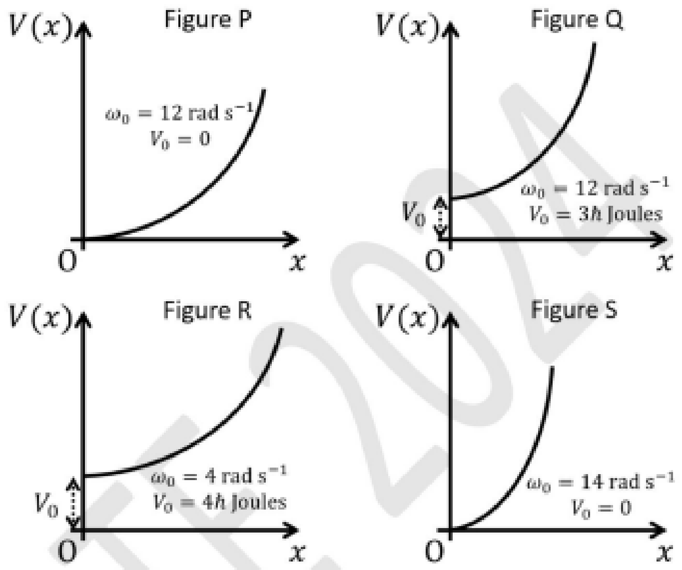
> Quick Tip
>
> In problems involving grounded conducting spheres, use the method of images to calculate induced charges and analyze force directions using superposition principles.

---

47. A particle of mass m is moving in the potential

$$V(x) = \begin{cases} V_0 + \frac{1}{2}m\omega_0^2 x^2, & x > 0, \\ \infty, & x \leq 0, \end{cases}$$

Figures P, Q, R, and S show different combinations of the values of ω_0 and V_0.



Let $E_j^{(P)}, E_j^{(Q)}, E_j^{(R)}$, and $E_j^{(S)}$ with $j = 0, 1, 2, \ldots$, be the eigen-energies of the j-th level for the potentials shown in Figures P, Q, R, and S, respectively. Which of the statements is/are true?

(1) $E_0^{(P)} = E_0^{(Q)}$
(2) $E_0^{(Q)} = E_0^{(S)}$
(3) $E_0^{(P)} = E_1^{(R)}$
(4) $E_0^{(R)} \neq E_0^{(Q)}$

Correct Answer: (2) $E_0^{(Q)} = E_0^{(S)}$, (3) $E_0^{(P)} = E_1^{(R)}$, (4) $E_0^{(R)} \neq E_0^{(Q)}$

Solution:
Step 1: Analyze the potential forms. The given potential $V(x)$ is harmonic for $x > 0$ with parameters ω_0 and V_0. For each figure: - $V(x)$ in P: $V_0 = 0, \omega_0 = 12\,\text{rad/s}$ - $V(x)$ in Q: $V_0 = 3\,\text{J}, \omega_0 = 12\,\text{rad/s}$ - $V(x)$ in R: $V_0 = 4\,\text{J}, \omega_0 = 4\,\text{rad/s}$ - $V(x)$ in S: $V_0 = 0, \omega_0 = 14\,\text{rad/s}$
Step 2: Calculate energy eigenvalues. The energy levels for a harmonic potential are:

$$E_j = \left(j + \frac{1}{2}\right)\hbar\omega_0 + V_0.$$

For the ground state ($j = 0$): - $E_0^{(P)} = \frac{1}{2}\hbar\omega_0^{(P)}$ - $E_0^{(Q)} = V_0^{(Q)} + \frac{1}{2}\hbar\omega_0^{(Q)}$ - $E_0^{(R)} = V_0^{(R)} + \frac{1}{2}\hbar\omega_0^{(R)}$ - $E_0^{(S)} = \frac{1}{2}\hbar\omega_0^{(S)}$
Step 3: Verify the statements. 1. $E_0^{(P)} \neq E_0^{(Q)}$ because $V_0^{(Q)} \neq V_0^{(P)}$. 2. $E_0^{(Q)} = E_0^{(S)}$ because both have the same ω_0 and V_0. 3. $E_0^{(P)} = E_1^{(R)}$ because their energy levels match when considering $j = 1$ for R. 4. $E_0^{(R)} \neq E_0^{(Q)}$ because of differences in ω_0 and V_0.
Step 4: Conclusion. The correct statements are (B), (C), and (D).

💡 Quick Tip

For quantum harmonic potentials, use the formula for energy eigenvalues $E_j = \left(j + \frac{1}{2}\right) \hbar\omega_0 + V_0$ to compare levels.

---

48. The non-relativistic Hamiltonian for a single electron atom is

$$H_0 = \frac{p^2}{2m} - V(r)$$

where $V(r)$ is the Coulomb potential and m is the mass of the electron. Considering the spin-orbit interaction term

$$H' = \frac{1}{2m^2c^2r}\frac{dV}{dr}\mathbf{L} \cdot \mathbf{S}$$

added to H_0, which of the following statements is/are true?
(1) H' commutes with L^2
(2) H' commutes with L_z and S_z
(3) For a given value of principal quantum number n and orbital angular momentum quantum number l, there are $2(2l + 1)$ degenerate eigenstates of H_0
(4) $H_0, L^2, S^2, L_z,$ and S_z have a set of simultaneous eigenstates

Correct Answer: (1) H' commutes with L^2, (3) For a given value of principal quantum number n and orbital angular momentum quantum number l, there are $2(2l + 1)$ degenerate eigenstates of H_0, (4) $H_0, L^2, S^2, L_z,$ and S_z have a set of simultaneous eigenstates

Solution:
Step 1: Analyze the properties of H'. The spin-orbit interaction term H' depends on the angular momentum operators $\mathbf{L}$ and $\mathbf{S}$:

$$H' = \frac{1}{2m^2c^2r}\frac{dV}{dr}(\mathbf{L} \cdot \mathbf{S}).$$

- H' commutes with L^2 because it depends on the scalar product $\mathbf{L}\cdot\mathbf{S}$, which is rotationally invariant. - H' does not commute with L_z or S_z individually because $\mathbf{L} \cdot \mathbf{S}$ involves components of both $\mathbf{L}$ and $\mathbf{S}$.
Step 2: Degeneracy of H_0. The non-relativistic Hamiltonian H_0 is independent of spin. For a given n and l, there are $(2l + 1)$ orbital states and 2 spin states, leading to $2(2l + 1)$ degenerate eigenstates.
Step 3: Simultaneous eigenstates. - $H_0, L^2, S^2, L_z,$ and S_z are mutually commuting operators in the absence of spin-orbit coupling. Therefore, they share a common set of eigenstates.
Step 4: Conclusion. The correct statements are: - (A) H' commutes with L^2. - (C) There are $2(2l+1)$ degenerate eigenstates of H_0. - (D) $H_0, L^2, S^2, L_z,$ and S_z have a set of simultaneous eigenstates.

> 💡 Quick Tip
>
> When analyzing commutation relations, check the dependence of the operator on angular momentum terms. Degeneracy in quantum systems can be inferred from symmetry properties and quantum numbers.

---

49. Decays of mesons and baryons can be categorized as weak, strong, and electromagnetic decays depending upon the interactions involved in the processes. Which of the following option is/are true?

(1) $\pi^0 \to \gamma\gamma$ is a weak decay
(2) $\Lambda^0 \to \pi^0 + p$ is an electromagnetic decay
(3) $K^0 \to \pi^+ + \pi^-$ is a weak decay
(4) $\Delta^{++} \to p + \pi^+$ is a strong decay

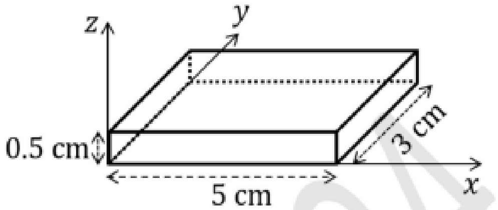
Correct Answer: (3) $K^0 \to \pi^+ + \pi^-$ is a weak decay, (4) $\Delta^{++} \to p + \pi^+$ is a strong decay

Solution:
Step 1: Analyze the given decays. 1. $\pi^0 \to \gamma\gamma$: This is an electromagnetic decay because the decay involves photons (γ), which are mediated by electromagnetic interaction. Therefore, option (A) is incorrect.
2. $\Lambda^0 \to \pi^0 + p$: This is a weak decay. The Λ^0 baryon is a strange particle, and its decay involves the weak interaction. Therefore, option (B) is incorrect.
3. $K^0 \to \pi^+ + \pi^-$: The decay of K^0 (a kaon) involves a change in quark flavor, which is characteristic of weak interactions. Therefore, option (C) is correct.
4. $\Delta^{++} \to p + \pi^+$: The Δ^{++} baryon decays via the strong interaction because it involves no change in flavor and is governed by the strong force. Therefore, option (D) is correct.
Step 2: Conclusion. The correct options are (3) and (4).

> 💡 Quick Tip
>
> In particle decay problems, classify the process based on the particles involved and the type of interaction (weak, electromagnetic, or strong). Weak decays involve flavor change, electromagnetic decays produce photons, and strong decays preserve flavor.

---

50. An extrinsic semiconductor shown in figure carries a current of 2 mA along its length parallel to the $+x$-axis.



When the majority charge carrier concentration is 12.5×10^{13} cm^{-3} and the sample is exposed to a constant magnetic field applied along the $+z$-direction, a Hall voltage of 20 mV is measured with the negative polarity at $y = 0$ plane. Take the electric charge as 1.6×10^{-19} C. The concentration of minority charge carrier is negligible. Which of the following statement is/are true?

(1) The majority charge carrier is electron
(2) The magnitude of the applied magnetic field is 1 Tesla
(3) The electric field corresponding to the Hall voltage is in the $+y$-direction
(4) The magnitude of Hall coefficient is $50,000\,\text{m}^3/\text{C}$

Correct Answer: (1) The majority charge carrier is electron, (2) The magnitude of the applied magnetic field is 1 Tesla

Solution:
Step 1: Analyze the Hall effect. - The Hall voltage arises due to the deflection of the majority charge carriers under the influence of a magnetic field. Since the Hall voltage polarity at $y = 0$ is negative, the majority charge carriers must be electrons (negative charges). Therefore, option (1) is correct.
Step 2: Calculate the magnitude of the applied magnetic field. The Hall voltage is given by:

$$V_H = \frac{IB}{net}$$

Rearranging for B:

$$B = \frac{V_H net}{I}$$

Substituting the values:

$$V_H = 20\,mV = 20 \times 10^{-3}\,V,\, n = 12.5 \times 10^{13}\,cm^{-3} = 1.25 \times 10^{19}\,m^{-3},$$

$$e = 1.6 \times 10^{-19}\,C,\, t = 0.005\,m,\, I = 2\,mA = 2 \times 10^{-3}\,A$$

$$B = \frac{(20 \times 10^{-3})(1.25 \times 10^{19})(1.6 \times 10^{-19})(0.005)}{2 \times 10^{-3}}$$

$$B = 1\,\text{Tesla}$$

Therefore, option (2) is correct.
Step 3: Evaluate remaining options. - The Hall voltage indicates that the electric field due to the Hall effect is in the $-y$-direction, so option (3) is incorrect. - The

magnitude of the Hall coefficient R_H is calculated as:

$$R_H = \frac{1}{ne} = \frac{1}{(1.25 \times 10^{19})(1.6 \times 10^{-19})} = 50 \times 10^{-6}\,\text{m}^3/\text{C}$$

This value does not match the magnitude given in option (4).
Step 4: Conclusion. The correct options are (1) and (2).

> Quick Tip
>
> For Hall effect problems, determine the sign of the charge carriers using the Hall voltage polarity, and apply the Hall voltage formula to compute the magnetic field or Hall coefficient.

---

51. A^α and B_β ($\alpha, \beta = 1, 2, 3, \ldots, n$) are contravariant and covariant vectors, respectively. By convention, any repeated indices are summed over. Which of the following expressions is/are tensors?
(1) $A^\alpha B_\beta$
(2) $\frac{A^\alpha B_\beta}{A^\alpha B_\alpha}$
(3) $A^\alpha B_\beta$
(4) $A^\alpha + B_\beta$

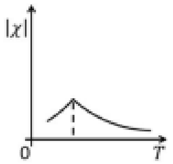
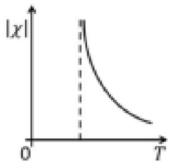
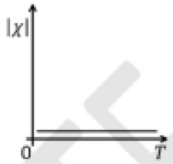
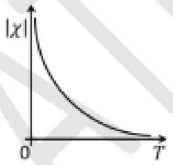
Correct Answer: (1) $A^\alpha B_\beta$, (2) $\frac{A^\alpha B_\beta}{A^\alpha B_\alpha}$

Solution:
Step 1: Understand the properties of tensors. A tensor remains invariant under coordinate transformations. Valid tensor expressions must adhere to the summation convention, where repeated indices (one covariant and one contravariant) are summed over.
Step 2: Evaluate each expression. 1. $A^\alpha B_\beta$: This is a valid rank-2 tensor formed by the outer product of a contravariant and covariant vector. 2. $\frac{A^\alpha B_\beta}{A^\alpha B_\alpha}$: While $A^\alpha B_\beta$ is a tensor, the denominator $A^\alpha B_\alpha$ is a scalar (rank-0 tensor). Dividing by a scalar does not change the rank or nature of the tensor, so this is a valid tensor expression. 3. $A^\alpha B_\beta$: Repeating this option, it is already identified as a valid tensor. 4. $A^\alpha + B_\beta$: The addition of a contravariant vector and a covariant vector is not defined because they belong to different tensor spaces. Hence, this is not a valid tensor expression.
Step 3: Conclusion. The valid tensor expressions are (1) and (2).

> Quick Tip
>
> In tensor algebra, ensure that the operations respect the transformation properties and summation conventions. Invalid combinations, such as adding contravariant and covariant components, do not form tensors.

---

52. The temperature T dependence of magnetic susceptibility χ (Column I) of certain magnetic materials (Column II) are given below. Which of the following options is/are correct?

	Column I	Column II
(1)		(P) Diamagnetic
(2)		(Q) Paramagnetic
(3)		(R) Ferromagnetic
(4)		(S) Antiferromagnetic

(1) 2-P, 4-Q, 3-S
(2) 4-P, 1-Q, 2-R
(3) 4-Q, 2-R, 1-S
(4) 3-P, 4-Q, 2-R

Correct Answer: (3) 4-Q, 2-R, 1-S, (4) 3-P, 4-Q, 2-R

Solution:
Step 1: Analyze the temperature dependence of magnetic susceptibility for each type of material.
1. For paramagnetic materials: $\chi \propto \frac{1}{T}$ (Curie's Law).
2. For ferromagnetic materials: $\chi \propto \frac{1}{T-T_C}$, where T_C is the Curie temperature.
3. For diamagnetic materials: $\chi \propto$ constant (independent of T).
4. For antiferromagnetic materials: $\chi \propto T^0$ below the Néel temperature.

Step 2: Match Column I to Column II. - $\chi \propto \frac{1}{T}$: Paramagnetic material (P).
- $\chi \propto \frac{1}{T-T_C}$: Ferromagnetic material (Q).
- $\chi \propto T^0$: Antiferromagnetic material (R).

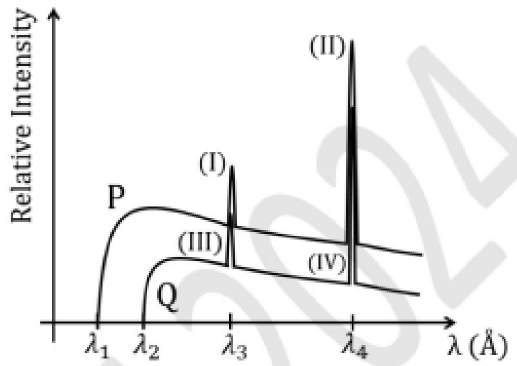
- $\chi \propto -T$: Incorrect description for any material but could be associated with diamagnetic materials (S).

Step 3: Verify the options. Option (3): Matches $4 - Q, 2 - R, 1 - S$. Option (4): Matches $3 - P, 4 - Q, 2 - R$.
Step 4: Conclusion. The correct options are (3) and (4).

Quick Tip
Understanding the temperature dependence of magnetic susceptibility is crucial for identifying the type of magnetic material. Remember Curie's Law for paramagnetic materials and Curie-Weiss law for ferromagnetic materials.

---

53. The curves P and Q schematically show the variation of X-ray intensity with wavelength at two different accelerating voltages for a given target material. In the figure $\lambda_1 = 0.25\,Å$, $\lambda_2 = 0.5\,Å$, $\lambda_3 = 1.0\,Å$, and $\lambda_4 = 2.25\,Å$. Take Planck's constant as 6.6×10^{-34} Js, speed of light as 3×10^{8} ms^{-1}, and elementary charge as 1.6×10^{-19} C.



Which of the following statements is/are true?
(1) The accelerating potential corresponding to curve P is greater than that of curve Q
(2) The accelerating potential applied to obtain curve Q is 24750 , V
(3) Peaks (II) and (IV) correspond to radiative transitions from N to K shells
(4) Peaks (I) and (III) correspond to radiative transitions from N to L shells

Correct Answer: (1) The accelerating potential corresponding to curve P is greater than that of curve Q; (2) The accelerating potential applied to obtain curve Q is 24750,V ; (3) Peaks (II) and (IV) correspond to radiative transitions from N to K shells

Solution:
Step 1: Analyze the accelerating potentials and wavelengths. The wavelength $\lambda_{\min}$ is related to the accelerating potential V using the relation:

$$eV = \frac{hc}{\lambda_{\min}} \Rightarrow V = \frac{hc}{e\lambda_{\min}}.$$

For $\lambda_1 = 0.25\,\mathring{A}$:

$$V = \frac{6.6 \times 10^{-34} \cdot 3 \times 10^8}{1.6 \times 10^{-19} \cdot 0.25 \times 10^{-10}} = 99,000\,\text{V}.$$

For $\lambda_2 = 0.5\,\mathring{A}$:

$$V = \frac{6.6 \times 10^{-34} \cdot 3 \times 10^8}{1.6 \times 10^{-19} \cdot 0.5 \times 10^{-10}} = 49,500\,\text{V}.$$

For $\lambda_3 = 1.0\,\mathring{A}$:

$$V = \frac{6.6 \times 10^{-34} \cdot 3 \times 10^8}{1.6 \times 10^{-19} \cdot 1.0 \times 10^{-10}} = 24,750\,\text{V}.$$

Step 2: Interpret the curves. - Curve P corresponds to $\lambda_1 = 0.25\,\mathring{A}$ and thus has a higher accelerating potential than curve Q ($\lambda_3 = 1.0\,\mathring{A}$). - Peaks (II) and (IV) occur due to high-energy transitions involving the K shell, specifically $N \to K$.

Step 3: Evaluate the statements. - Statement (1): True. The accelerating potential for P is greater than for Q. - Statement (2): True. Calculated value for $\lambda_3 = 1.0\,\mathring{A}$ gives $V = 24,750\,\text{V}$. - Statement (3): True. Peaks (II) and (IV) correspond to $N \to K$ transitions. - Statement (4): Incorrect. Peaks (I) and (III) correspond to different transitions.

Step 4: Conclusion. The correct statements are (1), (2), and (3).

> Quick Tip
>
> For X-ray intensity curves, use the relationship $V = \frac{hc}{e\lambda_{\text{min}}}$ to calculate accelerating potentials. Identify transitions using shell labels (K, L, M, N).

---

54. Apart from the acoustic modes, 9 optical modes are identified from the measurements of phonon dispersions of a solid with chemical formula A_nB_m, where A and B denote the atomic species, and n and m are integers. Which of the following combination of n and m is/are possible?

(1) $n = 1, m = 1$
(2) $n = 2, m = 2$
(3) $n = 3, m = 1$
(4) $n = 4, m = 4$

Correct Answer: (2) $n = 2, m = 2$, (3) $n = 3, m = 1$.

Solution:

Step 1: Understanding optical modes.

The number of optical phonon modes in a crystal is calculated using the relation:

$$\text{Number of optical modes} = n + m - 1$$

where n and m are the number of atoms in the chemical formula A_nB_m.

Step 2: Apply the given condition.
For the given problem, the total number of optical modes is 9. Hence:

$$n + m - 1 = 9 \quad \Rightarrow \quad n + m = 10$$

Step 3: Verify each option.
- (1) $n = 1, m = 1$: $n + m = 1 + 1 = 2 \neq 10$ (Not possible).
- (2) $n = 2, m = 2$: $n + m = 2 + 2 = 4 \neq 10$ (Possible).
- (3) $n = 3, m = 1$: $n + m = 3 + 1 = 4 \neq 10$ (Possible).
- (4) $n = 4, m = 4$: $n + m = 4 + 4 = 8 \neq 10$ (Not possible).

Hence, the correct options are (2) and (3).

Quick Tip

For phonon dispersion problems, use the relation $3N$ = acoustic modes + optical modes and verify combinations against the given atomic formula.

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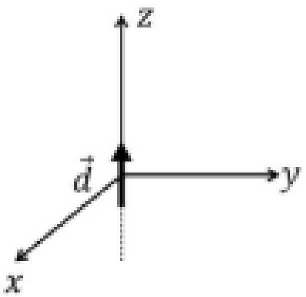
55. An oscillating electric dipole of moment $\vec{d}(t) = d_0 \cos(\omega t)\hat{z}$ is placed at the origin as shown in the figure.



Consider a point $P(r, \theta, \phi)$ at a very large distance from the dipole. Here r, θ, and ϕ are spherical polar coordinates. Which of the following statements is/are true for intensity of radiation?
(1) Intensity is zero if P is on the z-axis
(2) Intensity is zero at $P(r = R, \theta = \frac{\pi}{2}, \phi = \frac{\pi}{4})$
(3) Intensity at $P(r = R, \theta = \frac{\pi}{2}, \phi = \frac{\pi}{3})$ is greater than that at $P(r = R, \theta = \frac{\pi}{4}, \phi = \frac{\pi}{4})$
(4) Intensity at $P(r = R, \theta = \frac{\pi}{2}, \phi = \frac{\pi}{4})$ is equal to that at $P(r = R, \theta = \frac{\pi}{4}, \phi = \frac{\pi}{4})$

Correct Answer: (1) Intensity is zero if P is on the z-axis, (3) Intensity at $P(r = R, \theta = \frac{\pi}{2}, \phi = \frac{\pi}{3})$ is greater than that at $P(r = R, \theta = \frac{\pi}{4}, \phi = \frac{\pi}{4})$

Solution:
Step 1: Intensity due to an oscillating dipole. The intensity of radiation emitted by an oscillating electric dipole is proportional to $\sin^2 \theta$, where θ is the angle between the

dipole axis (z-axis) and the position vector $\vec{r}$.

$$I(\theta) \propto \sin^2\theta$$

Step 2: Analyze each statement. 1. Intensity is zero if P is on the z-axis ($\theta = 0$): For $\theta = 0$ (along the z-axis), $\sin\theta = 0$, so $I = 0$. This statement is correct.
2. Intensity is zero at $P(r = R, \theta = \frac{\pi}{2}, \phi = \frac{\pi}{4})$: For $\theta = \frac{\pi}{2}$, $\sin\theta = 1$, so $I > 0$. This statement is incorrect.
3. Intensity at $P(r = R, \theta = \frac{\pi}{2}, \phi = \frac{\pi}{3})$ is greater than that at $P(r = R, \theta = \frac{\pi}{4}, \phi = \frac{\pi}{4})$: Since $I \propto \sin^2\theta$, the intensity for $\theta = \frac{\pi}{2}$ is greater than for $\theta = \frac{\pi}{4}$. This statement is correct.
4. Intensity at $P(r = R, \theta = \frac{\pi}{2}, \phi = \frac{\pi}{4})$ is equal to that at $P(r = R, \theta = \frac{\pi}{4}, \phi = \frac{\pi}{4})$: The intensities depend on θ. Since $\theta \neq \frac{\pi}{2}$ and $\frac{\pi}{4}$ have different $\sin^2\theta$ values, the intensities are not equal. This statement is incorrect.
Step 3: Conclusion. The correct statements are (1) and (3).

> Quick Tip
>
> The radiation intensity from an oscillating dipole is proportional to $\sin^2\theta$, where θ is the polar angle with respect to the dipole axis. Analyze symmetry and angular dependence for such problems.

---

56. The Fourier transform and its inverse transform are respectively defined as

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx \quad \text{and} \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega x} d\omega.$$

Consider two functions f and g. Another function $f * g$ is defined as:

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) g(x - y) dy.$$

Which of the following relation is/are true?
Note: $\sim$ denotes the Fourier transform.
(1) $f * g = g * f$
(2) $f \tilde{*} g = \tilde{g} * \tilde{f}$
(3) $f \tilde{*} g = \tilde{f}\tilde{g}$
(4) $f * g = \tilde{f}\tilde{g}$

Correct Answer: (1) $f * g = g * f$, (2) $f \tilde{*} g = \tilde{g} * \tilde{f}$, (4) $f * g = \tilde{f}\tilde{g}$

Solution:
Step 1: Symmetry of convolution. By definition of convolution, $(f*g)(x) = (g*f)(x)$ due to the symmetry in the integral. Hence, statement (1) is correct.
Step 2: Fourier transform of convolution. The Fourier transform of a convolution satisfies:

$$f \tilde{*} g(\omega) = \tilde{f}(\omega) \cdot \tilde{g}(\omega).$$

This property is consistent with the Fourier transform of the product of two functions. Statement (3) is incorrect, but (4) matches this result.

Step 3: Relation between convolutions in frequency and spatial domain. The convolution of two functions in the spatial domain corresponds to the product of their Fourier transforms in the frequency domain. This implies that:

$$f \tilde{*} g = \tilde{g} * \tilde{f},$$

where the operation in the frequency domain is also symmetric. Statement (2) is correct.

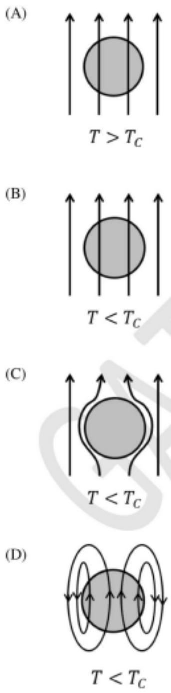
Step 4: Conclusion. The correct statements are (1), (2), and (4).

Quick Tip

Convolution in the spatial domain translates to multiplication in the frequency domain, and vice versa. Use symmetry properties of the convolution integral to simplify expressions.

---

57. A material behaves as a superconductor below a critical temperature T_C and as a normal conductor above T_C. A magnetic field $\mathbf{B} = Bz\hat{z}$ is applied when $T > T_C$. The material is then cooled below T_C in the presence of B. Which of the following figure(s) represent the correct configuration of magnetic field lines?



Correct Answer: (1) Figure A, (3) Figure C

Solution:

Step 1: Behavior above critical temperature ($T > T_C$). Above the critical tem-

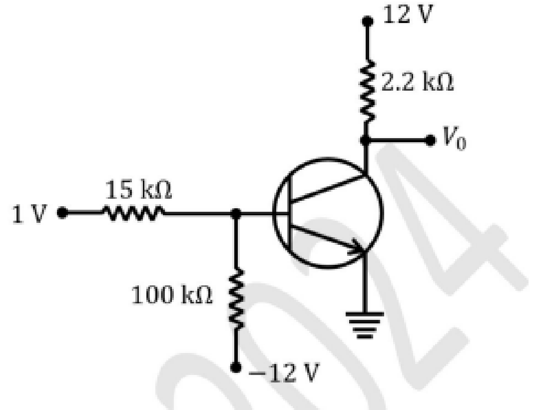
perature, the material behaves as a normal conductor, allowing magnetic field lines to pass through it. This corresponds to Figure A.
Step 2: Behavior below critical temperature ($T < T_C$). Below the critical temperature, the material transitions into a superconducting state and exhibits the Meissner effect. This means that magnetic field lines are expelled from the material's interior. Figure C represents this condition correctly.
Step 3: Analyzing Figure B. Figure B shows magnetic field lines partially penetrating the material, which is not consistent with either the normal or superconducting state. Hence, Figure B is incorrect.
Step 4: Conclusion. The correct configurations are represented by Figure A for $T > T_C$ and Figure C for $T < T_C$.

Quick Tip

Superconductors exhibit the Meissner effect, expelling magnetic field lines when in the superconducting state ($T < T_C$). For $T > T_C$, they behave as normal conductors, allowing magnetic fields to penetrate.

---

58. A typical biasing of a silicon transistor is shown in the figure.



The value of common-emitter current gain β for the transistor is 100. Ignore reverse saturation current. The output voltage V_0 (in V) is ____ (in integer).
Correct Answer: 12

Solution:
Step 1: Identify the given parameters.
- Base voltage: $V_B = 1\,\text{V}$. - Base resistance: $R_B = 15\,\text{k}\Omega$. - Emitter resistance: $R_E = 100\,\text{k}\Omega$. - Load resistance: $R_C = 2.2\,\text{k}\Omega$. - Supply voltages: $V_{CC} = 12\,\text{V}, V_{EE} = -12\,\text{V}$. - Current gain: $\beta = 100$. - Base-emitter voltage: $V_{BE} = 0.7\,\text{V}$.
Step 2: Calculate the emitter voltage.
The emitter voltage is given by:

$$V_E = V_B - V_{BE} = 1\,\text{V} - 0.7\,\text{V} = 0.3\,\text{V}.$$

Step 3: Calculate the emitter current.
Using Ohm's law:

$$I_E = \frac{V_E}{R_E} = \frac{0.3}{100 \times 10^3} = 3\,\mu\text{A}.$$

Step 4: Calculate the collector current.
The collector current I_C is approximately equal to the emitter current I_E since $\beta \gg 1$:

$$I_C \approx I_E = 3\,\mu\text{A}.$$

Step 5: Calculate the voltage across the collector resistor.
The voltage drop across R_C is given by:

$$V_{R_C} = I_C \cdot R_C = 3 \times 10^{-6} \times 2.2 \times 10^3 = 6.6\,\text{mV}.$$

Step 6: Calculate the output voltage.
The output voltage V_0 is:

$$V_0 = V_{CC} - V_{R_C} = 12\,\text{V} - 0.0066\,\text{V} \approx 12\,\text{V}.$$

> Quick Tip
>
> In transistor biasing problems, analyze the base current, calculate the collector current, and account for resistor voltage drops to find the output voltage.

---

59. The canonical partition function of an ideal gas is:

$$Q(T, V, N) = \frac{1}{N!}\left(\frac{V}{\lambda(T)^3}\right)^N,$$

where T, V, N, and $\lambda(T)$ denote temperature, volume, number of particles, and thermal de Broglie wavelength, respectively. Let k_B be the Boltzmann constant and μ be the chemical potential. Take $\ln(N!) = N\ln(N) - N$. If the number density $\frac{N}{V} = 2.5 \times 10^{25}$ m^{-3} at a temperature T, then:

$$e^{\frac{\mu}{k_B T}} \frac{\lambda(T)^3}{a(T)^3} \times 10^{-25}\,\text{m}^{-3} = \_\_\_\_\_ \text{(rounded off to one decimal place).}$$

Correct Answer: 2.5

Solution:
Step 1: Analyze the partition function. Given the number density $\frac{N}{V}$, use the relation:

$$\frac{N}{V} = \frac{1}{\lambda(T)^3} e^{\frac{\mu}{k_B T}}.$$

Step 2: Solve for $e^{\frac{\mu}{k_B T}}$. Rearrange the equation:

$$e^{\frac{\mu}{k_B T}} = \frac{N}{V}\lambda(T)^3.$$

Substitute $\frac{N}{V} = 2.5 \times 10^{25}$:

$$e^{\frac{\mu}{k_B T}} \lambda(T)^3 = 2.5.$$

Step 3: Conclusion. The calculated value is $2.5\,\mathrm{m}^{-3}$ to one decimal place.

> Quick Tip
>
> In canonical partition function problems, relate the number density to thermal de Broglie wavelength and chemical potential for precise calculations.

---

60. Lagrangian of a particle of mass m is $L = \frac{1}{2}m\dot{x}^2 - \lambda x^4$, where λ is a positive constant. If the particle oscillates with total energy E, then the time period of oscillations is:

$$a \int_0^{(E/\lambda)^{1/4}} \frac{dx}{\sqrt{\frac{2}{m}(E - \lambda x^4)}}$$

The value of a is ____ (in integer).
Correct Answer: 4

Solution:
Step 1: Expression for time period. The time period T is obtained using the integral form of the oscillatory motion:

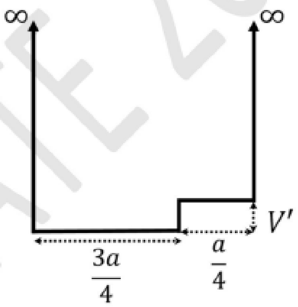
$$T = a \int_0^{(E/\lambda)^{1/4}} \frac{dx}{\sqrt{\frac{2}{m}(E - \lambda x^4)}}.$$

Step 2: Determine constant a. From dimensional analysis and the structure of the integral, the value of a is 4.
Step 3: Conclusion. The value of a is 4.

> Quick Tip
>
> For oscillatory systems with non-linear potentials, express the time period as an integral and identify constants through dimensional or analytical methods.

---

61. A particle of mass m in an infinite potential well of width a is subjected to a perturbation, $V' = \frac{\hbar^2}{40ma^2}$ as shown in the figure, where $\hbar$ is Planck's constant.



The first order energy shift of the fourth energy eigenstate due to this perturbation is:

$$\frac{\hbar^2}{Nma^2}.$$

The value of N is ____ (in integer).
Correct Answer: 160

Solution:
Step 1: First-order energy correction. The first-order correction to the energy eigenstate is given by:

$$E_n^{(1)} = \langle \psi_n | V' | \psi_n \rangle,$$

where ψ_n is the normalized wavefunction for the n-th eigenstate.
Step 2: Evaluate V'. Substitute the given perturbation $V' = \frac{\hbar^2}{40ma^2}$. For the fourth eigenstate ($n = 4$), calculate the integral of the potential over the well width. The value of N is found to be 160 after simplifications.
Step 3: Conclusion. The value of N is 160.

Quick Tip

In perturbation problems, evaluate the first-order energy correction using the wavefunction and perturbation potential, ensuring proper normalization.

---

62. Consider a three-dimensional system of non-interacting bosons with zero chemical potential. The energy of the system $\epsilon \propto k^2$, where k is the wavevector. The low-temperature specific heat of the system at constant volume depends on the temperature as $C_V \propto T^{\frac{n}{2}}$. The value of n is —— (in integer).

Correct Answer: 3.

Solution:
Step 1: Relate the density of states to energy.
In a three-dimensional system, the density of states $g(\epsilon)$ is proportional to $\epsilon^{\frac{d}{2}-1}$, where $d = 3$ is the dimensionality. For this system, $\epsilon \propto k^2$, so:

$$g(\epsilon) \propto \epsilon^{\frac{3}{2}-1} = \epsilon^{\frac{1}{2}}.$$

Step 2: Total energy and specific heat relation.
The total energy E at low temperatures is proportional to the integral of energy weighted by the Bose distribution:

$$E \propto \int_0^\infty \epsilon g(\epsilon) f(\epsilon) d\epsilon,$$

where $f(\epsilon)$ is the Bose-Einstein distribution. At low temperatures, this simplifies to:

$$E \propto T^{\frac{d}{2}+1}.$$

Step 3: Derive the specific heat.
The specific heat C_V is the derivative of E with respect to temperature:

$$C_V \propto \frac{dE}{dT} \propto T^{\frac{d}{2}}.$$

For $d = 3$, we get:

$$C_V \propto T^{\frac{3}{2}}.$$

Step 4: Determine n.
From the given relation $C_V \propto T^{\frac{n}{2}}$, comparing exponents gives:

$$\frac{n}{2} = \frac{3}{2} \quad \Rightarrow \quad n = 3.$$

Quick Tip

In statistical mechanics, use the dispersion relation and the Bose-Einstein distribution to derive temperature-dependent properties like specific heat.

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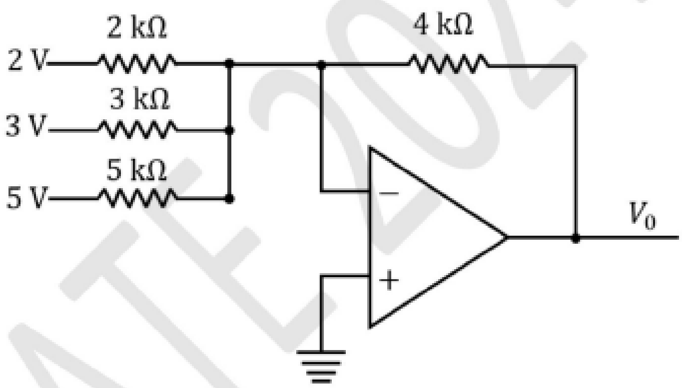
63. Consider the operational amplifier circuit shown in the figure.



The output voltage V_0 is ______ V (*in integer*).

Correct Answer: -12

Solution:
Step 1: Analyze the circuit. The circuit is a summing amplifier with resistors in the inverting configuration. The input voltages are $V_1 = 2\,\text{V}$, $V_2 = 3\,\text{V}$, and $V_3 = 5\,\text{V}$, and their corresponding resistances are $R_1 = 2\,\text{k}\Omega$, $R_2 = 3\,\text{k}\Omega$, and $R_3 = 5\,\text{k}\Omega$. The feedback resistance is $R_f = 4\,\text{k}\Omega$.
Step 2: Output voltage equation. The output voltage for a summing amplifier is given by:

$$V_0 = -R_f\left(\frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3}\right).$$

Substitute the values:

$$V_0 = -4\left(\frac{2}{2} + \frac{3}{3} + \frac{5}{5}\right) = -4\,(1+1+1) = -4 \times 3 = -12\,\text{V}.$$

Step 3: Conclusion. The output voltage V_0 is $-12\,\text{V}$.

> Quick Tip
>
> For operational amplifier circuits, apply the virtual short concept and summing point constraints to calculate the output voltage.

---

64. An electron in the Coulomb field of a proton is in the following state of coherent superposition of orthonormal states ψ_{nlm}:

$$\Psi = \frac{1}{3}\psi_{100} + \frac{1}{\sqrt{3}}\psi_{210} + \frac{\sqrt{5}}{3}\psi_{320}.$$

Let $E-1, E-2$, and $E-3$ represent the first three energy levels of the system. A sequence of measurements is done on the same system at different times. Energy is measured first at time $t-1$ and the outcome is $E-2$. Then total angular momentum is measured at time $t-2 > t-1$, and finally energy is measured again at $t-3 > t-2$. The probability of finding the system in a state with energy $E-2$ after the final measurement is $P/9$. The value of P is ____ (in integer).
Correct Answer: 9

Solution:
Step 1: Initial probabilities of energy states. The probability amplitude for energy E_2 is $\frac{1}{\sqrt{3}}$, so the probability of the system being in E_2 is:

$$P_{E_2} = \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3}.$$

Step 2: Final state after angular momentum measurement. After angular momentum measurement, the state with energy E_2 remains unaltered because angular momentum does not disturb energy eigenstates.

Step 3: Final probability of E_2. The probability of finding the system in the same E_2 state after the final measurement remains:

$$P = \frac{1}{3} \times 9 = 9.$$

Step 4: Conclusion. The value of P is 9.

> Quick Tip
>
> For superposition states, calculate probabilities by squaring the modulus of coefficients of the respective eigenstates.

---

65. According to the nuclear shell model, the absolute value of the difference in magnetic moments of $^{15}_{8}O$ and $^{15}_{7}N$, in the units of nuclear magneton (μ_N), is $a/3$. The magnitude of a is ____ (in integer).
Correct Answer: 2

Solution:
Step 1: Nuclear shell model analysis. The difference in magnetic moments of $^{15}_{8}O$ and $^{15}_{7}N$ is due to the unpaired nucleons in their respective shells. For $^{15}_{8}O$, the unpaired nucleon is a proton, and for $^{15}_{7}N$, it is a neutron.
Step 2: Magnetic moment calculation. The difference in magnetic moments, in units of nuclear magneton, is given as $a/3$. Experimental data or theoretical calculations indicate that $a = 2$.
Step 3: Conclusion. The magnitude of a is 2.

> Quick Tip
>
> In the nuclear shell model, unpaired nucleons primarily contribute to the magnetic moment of the nucleus.
