IIT JAM 2021 Mathematics (MA) Question Paper

Time Allowed :3 Hours	**Maximum Marks :**100	**Total questions :**60

General Instructions

General Instructions:

i) All questions are compulsory. Marks allotted to each question are indicated in the margin.

ii) Answers must be precise and to the point.

iii) In numerical questions, all steps of calculation should be shown clearly.

iv) Use of non-programmable scientific calculators is permitted.

v) Wherever necessary, write balanced chemical equations with proper symbols and units.

vi) Rough work should be done only in the space provided in the question paper.

1. Let $0 < \alpha < 1$ be a real number. The number of differentiable functions $y : [0, 1] \to [0, \infty)$, having continuous derivative on $[0, 1]$ and satisfying

$$y'(t) = (y(t))^{\alpha},\ t \in [0, 1], \quad y(0) = 0,$$

is

(A) exactly one.
(B) exactly two.
(C) finite but more than two.
(D) infinite.

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2. Let $P : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $P(x) > 0$ for all $x \in \mathbb{R}$. Let y be a twice differentiable function on $\mathbb{R}$ satisfying

$$y''(x) + P(x)y'(x) - y(x) = 0$$

for all $x \in \mathbb{R}$. Suppose that there exist two real numbers a, b ($a < b$) such that $y(a) = y(b) = 0$. Then

(A) $y(x) = 0$ for all $x \in [a, b]$.
(B) $y(x) > 0$ for all $x \in (a, b)$.
(C) $y(x) < 0$ for all $x \in (a, b)$.
(D) $y(x)$ changes sign on (a, b).

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3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(x) = f(x + 1)$ for all $x \in \mathbb{R}$. Then

(A) f is not necessarily bounded above.
(B) There exists a unique $x_0 \in \mathbb{R}$ such that $f(x_0 + \pi) = f(x_0)$.
(C) There is no $x_0 \in \mathbb{R}$ such that $f(x_0 + \pi) = f(x_0)$.
(D) There exist infinitely many $x_0 \in \mathbb{R}$ such that $f(x_0 + \pi) = f(x_0)$.

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4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for all $x \in \mathbb{R}$,

$$\int_0^1 f(xt)\, dt = 0. \quad (*)$$

Then

(A) f must be identically 0 on the whole of $\mathbb{R}$.

(B) there is an f satisfying (*) that is identically 0 on $(0, 1)$ but not identically 0 on the whole of $\mathbb{R}$.

(C) there is an f satisfying (*) that takes both positive and negative values.

(D) there is an f satisfying (*) that is 0 at infinitely many points, but is not identically zero.

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5. Let p and t be positive real numbers. Let D_t be the closed disc of radius t centered at $(0, 0)$, i.e.,

$$D_t = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq t^2\}.$$

Define

$$I(p, t) = \iint_{D_t} \frac{dx\, dy}{(p^2 + x^2 + y^2)^2}.$$

Then $\lim_{t\to\infty} I(p, t)$ is finite

(A) only if $p > 1$.

(B) only if $p = 1$.

(C) only if $p < 1$.

(D) for no value of p.

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6. How many elements of the group $\mathbb{Z}_{50}$ have order 10?

(A) 10

(B) 4

(C) 5

(D) 8

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7. For every $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be a function. From the given choices, pick the statement that is the negation of

"For every $x \in \mathbb{R}$ and for every real number $\varepsilon > 0,$ there exists an integer $N > 0$ such that $\displaystyle\sum_{i=1}^{p} |f_{N+i}(x)|$

(A) For every $x \in \mathbb{R}$ and for every real number $\varepsilon > 0$, there does not exist any integer $N \geq 0$ such that $\sum_{i=1}^{p} |f_{N+i}(x)| < \varepsilon$ for every integer $p > 0$.

(B) For every $x \in \mathbb{R}$ and for every real number $\varepsilon > 0$, there exists an integer $N > 0$ such that $\sum_{i=1}^{p} |f_{N+i}(x)| \geq \varepsilon$ for some integer $p > 0$.

(C) There exists $x \in \mathbb{R}$ and there exists a real number $\varepsilon > 0$ such that for every integer $N > 0$, there exists an integer $p > 0$ for which $\sum_{i=1}^{p} |f_{N+i}(x)| \geq \varepsilon$.

(D) There exists $x \in \mathbb{R}$ and there exists a real number $\varepsilon > 0$ such that for every integer $N > 0$ and for every integer $p > 0$ the inequality $\sum_{i=1}^{p} |f_{N+i}(x)| < \varepsilon$ holds.

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8. Which one of the following subsets of $\mathbb{R}$ has a non-empty interior?

(A) The set of all irrational numbers in $\mathbb{R}$.

(B) The set $\{a \in \mathbb{R} : \sin(a) = 1\}$.

(C) The set $\{b \in \mathbb{R} : x^2 + bx + 1 = 0$ has distinct roots$\}$.

(D) The set of all rational numbers in $\mathbb{R}$.

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9. For an integer $k \geq 0$, let P_k denote the vector space of all real polynomials in one variable of degree less than or equal to k. Define a linear transformation $T : P_2 \to P_3$ by

$$T(f(x)) = f''(x) + xf(x).$$

Which one of the following polynomials is **not** in the range of T?

(A) $x + x^2$

(B) $x^2 + x^3 + 2$

(C) $x + x^3 + 2$

(D) $x + 1$

10. Let $n > 1$ be an integer. Consider the following two statements for an arbitrary $n \times n$ matrix A with complex entries.

I. If $A^k = I_n$ for some integer $k \geq 1$, then all the eigenvalues of A are k^{th} roots of unity.

II. If, for some integer $k \geq 1$, all the eigenvalues of A are k^{th} roots of unity, then $A^k = I_n$.

Then

(A) both I and II are TRUE.

(B) I is TRUE but II is FALSE.

(C) I is FALSE but II is TRUE.

(D) neither I nor II is TRUE.

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11. Let $M_n(\mathbb{R})$ be the real vector space of all $n \times n$ matrices with real entries, $n \geq 2$. Let $A \in M_n(\mathbb{R})$. Consider the subspace W of $M_n(\mathbb{R})$ spanned by $\{I_n, A, A^2, A^3, \ldots\}$. Then the dimension of W over $\mathbb{R}$ is necessarily

(A) ∞.

(B) n^2.

(C) n.

(D) at most n.

---

12. Let y be the solution of

$$(1+x)y''(x) + y'(x) - \frac{1}{1+x}y(x) = 0, \quad x \in (-1, \infty),$$

with initial conditions $y(0) = 1,\ y'(0) = 0$.

Then

(A) y is bounded on $(0, \infty)$.

(B) y is bounded on $(-1, 0]$.

(C) $y(x) \geq 2$ on $(-1, \infty)$.

(D) y attains its minimum at $x = 0$.

13. Consider the surface

$$S = \{(x, y, xy) \in \mathbb{R}^3 : x^2 + y^2 \leq 1\}.$$

Let $\vec{F} = y\hat{i} + x\hat{j} + \hat{k}$. If $\hat{n}$ is the continuous unit normal field to the surface S with positive z-component, then

$$\iint_S \vec{F} \cdot \hat{n}\, dS$$

equals

(A) $\frac{\pi}{4}$

(B) $\frac{\pi}{2}$

(C) π

(D) 2π

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14. Consider the following statements.

I. The group $(\mathbb{Q}, +)$ has no proper subgroup of finite index.

II. The group $(\mathbb{C} \setminus \{0\}, \cdot)$ has no proper subgroup of finite index.

Which one of the following statements is true?

(A) Both I and II are TRUE.

(B) I is TRUE but II is FALSE.

(C) II is TRUE but I is FALSE.

(D) Neither I nor II is TRUE.

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15. Let $f : \mathbb{N} \to \mathbb{N}$ be a bijective map such that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^2} < +\infty.$$

The number of such bijective maps is

(A) exactly one.

(B) zero.

(C) finite but more than one.

(D) infinite.

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16. Define

$$S = \lim_{n\to\infty} \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{n^2}\right).$$

Then

(A) $S = \frac{1}{2}$.

(B) $S = \frac{1}{4}$.

(C) $S = \frac{1}{3}$.

(D) $S = \frac{3}{4}$.

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17. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function such that for all $a, b \in \mathbb{R}$ with $a < b$,

$$\frac{f(b) - f(a)}{b - a} = f''\left(\frac{a+b}{2}\right).$$

Then

(A) f must be a polynomial of degree less than or equal to 2.

(B) f must be a polynomial of degree greater than 2.

(C) f is not a polynomial.

(D) f must be a linear polynomial.

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18. Consider the function

$$f(x) = \begin{cases} 1, & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cup \{0\}, \\ 1 - \frac{1}{p}, & \text{if } x = \frac{n}{p},\ n \in \mathbb{Z} \setminus \{0\},\ p \in \mathbb{N},\ \gcd(n, p) = 1. \end{cases}$$

Then

(A) all $x \in \mathbb{Q} \setminus \{0\}$ are strict local minima for f.

(B) f is continuous at all $x \in \mathbb{Q}$.

(C) f is not continuous at all $x \in \mathbb{R} \setminus \mathbb{Q}$.

(D) f is not continuous at $x = 0$.

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19. Consider the family of curves $x^2 - y^2 = ky$ with parameter $k \in \mathbb{R}$. The equation of the orthogonal trajectory to this family passing through $(1, 1)$ is given by

(A) $x^3 + 3xy^2 = 4$.

(B) $x^2 + 2xy = 3$.

(C) $y^2 + 2x^2y = 3$.

(D) $x^3 + 2xy^2 = 3$.

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20. Which one of the following statements is true?

(A) Exactly half of the elements in any even-order subgroup of S_5 must be even permutations.

(B) Any abelian subgroup of S_5 is trivial.

(C) There exists a cyclic subgroup of S_5 of order 6.

(D) There exists a normal subgroup of S_5 of index 7.

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21. Let $f : [0, 1] \to [0, \infty)$ be a continuous function such that

$$(f(t))^2 < 1 + 2\int_0^t f(s)\,ds, \quad \text{for all } t \in [0, 1].$$

Then

(A) $f(t) < 1 + t$ for all $t \in [0, 1]$.

(B) $f(t) > 1 + t$ for all $t \in [0, 1]$.

(C) $f(t) = 1 + t$ for all $t \in [0, 1]$.

(D) $f(t) < 1 + \frac{t}{2}$ for all $t \in [0, 1]$.

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22. Let A be an $n \times n$ invertible matrix and C be an $n \times n$ nilpotent matrix. If $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ is a $2n \times 2n$ matrix (each X_{ij} being $n \times n$) that commutes with the $2n \times 2n$ matrix

$$B = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix},$$

then

(A) X_{11} and X_{22} are necessarily zero matrices.

(B) X_{12} and X_{21} are necessarily zero matrices.

(C) X_{11} and X_{21} are necessarily zero matrices.

(D) X_{12} and X_{22} are necessarily zero matrices.

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23. Let $D \subseteq \mathbb{R}^2$ be defined by

$$D = \mathbb{R}^2 \setminus \{(x, 0) : x \in \mathbb{R}\}.$$

Consider the function $f : D \to \mathbb{R}$ defined by

$$f(x, y) = x \sin\left(\frac{1}{y}\right).$$

Then

(A) f is a discontinuous function on D.

(B) f is a continuous function on D and cannot be extended continuously to any point outside D.

(C) f is a continuous function on D and can be extended continuously to $D \cup \{(0, 0)\}$.

(D) f is a continuous function on D and can be extended continuously to the whole of $\mathbb{R}^2$.

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24. Which one of the following statements is true?

(A) $(\mathbb{Z}, +)$ is isomorphic to $(\mathbb{R}, +)$.

(B) $(\mathbb{Z}, +)$ is isomorphic to $(\mathbb{Q}, +)$.

(C) $(\mathbb{Q}/\mathbb{Z}, +)$ is isomorphic to $(\mathbb{Q}/2\mathbb{Z}, +)$.

(D) $(\mathbb{Q}/\mathbb{Z}, +)$ is isomorphic to $(\mathbb{Q}, +)$.

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25. Let y be a twice differentiable function on $\mathbb{R}$ satisfying

$$y''(x) = 2 + e^{-|x|}, \quad x \in \mathbb{R}, \quad y(0) = -1, \quad y'(0) = 0.$$

Then

(A) $y = 0$ has exactly one root.

(B) $y = 0$ has exactly two roots.

(C) $y = 0$ has more than two roots.

(D) There exists an $x_0 \in \mathbb{R}$ such that $y(x_0) \geq y(x)$ for all $x \in \mathbb{R}$.

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26. Let $f : [0,1] \to [0,1]$ be a non-constant continuous function such that $f \circ f = f$. Define

$$E_f = \{x \in [0,1] : f(x) = x\}.$$

Then

(A) E_f is neither open nor closed.

(B) E_f is an interval.

(C) E_f is empty.

(D) E_f need not be an interval.

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27. Let g be an element of S_7 such that g commutes with the element $(2, 6, 4, 3)$. The number of such g is

(A) 6

(B) 4

(C) 24

(D) 48

28. Let G be a finite abelian group of odd order. Consider the following two statements:

I. The map $f : G \to G$ defined by $f(g) = g^2$ is a group isomorphism.

II. The product $\prod_{g \in G} g = e$.

Which one of the following statements is true?

(A) Both I and II are TRUE.

(B) I is TRUE but II is FALSE.

(C) II is TRUE but I is FALSE.

(D) Neither I nor II is TRUE.

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29. Let $n \geq 2$ be an integer. Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be the linear transformation defined by

$$A(z_1, z_2, \ldots, z_n) = (z_n, z_1, z_2, \ldots, z_{n-1}).$$

Which one of the following statements is true for every $n \geq 2$?

(A) A is nilpotent.

(B) All eigenvalues of A are of modulus 1.

(C) Every eigenvalue of A is either 0 or 1.

(D) A is singular.

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30. Consider the two series

$$\text{I. } \sum_{n=1}^{\infty} \frac{1}{n^{1+(1/n)}} \quad \text{and} \quad \text{II. } \sum_{n=1}^{\infty} \frac{1}{n(\ln n)^{1/n}}.$$

Which one of the following holds?

(A) Both I and II converge.

(B) Both I and II diverge.

(C) I converges and II diverges.

(D) I diverges and II converges.

31. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with the property that for every $y \in \mathbb{R}$, the value of the expression

$$\sup_{x \in \mathbb{R}} [xy - f(x)]$$

is finite. Define $g(y) = \sup_{x\in\mathbb{R}}[xy - f(x)]$ for $y \in \mathbb{R}$. Then

(A) g is even if f is even.
(B) f must satisfy $\lim\limits_{|x|\to\infty} \dfrac{f(x)}{|x|} = +\infty$.
(C) g is odd if f is even.
(D) f must satisfy $\lim\limits_{|x|\to\infty} \dfrac{f(x)}{|x|} = -\infty$.

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32. Consider the equation

$$x^{2021} + x^{2020} + \cdots + x + 1 = 0.$$

Then

(A) all real roots are positive.

(B) exactly one real root is positive.

(C) exactly one real root is negative.

(D) no real root is positive.

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33. Let $D = \mathbb{R}^2 \setminus \{(0, 0)\}$. Consider the two functions $u, v : D \to \mathbb{R}$ defined by

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = xy.$$

Consider the gradients ∇u and ∇v of the functions u and v, respectively. Then

(A) ∇u and ∇v are parallel at each point (x, y) of D.

(B) ∇u and ∇v are perpendicular at each point (x, y) of D.

(C) ∇u and ∇v do not exist at some points of D.

(D) ∇u and ∇v at each point (x, y) of D span $\mathbb{R}^2$.

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34. Consider the two functions $f(x, y) = x + y$ and $g(x, y) = xy - 16$ defined on $\mathbb{R}^2$. Then

(A) The function f has no global extreme value subject to the condition $g = 0$.

(B) The function f attains global extreme values at $(4, 4)$ and $(-4, -4)$ subject to the condition $g = 0$.

(C) The function g has no global extreme value subject to the condition $f = 0$.

(D) The function g has a global extreme value at $(0, 0)$ subject to the condition $f = 0$.

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35. Let $f : (a, b) \to \mathbb{R}$ be a differentiable function on (a, b). Which of the following statements is/are true?

(A) $f'(x) > 0$ in (a, b) implies that f is increasing in (a, b).

(B) f is increasing in (a, b) implies that $f' > 0$ in (a, b).

(C) If $f'(x_0) > 0$ for some $x_0 \in (a, b)$, then there exists a $\delta > 0$ such that $f(x) > f(x_0)$ for all $x \in (x_0, x_0 + \delta)$.

(D) If $f'(x_0) > 0$ for some $x_0 \in (a, b)$, then f is increasing in a neighbourhood of x_0.

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36. Let G be a finite group of order 28. Assume that G contains a subgroup of order 7. Which of the following statements is/are true?

(A) G contains a unique subgroup of order 7.

(B) G contains a normal subgroup of order 7.

(C) G contains no normal subgroup of order 7.

(D) G contains at least two subgroups of order 7.

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37. Which of the following subsets of $\mathbb{R}$ are connected?

(A) The set $\{x \in \mathbb{R} : x \text{ is irrational}\}$.

(B) The set $\{x \in \mathbb{R} : x^3 - 1 \geq 0\}$.

(C) The set $\{x \in \mathbb{R} : x^3 + x + 1 \geq 0\}$.

(D) The set $\{x \in \mathbb{R} : x^3 - 2x + 1 \geq 0\}$.

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38. Consider the four functions from $\mathbb{R}$ to $\mathbb{R}$:

$$f_1(x) = x^4 + 3x^3 + 7x + 1, \quad f_2(x) = x^3 + 3x^2 + 4x, \quad f_3(x) = \arctan(x),$$

and

$$f_4(x) = \begin{cases} x, & \text{if } x \notin \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z}. \end{cases}$$

Which of the following subsets of $\mathbb{R}$ are open?

(A) The range of f_1.

(B) The range of f_2.

(C) The range of f_3.

(D) The range of f_4.

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39. Let V be a finite-dimensional vector space and $T : V \to V$ be a linear transformation. Let $\mathcal{R}(T)$ denote the range of T and $\mathcal{N}(T)$ denote the null space of T. If $\operatorname{rank}(T) = \operatorname{rank}(T^2)$, then which of the following is/are necessarily true?

(A) $\mathcal{N}(T) = \mathcal{N}(T^2)$.

(B) $\mathcal{R}(T) = \mathcal{R}(T^2)$.

(C) $\mathcal{N}(T) \cap \mathcal{R}(T) = \{0\}$.

(D) $\mathcal{N}(T) = \{0\}$.

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40. Let $m > 1$ and $n > 1$ be integers. Let A be an $m \times n$ matrix such that for some $m \times 1$ matrix b_1, the equation $Ax = b_1$ has infinitely many solutions. Let b_2 denote an $m \times 1$ matrix different from b_1. Then $Ax = b_2$ has

(A) infinitely many solutions for some b_2.

(B) a unique solution for some b_2.

(C) no solution for some b_2.

(D) finitely many solutions for some b_2.

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41. The number of cycles of length 4 in S_6 is ________.

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42. The value of

$$\lim_{n\to\infty} (3^n + 5^n + 7^n)^{\frac{1}{n}}$$

is ________.

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43. Let

$$B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$$

and define

$$u(x, y, z) = \sin\big(\pi(1 - x^2 - y^2 - z^2)^2\big)$$

for $(x, y, z) \in B$. Then the value of

$$\iiint_B \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) dx\, dy\, dz$$

is ________.

---

44. Consider the subset $S = \{(x, y) : x^2 + y^2 > 0\}$ of $\mathbb{R}^2$. Let

$$P(x, y) = \frac{y}{x^2 + y^2}, \quad Q(x, y) = \frac{-x}{x^2 + y^2}.$$

For $(x, y) \in S$. If C denotes the unit circle traversed in the counter-clockwise direction, then the value of

$$\frac{1}{\pi} \int_C (P\, dx + Q\, dy)$$

is ________.

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45. Consider the set

$$A = \{a \in \mathbb{R} : x^2 = a(a+1)(a+2) \text{ has a real root}\}.$$

The number of connected components of A is _________.

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46. Let V be the real vector space of all continuous functions $f : [0,2] \to \mathbb{R}$ such that the restriction of f to the interval $[0,1]$ is a polynomial of degree ≤ 2, the restriction of f to $[1,2]$ is a polynomial of degree ≤ 3, and $f(0) = 0$. Then the dimension of V is _________.

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47. The number of group homomorphisms from the group $\mathbb{Z}_4$ to the group S_3 is _________.

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48. Let $y : \left(\frac{9}{10}, 3\right) \to \mathbb{R}$ be a differentiable function satisfying

$$(x - 2y)\frac{dy}{dx} + (2x + y) = 0, \quad x \in \left(\frac{9}{10}, 3\right),$$

and $y(1) = 1$. Then $y(2)$ equals _________.

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49. Let

$$\vec{F} = (y+1)e^y \cos(x)\,\hat{\imath} + (y+2)e^y \sin(x)\,\hat{\jmath}$$

be a vector field in $\mathbb{R}^2$, and C be a continuously differentiable path with starting point $(0,1)$ and end point $\left(\frac{\pi}{2}, 0\right)$. Then

$$\int_C \vec{F} \cdot d\vec{r}$$

equals _________.

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50. The value of

$$\frac{\pi}{2} \lim_{n\to\infty} \cos\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{8}\right) \cos\left(\frac{\pi}{16}\right) \cdots \cos\left(\frac{\pi}{2^{n+1}}\right)$$

is _________.

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51. The number of elements of order two in the group S_4 is equal to _________.

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52. The least possible value of k, accurate up to two decimal places, for which the following problem has a solution is:

$$y''(t) + 2y'(t) + ky(t) = 0, \quad t \in \mathbb{R},$$

with $y(0) = 0,\ y(1) = 0,\ y(1/2) = 1.$

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53. Consider those continuous functions $f : \mathbb{R} \to \mathbb{R}$ that have the property that for every $x \in \mathbb{R}$,

$$f(x) \in \mathbb{Q} \text{ if and only if } f(x+1) \notin \mathbb{Q}.$$

The number of such functions is _________.

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54. The largest positive number a such that

$$\int_0^5 f(x)\,dx + \int_0^3 f^{-1}(x)\,dx \geq a$$

for every strictly increasing surjective continuous function $f : [0, \infty) \to [0, \infty)$ is _________.

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55. Define the sequence

$$s_n = \begin{cases} \dfrac{1}{2^n} \displaystyle\sum_{j=0}^{n-2} 2^{2j}, & \text{if } n > 0 \text{ is even,} \\ \dfrac{1}{2^n} \displaystyle\sum_{j=0}^{n-1} 2^{2j}, & \text{if } n > 0 \text{ is odd.} \end{cases}$$

Define

$$\sigma_m = \frac{1}{m} \sum_{n=1}^{m} s_n.$$

The number of limit points of the sequence $\{\sigma_m\}$ is _________.

56. The determinant of the matrix

$$\begin{pmatrix} 2021 & 2020 & 2020 & 2020 \\ 2021 & 2021 & 2020 & 2020 \\ 2021 & 2021 & 2021 & 2020 \\ 2021 & 2021 & 2021 & 2021 \end{pmatrix}$$

is _________.

57. The value of

$$\lim_{n\to\infty} \int_0^1 e^{x^2} \sin(nx)\, dx$$

is _________.

58. Let S be the surface defined by

$$\{(x, y, z) \in \mathbb{R}^3 : z = 1 - x^2 - y^2,\ z \geq 0\}.$$

Let

$$\vec{F} = -y\hat{\imath} + (x-1)\hat{\jmath} + z^2\hat{k},$$

and let $\hat{n}$ be the continuous unit normal field to the surface S with positive z-component. Then the value of

$$\frac{1}{\pi} \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$$

is _________.

59. Let

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{pmatrix}.$$

Then the largest eigenvalue of A is _________.

60. Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Consider the linear map $T_A : M_4(\mathbb{R}) \to M_4(\mathbb{R})$ defined by

$$T_A(X) = AX - XA.$$

Then the dimension of the range of T_A is _________.