IIT JAM 2021 Mathematical Statistics (MS) Question Paper

Time Allowed : 3 Hours	**Maximum Marks :** 100	**Total questions :** 60

General Instructions

General Instructions:

i) All questions are compulsory. Marks allotted to each question are indicated in the margin.

ii) Answers must be precise and to the point.

iii) In numerical questions, all steps of calculation should be shown clearly.

iv) Use of non-programmable scientific calculators is permitted.

v) Wherever necessary, write balanced chemical equations with proper symbols and units.

vi) Rough work should be done only in the space provided in the question paper.

1. The value of the limit

$$\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right)^{\frac{1}{n}}$$

is equal to:

(A) e

(B) $\frac{1}{e}$

(C) $\frac{3}{e}$

(D) $\frac{4}{e}$

---

2. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=x^7+5x^3+11x+15$. Then, which of the following statements is TRUE?

(A) f is both one-one and onto

(B) f is neither one-one nor onto

(C) f is one-one but NOT onto

(D) f is onto but NOT one-one

---

3. The value of the limit

$$\lim_{x\to 0}\frac{e^{-3x}-e^{x}+4x}{5(1-\cos x)}$$

is equal to:

(A) 1

(B) 0

(C) $\frac{2}{5}$

(D) $\frac{8}{5}$

---

4. The value of the limit

$$\lim_{n\to\infty}\sum_{k=0}^{n}\binom{2n}{k}\frac{1}{4^n}$$

is equal to:

(A) 1

(B) $\frac{1}{2}$

(C) 0

(D) $\frac{1}{4}$

---

5. Let $\{X_n\}_{n\geq 1}$ be i.i.d. random variables with

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Then, the value of the limit

$$\lim_{n\to\infty} P\left(-\frac{1}{n}\sum_{i=1}^{n} \ln X_i \leq 1 + \frac{1}{\sqrt{n}}\right)$$

is equal to:

(A) $\frac{1}{2}$

(B) $\Phi(1)$

(C) 0

(D) $\Phi(2)$

---

6. Let X be a $U(0,1)$ random variable and $Y = X^2$. If ρ is the correlation coefficient between X and Y, then $48\rho^2$ is equal to:

(A) 48

(B) 45

(C) 35

(D) 30

---

7. Let M be a 3×3 real matrix. Let

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -1 \\ \alpha \end{pmatrix}$$

be eigenvectors of M corresponding to three distinct eigenvalues. Then, which of the following is NOT a possible value of α?

(A) 0

(B) 1

(C) -2

(D) 2

---

8. If the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then which of the following series diverges?

(A) $\sum_{n=1}^{\infty} |a_{2n}|$

(B) $\sum_{n=1}^{\infty} \frac{a_n + a_{n+1}}{2}$

(C) $\sum_{n=1}^{\infty} (a_n)^3$

(D) $\sum_{n=2}^{\infty} \left(\frac{1}{(\ln n)^2} + a_n \right)$

---

9. There are three urns labeled 1, 2, 3.

Urn 1: 2 white, 2 black; Urn 2: 1 white, 3 black; Urn 3: 3 white, 1 black.

Two coins are tossed independently, each with $P(\text{head}) = 0.2$.

Urn 1 is selected if 2 heads occur, Urn 3 if 2 tails occur, otherwise Urn 2 is selected. A ball is drawn at random from the chosen urn. Find

$$P(\text{Urn 1 is selected} \mid \text{Ball drawn is white})$$

(A) $\frac{6}{109}$

(B) $\frac{12}{109}$

(C) $\frac{1}{18}$

(D) $\frac{1}{9}$

---

10. Let X be a random variable with

$$f(x) = \frac{1}{2}e^{-|x|}, \quad -\infty < x < \infty$$

Then, which of the following statements is FALSE?

(A) $E(X|X|) = 0$

(B) $E(X|X|^2) = 0$

(C) $E(|X| \sin(\frac{X}{|X|})) = 0$

(D) $E(|X| \sin^2(\frac{X}{|X|})) = 0$

---

11. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x,y) = \begin{cases} \dfrac{y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Let $f_x(x,y)$ and $f_y(x,y)$ denote the first-order partial derivatives of $f(x,y)$ with respect to x and y respectively. Then, which of the following statements is FALSE?

(A) $f_x(x,y)$ exists and is bounded at every $(x,y) \in \mathbb{R}^2$

(B) $f_y(x,y)$ exists and is bounded at every $(x,y) \in \mathbb{R}^2$

(C) $f_y(0,0)$ exists and $f_y(x,y)$ is continuous at (0,0)

(D) f is NOT differentiable at (0,0)

---

12. Let $\{X_n\}_{n\geq 1}$ be i.i.d. random variables distributed as $N(0,1)$. Then find

$$\lim_{n\to\infty} P\left(\frac{\sum_{i=1}^{n} X_i^2 - 3n}{\sqrt{32n}} \leq \sqrt{6}\right)$$

is equal to:

(A) $\frac{1}{2}$

(B) $\Phi(\sqrt{2})$

(C) 0

(D) $\Phi(1)$

---

13. Consider independent Bernoulli trials with success probability $p = \frac{1}{3}$. The probability that three successes occur before four failures is:

(A) $\frac{179}{243}$

(B) $\frac{179}{841}$

(C) $\frac{233}{729}$

(D) $\frac{179}{1215}$

---

14. Let X and Y be independent $N(0,1)$ random variables and $Z = \left|\frac{X}{Y}\right|$. Then, which of the following expectations is finite?

(A) $E\left(\frac{1}{\sqrt{Z}}\right)$

(B) $E(Z\sqrt{Z})$

(C) $E(Z)$

(D) $E\left(\frac{1}{Z\sqrt{Z}}\right)$

---

15. Three coins have probabilities of head in a single toss as $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ respectively. A player selects one coin at random and tosses it five times. The probability of obtaining two tails in five tosses is:

(A) $\frac{85}{384}$

(B) $\frac{255}{384}$

(C) $\frac{125}{384}$

(D) $\frac{64}{384}$

---

16. Let X be a random variable with pdf

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

Define $Y = [X]$, the greatest integer less than or equal to X. Then $E(Y^2)$ is equal to:

(A) $\frac{e(e+1)}{e-1}$

(B) $\frac{e+1}{(e-1)^2}$

(C) $\frac{(e+1)^2}{e-1}$

(D) $\frac{(e+1)^2}{(e-1)^2}$

---

17. Let X be a continuous random variable having the moment generating function

$$M(t) = \frac{e^t - 1}{t}, \quad t \neq 0.$$

Let $\alpha = P(48X^2 - 40X + 3 > 0)$ and $\beta = P((\ln X)^2 + 2\ln X - 3 > 0)$. Then, the value of $\alpha - 2\ln\beta$ is equal to:

(A) $\frac{10}{3}$

(B) $\frac{19}{3}$

(C) $\frac{13}{3}$

(D) $\frac{17}{3}$

---

18. Let $X_1, X_2, ..., X_n$ ($n \geq 3$) be a random sample from Poisson(θ), where $\theta > 0$ is unknown, and let $T = \sum_{i=1}^{n} X_i$. Then, the uniformly minimum variance unbiased estimator (UMVUE) of $e^{-2\theta}\theta^3$ is:

(A) $\frac{T}{n}\left(\frac{T}{n} - 1\right)\left(\frac{T}{n} - 2\right)\left(1 - \frac{2}{n}\right)^{T-3}$

(B) $\frac{T(T-1)(T-2)(n-2)^{T-3}}{n^T}$

(C) does NOT exist

(D) $e^{-2T/n}\left(\frac{T}{n}\right)^3$

19. Let $X_1, X_2, ..., X_n$ ($n \geq 2$) be a random sample from $U(\theta - 5, \theta + 5)$, where $\theta \in (0, \infty)$ is unknown. Let $T = \max(X_1, ..., X_n)$ and $U = \min(X_1, ..., X_n)$. Then, which of the following statements is TRUE?

(A) $\frac{T+U}{2}$ is the unique MLE of θ

(B) $\frac{2}{T+U}$ is an MLE of $\frac{1}{\theta}$

(C) MLE of θ does NOT exist

(D) $U + 8$ is an MLE of θ

---

20. Let X and Y be random variables having chi-square distributions with 6 and 3 degrees of freedom respectively. Then, which of the following statements is TRUE?

(A) $P(X > 0.7) > P(Y > 0.7)$

(B) $P(X > 0.7) < P(Y > 0.7)$

(C) $P(X > 3) < P(Y > 3)$

(D) $P(X < 6) > P(Y < 6)$

---

21. Let (X, Y) be a random vector with joint moment generating function

$$M(t_1, t_2) = \frac{1}{(1 - (t_1 + t_2))(1 - t_2)}, \quad -\infty < t_1, t_2 < \min(1, 1 - t_2)$$

Let $Z = X + Y$. Then, $\mathrm{Var}(Z)$ is equal to:

(A) 3

(B) 4

(C) 5

(D) 6

---

22. Let X be a continuous random variable with CDF

$$F(x) = \begin{cases} 0, & x < 0, \\ ax^2, & 0 \leq x < 2, \\ 1, & x \geq 2, \end{cases}$$

for some real constant a. Then $E(X)$ is equal to:

(A) $\frac{4}{3}$

(B) $\frac{1}{4}$

(C) 1

(D) 0

---

23. Let $X_1, X_2, ..., X_n$ be a random sample from an exponential distribution with probability density function

$$f(x;\theta) = \begin{cases} \theta e^{-\theta x}, & x > 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $\theta \in (0, \infty)$ is unknown. Let $\alpha \in (0, 1)$ be fixed and let β be the power of the most powerful test of size α for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$. Consider the critical region

$$R = \left\{ (x_1, x_2, ..., x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i > \frac{1}{2}\chi^2_{2n}(1-\alpha) \right\},$$

where for any $\gamma \in (0, 1)$, $\chi^2_{2n}(\gamma)$ is a fixed point such that $P(\chi^2_{2n} > \chi^2_{2n}(\gamma)) = \gamma$. Then, the critical region R corresponds to the

(A) most powerful test of size α for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$

(B) most powerful test of size $1 - \alpha$ for testing $H_0 : \theta = 2$ against $H_1 : \theta = 1$

(C) most powerful test of size β for testing $H_0 : \theta = 2$ against $H_1 : \theta = 1$

(D) most powerful test of size $1 - \beta$ for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$

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24. Let

$$S = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} \left(\frac{1}{4}\right)^k, \quad T = \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{1}{5}\right)^k.$$

Then, which of the following statements is TRUE?

(A) $S - T = 0$

(B) $5S - 4T = 0$

(C) $4S - 5T = 0$

(D) $16S - 25T = 0$

---

25. Let E_1, E_2, E_3 and E_4 be four events such that

$P(E_i|E_4) = \dfrac{2}{3},\ i = 1, 2, 3;\quad P(E_i \cap E_j^c|E_4) = \dfrac{1}{6},\ i, j = 1, 2, 3;\ i \neq j;\quad P(E_1 \cap E_2 \cap E_3^c|E_4) = \dfrac{1}{6}.$

Then, $P(E_1 \cup E_2 \cup E_3|E_4)$ is equal to

(A) $\frac{1}{2}$

(B) $\frac{2}{3}$

(C) $\frac{5}{6}$

(D) $\frac{7}{12}$

---

26. Let $a_1 = 5$ and define recursively

$$a_{n+1} = \frac{1}{3}\,(a_n)^{\frac{3}{4}}, \quad n \geq 1.$$

Then, which of the following statements is TRUE?

(A) $\{a_n\}$ is monotone increasing, and $\lim_{n\to\infty} a_n = 3$

(B) $\{a_n\}$ is monotone decreasing, and $\lim_{n\to\infty} a_n = 3$

(C) $\{a_n\}$ is non-monotone, and $\lim_{n\to\infty} a_n = 3$

(D) $\{a_n\}$ is decreasing, and $\lim_{n\to\infty} a_n = 0$

---

27. Consider the problem of testing $H_0 : X \sim f_0$ against $H_1 : X \sim f_1$ based on a sample of size 1, where

$$f_0(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ 0, & \text{otherwise}, \end{cases} \qquad f_1(x) = \begin{cases} 2x, & 0 \leq x \leq 1, \\ 0, & \text{otherwise}. \end{cases}$$

Then, the probability of Type II error of the most powerful test of size $\alpha = 0.1$ is equal to

(A) 0.81

(B) 0.91

(C) 0.1

(D) 1

---

28. For $a \in \mathbb{R}$, consider the system of linear equations

$$\begin{cases} ax + ay = a + 2, \\ x + ay + (a-1)z = a - 4, \\ ax + ay + (a-2)z = -8, \end{cases}$$

in the unknowns x, y, z. Then, which of the following statements is TRUE?

(A) The given system has a unique solution for $a = 1$

(B) The given system has infinitely many solutions for $a = 2$

(C) The given system has a unique solution for $a = -2$

(D) The given system has infinitely many solutions for $a = -2$

---

29. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that $a_n \geq 1$, for all $n \geq 1$. Then, which of the following conditions imply the divergence of $\{a_n\}_{n\geq 1}$?

(A) $\{a_n\}_{n\geq 1}$ is non-increasing

(B) $\sum_{n=1}^{\infty} b_n$ converges, where $b_1 = a_1$ and $b_n = a_{n+1} - a_n$ for all $n > 1$

(C) $\lim_{n\to\infty} \frac{a_{2n+1}}{a_{2n}} = \frac{1}{2}$

(D) $\{\sqrt{a_n}\}_{n\geq 1}$ converges

---

30. Let E_1, E_2 and E_3 be three events such that $P(E_1) = \frac{4}{5}, P(E_2) = \frac{1}{2}$ and $P(E_3) = \frac{9}{10}$. Then, which of the following statements is FALSE?

(A) $P(E_1 \cup E_2 \cup E_3) \geq \frac{9}{10}$

(B) $P(E_2 \cup E_3) \geq \frac{9}{10}$

(C) $P(E_1 \cap E_2 \cap E_3) \leq \frac{1}{6}$

(D) $P(E_1 \cup E_2) \leq \frac{4}{5}$

---

31. Consider the linear system $Ax = b$, where A is an $m \times n$ matrix, x is an $n \times 1$ vector of unknowns and b is an $m \times 1$ vector. Further, suppose there exists an $m \times 1$ vector c such that the linear system $Ax = c$ has **NO** solution. Then, which of the following statements is/are necessarily TRUE?

(A) If $m \leq n$ and d is the first column of A, then the linear system $Ax = d$ has a unique solution

(B) If $m \geq n$, then Rank$(A) < n$

(C) Rank$(A) < m$

(D) If $m > n$, then the linear system $Ax = 0$ has a solution other than $x = 0$

---

32. Let A be a 3×3 real matrix such that $A \neq I_3$ and the sum of the entries in each row of A is 1. Then, which of the following statements is/are necessarily TRUE?

(A) $A - I_3$ is an invertible matrix

(B) The set $\{x \in \mathbb{R}^3 : (A - I_3)x = 0\}$ has at least two elements (x is a column vector)

(C) The characteristic polynomial, $p(\lambda)$, of $A + 2A^2 + A^3$ has $(\lambda - 4)$ as a factor

(D) A cannot be an orthogonal matrix

---

33. Let $X_1, X_2, ..., X_n$ be a random sample from $N(\theta, 1)$, where $\theta \in (-\infty, \infty)$ is unknown. Consider the problem of testing $H_0 : \theta \leq 0$ against $H_1 : \theta > 0$. Let $\beta(\theta)$ denote the power function of the likelihood ratio test of size α $(0 < \alpha < 1)$ for testing H_0 against H_1. Then, which of the following statements is/are TRUE?

(A) $\beta(\theta) > \beta(0)$, for all $\theta > 0$

(B) $\beta(\theta) < \beta(0)$, for all $\theta > 0$

(C) The critical region of the likelihood test of size α is

$$\left\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : \sqrt{n}\frac{\sum_{i=1}^{n} x_i}{n} > \tau_{\alpha/2}\right\},$$

where $\tau_{\alpha/2}$ is a fixed point such that $P(Z > \tau_{\alpha/2}) = \frac{\alpha}{2}$, $Z \sim N(0,1)$.

(D) The critical region of the likelihood test of size α is

$$\left\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : \sqrt{n}\frac{\sum_{i=1}^{n} x_i}{n} > \tau_{\alpha}\right\},$$

where τ_α is a fixed point such that $P(Z > \tau_\alpha) = \alpha$, $Z \sim N(0,1)$.

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34. Consider the function

$$f(x,y) = 3x^2 + 4xy + y^2, \quad (x,y) \in \mathbb{R}^2.$$

If $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, then which of the following statements is/are TRUE?

(A) The maximum value of f on S is $3 + \sqrt{5}$

(B) The minimum value of f on S is $3 - \sqrt{5}$

(C) The maximum value of f on S is $2 + \sqrt{5}$

(D) The minimum value of f on S is $2 - \sqrt{5}$

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35. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Then, which of the following statements is/are necessarily TRUE?

(A) f'' is continuous

(B) If $f'(0) = f'(1)$, then $f''(x) = 0$ has a solution in $(0,1)$

(C) f' is bounded on $[8,10]$

(D) f'' is bounded on $(0,1)$

---

36. Let $X_1, X_2, ..., X_n$ $(n \geq 2)$ be independent and identically distributed random variables with probability density function

$$f(x) = \begin{cases} \frac{1}{x^2}, & x \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, which of the following random variables has/have finite expectation?

(A) X_1

(B) $\frac{1}{X_2}$

(C) $\sqrt{X_1}$

(D) $\min\{X_1, ..., X_n\}$

---

37. A sample of size n is drawn randomly (without replacement) from an urn containing $5n^2$ balls, of which $2n^2$ are red balls and $3n^2$ are black balls. Let X_n denote the number of red balls in the selected sample. If $\ell = \lim_{n\to\infty} \frac{E(X_n)}{n}$ and $m = \lim_{n\to\infty} \frac{\text{Var}(X_n)}{n}$, then which of the following statements is/are TRUE?

(A) $\ell + m = \frac{16}{25}$

(B) $\ell - m = \frac{3}{25}$

(C) $\ell m = \frac{14}{125}$

(D) $\frac{\ell}{m} = \frac{5}{3}$

---

38. Let $X_1, X_2, ..., X_n$ $(n \geq 2)$ be a random sample from a distribution with probability density function

$$f(x;\theta) = \begin{cases} \frac{1}{2\theta}, & -\theta \leq x \leq \theta, \\ 0, & |x| > \theta, \end{cases}$$

where $\theta \in (0, \infty)$ is unknown. If $R = \min\{X_1, X_2, ..., X_n\}$ and $S = \max\{X_1, X_2, ..., X_n\}$, then which of the following statements is/are TRUE?

(A) (R, S) is jointly sufficient for θ

(B) S is an MLE of θ

(C) $\max\{|X_1|, |X_2|, ..., |X_n|\}$ is a complete and sufficient statistic for θ

(D) Distribution of $\frac{R}{S}$ does NOT depend on θ

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39. Let $X_1, X_2, ..., X_n$ $(n \geq 2)$ be a random sample from a distribution with probability density function

$$f(x;\theta) = \begin{cases} \dfrac{3x^2}{\theta} e^{-x^3/\theta}, & x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta \in (0, \infty)$ is unknown. If $T = \sum_{i=1}^{n} X_i^3$, then which of the following statements is/are TRUE?

(A) $\dfrac{n-1}{T}$ is the unique uniformly minimum variance unbiased estimator (UMVUE) of $\dfrac{1}{\theta}$

(B) $\dfrac{n}{T}$ is the unique uniformly minimum variance unbiased estimator of $\dfrac{1}{\theta}$

(C) $(n-1)\sum_{i=1}^{n} \dfrac{1}{X_i^3}$ is the unique uniformly minimum variance unbiased estimator of $\dfrac{1}{\theta}$

(D) $\dfrac{n}{T}$ is the MLE of $\dfrac{1}{\theta}$

---

40. Let $X_1, X_2, ..., X_n$ $(n \geq 2)$ be a random sample from a distribution with probability density function

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1}, & 0 \leq x \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta \in (0, \infty)$ is unknown. Then, which of the following statements is/are TRUE?

(A) Cramer-Rao lower bound, based on $X_1, X_2, ..., X_n$, for the estimator θ^3 is $\dfrac{9\theta^6}{n}$

(B) Cramer-Rao lower bound, based on $X_1, X_2, ..., X_n$, for the estimator θ^3 is $\dfrac{9\theta^4}{n}$

(C) There does NOT exist any unbiased estimator of $\dfrac{1}{\theta}$ which attains the Cramer-Rao lower bound

(D) There exists an unbiased estimator of $\dfrac{1}{\theta}$ which attains the Cramer-Rao lower bound

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41. Let α, β and γ be the eigenvalues of

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 3 & 3 \\ -1 & 2 & 2 \end{bmatrix}.$$

If $\gamma = 1$ and $\alpha > \beta$, then the value of $2\alpha + 3\beta$ is ....

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42. Let

$$M = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$$

be a 2×2 matrix. If $\alpha = \det(M^4 - 6I_2)$, then the value of α^2 is ....

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43. Let $S = \{(x, y) \in \mathbb{R}^2 : 2 \leq x \leq y \leq 4\}$. Then, the value of the integral

$$\iint_S \frac{1}{4 - x} \, dx \, dy$$

is ....

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44. Let $A = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq y \leq z \leq 1\}$. Let α be the value of the integral

$$\iiint_A xyz \, dx \, dy \, dz.$$

Then, 384α is equal to ....

---

45. Let f_0 and f_1 be the probability mass functions given by:

x	1	2	3	4	5	6
$f_0(x)$	0.1	0.1	0.1	0.1	0.1	0.5
$f_1(x)$	0.1	0.1	0.2	0.2	0.2	0.2

Consider the problem of testing the null hypothesis $H_0 : X \sim f_0$ against $H_1 : X \sim f_1$ based on a single sample X. If α and β, respectively, denote the size and power of the test with critical region $\{x \in \mathbb{R} : x > 3\}$, then $10(\alpha + \beta)$ is equal to ....

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46. Let $5, 10, 4, 15, 6$ be an observed random sample of size 5 from a distribution with probability density function

$$f(x;\theta) = \begin{cases} e^{-(x-\theta)}, & x \geq \theta, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta \in (-\infty, 3]$ is unknown. Then, the maximum likelihood estimate (MLE) of θ based on the observed sample is equal to ....

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47. Let

$$\alpha = \lim_{n\to\infty} \sum_{m=n^2}^{2n^2} \frac{1}{\sqrt{5n^4 + n^3 + m}}.$$

Then, $10\sqrt{5}\,\alpha$ is equal to ....

---

48. Let X be a random variable having the probability density function

$$f(x) = \frac{1}{8\sqrt{2\pi}} \left(2e^{-x^2/2} + 3e^{-x^2/8}\right), \quad x \in \mathbb{R}.$$

Then, $4E(X^4)$ is equal to ....

---

49. Let X be a random variable with moment generating function

$$M_X(t) = \frac{1}{12} + \frac{1}{6}e^t + \frac{1}{3}e^{2t} + \frac{1}{4}e^{-t} + \frac{1}{6}e^{-2t}, \quad t \in \mathbb{R}.$$

Then, $8E(X)$ is equal to ....

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50. Let B denote the length of the curve $y = \ln(\sec x)$ from $x = 0$ to $x = \frac{\pi}{4}$. Then, the value of $3\sqrt{2}(e^B - 1)$ is equal to ....

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51. Let $S \subseteq \mathbb{R}^2$ be the region bounded by the parallelogram with vertices at the points $(1,0), (3,2), (3,5)$ and $(1,3)$. Then, the value of the integral

$$\iint_S (x+2y)\, dx\, dy$$

is equal to ....

---

52. Let

$$A = \left\{(x,y) \in \mathbb{R}^2 : x^2 - \frac{1}{2\sqrt{\pi}} < y < x^2 + \frac{1}{2\sqrt{\pi}}\right\}$$

and let the joint probability density function of (X,Y) be

$$f(x,y) = \begin{cases} e^{-(x-1)^2}, & (x,y) \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the covariance between the random variables X and Y is equal to ....

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53. Let X_1 and X_2 be independent $N(0,1)$ random variables. Define

$$\text{sgn}(u) = \begin{cases} -1, & \text{if } u < 0, \\ 0, & \text{if } u = 0, \\ 1, & \text{if } u > 0. \end{cases}$$

Let $Y_1 = X_1 \,\text{sgn}(X_2)$ and $Y_2 = X_2 \,\text{sgn}(X_1)$. If the correlation coefficient between Y_1 and Y_2 is α, then $\pi\alpha$ is equal to ....

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54. Let

$$a_n = \sum_{k=2}^{n} \binom{n}{k} \frac{2^k (n-2)^{n-k}}{n^n}, \quad n = 2, 3, \ldots$$

Then,

$$e^2 \lim_{n\to\infty} (1 - a_n)$$

is equal to ....

55. Let E_1, E_2, E_3 and E_4 be four independent events such that

$$P(E_1) = \frac{1}{2}, \quad P(E_2) = \frac{1}{3}, \quad P(E_3) = \frac{1}{4}, \quad P(E_4) = \frac{1}{5}.$$

Let p be the probability that at most two events among E_1, E_2, E_3, E_4 occur. Then, $240p$ is equal to ....

56. Let the random vector (X, Y) have the joint probability mass function

$$f(x,y) = \begin{cases} \binom{10}{x}\binom{5}{y}\left(\frac{1}{4}\right)^{x-y+5}\left(\frac{3}{4}\right)^{y-x+10}, & x = 0, 1, \ldots, 10;\ y = 0, 1, \ldots, 5, \\ 0, & \text{otherwise.} \end{cases}$$

Let $Z = Y - X + 10$. If $\alpha = E(Z)$ and $\beta = \text{Var}(Z)$, then $8\alpha + 48\beta$ is equal to ....

57. Let

$$S = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq \pi, \min(\sin x, \cos x) \leq y \leq \max(\sin x, \cos x)\}.$$

If α is the area of S, then the value of $2\sqrt{2}\,\alpha$ is equal to ....

58. The number of real roots of the polynomial

$$f(x) = x^{11} - 13x + 5$$

is ....

59. Let

$$\alpha = \lim_{n\to\infty}\left(1 + n\sin\frac{3}{n^2}\right)^{2n}.$$

Then, $\ln \alpha$ is equal to ....

60. Let $\phi : (-1, 1) \to \mathbb{R}$ be defined by

$$\phi(x) = \int_{x^7}^{x^4} \frac{1}{1 + t^3}\, dt.$$

If

$$\alpha = \lim_{x \to 0} \frac{\phi(x)}{e^{x^4} - 1},$$

then 42α is equal to ....
