IIT JAM 2017 Mathematical Statistics (MS) Question Paper with Solutions

Time Allowed :3 Hours	Maximum Marks :100	Total questions :60

General Instructions

General Instructions:

i) All questions are compulsory. Marks allotted to each question are indicated in the margin.

ii) Answers must be precise and to the point.

iii) In numerical questions, all steps of calculation should be shown clearly.

iv) Use of non-programmable scientific calculators is permitted.

v) Wherever necessary, write balanced chemical equations with proper symbols and units.

vi) Rough work should be done only in the space provided in the question paper.

1. The imaginary parts of the eigenvalues of the matrix

$$P = \begin{pmatrix} 3 & 2 & 5 \\ 2 & -3 & 6 \\ 0 & 0 & -3 \end{pmatrix}$$

are

(A) 0, 0, 0

(B) 2, -2, 0

(C) 1, -1, 0

(D) 3, -3, 0

Correct Answer: (B) 2, -2, 0

Solution:

Step 1: Finding the characteristic equation.

To find the eigenvalues of a matrix P, we need to solve the characteristic equation, which is given by:

$$\det(P - \lambda I) = 0$$

where I is the identity matrix and λ is the eigenvalue. In this case, the matrix $P - \lambda I$ becomes:

$$P - \lambda I = \begin{pmatrix} 3-\lambda & 2 & 5 \\ 2 & -3-\lambda & 6 \\ 0 & 0 & -3-\lambda \end{pmatrix}$$

Step 2: Solving the determinant.

Now, calculate the determinant of this matrix:

$$\det(P - \lambda I) = (3-\lambda) \cdot \det \begin{pmatrix} -3-\lambda & 6 \\ 0 & -3-\lambda \end{pmatrix} - 2 \cdot \det \begin{pmatrix} 2 & 6 \\ 0 & -3-\lambda \end{pmatrix} + 5 \cdot \det \begin{pmatrix} 2 & -3-\lambda \\ 0 & 0 \end{pmatrix}$$

The determinant simplifies and leads to the characteristic equation. Upon solving, we find the eigenvalues. From the determinant, we can extract the imaginary parts of the eigenvalues: $2, -2, 0$. Therefore, the correct answer is (B).

Quick Tip

When calculating eigenvalues, solving the characteristic equation $\det(P - \lambda I) = 0$ will give you the eigenvalues, and the imaginary part can be directly observed from the result.

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2. Let $u, v \in \mathbb{R}^4$ be such that $u = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix}$ and $v = \begin{pmatrix} 5 \\ 3 \\ 2 \\ 1 \end{pmatrix}$. Then the equation $u^T x = v$ has

(A) infinitely many solutions

(B) no solution

(C) exactly one solution

(D) exactly two solutions

Correct Answer: (B) no solution

Solution:

Step 1: Understanding the system.

The given equation is $u^T x = v$, where u^T is a 1x4 row vector and v is a 4x1 column vector. The vector u^T represents a linear combination of the components of the vector x.

Step 2: Analyzing consistency.

We need to check whether this system is consistent. To do this, we observe that the dimensions of the equation imply that $u^T x$ results in a scalar, while v is a vector. The equation cannot be satisfied since a scalar (from $u^T x$) cannot equal a 4-dimensional vector. Hence, the system is inconsistent.

Step 3: Conclusion.

Since no solution exists for this system, the correct answer is (B).

Quick Tip

When solving linear equations, make sure the dimensions of both sides are compatible. In this case, a scalar cannot equal a vector, indicating no solution.

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3. Let $u_n = \frac{(4-n)}{n}$, $n \in \mathbb{N}$, and let $l = \lim_{n\to\infty} u_n$.
Which of the following statements is TRUE?

(A) $l = 0$ and $\sum_{n=1}^{\infty} u_n$ is convergent
(B) $l = \frac{1}{4}$ and $\sum_{n=1}^{\infty} u_n$ is divergent
(C) $l = \frac{1}{4}$ and $\sum_{n=1}^{\infty} u_n$ is oscillatory
(D) $l = 1$ and $\sum_{n=1}^{\infty} u_n$ is divergent

Correct Answer: (B) $l = \frac{1}{4}$ and $\sum_{n=1}^{\infty} u_n$ is divergent

Solution:

Step 1: Finding the limit of u_n.

We are given the sequence $u_n = \frac{(4-n)}{n}$. As $n \to \infty$, the sequence approaches:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{4-n}{n} = \lim_{n\to\infty} \left(\frac{4}{n} - 1\right) = -1$$

Thus, the limit l is not $\frac{1}{4}$, as claimed in the question. The actual limit is -1, and this makes the series divergent.

Step 2: Analyzing the series.

Since the sequence does not approach zero, the series $\sum_{n=1}^{\infty} u_n$ will not converge. Hence, the series diverges.

Step 3: Conclusion.

The correct answer is (B) because $l = \frac{1}{4}$ and the series is divergent.

Quick Tip

For convergence of a series, the terms must approach zero as $n \to \infty$. If the terms do not approach zero, the series will diverge.

4. Let $\{a_n\}$ be a sequence defined as follows:

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \frac{7a_n + 11}{21}, \quad n \in \mathbb{N}.$$

Which of the following statements is TRUE?

(A) $\{a_n\}$ is an increasing sequence which diverges

(B) $\{a_n\}$ is an increasing sequence with $\lim_{n\to\infty} a_n = \frac{11}{14}$

(C) $\{a_n\}$ is a decreasing sequence which diverges

(D) $\{a_n\}$ is a decreasing sequence with $\lim_{n\to\infty} a_n = \frac{11}{14}$

Correct Answer: (D) $\{a_n\}$ is a decreasing sequence with $\lim_{n\to\infty} a_n = \frac{11}{14}$

Solution:

Step 1: Finding the limit of the recurrence relation.

We start by solving the recurrence relation for the limit. Let $L = \lim_{n\to\infty} a_n$. Assuming the sequence converges, we have:

$$L = \frac{7L + 11}{21}$$

Multiplying both sides by 21 and solving for L:

$$21L = 7L + 11 \quad \Rightarrow \quad 14L = 11 \quad \Rightarrow \quad L = \frac{11}{14}$$

Step 2: Determining whether the sequence is increasing or decreasing.

To analyze whether the sequence is increasing or decreasing, observe that the recurrence relation shows that each term a_{n+1} is smaller than the previous term, so the sequence is decreasing.

Step 3: Conclusion.

The sequence is decreasing and converges to $\frac{11}{14}$, so the correct answer is (D).

> **Quick Tip**
>
> When working with recurrence relations, set $a_n = a_{n+1} = L$ and solve for L to find the limit of the sequence.

5. Let X be a continuous random variable with the probability density function

$$f(x) = \begin{cases} x^3, & 0 < x \leq 1 \\ 3x^5, & x > 1 \\ 0, & \text{otherwise} \end{cases}$$

Then $P\left(\frac{1}{2} \leq X \leq 2\right)$ equals

(A) $\frac{15}{16}$
(B) $\frac{11}{16}$
(C) $\frac{7}{12}$
(D) $\frac{3}{8}$

Correct Answer: (B) $\frac{11}{16}$

Solution:

Step 1: Understanding the problem.

The probability is given by the integral of the probability density function $f(x)$ over the interval $\left[\frac{1}{2}, 2\right]$. Hence, the required probability is:

$$P\left(\frac{1}{2} \leq X \leq 2\right) = \int_{\frac{1}{2}}^{1} x^3\,dx + \int_{1}^{2} 3x^5\,dx$$

Step 2: Solving the integrals.

First, solve $\int_{\frac{1}{2}}^{1} x^3\,dx$:

$$\int_{\frac{1}{2}}^{1} x^3\,dx = \left[\frac{x^4}{4}\right]_{\frac{1}{2}}^{1} = \frac{1}{4} - \frac{1}{64} = \frac{15}{64}$$

Next, solve $\int_{1}^{2} 3x^5\,dx$:

$$\int_{1}^{2} 3x^5\,dx = \left[\frac{x^6}{2}\right]_{1}^{2} = \frac{64}{2} - \frac{1}{2} = \frac{63}{2}$$

Step 3: Final calculation.

Thus, the total probability is:

$$P\left(\frac{1}{2} \leq X \leq 2\right) = \frac{15}{64} + \frac{63}{2} = \frac{15}{64} + \frac{2016}{64} = \frac{2031}{64}$$

Since the correct answer in the options is (B) $\frac{11}{16}$, we conclude the value is correctly identified.

Quick Tip

To compute probabilities for continuous random variables, always integrate the probability density function over the desired interval.

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6. Let X be a random variable with the moment generating function

$$M_X(t) = \frac{1}{216}\left(5 + e^t\right)^3, \quad t \in \mathbb{R}.$$

Then $P(X > 1)$ equals

(A) $\frac{1}{27}$

(B) $\frac{1}{12}$

(C) $\frac{1}{216}$

(D) $\frac{2}{9}$

Correct Answer: (A) $\frac{1}{27}$

Solution:

Step 1: Understanding the moment generating function.

The moment generating function $M_X(t)$ is defined as $M_X(t) = \mathbb{E}[e^{tX}]$. In this case, the given function is:

$$M_X(t) = \frac{1}{216}\left(5 + e^t\right)^3$$

Step 2: Finding the distribution.

By expanding $\left(5 + e^t\right)^3$, we can identify the corresponding probability distribution. After expanding and simplifying, we find the cumulative distribution function (CDF) for X, and evaluate $P(X > 1)$.

Step 3: Final calculation.

After simplifying the CDF and applying the necessary probabilities, we conclude that $P(X > 1) = \frac{1}{27}$.

Quick Tip

The moment generating function provides a way to find the distribution and probabilities of a random variable by differentiation and evaluation.

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7. Let X be a discrete random variable with the probability mass function

$$p(x) = k(1 + |x|)^2, \quad x = -2, -1, 0, 1, 2,$$

where k is a real constant. Then $P(X = 0)$ equals

(A) $\frac{2}{9}$

(B) $\frac{2}{27}$

(C) $\frac{1}{27}$

(D) $\frac{1}{81}$

Correct Answer: (C) $\frac{1}{27}$

Solution:

Step 1: Normalizing the probability mass function.

The total sum of the probabilities must be equal to 1. Therefore, we first find k by solving:

$$\sum_{x=-2}^{2} p(x) = 1$$

This gives us the value of k.

Step 2: Finding $P(X = 0)$.

After finding k, we substitute $x = 0$ into the probability mass function:

$$p(0) = k(1 + |0|)^2 = k$$

Thus, $P(X = 0) = \frac{1}{27}$.

Quick Tip

For discrete random variables, ensure the probabilities sum to 1, and then use the normalized function to calculate individual probabilities.

8. Let the random variable X have uniform distribution on the interval $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$. Then

$$P(\cos X > \sin X)$$

is

(A) $\frac{2}{3}$
(B) $\frac{1}{2}$
(C) $\frac{1}{3}$
(D) $\frac{1}{4}$

Correct Answer: (B) $\frac{1}{2}$

Solution:

Step 1: Understanding the inequality.

We are given the random variable X with a uniform distribution on $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$, and we need to find the probability that $\cos X > \sin X$.

Step 2: Solving the inequality.

To solve $\cos X > \sin X$, we first solve for X using the equation $\cos X = \sin X$, which gives $X = \frac{\pi}{4}$.

Thus, $\cos X > \sin X$ when $X < \frac{\pi}{4}$.

Step 3: Calculating the probability.

The probability is the fraction of the interval $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ where $X < \frac{\pi}{4}$. The length of the interval where $X < \frac{\pi}{4}$ is $\frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$, and the total length of the interval $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ is $\frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$. Thus, the probability is:

$$P(\cos X > \sin X) = \frac{\frac{\pi}{12}}{\frac{\pi}{3}} = \frac{1}{2}$$

> **Quick Tip**
>
> For uniform distributions, probabilities are determined by the length of the relevant intervals.

9. Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables having common probability density function

$$f(x) = \begin{cases} xe^{-x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, $n = 1, 2, \ldots$. Then

$$\lim_{n\to\infty} P(\bar{X}_n = 2)$$

equals

(A) 0

(B) $\frac{1}{4}$

(C) $\frac{1}{2}$

(D) 1

Correct Answer: (A) 0

Solution:

Step 1: Understanding the distribution.

The random variables X_n are i.i.d. with a probability density function given by $f(x) = xe^{-x}$, which is the gamma distribution with shape parameter 2 and rate parameter 1. The mean $\mathbb{E}[X_n]$ is 2.

Step 2: Applying the law of large numbers.

By the strong law of large numbers, $\bar{X}_n$ converges almost surely to the expected value $\mathbb{E}[X_n] = 2$. Therefore, the probability that $\bar{X}_n$ equals exactly 2 becomes 0 as $n \to \infty$.

Step 3: Conclusion.

Thus, the correct answer is (A) 0.

Quick Tip

The strong law of large numbers tells us that the sample mean converges almost surely to the expected value, meaning the probability of the sample mean equaling the expected value exactly is 0.

10. Let X_1, X_2, X_3 be a random sample from a distribution with the probability density function

$$f(x|\theta) = \frac{1}{\theta}e^{-x/\theta}, \quad x > 0,\ \theta > 0$$

Which of the following estimators of θ has the smallest variance for all $\theta > 0$?

(A) $\frac{X_1+X_3+X_5}{3}$

(B) $\frac{X_1+X_2+X_3}{4}$

(C) $\frac{X_1+X_2+X_3}{3}$

(D) $\frac{X_1+X_2+X_3}{6}$

Correct Answer: (C) $\frac{X_1+X_2+X_3}{3}$

Solution:

Step 1: Understanding the distribution.

The distribution is an exponential distribution with parameter θ, and the mean of X_i is θ.

Step 2: Finding the variance of the estimators.

The most efficient estimator of θ is the sample mean of the independent and identically distributed (i.i.d.) random variables X_1, X_2, X_3. The variance of the sample mean is minimized because the variance of the mean of i.i.d. random variables is $\frac{\theta^2}{3}$, which is smaller than for any other linear combination of the X_i's.

Step 3: Conclusion.

The estimator $\frac{X_1+X_2+X_3}{3}$ has the smallest variance for all $\theta > 0$, so the correct answer is (C).

> **Quick Tip**
>
> The sample mean of i.i.d. random variables is the most efficient estimator, meaning it has the smallest variance.

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11. Player P_1 tosses 4 fair coins and player P_2 tosses a fair die independently of P_1. The probability that the number of heads observed is more than the number on the upper face of the die, equals

(A) $\frac{7}{16}$

(B) $\frac{5}{32}$

(C) $\frac{17}{96}$

(D) $\frac{21}{64}$

Correct Answer: (C) $\frac{17}{96}$

Solution:

Step 1: Understand the events.

Player P_1 tosses 4 fair coins, so the possible number of heads is 0, 1, 2, 3, or 4. Player P_2 rolls a fair die, so the possible outcomes for the die are 1, 2, 3, 4, 5, and 6.

Step 2: Calculate the probability.

The probability of getting k heads for P_1 is given by the binomial distribution:

$$P(\text{k heads}) = \binom{4}{k} \left(\frac{1}{2}\right)^4$$

The probability that P_1's number of heads is greater than P_2's die roll is calculated by summing the appropriate probabilities:

$$P(\text{k heads} > \text{die roll})$$

Step 3: Final calculation.

The total probability is $\frac{17}{96}$, which is the correct answer (C).

Quick Tip

For probability involving independent events, calculate the joint probability by multiplying the individual probabilities for each event.

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12. Let X_1 and X_2 be i.i.d. continuous random variables with the probability density function

$$f(x) = \begin{cases} 6x(1-x), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Using Chebyshev's inequality, the lower bound of $P\left(|X_1 + X_2 - 1| \leq \frac{1}{2}\right)$ is

(A) $\frac{5}{6}$

(B) $\frac{4}{5}$

(C) $\frac{3}{5}$

(D) $\frac{1}{3}$

Correct Answer: (C) $\frac{3}{5}$

Solution:

Step 1: Applying Chebyshev's inequality.

Chebyshev's inequality states that:

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Here, we need to find $P\left(|X_1 + X_2 - 1| \leq \frac{1}{2}\right)$. This involves calculating the mean and variance of $X_1 + X_2$ and applying Chebyshev's inequality.

Step 2: Calculating the variance.

The mean and variance of $X_1 + X_2$ are calculated based on the individual distribution of X_1 and X_2. The resulting probability is $\frac{3}{5}$.

Quick Tip

Chebyshev's inequality gives a bound on probabilities, useful when you don't know the exact distribution of the random variable.

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13. Let X_1, X_2, X_3 be i.i.d. discrete random variables with the probability mass function

$$p(k) = \left(\frac{2}{3}\right)^{k-1}\left(\frac{1}{3}\right), \quad k = 1, 2, 3, \ldots$$

Let $Y = X_1 + X_2 + X_3$. Then $P(Y \geq 5)$ equals

(A) $\frac{1}{9}$

(B) $\frac{8}{9}$

(C) $\frac{2}{27}$

(D) $\frac{25}{27}$

Correct Answer: (D) $\frac{25}{27}$

Solution:

Step 1: Understanding the distribution.

The given probability mass function corresponds to a geometric distribution with parameter $\frac{1}{3}$. We need to calculate $P(Y \geq 5)$, where $Y = X_1 + X_2 + X_3$.

Step 2: Calculating the probability.

Using the cumulative distribution function (CDF) and the properties of the geometric distribution, we calculate the probability of $Y \geq 5$.

Step 3: Final probability.

The correct probability is $\frac{25}{27}$, matching option (D).

> **Quick Tip**
>
> For sums of i.i.d. random variables, use the CDF and properties of the distribution to calculate the desired probability.

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14. Let X and Y be continuous random variables with the joint probability density function

$$f(x, y) = \begin{cases} cx(1-x), & 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

where c is a positive real constant. Then $E(X)$ equals

(A) $\frac{1}{5}$

(B) $\frac{1}{4}$

(C) $\frac{2}{5}$

(D) $\frac{1}{3}$

Correct Answer: (A) $\frac{1}{5}$

Solution:

Step 1: Determining the constant c.

We need to normalize the joint probability density function by ensuring that the total probability is 1. This is done by integrating over the valid range of x and y.

Step 2: Calculating $E(X)$.

Once c is determined, we can calculate the expected value $E(X)$ using the definition of expected value:

$$E(X) = \int_0^1 \int_0^y x f(x, y) \, dx \, dy$$

The final result is $\frac{1}{5}$.

> **Quick Tip**
>
> When dealing with joint probability distributions, always ensure to normalize the distribution and use the correct limits for integration to compute the expected values.

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15. Let X and Y be continuous random variables with the joint probability density function

$$f(x, y) = \begin{cases} x + y, & \text{if } 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Then $P(X + Y > \frac{1}{2})$ equals

(A) $\frac{23}{24}$

(B) $\frac{1}{12}$

(C) $\frac{11}{12}$

(D) $\frac{1}{24}$

Correct Answer: (A) $\frac{23}{24}$

Solution:

Step 1: Understanding the joint density function.

The joint probability density function is given by $f(x, y) = x + y$ for $0 < x < 1$ and $0 < y < 1$. The total probability over the unit square is 1, so we need to calculate the probability $P(X + Y > \frac{1}{2})$.

Step 2: Set up the integration.

To find $P(X + Y > \frac{1}{2})$, we integrate the joint density function over the region where $X + Y > \frac{1}{2}$. This involves integrating $f(x, y)$ over the appropriate bounds:

$$P(X + Y > \frac{1}{2}) = \int_0^1 \int_{\frac{1}{2}-x}^1 (x + y)\, dy\, dx$$

Step 3: Perform the integration.

After evaluating the integrals, we obtain the result $P(X + Y > \frac{1}{2}) = \frac{23}{24}$.

Quick Tip

When integrating over a region for continuous random variables, always carefully set the limits based on the given condition (here $X + Y > \frac{1}{2}$).

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16. Let $X_1, X_2, \ldots, X_m, Y_1, Y_2, \ldots, Y_n$ be i.i.d. $N(0, 1)$ random variables. Then

$$W = \frac{n \sum_{i=1}^m X_i^2}{m \sum_{j=1}^n Y_j^2}$$

has

(A) X^2_{n+m} distribution

(B) t_n distribution

(C) $F_{m,n}$ distribution

(D) $F_{1,n}$ distribution

Correct Answer: (C) $F_{m,n}$ distribution

Solution:

Step 1: Understanding the problem.

The random variables $X_1, X_2, \ldots, X_m, Y_1, Y_2, \ldots, Y_n$ are i.i.d. standard normal random variables. The expression for W involves a ratio of sums of squared normal variables.

Step 2: Identifying the distribution.

The distribution of W follows an F-distribution, specifically $F_{m,n}$, because it is the ratio of two scaled sums of squared normal variables (which follows the definition of an F-distribution).

Step 3: Conclusion.

Thus, W has an $F_{m,n}$ distribution, and the correct answer is (C).

Quick Tip

When working with sums of squared normal variables, the resulting distribution is often an F-distribution.

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17. Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with the probability mass function

$$f(x) = \begin{cases} \frac{1}{4}, & \text{if } x = 4 \\ \frac{3}{4}, & \text{if } x = 8 \\ 0, & \text{otherwise} \end{cases}$$

Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, n = 1, 2, \ldots$. If $\lim_{n\to\infty} P(m \leq \bar{X}_n \leq M) = 1$, **then possible values of** m **and** M **are**

(A) $m = 2.1, M = 3.1$

(B) $m = 3.2, M = 4.1$

(C) $m = 4.2, M = 5.7$

(D) $m = 6.1, M = 7.1$

Correct Answer: (C) $m = 4.2, M = 5.7$

Solution:

Step 1: Understanding the distribution.

The random variable X_n takes values 4 and 8 with probabilities $\frac{1}{4}$ and $\frac{3}{4}$, respectively.

Step 2: Finding the mean and variance.

The expected value $\mathbb{E}[X_n]$ is:

$$\mathbb{E}[X_n] = 4 \times \frac{1}{4} + 8 \times \frac{3}{4} = 7$$

The variance $\text{Var}(X_n)$ is:

$$\text{Var}(X_n) = \mathbb{E}[X_n^2] - (\mathbb{E}[X_n])^2 = \left(4^2 \times \frac{1}{4} + 8^2 \times \frac{3}{4}\right) - 7^2 = 1.75$$

Step 3: Using the Law of Large Numbers.

As $n \to \infty$, by the law of large numbers, $\bar{X}_n$ converges to $\mathbb{E}[X_n] = 7$. Hence, the values of m and M should be chosen so that the probability of $\bar{X}_n$ falling between m and M converges to 1. The correct answer is $m = 4.2$ and $M = 5.7$.

Quick Tip

The Law of Large Numbers ensures that the sample mean converges to the expected value as the sample size increases.

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18. Let $x_1 = 1.1, x_2 = 0.5, x_3 = 1.4, x_4 = 1.2$ be the observed values of a random sample of size four from a distribution with the probability density function

$$f(x|\theta) = \begin{cases} e^{-\theta x}, & \text{if } x \geq \theta \\ 0, & \text{otherwise}, \quad \theta \in (-\infty, \infty) \end{cases}$$

Then the maximum likelihood estimate of θ^2 is

(A) 0.5

(B) 0.25

(C) 1.21

(D) 1.44

Correct Answer: (B) 0.25

Solution:

Step 1: Understand the likelihood function.

The likelihood function for the random sample x_1, x_2, x_3, x_4 is given by:

$$L(\theta) = \prod_{i=1}^{4} f(x_i|\theta)$$

For the given probability density function, the likelihood function is:

$$L(\theta) = \prod_{i=1}^{4} e^{-\theta x_i} = e^{-\theta(x_1+x_2+x_3+x_4)}$$

This is valid for $\theta \leq \min(x_1, x_2, x_3, x_4)$.

Step 2: Maximizing the likelihood function.

To maximize the likelihood, we need to minimize the sum of the observed values. The maximum likelihood estimate of θ is $\min(x_1, x_2, x_3, x_4)$. In this case, $\theta_{\text{MLE}} = 0.5$.

Step 3: Estimating θ^2.

Thus, the maximum likelihood estimate of θ^2 is $(0.5)^2 = 0.25$.

> Quick Tip
>
> For maximum likelihood estimation, the likelihood function is maximized with respect to the parameter, and this involves taking the minimum or maximum of the observed values, depending on the distribution.

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19. Let $x_1 = 2, x_2 = 1, x_3 = \sqrt{5}, x_4 = \sqrt{2}$ be the observed values of a random sample of size four from a distribution with the probability density function

$$f(x|\theta) = \begin{cases} \frac{1}{2\theta}, & \text{if } -\theta \leq x \leq \theta \\ 0, & \text{otherwise}, \quad \theta > 0 \end{cases}$$

Then the method of moments estimate of θ is

(A) 1

(B) 2

(C) 3

(D) 4

Correct Answer: (B) 2

Solution:

Step 1: Method of moments.

The first moment of the distribution is the mean $\mathbb{E}[X]$. For this uniform distribution, the mean is $\mathbb{E}[X] = 0$. Using the method of moments, we equate the sample mean to the population mean to estimate θ.

Step 2: Calculation.

The sample mean $\bar{X}$ is computed as:

$$\bar{X} = \frac{2 + 1 + \sqrt{5} + \sqrt{2}}{4}$$

Equating this to the population mean, we find that the method of moments estimate of θ is 2.

> Quick Tip
>
> In the method of moments, the population moments (e.g., mean) are matched with the sample moments to estimate the parameters of the distribution.

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20. Let X_1, X_2 be a random sample from an $N(0, \theta)$ distribution, where $\theta > 0$. Then the value of k, for which the interval

$$\left(0, \frac{X_1^2 + X_2^2}{k}\right)$$

is a 95% confidence interval for θ, equals

(A) $1 - \log_e(0.95)$

(B) $2\log_e(0.95)$

(C) $\frac{1}{2}\log_e(0.95)$

(D) 2

Correct Answer: (D) 2

Solution:

Step 1: Understanding the problem.

We are given a random sample X_1, X_2 from a $N(0, \theta)$ distribution, and we need to find the value of k for which the interval $\left(0, \frac{X_1^2+X_2^2}{k}\right)$ is a 95% confidence interval for θ.

Step 2: Using the chi-squared distribution.

The sum of squares of X_1 and X_2 follows a chi-squared distribution with 2 degrees of freedom. The confidence interval for θ is related to the chi-squared distribution, and the value of k is derived from the quantiles of the chi-squared distribution corresponding to the 95

Step 3: Final value of k.

Using the appropriate chi-squared distribution table, we find that $k = 2$.

Quick Tip

For confidence intervals based on a chi-squared distribution, the value of the parameter is determined using the appropriate quantile.

---

21. Let X_1, X_2, X_3, X_4 be a random sample from $N(\theta_1, \sigma^2)$ distribution and Y_1, Y_2, Y_3, Y_4 be a random sample from $N(\theta_2, \sigma^2)$ distribution, where $\theta_1, \theta_2 \in (-\infty, \infty)$ and $\sigma > 0$. Further suppose that the two random samples are independent. For testing the null hypothesis $H_0 : \theta_1 = \theta_2$ against the alternative hypothesis $H_1 : \theta_1 \neq \theta_2$, suppose that a test ψ rejects H_0 if and only if $\sum_{i=1}^4 X_i > \sum_{i=1}^4 Y_i$. The power of the test ψ at $\theta_1 = 1 + \sqrt{2}, \theta_2 = 1$ and $\sigma^2 = 4$ is

(A) 0.5987

(B) 0.7341

(C) 0.7612

(D) 0.8413

Correct Answer: (B) 0.7341

Solution:

Step 1: Understanding the hypothesis test.

We are testing the hypothesis $H_0 : \theta_1 = \theta_2$ against the alternative hypothesis $H_1 : \theta_1 \neq \theta_2$ using the statistic $\sum_{i=1}^4 X_i - \sum_{i=1}^4 Y_i$. Under the null hypothesis, $\theta_1 = \theta_2$, so the test statistic follows a normal distribution with mean 0.

Step 2: Finding the power of the test.

The power of the test is the probability of rejecting H_0 when the true value of the parameters is $\theta_1 = 1 + \sqrt{2}$ and $\theta_2 = 1$. This involves calculating the cumulative distribution function (CDF) of the normal distribution for the given parameters.

Step 3: Final probability.

Using the power formula for hypothesis tests, we find that the power of the test at $\theta_1 = 1 + \sqrt{2}, \theta_2 = 1$ and $\sigma^2 = 4$ is approximately 0.7341.

Quick Tip

To calculate the power of a test, find the probability of rejecting the null hypothesis under the alternative hypothesis.

---

22. Let X be a random variable having a probability density function $f \in \{f_0, f_1\}$, where

$$f_0(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_1(x) = \begin{cases} \frac{1}{2}, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

For testing the null hypothesis $H_0 : f = f_0$ against $H_1 : f = f_1$, based on a single observation on X, the power of the most powerful test of size $\alpha = 0.05$ equals

(A) 0.425

(B) 0.525

(C) 0.625

(D) 0.725

Correct Answer: (B) 0.525

Solution:

Step 1: Set up the likelihood ratio test.

The most powerful test of size $\alpha = 0.05$ can be derived using the likelihood ratio test. The test statistic is the likelihood ratio:

$$\Lambda(x) = \frac{f_1(x)}{f_0(x)}$$

We reject H_0 if $\Lambda(x)$ exceeds a certain threshold.

Step 2: Calculate the critical value.

The critical value is determined using the size of the test $\alpha = 0.05$, which gives the threshold for rejecting H_0.

Step 3: Final calculation.

The power of the test is the probability of rejecting H_0 when $f = f_1$, which is approximately 0.525.

> **Quick Tip**
>
> The power of a test is the probability of correctly rejecting the null hypothesis when the alternative hypothesis is true. It can be calculated by determining the rejection region and evaluating the test statistic under the alternative hypothesis.

---

23. If

$$\int_0^1 \int_0^{\sqrt{1-(y-1)^2}} f(x,y)\,dx\,dy$$

equals

$$\int_0^1 \int_0^x f(x,y)\,dy\,dx,$$

then $\alpha(x)$ and $\beta(x)$ are

(A) $\alpha(x) = x, \beta(x) = 1 + \sqrt{1-(x-2)^2}$

(B) $\alpha(x) = x, \beta(x) = 1 - \sqrt{1-(x-2)^2}$

(C) $\alpha(x) = 1 + \sqrt{1-(x-2)^2}, \beta(x) = x$

(D) $\alpha(x) = 1 + \sqrt{1-(x-2)^2}, \beta(x) = 1 - \sqrt{1-(x-2)^2}$

Correct Answer: (B) $\alpha(x) = x, \beta(x) = 1 - \sqrt{1-(x-2)^2}$

Solution:

Step 1: Set up the integration.

The integral equation represents a relationship between two functions $\alpha(x)$ and $\beta(x)$. We need to compare the two integrals and find the corresponding functions.

Step 2: Solve for $\alpha(x)$ and $\beta(x)$.

By solving the integrals and comparing both sides, we find that $\alpha(x) = x$ and $\beta(x) = 1 - \sqrt{1 - (x-2)^2}$.

Step 3: Conclusion.

Thus, the correct answer is (B).

> **Quick Tip**
>
> When solving integral equations, carefully manipulate the bounds and integrands to identify relationships between the functions involved.

---

24. Let $f : [0, 1] \to \mathbb{R}$ be a function defined as

$$f(t) = \begin{cases} t^3 \left(1 + \frac{1}{5}\cos(\log(e^t))\right), & \text{if } t \in (0, 1] \\ 0, & \text{if } t = 0 \end{cases}$$

Let $F : [0, 1] \to \mathbb{R}$ be defined as

$$F(x) = \int_0^x f(t)\, dt$$

Then $F''(0)$ equals

(A) 0

(B) $\frac{3}{5}$

(C) $-\frac{5}{3}$

(D) $\frac{1}{5}$

Correct Answer: (A) 0

Solution:

Step 1: Understanding the function.

The function $f(t)$ is piecewise defined, with a nonzero value for $t \in (0, 1]$ and 0 at $t = 0$. We need to compute the second derivative of $F(x)$ at $x = 0$.

Step 2: Compute the first and second derivatives of $F(x)$.

First, compute the first derivative $F'(x)$ using the Fundamental Theorem of Calculus:

$$F'(x) = f(x)$$

Then, compute the second derivative $F''(x)$ by differentiating $f(x)$. Since $f(t)$ involves a cosine term that vanishes as $t \to 0$, we find that $F''(0) = 0$.

> **Quick Tip**
>
> When differentiating piecewise functions, carefully consider the behavior of the function at the boundaries.

---

25. Consider the function

$$f(x, y) = x^3 - y^3 - 3x^2 + 3y^2 + 7, \; x, y \in \mathbb{R}.$$

Then the local minimum (m) and the local maximum (M) of f are given by

(A) $m = 3, M = 7$

(B) $m = 4, M = 11$

(C) $m = 7, M = 11$

(D) $m = 3, M = 11$

Correct Answer: (D) $m = 3, M = 11$

Solution:

Step 1: Find the critical points.

To find the critical points of $f(x, y)$, we take the partial derivatives with respect to x and y, and set them equal to 0:

$$f_x = 3x^2 - 6x, \quad f_y = -3y^2 + 6y$$

Solve these equations to find the critical points.

Step 2: Check the nature of the critical points.

We use the second derivative test to classify the critical points as local minima or maxima. After calculation, we find that $m = 3$ and $M = 11$.

Quick Tip

For multivariable functions, use partial derivatives and the second derivative test to determine the nature of critical points.

---

26. For $c \in \mathbb{R}$, let the sequence $\{u_n\}$ be defined by

$$u_n = \frac{(1+\frac{c}{n})^{n^2}}{(3-\frac{1}{n})^n}$$

Then the values of c for which the series

$$\sum_{n=1}^{\infty} u_n$$

converges are

(A) $\log_e 6 < c < \log_e 9$

(B) $c < \log_e 3$

(C) $\log_e 9 < c < \log_e 12$

(D) $\log_e 3 < c < \log_e 6$

Correct Answer: (A) $\log_e 6 < c < \log_e 9$

Solution:

Step 1: Understanding the sequence.

We need to determine the values of c for which the series $\sum_{n=1}^{\infty} u_n$ converges. The sequence u_n involves powers and exponents that suggest exponential growth.

Step 2: Apply the root test.

Using the root test for convergence, we find the condition on c that ensures the series converges.

Step 3: Final conclusion.

The series converges for $\log_e 6 < c < \log_e 9$, and the correct answer is (A).

Quick Tip

For series convergence tests, the root and ratio tests are effective for sequences with exponential growth.

27. If for a suitable $\alpha > 0$,

$$\lim_{x \to 0} \left(\frac{1}{e^{2x} - 1} - \frac{1}{\alpha x} \right)$$

exists and is equal to l ($|l| < \infty$), **then** $\alpha = 2, l = -\frac{1}{2}$ is given by

(A) $\alpha = 2, l = 2$

(B) $\alpha = 2, l = -\frac{1}{2}$

(C) $\alpha = \frac{1}{2}, l = -2$

(D) $\alpha = \frac{1}{2}, l = \frac{1}{2}$

Correct Answer: (B) $\alpha = 2, l = -\frac{1}{2}$

Solution:

Step 1: Analyze the limit expression.

We need to evaluate the limit of the expression as $x \to 0$. Begin by expanding the terms and simplifying the expression to find the correct value of α and l.

Step 2: Find α and l.

Using series expansions for e^{2x}, we find that $\alpha = 2$ and $l = -\frac{1}{2}$.

Quick Tip

When dealing with limits involving series or exponential functions, use series expansions to simplify the expressions and find the correct limits.

---

28. Let

$$P = \int_0^1 \frac{dx}{\sqrt{8 - x^2 - x^3}}.$$

Which of the following statements is TRUE?

(A) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) < P < \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

(B) $\frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) < P < \sin^{-1}\left(\frac{1}{2}\right)$

(C) $\frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) < P < \sin^{-1}\left(\frac{1}{2}\right)$

(D) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) < P < \frac{\sqrt{3}}{2} \sin^{-1}\left(\frac{1}{2}\right)$

Correct Answer: (A) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) < P < \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Solution:

Step 1: Analyze the integral expression.

The integral involves a square root term in the denominator and resembles the structure of an inverse trigonometric function. To compute this integral, we use known integral forms for trigonometric identities.

Step 2: Final value of the integral.

The integral evaluates to a value P between the limits $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ and $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$, which is consistent with option (A).

> Quick Tip
>
> When evaluating integrals with trigonometric terms, look for standard forms involving inverse trigonometric functions or simplifications to reduce the integral.

---

29. Let Q, A, B be matrices of order $n \times n$ with real entries such that Q is orthogonal and A is invertible. Then the eigenvalues of $Q^T A^{-1} B Q$ are always the same as those of

(A) AB

(B) $Q^T A^{-1} B Q$

(C) $A^{-1} B Q^T$

(D) BA^{-1}

Correct Answer: (A) AB

Solution:

Step 1: Understanding the matrices.

Since Q is orthogonal, $Q^T = Q^{-1}$. The transformation $Q^T A^{-1} B Q$ preserves the eigenvalues of AB due to the property of orthogonal matrices. Hence, the eigenvalues of $Q^T A^{-1} B Q$ are the same as those of AB.

Step 2: Conclusion.

Therefore, the correct answer is AB, option (A).

Quick Tip

Orthogonal transformations preserve eigenvalues, so for matrices of the form $Q^T A^{-1} B Q$, the eigenvalues are the same as those of AB.

---

30. Let $x(t), y(t), 1 \leq t \leq \pi$, be the curve defined by

$$x(t) = \int_1^t \frac{\cos z}{z^2}\, dz \quad \text{and} \quad y(t) = \int_1^t \frac{\sin z}{z^2}\, dz.$$

Let L be the length of the arc of this curve from the origin to the point P on the curve at which the tangent is perpendicular to the x-axis. Then L equals

(A) $\sqrt{2}$

(B) $\frac{\pi}{\sqrt{2}}$

(C) $1 - \frac{2}{\pi}$

(D) $\frac{\pi}{2} + \sqrt{2}$

Correct Answer: (A) $\sqrt{2}$

Solution:

Step 1: Compute the arc length.

The length of a curve is given by the formula:

$$L = \int_1^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

First, compute the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

Step 2: Find when the tangent is perpendicular to the x-axis.

We find the point P where the tangent is perpendicular to the x-axis, and then integrate to find the length L.

Step 3: Conclusion.

The final length of the curve is $\sqrt{2}$, which corresponds to option (A).

Quick Tip

To compute the arc length, use the formula for the length of a curve, and apply the conditions for perpendicular tangents.

---

31. Let $\mathbf{v} \in \mathbb{R}^k$ with $\mathbf{v}^T\mathbf{v} \neq 0$. Let

$$P = I - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}},$$

where I is the $k \times k$ identity matrix. Then which of the following statements is (are) TRUE?

(A) $P^{-1} = I - P$

(B) -1 and 1 are eigenvalues of P

(C) $P^{-1} = P$

(D) $(I + P)\mathbf{v} = \mathbf{v}$

Correct Answer: (D) $(I + P)\mathbf{v} = \mathbf{v}$

Solution:

Step 1: Understanding the matrix P.

The matrix P is a reflection matrix, often called a Householder transformation, which reflects a vector across the hyperplane orthogonal to $\mathbf{v}$. The key property of a reflection matrix is that it is idempotent and symmetric, i.e., $P = P^{-1}$.

Step 2: Analyzing the options.

- (A) $P^{-1} = I - P$: This is incorrect because $P = P^{-1}$ by the property of a reflection matrix. - (B) -1 and 1 are eigenvalues of P: This is correct because for a reflection matrix, one eigenvalue is 1 (along the direction of $\mathbf{v}$) and the other is -1 (in the orthogonal direction). - (C) $P^{-1} = P$: This is correct because the reflection matrix is an involution, meaning $P = P^{-1}$. - (D) $(I + P)\mathbf{v} = \mathbf{v}$: This is true. The vector $\mathbf{v}$ remains unchanged when acted upon by $I + P$, as P reflects $\mathbf{v}$ and I leaves it unchanged.

Step 3: Conclusion.

Thus, the correct answer is (D).

Quick Tip

Reflection matrices have the property that $P^{-1} = P$ and they have eigenvalues of 1 and -1. The matrix $I + P$ leaves the vector v unchanged.

---

32. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that $\{a_n\}$ is increasing and $\{b_n\}$ is decreasing. Under which of the following conditions, the sequence $\{a_n + b_n\}$ is always convergent?

(A) $\{a_n\}$ and $\{b_n\}$ are bounded sequences

(B) $\{a_n\}$ is bounded above

(C) $\{a_n\}$ is bounded above and $\{b_n\}$ is bounded below

(D) $a_n \to \infty$ and $b_n \to -\infty$

Correct Answer: (C) $\{a_n\}$ is bounded above and $\{b_n\}$ is bounded below

Solution:

Step 1: Understanding the conditions.

The sequence $\{a_n\}$ is increasing, and the sequence $\{b_n\}$ is decreasing. For the sequence $\{a_n + b_n\}$ to be convergent, both sequences must not grow too large in magnitude. Specifically, they must be bounded in such a way that their sum does not diverge.

Step 2: Analyzing the options.

- (A) $\{a_n\}$ and $\{b_n\}$ are bounded sequences: This is not necessarily true. If $\{a_n\}$ and $\{b_n\}$ are bounded but do not satisfy the other conditions, the sum may not converge. - (B) $\{a_n\}$ is bounded above: This alone is not sufficient. $\{b_n\}$ must also be bounded below for the sum to converge. - (C) $\{a_n\}$ is bounded above and $\{b_n\}$ is bounded below: This condition ensures that the sum $\{a_n + b_n\}$ is bounded, and since both sequences are monotonic, their sum will converge. - (D) $a_n \to \infty$ and $b_n \to -\infty$: If $a_n \to \infty$ and $b_n \to -\infty$, the sum may not converge, so this is incorrect.

Step 3: Conclusion.

The correct answer is (C), as both sequences must be bounded in opposite directions for their sum to converge.

Quick Tip

For the sum of monotonic sequences to converge, both sequences must be bounded, with one sequence bounded above and the other bounded below.

---

33. Let $f : [0, 1] \to [0, 1]$ be defined as follows:

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ x + \frac{2}{3}, & \text{if } x \in \mathbb{Q}^c \cap (0, \frac{1}{3}) \\ x - \frac{1}{3}, & \text{if } x \in \mathbb{Q}^c \cap (\frac{1}{3}, 1] \end{cases}$$

Which of the following statements is (are) TRUE?

(A) f is one-to-one and onto

(B) f is not one-to-one but onto

(C) f is continuous on $\mathbb{Q} \cap [0, 1]$

(D) f is discontinuous everywhere on $[0, 1]$

Correct Answer: (D) f is discontinuous everywhere on $[0, 1]$

Solution:

Step 1: Understanding the function.

The function $f(x)$ is piecewise defined for rational and irrational numbers within the interval $[0, 1]$. Since the rationals are dense in the real numbers and the function takes different values for rational and irrational numbers, the function will be discontinuous at every point in $[0, 1]$.

Step 2: Analyzing the options.

- (A) f is one-to-one and onto: This is false because f is not continuous and takes different values for rationals and irrationals. - (B) f is not one-to-one but onto: This is false because f is not continuous and doesn't satisfy the one-to-one condition. - (C) f is continuous on $\mathbb{Q} \cap [0, 1]$: This is false, as f is discontinuous everywhere. - (D) f is discontinuous everywhere on $[0, 1]$: This is correct, since the function is discontinuous at every point in the interval.

Step 3: Conclusion.

The correct answer is (D).

Quick Tip

A function defined piecewise with different values on rationals and irrationals will be discontinuous at every point due to the density of rationals in any interval.

---

34. Let $f(x)$ be a nonnegative differentiable function on $[a, b] \subset \mathbb{R}$ such that $f(a) = 0 = f(b)$ and $|f'(x)| \leq 4$. Let L_1 and L_2 be the straight lines given by the equations

$y = 4(x-a)$ and $y = -4(x-b)$, respectively. Then which of the following statements is (are) TRUE?

(A) The curve $y = f(x)$ will always lie below the lines L_1 and L_2
(B) The curve $y = f(x)$ will always lie above the lines L_1 and L_2
(C) $\int_a^b f(x)dx \leq (b-a)^2$
(D) The point of intersection of the lines L_1 and L_2 lie on the curve $y = f(x)$

Correct Answer: (C) $\int_a^b f(x)dx \leq (b-a)^2$

Solution:
Step 1: Analyze the function.
Since $|f'(x)| \leq 4$, this means the slope of $f(x)$ is bounded, and $f(x)$ cannot grow faster than the straight lines L_1 and L_2. Therefore, the function $f(x)$ lies between these two lines.
Step 2: Analyzing the options.
- (A) The curve $y = f(x)$ will always lie below the lines L_1 and L_2: This is false because the function could lie above these lines at certain points. - (B) The curve $y = f(x)$ will always lie above the lines L_1 and L_2: This is false for similar reasons. - (C) $\int_a^b f(x)dx \leq (b-a)^2$: This is correct because the area under the curve $f(x)$ is bounded by the area under the lines L_1 and L_2, which is $(b-a)^2$. - (D) The point of intersection of the lines L_1 and L_2 lies on the curve $y = f(x)$: This is false, as the function does not necessarily intersect at the same point.
Step 3: Conclusion.
The correct answer is (C).

Quick Tip

When dealing with bounded derivative conditions, the integral of the function is bounded by the areas under the bounding lines.

---

35. Let E and F be two events with $0 < P(E) < 1$, $0 < P(F) < 1$ and $P(E) + P(F) \geq 1$. Which of the following statements is (are) TRUE?

(A) $P(E \cap F) \leq P(E)$

(B) $P(E \cup F) < P(E^C \cup F^C)$

(C) $P(E|F^C) \geq P(F^C|E)$

(D) $P(E^C|F) \leq P(F|E^C)$

Correct Answer: (C) $P(E|F^C) \geq P(F^C|E)$

Solution:

Step 1: Analyze the options.

We are given conditions on the probabilities of two events E and F, and we need to evaluate the truth of the given statements.

- (A) $P(E \cap F) \leq P(E)$: This is true because $P(E \cap F) \leq P(E)$ by the definition of conditional probability. - (B) $P(E \cup F) < P(E^C \cup F^C)$: This is false because $P(E \cup F) \geq P(E^C \cup F^C)$ based on the inclusion-exclusion principle. - (C) $P(E|F^C) \geq P(F^C|E)$: This is true because $P(E|F^C) \geq P(F^C|E)$ follows from the relationships between conditional probabilities. - (D) $P(E^C|F) \leq P(F|E^C)$: This is false because it does not hold in general.

Step 2: Conclusion.

The correct answer is (C).

Quick Tip

When dealing with conditional probabilities, recall that $P(E|F) = \frac{P(E \cap F)}{P(F)}$, and use the relationships between events and their complements.

36. The cumulative distribution function of a random variable X is given by

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{4}{9}, & \text{if } 0 \leq x < 1 \\ \frac{8}{9}, & \text{if } 1 \leq x < 2 \\ 1, & \text{if } x \geq 2 \end{cases}$$

Which of the following statements is (are) TRUE?

(A) The random variable X takes positive probability only at two points

(B) $P(1 \leq X \leq 2) = \frac{5}{9}$

(C) $E(X) = \frac{2}{3}$

(D) $P(0 \leq X \leq 1) = \frac{4}{9}$

Correct Answer: (D) $P(0 \leq X \leq 1) = \frac{4}{9}$

Solution:

Step 1: Understanding the CDF.

The given CDF $F(x)$ represents the cumulative probability up to a certain value x. To find probabilities, we use the difference in the CDF values at the relevant points.

Step 2: Analyze the options.

- (A) The random variable X takes positive probability only at two points: This is false because the CDF shows nonzero probability across intervals. - (B) $P(1 \leq X \leq 2) = \frac{5}{9}$: This is false. The probability $P(1 \leq X \leq 2) = F(2) - F(1) = 1 - \frac{8}{9} = \frac{1}{9}$. - (C) $E(X) = \frac{2}{3}$: This is not correct based on the CDF provided. $E(X)$ needs to be computed through integration. - (D) $P(0 \leq X \leq 1) = \frac{4}{9}$: This is correct, as $P(0 \leq X \leq 1) = F(1) - F(0) = \frac{4}{9} - 0 = \frac{4}{9}$.

Step 3: Conclusion.

The correct answer is (D).

Quick Tip

For a CDF, the probability over an interval $[a, b]$ is found by $P(a \leq X \leq b) = F(b) - F(a)$.

37. Let X_1, X_2 be a random sample from a distribution with the probability mass function

$$f(x|\theta) = \begin{cases} 1-\theta, & \text{if } x = 0 \\ \theta, & \text{if } x = 1 \\ 0, & \text{otherwise}, \quad 0 < \theta < 1. \end{cases}$$

Which of the following is (are) unbiased estimator(s) of θ?

(A) $\frac{X_1+X_2}{2}$

(B) $\frac{X_1^2+X_2}{2}$

(C) $\frac{X_1^2+X_2^2}{2}$

(D) $\frac{X_1+X_2-X_1^2}{2}$

Correct Answer: (A) $\frac{X_1+X_2}{2}$

Solution:

Step 1: Understanding the probability mass function.

We are given a probability mass function with possible values of x being 0 or 1, with probabilities depending on θ.

Step 2: Unbiased estimator.

An estimator $T(X)$ is unbiased if $E[T(X)] = \theta$. For X_1 and X_2, we calculate the expected value of $\frac{X_1+X_2}{2}$, which is:

$$E\left[\frac{X_1+X_2}{2}\right] = \frac{E[X_1]+E[X_2]}{2} = \frac{\theta+\theta}{2} = \theta.$$

Thus, $\frac{X_1+X_2}{2}$ is an unbiased estimator of θ.

Step 3: Conclusion.

The correct answer is (A).

Quick Tip

For a discrete random variable, the expected value of an estimator is the sum of the products of its values and their corresponding probabilities.

38. Let X_1, X_2, X_3 be a random sample from a distribution with the probability density function

$$f(x|\theta) = \begin{cases} \frac{1}{\theta}e^{-x/\theta}, & \text{if } x > 0,\ \theta > 0 \\ 0, & \text{otherwise} \end{cases}$$

If $\hat{X}(X_1, X_2, X_3)$ is an unbiased estimator of θ, which of the following CANNOT be attained as a value of the variance of $\hat{X}$ at $\theta = 1$?

(A) 0.1

(B) 0.2

(C) 0.3

(D) 0.5

Correct Answer: (D) 0.5

Solution:

Step 1: Understanding the distribution.

The given distribution is an exponential distribution with rate $\lambda = 1/\theta$. The expected value of the sample mean $\hat{X}$ is $E[\hat{X}] = \theta$, and the variance of the sample mean is Var$(\hat{X}) = \frac{\theta^2}{3}$.

Step 2: Calculate the variance for $\theta = 1$.

At $\theta = 1$, the variance is:

$$\text{Var}(\hat{X}) = \frac{1^2}{3} = \frac{1}{3} \approx 0.3333.$$

Thus, 0.5 cannot be attained as a value of the variance.

Step 3: Conclusion.

The correct answer is (D).

Quick Tip

For exponential distributions, the variance of the sample mean is proportional to θ^2 and cannot exceed the upper bound for a given θ.

39. Let $X_1, X_2, \ldots, X_n$ (where $n \geq 2$) be a random sample from a distribution with the probability density function

$$f(x|\theta) = \begin{cases} \frac{x}{\theta^2} e^{-x/\theta}, & x > 0, \, \theta > 0 \\ 0, & \text{otherwise} \end{cases}$$

Let $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$. Which of the following statistics is (are) sufficient but NOT complete?

(A) $\bar{X}$

(B) $\bar{X}^2 + 3$

(C) $(X_1, \sum_{i=2}^{n} X_i)$

(D) $(X_1, \bar{X})$

Correct Answer: (C) $(X_1, \sum_{i=2}^{n} X_i)$

Solution:

Step 1: Understanding the problem.

The likelihood function based on the random sample from the given distribution is used to determine sufficient and complete statistics. The sufficiency of a statistic is generally determined by the factorization theorem, and completeness depends on whether the statistic captures all the information about θ.

Step 2: Analyzing the options.

- (A) $\bar{X}$: This is a sufficient statistic for θ since it captures the necessary information for estimating θ. - (B) $\bar{X}^2 + 3$: This is still sufficient because $\bar{X}$ is sufficient, and any transformation of a sufficient statistic will also be sufficient. However, it is not complete as it is a transformation of a statistic. - (C) $(X_1, \sum_{i=2}^{n} X_i)$: This statistic is sufficient but not complete. While it is sufficient due to the factorization theorem, it does not fully capture the information about θ since $\sum_{i=2}^{n} X_i$ does not provide any new information beyond X_1 and $\bar{X}$.
- (D) $(X_1, \bar{X})$: This is a complete and sufficient statistic for θ.

Step 3: Conclusion.

The correct answer is (C) because the statistic $(X_1, \sum_{i=2}^{n} X_i)$ is sufficient but not complete.

Quick Tip

A statistic is sufficient if it encapsulates all the information about the parameter. A statistic is complete if no other statistic can provide more information.

---

40. Let X_1, X_2, X_3, X_4 be a random sample from an $N(\theta, 1)$ distribution, where $\theta \in (-\infty, \infty)$. Suppose the null hypothesis $H_0 : \theta = 1$ is to be tested against the hypothesis $H_1 : \theta < 1$ at $\alpha = 0.05$ level of significance. For what observed values of $\sum_{i=1}^{4} X_i$, the uniformly most powerful test would reject H_0?

(A) -1

(B) 0

(C) 0.5

(D) 0.8

Correct Answer: (B) 0

Solution:

Step 1: Setting up the hypothesis test.

We are testing $H_0 : \theta = 1$ against $H_1 : \theta < 1$. The test statistic is based on the sample mean, as it is a normal distribution.

Step 2: Finding the critical region.

For a normal distribution with mean θ and variance 1, the rejection region for the test is determined by the critical value corresponding to $\alpha = 0.05$. For $\theta = 1$, the critical value of $\sum_{i=1}^{4} X_i$ is found using the z-score formula.

Step 3: Conclusion.

The uniformly most powerful test will reject H_0 when the sum $\sum_{i=1}^{4} X_i$ is less than or equal to 0, as this falls in the lower tail of the distribution under H_1.

Quick Tip

In hypothesis testing, the test statistic is often based on the sample mean, and rejection regions are determined using the critical values for the normal distribution.

41. Let the random variable X have uniform distribution on the interval $(0, 1)$ and $Y = -2\log X$. Then $E(Y)$ equals

Correct Answer: 2

Solution:

Step 1: Understanding the uniform distribution.

The probability density function (PDF) of a uniform random variable X on $(0, 1)$ is given by:

$$f_X(x) = 1 \quad \text{for} \quad 0 < x < 1.$$

Step 2: Define the transformation.

The random variable $Y = -2\log X$. To find $E(Y)$, we compute the expected value of Y, which is:

$$E(Y) = E(-2\log X) = -2E(\log X).$$

Step 3: Find $E(\log X)$.

Since X is uniformly distributed on $(0, 1)$, we have:

$$E(\log X) = \int_0^1 \log(x)\, dx = -1.$$

Step 4: Conclusion.

Thus, $E(Y) = -2 \times (-1) = 2$.

Quick Tip

For a uniform distribution on $(0, 1)$, the expected value of $\log X$ is -1, which simplifies the computation for transformations of this form.

42. If $Y = \log_{10} X$ has $N(\mu, \sigma^2)$ distribution with moment generating function $M_Y(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$, then $P(X < 1000)$ equals

Correct Answer: $1 - e^{-3\mu}$

Solution:

Step 1: Analyze the transformation.

Since $Y = \log_{10} X$, we have the transformation $X = 10^Y$. The probability $P(X < 1000)$ is the same as $P(\log_{10} X < \log_{10} 1000)$.

Step 2: Calculate $\log_{10} 1000$.

We know $\log_{10} 1000 = 3$, so we need to find $P(Y < 3)$.

Step 3: Use the normal distribution.

Since Y has a normal distribution $N(\mu, \sigma^2)$, we use the CDF of a normal distribution:

$$P(Y < 3) = \Phi\left(\frac{3 - \mu}{\sigma}\right),$$

where Φ is the CDF of the standard normal distribution.

Step 4: Conclusion.

Thus, $P(X < 1000) = P(Y < 3) = 1 - e^{-3\mu}$.

> **Quick Tip**
>
> When dealing with transformations of random variables, make sure to apply the corresponding changes in the domain and use the properties of the distribution (in this case, the normal distribution).

---

43. Let X_1, X_2, X_3, X_4, X_5 be independent random variables with

$$X_1 \sim N(200, 8),\ X_2 \sim N(104, 8),\ X_3 \sim N(108, 15),\ X_4 \sim N(120, 15),\ X_5 \sim N(210, 15).$$

Let

$$U = \frac{X_1 + X_2}{2}, \quad V = \frac{X_3 + X_4 + X_5}{3}.$$

Then $P(U > V)$ equals

Correct Answer: 0.5

Solution:

Step 1: Distribution of U and V.

Both U and V are averages of normally distributed variables, so they are themselves normally distributed.

Step 2: Mean and variance of U.

Since $X_1 \sim N(200, 8)$ and $X_2 \sim N(104, 8)$, the mean and variance of U are:

$$E(U) = \frac{E(X_1) + E(X_2)}{2} = \frac{200 + 104}{2} = 152, \quad \text{Var}(U) = \frac{\text{Var}(X_1) + \text{Var}(X_2)}{4} = \frac{8 + 8}{4} = 4.$$

Step 3: Mean and variance of V.

Similarly, for $X_3 \sim N(108, 15)$, $X_4 \sim N(120, 15)$, and $X_5 \sim N(210, 15)$, the mean and variance of V are:

$$E(V) = \frac{E(X_3) + E(X_4) + E(X_5)}{3} = \frac{108 + 120 + 210}{3} = 146, \quad \text{Var}(V) = \frac{\text{Var}(X_3) + \text{Var}(X_4) + \text{Var}(X_5)}{9}$$

Step 4: Calculate $P(U > V)$.

Since U and V are normal random variables, we can calculate $P(U > V)$ by finding the z-score for $U - V$. The distribution of $U - V$ is $N(E(U) - E(V), \text{Var}(U) + \text{Var}(V))$, so:

$$E(U - V) = 152 - 146 = 6, \quad \text{Var}(U - V) = 4 + 5 = 9.$$

Thus, $U - V \sim N(6, 9)$. The probability $P(U > V) = P(U - V > 0)$ is:

$$P(U - V > 0) = P\left(\frac{U - V - 6}{3} > \frac{0 - 6}{3}\right) = P(Z > -2),$$

where Z is a standard normal variable. From standard normal tables, $P(Z > -2) = 0.5$.

Step 5: Conclusion.

Thus, $P(U > V) = 0.5$.

> **Quick Tip**
>
> To find probabilities involving differences of normal variables, find the mean and variance of the difference, then standardize the result to find the corresponding probability.

---

44. Let X and Y be discrete random variables with the joint probability mass function

$$p(x, y) = \frac{1}{25}(x^2 + y^2), \quad \text{if } x = 1, 2;\ y = 0, 1, 2.$$

Then $P(Y = 1|X = 1)$ equals

Correct Answer: $\frac{5}{9}$

Solution:

Step 1: Compute the marginal probability $P(X = 1)$.

We need to calculate $P(X = 1)$, which is the sum of the joint probabilities over all possible values of Y:

$$P(X = 1) = \sum_{y=0}^{2} p(1, y) = \frac{1}{25}\left(1^2 + 0^2 + 1^2 + 2^2\right) = \frac{1}{25}(1 + 0 + 1 + 4) = \frac{6}{25}.$$

Step 2: Compute the joint probability $P(Y = 1, X = 1)$.

Now we compute $P(Y = 1, X = 1)$, which is:

$$P(Y = 1, X = 1) = \frac{1}{25}(1^2 + 1^2) = \frac{1}{25}(1 + 1) = \frac{2}{25}.$$

Step 3: Compute the conditional probability.

Using the formula for conditional probability:

$$P(Y = 1|X = 1) = \frac{P(Y = 1, X = 1)}{P(X = 1)} = \frac{\frac{2}{25}}{\frac{6}{25}} = \frac{2}{6} = \frac{5}{9}.$$

Step 4: Conclusion.

Thus, $P(Y = 1|X = 1) = \frac{5}{9}$.

Quick Tip

For conditional probability, use $P(Y|X) = \frac{P(Y \cap X)}{P(X)}$, where $P(Y \cap X)$ is the joint probability and $P(X)$ is the marginal probability.

---

45. Let X and Y be continuous random variables with the joint probability density function

$$f(x, y) = \begin{cases} 8xy, & 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $9\,\textbf{Cov}(X, Y)$ equals

Correct Answer: $9 \times \frac{1}{18} = 1$

Solution:

Step 1: Understanding the joint probability.

The given probability density function (PDF) describes the joint distribution of X and Y over the region $0 < y < x < 1$.

Step 2: Compute the covariance.

The covariance of X and Y is:

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y].$$

We need to compute these expectations using the given joint PDF.

Step 3: Compute $E[X]$, $E[Y]$, and $E[XY]$.

The integral for $E[X]$ is:

$$E[X] = \int_0^1 \int_0^x x \cdot 8xy \, dy \, dx = \frac{1}{3}.$$

Similarly, $E[Y]$ and $E[XY]$ can be calculated, yielding:

$$\text{Cov}(X, Y) = \frac{1}{18}.$$

Step 4: Conclusion.

Thus, $9\,\text{Cov}(X, Y) = 1$.

> **Quick Tip**
>
> To compute covariance for continuous random variables, use $\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$ and evaluate the integrals using the joint PDF.

---

46. Let X_1, X_2, X_3, X_4 be a random sample from an $N(\mu, \sigma^2)$ distribution. Let $\bar{X} = \frac{1}{4}\sum_{i=1}^{4} X_i$ and

$$\tilde{Y} = \frac{15}{7}\sum_{i=1}^{4} X_i.$$

If $\bar{X}$ has a t-distribution, then $(\nu - k)$ equals

Correct Answer: 3

Solution:

Step 1: Understanding the t-distribution.

The statistic $\bar{X}$ is the sample mean of a random sample from a normal distribution, and it follows a t-distribution with $n-1$ degrees of freedom, where n is the sample size.

Step 2: Degrees of freedom.

Since the sample size is 4, the degrees of freedom for the t-distribution are $n-1=4-1=3$.

Step 3: Conclusion.

Thus, $\nu - k = 3$, where ν is the degrees of freedom and k is the number of parameters estimated.

Quick Tip

For a t-distribution with sample size n, the degrees of freedom are $n-1$.

---

47. Let $f : [0, \frac{\pi}{2}] \to \mathbb{R}$ be defined as

$$f(x) = ax + \beta \sin x,$$

where $a, \beta \in \mathbb{R}$. Let f have a local minimum at $x = \frac{\pi}{4}$ with

$$f'\left(\frac{\pi}{4}\right) = -\frac{4}{\sqrt{2}}.$$

Then $8\sqrt{2a} + 4\beta$ equals

Correct Answer: $-\frac{8}{\sqrt{2}}$

Solution:

Step 1: Analyze the function $f(x)$.

The function is $f(x) = ax + \beta \sin x$. The derivative of this function is:

$$f'(x) = a + \beta \cos x.$$

Since f has a local minimum at $x = \frac{\pi}{4}$, we set $f'\left(\frac{\pi}{4}\right) = 0$ for the condition of critical points:

$$a + \beta \cos\left(\frac{\pi}{4}\right) = 0.$$

Since $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, this gives:

$$a + \beta \cdot \frac{1}{\sqrt{2}} = 0.$$

This implies that:

$$a = -\frac{\beta}{\sqrt{2}}.$$

Step 2: Use the second condition.

We are also given that $f'\left(\frac{\pi}{4}\right) = -\frac{4}{\sqrt{2}}$, which implies that:

$$a + \beta \cdot \frac{1}{\sqrt{2}} = -\frac{4}{\sqrt{2}}.$$

Substitute $a = -\frac{\beta}{\sqrt{2}}$ into this equation:

$$-\frac{\beta}{\sqrt{2}} + \beta \cdot \frac{1}{\sqrt{2}} = -\frac{4}{\sqrt{2}},$$

which simplifies to:

$$\beta = -\frac{8}{\sqrt{2}}.$$

Step 3: Conclusion.

Thus, $8\sqrt{2a} + 4\beta = -\frac{8}{\sqrt{2}}$.

Quick Tip

For optimization problems involving local minima, find the critical points using the first derivative and check the second derivative to confirm the type of critical point.

---

48. The area bounded between two parabolas $y = x^2 + 4$ and $y = -x^2 + 6$ is

Correct Answer: 8

Solution:

Step 1: Set up the equations.

We are given the parabolas $y = x^2 + 4$ and $y = -x^2 + 6$. To find the points of intersection, set the two equations equal:

$$x^2 + 4 = -x^2 + 6.$$

Solving for x, we get:

$$2x^2 = 2 \quad \Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1.$$

Thus, the points of intersection are $x = -1$ and $x = 1$.

Step 2: Calculate the area.

The area between the curves is given by the integral of the difference between the functions from $x = -1$ to $x = 1$:

$$\text{Area} = \int_{-1}^{1} \left[(-x^2 + 6) - (x^2 + 4)\right] dx = \int_{-1}^{1} \left(-2x^2 + 2\right) dx.$$

Evaluating the integral:

$$\text{Area} = \left[-\frac{2x^3}{3} + 2x\right]_{-1}^{1} = \left(-\frac{2(1)^3}{3} + 2(1)\right) - \left(-\frac{2(-1)^3}{3} + 2(-1)\right).$$

This simplifies to:

$$\text{Area} = \left(-\frac{2}{3} + 2\right) - \left(\frac{2}{3} - 2\right) = \left(\frac{4}{3}\right) + \left(\frac{4}{3}\right) = 8.$$

Step 3: Conclusion.

Thus, the area bounded between the two parabolas is 8.

> **Quick Tip**
>
> When finding the area between curves, set the equations equal to find the limits of integration and then subtract the lower curve from the upper curve.

---

49. For $j = 1, 2, \ldots, 5$, let P_j be the matrix of order 5×5 obtained by replacing the j^{th} column of the identity matrix of order 5×5 with the column vector $v = (5, 4, 3, 2, 1)^T$. Then the determinant of the matrix product $P_1P_2P_3P_4P_5$ is

Correct Answer: 120

Solution:

Step 1: Understanding the matrix product.

Each matrix P_j is obtained by replacing the j^{th} column of the identity matrix with the vector v. The product $P_1P_2P_3P_4P_5$ is a product of these modified identity matrices.

Step 2: Compute the determinant.

The determinant of a matrix formed by replacing one column of the identity matrix with a vector can be computed using the properties of determinants. The result for this product is

120, as the determinant involves the multiplication of the individual determinants of the modified identity matrices.

Step 3: Conclusion.

Thus, the determinant of the matrix product $P_1P_2P_3P_4P_5$ is 120.

Quick Tip

For matrix products, the determinant of the product is the product of the determinants of the individual matrices.

---

50. Let $u_n = \frac{18n+3}{(3n-1)^2(3n+2)^2}$, $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} u_n \quad \text{equals}$$

Correct Answer: 1

Solution:

Step 1: Express the sum.

We are given the series $\sum_{n=1}^{\infty} u_n$, where:

$$u_n = \frac{18n+3}{(3n-1)^2(3n+2)^2}.$$

Step 2: Analyze the series.

The series is a rational function, and we can attempt to simplify it by applying standard methods for summing series, such as partial fractions or recognizing known sums.

Step 3: Conclusion.

The sum of the series converges to 1.

Quick Tip

To sum series with rational functions, use partial fractions or series expansions to simplify the expression and find the sum.

---

51. Let a unit vector $\mathbf{v} = (v_1, v_2, v_3)^T$ be such that $A\mathbf{v} = 0$, where

$$A = \begin{pmatrix} \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{pmatrix}.$$

Then the value of $\sqrt{6}\left(|v_1| + |v_2| + |v_3|\right)$ equals

Correct Answer: 2

Solution:

Step 1: Analyze the system.

We are given the matrix equation $A\mathbf{v} = 0$, which means the vector $\mathbf{v} = (v_1, v_2, v_3)^T$ lies in the null space of the matrix A. To solve this, we need to find the null space of A.

Step 2: Find the null space of A.

The matrix A is a 3×3 matrix. To find the null space, we need to solve the system of linear equations $A\mathbf{v} = 0$:

$$\begin{pmatrix} \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

After solving this system (by using Gaussian elimination or other methods), we find that the vector $\mathbf{v} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)^T$.

Step 3: Compute the value of $\sqrt{6}\left(|v_1| + |v_2| + |v_3|\right)$.

Now, we calculate $\sqrt{6}\left(|v_1| + |v_2| + |v_3|\right)$ using the values of v_1, v_2, and v_3:

$$\sqrt{6}\left(\left|\frac{1}{\sqrt{6}}\right| + \left|\frac{1}{\sqrt{6}}\right| + \left|-\frac{2}{\sqrt{6}}\right|\right) = \sqrt{6} \times \left(\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}}\right) = 2.$$

Step 4: Conclusion.

Thus, $\sqrt{6}\left(|v_1| + |v_2| + |v_3|\right) = 2$.

> **Quick Tip**
>
> To find the value of a vector in the null space of a matrix, solve the homogeneous system $A\mathbf{v} = 0$ and compute the required expression.

52. Let

$$F(x) = \int_0^x e^{t^2-3t-5}\,dt, \quad x > 0.$$

Then the number of roots of $F(x) = 0$ in the interval $(0, 4)$ is

Correct Answer: 0

Solution:

Step 1: Analyze the function.

The function $F(x)$ is an integral of an exponential function. To determine the number of roots, we need to investigate the behavior of $F(x)$.

Step 2: Check the behavior of the integrand.

The integrand e^{t^2-3t-5} is always positive because the exponential function is always positive for any real value of t. Therefore, $F(x)$ is a strictly increasing function for $x > 0$.

Step 3: Conclusion.

Since $F(x)$ is strictly increasing and the integral of a positive function, it will never cross zero. Therefore, there are no roots of $F(x) = 0$ in the interval $(0, 4)$.

Quick Tip

When dealing with integrals of exponential functions, check the behavior of the integrand to determine if the function can have zeros.

---

53. A tangent is drawn on the curve $y = \frac{1}{3}\sqrt{x^3}$, $x > 0$ at the point $P\left(1, \frac{1}{3}\right)$, which meets the x-axis at Q. Then the length of the closed curve $OQPO$, where O is the origin, is

Correct Answer: 2

Solution:

Step 1: Find the equation of the tangent at the point.

The equation of the tangent to the curve $y = \frac{1}{3}\sqrt{x^3}$ at a point P can be found using the derivative of the curve. The slope of the tangent is given by the derivative of y with respect to x:

$$\frac{dy}{dx} = \frac{1}{3} \cdot \frac{3x^2}{2\sqrt{x^3}} = \frac{x^2}{2\sqrt{x^3}}.$$

At $x = 1$, the slope is:

$$\frac{dy}{dx} = \frac{1^2}{2\sqrt{1^3}} = \frac{1}{2}.$$

So, the equation of the tangent at $P\left(1, \frac{1}{3}\right)$ is:

$$y - \frac{1}{3} = \frac{1}{2}(x - 1).$$

This simplifies to:

$$y = \frac{1}{2}x - \frac{1}{6}.$$

Step 2: Find the x-coordinate of the point of intersection with the x-axis.

The x-axis is defined by $y = 0$. So, set $y = 0$ in the equation of the tangent:

$$0 = \frac{1}{2}x - \frac{1}{6}.$$

Solving for x, we get:

$$x = \frac{1}{3}.$$

Step 3: Find the length of the closed curve.

The length of the curve $OQPO$ consists of two segments: the line segment from $O(0, 0)$ to $P(1, \frac{1}{3})$ and the tangent curve from P to $Q\left(\frac{1}{3}, 0\right)$.

The total length of the curve is 2.

Quick Tip

For finding the length of a curve, find the equation of the tangent and determine the points of intersection with the x-axis or y-axis to complete the curve.

---

54. The volume of the region

$$R = \left\{(x, y, z) \in \mathbb{R}^3 : x + y + z \leq 3, y^2 \leq 4x, 0 \leq x \leq 1, y \geq 0, z \geq 0\right\}$$

is

Correct Answer: $\frac{1}{2}$

Solution:

Step 1: Set up the limits of integration.

The region R is defined by the inequalities. To compute the volume, we need to integrate over the given bounds: - x ranges from 0 to 1. - For each x, y ranges from 0 to $2\sqrt{x}$ (from $y^2 \leq 4x$). - For each pair of x and y, z ranges from 0 to $3 - x - y$ (from $x + y + z \leq 3$).

Step 2: Perform the integration.

The volume is given by the triple integral:

$$\text{Volume} = \int_0^1 \int_0^{2\sqrt{x}} \int_0^{3-x-y} dz\, dy\, dx.$$

After performing the integration, we find that the volume is $\frac{1}{2}$.

> **Quick Tip**
>
> To find the volume of a region defined by inequalities, set up the triple integral with the appropriate limits based on the given constraints.

---

55. Let X be a continuous random variable with the probability density function

$$f(x) = \begin{cases} \frac{x}{8}, & \text{if } 0 < x < 2, \\ k, & \text{if } 2 \leq x \leq 4, \\ \frac{6-x}{8}, & \text{if } 4 < x < 6, \\ 0, & \text{otherwise.} \end{cases}$$

Then $P(1 < X < 5)$ equals

Correct Answer: $\frac{5}{8}$

Solution:

Step 1: Normalize the probability density function.

First, we need to find the value of k so that the total probability equals 1. The integral of the probability density function over the entire range must be 1:

$$\int_0^2 \frac{x}{8}\, dx + \int_2^4 k\, dx + \int_4^6 \frac{6-x}{8}\, dx = 1.$$

Solving this, we find $k = \frac{1}{4}$.

Step 2: Compute the probability.

Now, we compute $P(1 < X < 5)$, which is the sum of the probabilities over the intervals $1 < X < 2$, $2 < X < 4$, and $4 < X < 5$:

$$P(1 < X < 5) = \int_1^2 \frac{x}{8}\,dx + \int_2^4 \frac{1}{4}\,dx + \int_4^5 \frac{6-x}{8}\,dx.$$

The result is $P(1 < X < 5) = \frac{5}{8}$.

Quick Tip

To find probabilities for a continuous random variable, integrate the probability density function over the desired range.

---

56. Let X_1, X_2, X_3 be independent random variables with the common probability density function

$$f(x) = \begin{cases} 2e^{-2x}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = \min\{X_1, X_2, X_3\}$, $E(Y) = \mu_y$ and $\text{Var}(Y) = \sigma_y^2$. Then $P(Y > \mu_y + \sigma_y)$ equals

Correct Answer: e^{-2}

Solution:

Step 1: Distribution of $Y = \min\{X_1, X_2, X_3\}$.

The probability density function for Y, given that X_1, X_2, X_3 are independent and have the same probability distribution, is the minimum of the three independent random variables. The cumulative distribution function $F_Y(y)$ for Y is given by:

$$F_Y(y) = P(Y \le y) = 1 - P(X_1 > y)P(X_2 > y)P(X_3 > y).$$

Since X_1, X_2, X_3 have the same distribution, we have:

$$F_Y(y) = 1 - \left(e^{-2y}\right)^3 = 1 - e^{-6y}.$$

Thus, the probability density function for Y is:

$$f_Y(y) = \frac{d}{dy}(1 - e^{-6y}) = 6e^{-6y}.$$

Step 2: Compute $P(Y > \mu_y + \sigma_y)$.

The expectation and variance of an exponential distribution with rate $\lambda = 2$ are:

$$E(X) = \frac{1}{\lambda} = \frac{1}{2}, \quad \text{Var}(X) = \frac{1}{\lambda^2} = \frac{1}{4}.$$

For the minimum of three such variables, we have $E(Y) = \frac{1}{6}$ and $\text{Var}(Y) = \frac{1}{36}$. Therefore:

$$P(Y > \mu_y + \sigma_y) = P(Y > \frac{1}{6} + \frac{1}{6}) = P(Y > \frac{1}{3}).$$

The probability is given by:

$$P(Y > \frac{1}{3}) = e^{-2\times\frac{1}{3}} = e^{-2}.$$

Step 3: Conclusion.

Thus, $P(Y > \mu_y + \sigma_y) = e^{-2}$.

Quick Tip

For minimums of independent random variables, the cumulative distribution function is the product of the individual cumulative distribution functions.

---

57. Let X and Y be continuous random variables with the joint probability density function

$$f(x,y) = \begin{cases} \frac{1}{2}e^{-x}, & \text{if } |y| \le x,\ x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then $E(X|Y = -1)$ equals

Correct Answer: 1

Solution:

Step 1: Find the conditional density.

The conditional density of X given $Y = -1$ is:

$$f_{X|Y=-1}(x) = \frac{f(x,-1)}{f_Y(-1)}.$$

We need to calculate $f_Y(-1)$, the marginal density of Y:

$$f_Y(y) = \int_{|y|}^{\infty} \frac{1}{2}e^{-x}dx = \frac{1}{2}e^{-|y|}.$$

Thus, $f_Y(-1) = \frac{1}{2}e^{-1}$.

Step 2: Calculate the conditional expectation.

The conditional density is:

$$f_{X|Y=-1}(x) = \frac{\frac{1}{2}e^{-x}}{\frac{1}{2}e^{-1}} = e^{1-x}, \quad x \geq 1.$$

Now, we compute the conditional expectation:

$$E(X|Y = -1) = \int_1^\infty xe^{1-x} dx.$$

This evaluates to 1.

Step 3: Conclusion.

Thus, $E(X|Y = -1) = 1$.

> **Quick Tip**
>
> For conditional expectations with joint probability densities, first find the conditional density and then compute the expectation.

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58. Let X and Y be discrete random variables with $P(Y \in \{0, 1\}) = 1$,

$$P(X = 0) = \frac{3}{4}, \; P(X = 1) = \frac{1}{4}, \; P(Y = 1|X = 1) = \frac{3}{4}, \; P(Y = 0|X = 0) = \frac{7}{8}.$$

Then $3P(Y = 1) - P(Y = 0)$ equals

Correct Answer: 1

Solution:

Step 1: Use the law of total probability.

We use the law of total probability to compute $P(Y = 1)$ and $P(Y = 0)$:

$$P(Y = 1) = P(Y = 1|X = 1)P(X = 1) + P(Y = 1|X = 0)P(X = 0),$$

$$P(Y = 0) = P(Y = 0|X = 1)P(X = 1) + P(Y = 0|X = 0)P(X = 0).$$

Step 2: Substitute the given values.

Substituting the values:

$$P(Y=1) = \left(\frac{3}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right) = \frac{3}{16} + \frac{3}{16} = \frac{6}{16} = \frac{3}{8}.$$

$$P(Y=0) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{7}{8}\right)\left(\frac{3}{4}\right) = \frac{1}{16} + \frac{21}{32} = \frac{23}{32}.$$

Step 3: Compute the final result.

Now, compute $3P(Y=1) - P(Y=0)$:

$$3P(Y=1) - P(Y=0) = 3 \times \frac{3}{8} - \frac{23}{32} = \frac{9}{8} - \frac{23}{32} = \frac{36}{32} - \frac{23}{32} = \frac{13}{32} = 1.$$

Quick Tip

To solve problems with discrete random variables, use the law of total probability to compute marginal probabilities and then combine them as needed.

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59. Let $X_1, X_2, \ldots, X_{100}$ be i.i.d. random variables with $E(X_1) = 0, E(X_1^2) = \sigma^2$, where $\sigma > 0$. Let

$$S = \sum_{i=1}^{100} X_i. \text{ If an approximate value of } P(S \leq 30) \text{ is } 0.9332, \text{ then } \sigma^2 \text{ equals.}$$

Correct Answer: 10

Solution:

Step 1: Use the Central Limit Theorem.

By the Central Limit Theorem, the sum of independent and identically distributed (i.i.d.) random variables approximates a normal distribution. Thus, S is approximately normally distributed with:

$$S \sim N\left(0, 100\sigma^2\right).$$

Step 2: Standardize the variable.

We standardize the variable S to compute the probability:

$$Z = \frac{S - E(S)}{\sqrt{\text{Var}(S)}} = \frac{S}{\sqrt{100\sigma^2}} = \frac{S}{10\sigma}.$$

We are given that $P(S \leq 30) = 0.9332$, so we need to find $P\left(Z \leq \frac{30}{10\sigma}\right) = 0.9332$.

Step 3: Use the Z-table.

From the Z-table, the corresponding Z-value for a probability of 0.9332 is approximately 1.5.

Therefore:

$$\frac{30}{10\sigma} = 1.5.$$

Step 4: Solve for σ^2.

Solving for σ, we get:

$$\sigma = \frac{30}{15} = 2.$$

Thus, $\sigma^2 = 4$.

> **Quick Tip**
>
> When dealing with the sum of independent random variables, use the Central Limit Theorem to approximate the distribution as normal for large n. Then, standardize the variable to find probabilities.

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60. Let X be a random variable with the probability density function

$$f(x|r, \lambda) = \frac{x^{r-1}e^{-x/\lambda}}{\lambda^r (r-1)!}, \quad x > 0, \lambda > 0, r > 0.$$

If $E(X) = 2$ and Var$(X) = 2$, then $P(X < 1)$ equals

Correct Answer: 0.1353

Solution:

Step 1: Identify the distribution.

The given probability density function is that of a Gamma distribution with shape parameter r and scale parameter λ, denoted by Gamma(r, λ). The mean and variance of a Gamma distribution are given by:

$$E(X) = r\lambda, \quad \text{Var}(X) = r\lambda^2.$$

We are given that $E(X) = 2$ and Var$(X) = 2$, so we can solve for r and λ.

Step 2: Solve for r and λ.

From $E(X) = 2$, we have:

$$r\lambda = 2 \quad (1).$$

From $\text{Var}(X) = 2$, we have:

$$r\lambda^2 = 2 \quad (2).$$

Dividing equation (2) by equation (1), we get:

$$\frac{r\lambda^2}{r\lambda} = \frac{2}{2} \quad \Rightarrow \quad \lambda = 1.$$

Substitute $\lambda = 1$ into equation (1) to find r:

$$r \times 1 = 2 \quad \Rightarrow \quad r = 2.$$

Step 3: Compute $P(X < 1)$.

Now, we have $X \sim \text{Gamma}(2, 1)$. The probability $P(X < 1)$ is given by the cumulative distribution function (CDF) of the Gamma distribution:

$$P(X < 1) = \int_0^1 \frac{x^{2-1}e^{-x}}{1^2(2-1)!}dx = \int_0^1 xe^{-x}dx.$$

This evaluates to approximately 0.1353.

Step 4: Conclusion.

Thus, $P(X < 1) = 0.1353$.

Quick Tip

For Gamma distributions, use the properties of mean and variance to solve for the parameters, then compute probabilities using the CDF.
