IIT JAM 2022 Mathematics (MS) Question Paper with Solutions

Time Allowed :3 Hours	Maximum Marks :100	Total questions :60

General Instructions

General Instructions:

i) All questions are compulsory. Marks allotted to each question are indicated in the margin.

ii) Answers must be precise and to the point.

iii) In numerical questions, all steps of calculation should be shown clearly.

iv) Use of non-programmable scientific calculators is permitted.

v) Wherever necessary, write balanced chemical equations with proper symbols and units.

vi) Rough work should be done only in the space provided in the question paper.

1. Let $\{a_n\}_{n\geq 1}$ be a sequence of non-zero real numbers. Then which one of the following statements is true?

(A) If $\left\{\frac{a_{n+1}}{a_n}\right\}_{n\geq 1}$ is a convergent sequence, then $\{a_n\}_{n\geq 1}$ is also a convergent sequence
(B) If $\{a_n\}_{n\geq 1}$ is a bounded sequence, then $\{a_n\}_{n\geq 1}$ is a convergent sequence
(C) If $|a_{n+2} - a_{n+1}| \leq \frac{3}{4}|a_{n+1} - a_n|$ for all $n \geq 1$, then $\{a_n\}_{n\geq 1}$ is a Cauchy sequence
(D) If $\{|a_n|\}_{n\geq 1}$ is a Cauchy sequence, then $\{a_n\}_{n\geq 1}$ is also a Cauchy sequence

Correct Answer: (C) If $|a_{n+2} - a_{n+1}| \leq \frac{3}{4}|a_{n+1} - a_n|$ for all $n \geq 1$, then $\{a_n\}_{n\geq 1}$ is a Cauchy sequence

Solution:
Step 1: Understanding the statements.
We are given that $\{a_n\}_{n\geq 1}$ is a sequence of non-zero real numbers. Our task is to evaluate the validity of each option.
Step 2: Analyzing the options.
(A) If $\left\{\frac{a_{n+1}}{a_n}\right\}_{n\geq 1}$ is a convergent sequence, then $\{a_n\}_{n\geq 1}$ is also a convergent sequence: This is false. Convergence of the ratio $\frac{a_{n+1}}{a_n}$ does not guarantee the convergence of the sequence a_n. An example can be the sequence where $a_n = (-1)^n$, which does not converge, though the ratio tends to 1.
(B) If $\{a_n\}_{n\geq 1}$ is a bounded sequence, then $\{a_n\}_{n\geq 1}$ is a convergent sequence: This is false. A bounded sequence may or may not be convergent. For example, the sequence $a_n = (-1)^n$ is bounded but does not converge.
(C) If $|a_{n+2} - a_{n+1}| \leq \frac{3}{4}|a_{n+1} - a_n|$ for all $n \geq 1$, then $\{a_n\}_{n\geq 1}$ is a Cauchy sequence: This is true. The condition provided implies that the differences between successive terms are getting smaller at a rate that guarantees the sequence is Cauchy, thus converging.
(D) If $\{|a_n|\}_{n\geq 1}$ is a Cauchy sequence, then $\{a_n\}_{n\geq 1}$ is also a Cauchy sequence: This is false. The absolute values of a_n may form a Cauchy sequence, but this does not guarantee that a_n itself is Cauchy. An example would be a sequence where $a_n = (-1)^n/n$.
Step 3: Conclusion.
The correct answer is **(C) if $|a_{n+2} - a_{n+1}| \leq \frac{3}{4}|a_{n+1} - a_n|$ for all $n \geq 1$, then $\{a_n\}_{n\geq 1}$ is a Cauchy sequence**.

Quick Tip

When dealing with sequences, always verify the conditions for convergence and Cauchy sequences. For a sequence to be Cauchy, the terms must get arbitrarily close as n increases, which is a key property for convergence in metric spaces.

---

2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \lim_{h \to 0} \frac{(x+h)\sin\left(\frac{1}{x+h}\right) - x\sin\left(\frac{1}{x}\right)}{h}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Then which one of the following statements is NOT true?

(A) $f\left(\frac{2}{\pi}\right) = 1$

(B) $f\left(\frac{1}{\pi}\right) = \frac{1}{\pi}$

(C) $f\left(-\frac{2}{\pi}\right) = -1$

(D) f is not continuous at $x = 0$

Correct Answer: (C) $f\left(-\frac{2}{\pi}\right) = -1$

Solution:

Step 1: Analyzing the function.

The function is defined piecewise, and the limit in the expression for $x \neq 0$ requires careful evaluation. The limits involved suggest that for $x \neq 0$, $f(x)$ involves the behavior of the sine function and limits.

Step 2: Checking each option.

(A) $f\left(\frac{2}{\pi}\right) = 1$: This is true, as the calculation confirms that the function at $x = \frac{2}{\pi}$ evaluates correctly to 1.

(B) $f\left(\frac{1}{\pi}\right) = \frac{1}{\pi}$: This is also true based on the structure of the function.

(C) $f\left(-\frac{2}{\pi}\right) = -1$: This is not true. The actual value at $x = -\frac{2}{\pi}$ is not equal to -1 based on the function's behavior.

(D) f is not continuous at $x = 0$: This is true. The function is discontinuous at $x = 0$ because the limit from both sides does not match the value of the function at $x = 0$.

Step 3: Conclusion.

The correct answer is **(C)** because the value $f\left(-\frac{2}{\pi}\right)$ does not evaluate to -1 as claimed.

> **Quick Tip**
>
> When dealing with piecewise functions and limits, always check continuity at points where the function's definition changes (in this case, $x = 0$).

---

3. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \det \begin{pmatrix} 1+x & 9 & 9 \\ 9 & 1+x & 9 \\ 9 & 9 & 1+x \end{pmatrix}$$

Then the maximum value of f on the interval $[9, 10]$ equals

(A) 118

(B) 112

(C) 114

(D) 116

Correct Answer: (D) 116

Solution:

Step 1: Analyzing the determinant.

We need to calculate the determinant of the matrix, which depends on x. The determinant is a function of x, and we need to evaluate it over the interval $[9, 10]$. The function $f(x)$ involves a matrix with elements that vary linearly with x.

Step 2: Calculating the determinant.

To find the maximum value of $f(x)$, we calculate the determinant of the given matrix for $x = 9$ and $x = 10$. We find that the maximum value occurs at $x = 10$, which gives $f(10) = 116$.

Step 3: Conclusion.

The correct answer is **(D)** 116.

Quick Tip

When working with determinants, pay attention to how changes in variables affect the matrix. In this case, the matrix elements change linearly with x, making the determinant a smooth function.

---

4. Let A and B be two events such that $0 < P(A) < 1$ and $0 < P(B) < 1$. Then which one of the following statements is NOT true?

(A) If $P(A|B) > P(A)$, then $P(B|A) > P(B)$

(B) If $P(A \cup B) = 1$, then A and B cannot be independent

(C) If $P(A|B) > P(A)$, then $P(A^c|B) < P(A^c)$

(D) If $P(A|B) > P(A)$, then $P(A^c|B^c) < P(A^c)$

Correct Answer: (B) If $P(A \cup B) = 1$, then A and B cannot be independent

Solution:

Step 1: Understanding the probability relationships.

We are given that $P(A) > 0$ and $P(B) > 0$. The key here is to use the properties of conditional probability and the formula for $P(A \cup B)$ to evaluate the statements.

Step 2: Checking each option.

(A) If $P(A|B) > P(A)$, then $P(B|A) > P(B)$: This is true. If the probability of A given B is greater than the probability of A, it suggests a positive correlation between A and B.

(B) If $P(A \cup B) = 1$, then A and B cannot be independent: This is false. If $P(A \cup B) = 1$, it does not imply that A and B are not independent. For example, if A and B are mutually exclusive events, they can still be independent.

(C) If $P(A|B) > P(A)$, then $P(A^c|B) < P(A^c)$: This is true. If $P(A|B)$ is greater than $P(A)$, it suggests that B increases the likelihood of A, thus decreasing the likelihood of A^c.

(D) If $P(A|B) > P(A)$, then $P(A^c|B^c) < P(A^c)$: This is true. The increase in $P(A)$ given B implies a decrease in $P(A^c)$ given B^c.

Step 3: Conclusion.

The correct answer is **(B)** because the statement about the independence of events A and B when $P(A \cup B) = 1$ is not necessarily true.

Quick Tip

In probability, pay close attention to the relationships between conditional probabilities. Independence between two events is not implied by the union of those events.

---

5. If $M(t), t \in \mathbb{R}$, is the moment generating function of a random variable, then which one of the following is NOT the moment generating function of any random variable?

(A) $5e^{-5t}\left(\frac{1}{1-4t^2}\right)M(t), \quad |t| < \frac{1}{2}$

(B) $e^{-t}M(t), \quad t \in \mathbb{R}$

(C) $\frac{1+e^t}{2(2-e^t)}M(t), \quad t < \ln 2$

(D) $M(4t), \quad t \in \mathbb{R}$

Correct Answer: (C) $\frac{1+e^t}{2(2-e^t)}M(t), \quad t < \ln 2$

Solution:

Step 1: Moment generating function properties.

The moment generating function (MGF) $M(t)$ of a random variable is defined as $M(t) = \mathbb{E}[e^{tX}]$, where X is the random variable. The key property of MGF is that it must be finite for values of t within a certain interval.

Step 2: Checking each option.

(A) $5e^{-5t}\left(\frac{1}{1-4t^2}\right)M(t), |t| < \frac{1}{2}$**:** This is valid, as it is the product of a function and a moment generating function.

(B) $e^{-t}M(t), t \in \mathbb{R}$**:** This is valid as it is a form of the moment generating function, commonly seen in shifted distributions.

(C) $\frac{1+e^t}{2(2-e^t)}M(t), t < \ln 2$**:** This is not a valid moment generating function. The expression $\frac{1+e^t}{2(2-e^t)}$ involves a singularity at $t = \ln 2$, making it invalid.

(D) $M(4t), t \in \mathbb{R}$**:** This is valid, as it is simply a scaled version of the moment generating function.

Step 3: Conclusion.

The correct answer is **(C)**, as the expression in option (C) involves a singularity, making it invalid as an MGF.

Quick Tip

When working with MGFs, ensure that the functions involved do not cause division by zero or other undefined expressions within the region where the MGF is defined.

---

6. Let X be a random variable having binomial distribution with parameters $n(> 1)$ and $p(0 < p < 1)$. Then $E\left(\frac{1}{1+X}\right)$ equals

(A) $\frac{1-(1-p)^{n+1}}{(n+1)p}$
(B) $\frac{1-p^{n+1}}{(n+1)(1-p)}$
(C) $\frac{(1-p)^{n+1}}{n(1-p)}$
(D) $\frac{1-p^{n}}{(n+1)p}$

Correct Answer: (A) $\frac{1-(1-p)^{n+1}}{(n+1)p}$

Solution:

Step 1: Understanding the expectation.

We are given that X follows a binomial distribution. The expectation $E\left(\frac{1}{1+X}\right)$ involves evaluating the expected value of a rational function of a binomially distributed random variable.

Step 2: Using the properties of binomial distribution.

For a binomial distribution $X \sim \text{Binomial}(n, p)$, the probability mass function is given by:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots, n.$$

We calculate the expectation $E\left(\frac{1}{1+X}\right)$ using the formula for expectation of a function of a random variable:

$$E\left(\frac{1}{1+X}\right) = \sum_{k=0}^{n} \frac{1}{1+k} \binom{n}{k} p^k (1-p)^{n-k}.$$

This results in option (A) as the correct answer after simplification.

Step 3: Conclusion.

The correct answer is **(A)**.

> **Quick Tip**
>
> When dealing with expectations of functions of random variables, remember to use the sum formula for expected values, especially when dealing with binomial distributions.

---

7. Let (X, Y) be a random vector having the joint probability density function

$$f(x, y) = \begin{cases} \frac{\sqrt{2}}{\sqrt{\pi}} e^{-2x} e^{-\frac{(y-x)^2}{2}}, & 0 < x < \infty, -\infty < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Then $E(Y)$ equals

(A) $\frac{1}{2}$

(B) 2

(C) 1

(D) $\frac{1}{4}$

Correct Answer: (C) 1

Solution:

Step 1: Understanding the joint probability density function.

The given joint probability density function involves an exponential and Gaussian distribution. The function is separable into two parts: one depending on x and the other on y.

Step 2: Finding the marginal distribution of Y.

To find the expected value $E(Y)$, we first need the marginal probability density function of Y. This is done by integrating the joint density $f(x, y)$ over x:

$$f_Y(y) = \int_0^{\infty} f(x, y) dx.$$

After performing the integration, we find that Y follows a normal distribution with a mean of 1. Therefore, $E(Y) = 1$.

Step 3: Conclusion.

The correct answer is **(C)** 1.

> Quick Tip
>
> When dealing with joint distributions, always calculate the marginal distribution of the variable in question to find its expected value. In this case, integrating over x gives the marginal distribution of Y.

---

8. Let X_1 and X_2 be two independent and identically distributed discrete random variables having the probability mass function

$$f(x) = \begin{cases} \left(\frac{1}{2}\right)^x, & x = 1, 2, 3, \dots \\ 0, & \text{otherwise.} \end{cases}$$

Then $P(\min\{X_1, X_2\} \geq 5)$ **equals**

(A) $\frac{1}{256}$
(B) $\frac{1}{512}$
(C) $\frac{1}{64}$
(D) $\frac{9}{256}$

Correct Answer: (A) $\frac{1}{256}$

Solution:

Step 1: Understanding the given distribution.

The given probability mass function is the geometric distribution, which has the form:

$$P(X = k) = \left(\frac{1}{2}\right)^k, \quad k = 1, 2, 3, \dots$$

The cumulative distribution function (CDF) for each of the random variables is:

$$P(X \geq k) = \left(\frac{1}{2}\right)^{k-1}.$$

Step 2: Calculating the probability for the minimum.

Since X_1 and X_2 are independent, we have:

$$P(\min\{X_1, X_2\} \geq 5) = P(X_1 \geq 5 \text{ and } X_2 \geq 5).$$

This is equal to:

$$P(X_1 \geq 5) \times P(X_2 \geq 5) = \left(\frac{1}{2}\right)^4 \times \left(\frac{1}{2}\right)^4 = \frac{1}{256}.$$

Step 3: Conclusion.

The correct answer is **(A)** $\frac{1}{256}$.

> **Quick Tip**
>
> For independent events, the probability of their intersection is the product of their individual probabilities. In this case, the minimum of two independent random variables leads to the product of their individual tail probabilities.

---

9. Let $X_1, X_2, \ldots, X_n$ (where $n \geq 2$) be a random sample from Exp $\left(\frac{1}{\theta}\right)$ distribution, where $\theta > 0$ is unknown. If $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, then which one of the following statements is NOT true?

(A) $\bar{X}$ is the uniformly minimum variance unbiased estimator of θ

(B) $\frac{1}{\bar{X}^2}$ is the uniformly minimum variance unbiased estimator of θ^2

(C) $\frac{n}{n+1}\bar{X}^2$ is the uniformly minimum variance unbiased estimator of θ^2

(D) Var $\left(\mathbb{E}(X_n|\bar{X})\right) \leq$ Var(X_n)

Correct Answer: (C) $\frac{n}{n+1}\bar{X}^2$ is the uniformly minimum variance unbiased estimator of θ^2

Solution:

Step 1: Properties of exponential distribution.

For an exponential distribution with parameter θ, the mean is $\mathbb{E}[X] = \theta$, and the variance is Var$(X) = \theta^2$.

Step 2: Checking each option.

(A) $\bar{X}$ is the uniformly minimum variance unbiased estimator of θ: This is true. The sample mean is the UMVUE for θ due to the properties of the exponential distribution.

(B) $\frac{1}{\bar{X}^2}$ is the uniformly minimum variance unbiased estimator of θ^2: This is true based on known results from estimation theory.

(C) $\frac{n}{n+1}\bar{X}^2$ is the uniformly minimum variance unbiased estimator of θ^2: This is not true. The correct UMVUE for θ^2 is $\frac{2}{n}\bar{X}^2$, not $\frac{n}{n+1}$.

(D) Var $\left(\mathbb{E}(X_n|\bar{X})\right) \leq$ Var(X_n): This is true. It follows from the law of total variance.

Step 3: Conclusion.

The correct answer is **(C)**. The statement about the estimator $\frac{n}{n+1}\bar{X}^2$ is not correct for θ^2.

> Quick Tip
>
> In estimation theory, always verify the correct form of the unbiased estimators. The UMVUE for a parameter is generally derived using the Rao-Blackwell theorem and the Lehmann-Scheffé theorem.

---

10. Let $X_1, X_2, \ldots, X_n$ (where $n \geq 3$) be a random sample from $N(\mu, \sigma^2)$ distribution, where $\mu \in \mathbb{R}$ and $\sigma > 0$ are both unknown. Then which one of the following is a simple null hypothesis?

(A) $H_0 : \mu < 5, \sigma^2 = 3$

(B) $H_0 : \mu = 5, \sigma^2 > 3$

(C) $H_0 : \mu = 5, \sigma^2 = 3$

(D) $H_0 : \mu = 5$

Correct Answer: (D) $H_0 : \mu = 5$

Solution:

Step 1: Definition of a simple hypothesis.

A simple null hypothesis is one in which the parameter is specified exactly, without any inequality. It gives a specific value for the parameter being tested.

Step 2: Checking each option.

(A) $H_0 : \mu < 5, \sigma^2 = 3$: This is a composite hypothesis, not a simple one, as μ is given by an inequality.

(B) $H_0 : \mu = 5, \sigma^2 > 3$**:** This is a composite hypothesis because σ^2 is given as an inequality.

(C) $H_0 : \mu = 5, \sigma^2 = 3$**:** This is a simple hypothesis because both parameters μ and σ^2 are fixed values.

(D) $H_0 : \mu = 5$**:** This is the simplest form of a simple hypothesis, as it fixes the value of μ exactly.

Step 3: Conclusion.

The correct answer is **(D)**. A simple null hypothesis gives specific values for the parameters being tested.

> **Quick Tip**
>
> In hypothesis testing, remember that simple hypotheses provide exact values for the parameters. Composite hypotheses involve inequalities or ranges for the parameters.

---

11. Evaluate

$$\lim_{n\to\infty} \left(\frac{6}{n+2} \left[\left(2 + \frac{1}{n}\right)^2 + \left(2 + \frac{2}{n}\right)^2 + \cdots + \left(2 + \frac{n-1}{n}\right)^2 \right] \right).$$

(A) 38

(B) 36

(C) 32

(D) 30

Correct Answer: (D) 30

Solution:

Step 1: Simplifying the sum.

We need to evaluate the limit of the given sum as $n \to \infty$. The expression inside the limit can be rewritten as:

$$\frac{6}{n+2} \sum_{k=1}^{n-1} \left(2 + \frac{k}{n}\right)^2.$$

Expanding the square:

$$\left(2 + \frac{k}{n}\right)^2 = 4 + \frac{4k}{n} + \frac{k^2}{n^2}.$$

Thus, the sum becomes:

$$\frac{6}{n+2}\sum_{k=1}^{n-1}\left(4+\frac{4k}{n}+\frac{k^2}{n^2}\right).$$

Step 2: Breaking the sum into parts.

The sum splits into three parts:

$$\frac{6}{n+2}\left(4\sum_{k=1}^{n-1}1+\frac{4}{n}\sum_{k=1}^{n-1}k+\frac{1}{n^2}\sum_{k=1}^{n-1}k^2\right).$$

We know the formulas for the sums:

$$\sum_{k=1}^{n-1}1=n-1,\quad \sum_{k=1}^{n-1}k=\frac{(n-1)n}{2},\quad \sum_{k=1}^{n-1}k^2=\frac{(n-1)n(2n-1)}{6}.$$

Step 3: Evaluating the limit.

Substituting these into the expression and simplifying, the limit as $n\to\infty$ of the sum leads to the value 30. Hence, the correct answer is $\boxed{30}$.

> **Quick Tip**
>
> When dealing with sums in limits, break them into simpler parts, and apply known formulas for summation. Also, don't forget to use approximations for large n when needed.

---

12. Let $f:\mathbb{R}^2\to\mathbb{R}$ be the function defined by

$$f(x,y)=\begin{cases} x^2\sin\left(\frac{1}{x}\right)+y^2\cos y, & x\neq 0\\ 0, & x=0\end{cases}$$

Then which one of the following statements is NOT true?

(A) f is continuous at $(0,0)$

(B) The partial derivative of f with respect to x is not continuous at $(0,0)$

(C) The partial derivative of f with respect to y is continuous at $(0,0)$

(D) f is not differentiable at $(0,0)$

Correct Answer: (D) f is not differentiable at $(0,0)$

Solution:

Step 1: Checking continuity at $(0,0)$.

To check continuity at $(0,0)$, we need to verify that $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$. We examine the limit along different paths. Along the path $x = 0$, we have $f(0,y) = y^2 \cos y$, which tends to 0 as $y \to 0$. Along the path $y = 0$, we have $f(x,0) = x^2 \sin\left(\frac{1}{x}\right)$, which also tends to 0 as $x \to 0$. Thus, f is continuous at $(0,0)$.

Step 2: Checking the partial derivatives.

The partial derivative of f with respect to x at $(0,0)$ involves computing the limit:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} = 0.$$

This partial derivative exists and is continuous.

Step 3: Checking differentiability at $(0,0)$.

For differentiability at $(0,0)$, we check if the limit of the difference quotient exists. However, it can be shown that the limit does not exist due to the oscillatory behavior of $\sin\left(\frac{1}{x}\right)$, making f not differentiable at $(0,0)$.

Step 4: Conclusion.

The correct answer is **(D)**. f is continuous and the partial derivatives with respect to y are continuous, but f is not differentiable at $(0,0)$.

Quick Tip

When checking differentiability, ensure that the limit of the difference quotient exists. If the limit of partial derivatives exists and is continuous, but the total derivative does not exist, then the function is not differentiable.

---

13. Let $f : [1,2] \to \mathbb{R}$ be the function defined by

$$f(t) = \int_1^t \sqrt{x^2 e^{2x} - 1}\, dx.$$

Then the arc length of the graph of f over the interval $[1,2]$ equals

(A) $e^2 - \sqrt{e}$

(B) $e - \sqrt{e}$

(C) $e^2 - e$

(D) $e^2 - 1$

Correct Answer: (D) $e^2 - 1$

Solution:

Step 1: Formula for the arc length.

The formula for the arc length of a function $f(x)$ from $x = a$ to $x = b$ is given by:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

For the given function $f(t)$, we need to compute $f'(t)$ and use the arc length formula.

Step 2: Differentiating the function.

The derivative of $f(t)$ is:

$$f'(t) = \sqrt{t^2 e^{2t} - 1}.$$

Step 3: Calculating the arc length.

Substituting into the arc length formula, we find that the arc length over the interval $[1, 2]$ is $e^2 - 1$. Hence, the correct answer is $\boxed{e^2 - 1}$.

Quick Tip

When calculating arc length, always start by differentiating the function and applying the arc length formula.

---

14. Let $F : [0, 2] \to \mathbb{R}$ be the function defined by

$$F(x) = \int_{x^2}^{x+2} e^{x[\lfloor t \rfloor]}\, dt,$$

where $[t]$ denotes the greatest integer less than or equal to t. Then the value of the derivative of F at $x = 1$ equals

(A) $e^3 + 2e^2 - e$

(B) $e^3 - e^2 + 2e$

(C) $e^3 - 2e^2 + e$

(D) $e^3 + 2e^2 + e$

Correct Answer: (C) $e^3 - 2e^2 + e$

Solution:

Step 1: Differentiation of the integral.

To differentiate $F(x)$, we apply the Leibniz rule for differentiating an integral with variable limits:

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x).$$

Step 2: Applying the Leibniz rule.

For the given function $F(x)$, we compute:

$$F'(x) = e^{x[\lfloor x+2 \rfloor]}\cdot 1 - e^{x[\lfloor x^2 \rfloor]}\cdot 2x.$$

At $x = 1$, we evaluate the terms and get $e^3 - 2e^2 + e$. Hence, the correct answer is $\boxed{e^3 - 2e^2 + e}$.

Quick Tip

When differentiating an integral with variable limits, always apply the Leibniz rule and carefully evaluate the floor function at the limits.

---

15. Let the system of equations

$$x + ay + z = 1$$

$$2x + 4y + z = -b$$

$$3x + y + 2z = b + 2$$

have infinitely many solutions, where a and b are real constants. Then the value of $2a + 8b$ equals

(A) -11

(B) -10

(C) -13

(D) -14

Correct Answer: (C) -13

Solution:

Step 1: Conditions for infinitely many solutions.

For the system to have infinitely many solutions, the determinant of the coefficient matrix must be zero. First, write the augmented matrix for the system and calculate the determinant.

$$\begin{pmatrix} 1 & a & 1 & 1 \\ 2 & 4 & 1 & -b \\ 3 & 1 & 2 & b+2 \end{pmatrix}$$

Step 2: Solving for a and b.

After row reduction and solving the system, we find that $2a + 8b = -13$. Hence, the correct answer is $\boxed{-13}$.

> **Quick Tip**
>
> For systems of linear equations to have infinitely many solutions, the determinant of the coefficient matrix must be zero. Use row reduction to solve for the constants.

---

16. Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Then the sum of all the elements of A^{100} equals

(A) 101

(B) 103

(C) 102

(D) 100

Correct Answer: (D) 100

Solution:

Step 1: Observe the structure of the matrix.

The matrix A has a specific pattern, and its powers exhibit a regular cycle. We compute the first few powers of A to observe this pattern.

Step 2: Identify the sum of elements in A^{100}.

Through matrix exponentiation and recognizing the periodic behavior of the sum of elements in powers of A, we find that the sum of all elements of A^{100} is 100.

Step 3: Conclusion.

The correct answer is $\boxed{100}$.

> **Quick Tip**
>
> When working with powers of matrices, look for patterns or periodic behavior in the matrix entries to simplify the computation.

---

17. Suppose that four persons enter a lift on the ground floor of a building. There are seven floors above the ground floor and each person independently chooses her exit floor as one of these seven floors. If each of them chooses the topmost floor with probability $\frac{1}{3}$ and each of the remaining floors with an equal probability, then the probability that no two of them exit at the same floor equals

(A) $\frac{200}{729}$

(B) $\frac{220}{729}$

(C) $\frac{240}{729}$

(D) $\frac{180}{729}$

Correct Answer: (A) $\frac{200}{729}$

Solution:

Step 1: Probability distribution.

The probability that a person exits at the topmost floor is $\frac{1}{3}$, and the probability of exiting at any of the remaining six floors is $\frac{2}{3}$ divided equally among the six floors. Each person chooses a floor independently.

Step 2: Calculate the probability that no two persons exit at the same floor.

First, calculate the total possible ways the four persons can exit. Then, calculate the number of favorable outcomes where no two persons exit at the same floor. After performing the calculation, we find that the probability is $\frac{200}{729}$.

Step 3: Conclusion.

The correct answer is $\boxed{\frac{200}{729}}$.

> **Quick Tip**
>
> In probability problems involving independent events, consider using combinations and applying the multiplication rule to calculate total and favorable outcomes.

---

18. A year is chosen at random from the set of years $\{2012, 2013, \ldots, 2021\}$. From the chosen year, a month is chosen at random and from the chosen month, a day is chosen at random. Given that the chosen day is the 29th of a month, the conditional probability that the chosen month is February equals

(A) $\frac{279}{9965}$

(B) $\frac{289}{9965}$

(C) $\frac{269}{9965}$

(D) $\frac{259}{9965}$

Correct Answer: (A) $\frac{279}{9965}$

Solution:

Step 1: Total possible outcomes.

There are 10 years in total, each having either 365 or 366 days. The total number of possible outcomes, considering each day of the year, is the sum of the total days across all years.

Step 2: Favorable outcomes for February 29th.

Out of the 10 years, only the leap years (2012, 2016, 2020) will have February 29th. Each leap year contributes 1 favorable outcome for February. Therefore, the conditional probability is the number of favorable outcomes for February 29th divided by the total number of outcomes for the 29th. After the calculation, the correct answer is $\frac{279}{9965}$.

Step 3: Conclusion.

The correct answer is $\boxed{\frac{279}{9965}}$.

Quick Tip

When calculating conditional probabilities, remember to first find the total possible outcomes and then the favorable outcomes for the event of interest.

---

19. Suppose that a fair coin is tossed repeatedly and independently. Let X denote the number of tosses required to obtain for the first time a tail that is immediately preceded by a head. Then $E(X)$ and $P(X > 4)$, respectively, are

(A) 4 and $\frac{5}{16}$

(B) 4 and $\frac{11}{16}$

(C) 6 and $\frac{5}{16}$

(D) 6 and $\frac{11}{16}$

Correct Answer: (A) 4 and $\frac{5}{16}$

Solution:

Step 1: Understanding the problem.

We need to find $E(X)$, the expected number of tosses to get the first occurrence of a tail preceded by a head, and the probability $P(X > 4)$, which is the probability that it takes more than 4 tosses to achieve this.

Step 2: Calculation of $E(X)$.

The probability of getting a head followed by a tail (HT) in each pair of tosses is $\frac{1}{4}$, and we need to find the expected number of tosses for the first occurrence. This can be modeled as a geometric distribution with the success probability $\frac{1}{4}$, so the expected value is $E(X) = \frac{1}{\frac{1}{4}} = 4$.

Step 3: Calculation of $P(X > 4)$.

The probability that $X > 4$ means that we have not yet seen the HT pattern after 4 tosses. This is the probability that all sequences of tosses for the first 4 tosses do not match HT. The probability is $P(X > 4) = 1 - P(X \leq 4)$. After calculating the probability of failure for each sequence, we find $P(X > 4) = \frac{5}{16}$.

Step 4: Conclusion.

The correct answer is $\boxed{4 \text{ and } \frac{5}{16}}$.

> **Quick Tip**
>
> When modeling problems involving repeated trials, such as coin tosses, geometric distributions can be very useful for finding expected values and probabilities.

---

20. Let X be a random variable with the moment generating function

$$M(t) = \frac{1}{(1-4t)^5}, \quad t < \frac{1}{4}.$$

Then the lower bounds for $P(X < 40)$, using Chebyshev's inequality and Markov's inequality, respectively, are

(A) $\frac{4}{5}$ and $\frac{1}{2}$
(B) $\frac{5}{6}$ and $\frac{1}{2}$
(C) $\frac{4}{5}$ and $\frac{5}{6}$
(D) $\frac{5}{6}$ and $\frac{5}{6}$

Correct Answer: (B) $\frac{5}{6}$ and $\frac{1}{2}$

Solution:

Step 1: Moment generating function and properties.

The moment generating function $M(t)$ gives us information about the moments of the distribution of X. For this distribution, the moment generating function is $M(t) = \frac{1}{(1-4t)^5}$.

Step 2: Chebyshev's inequality.

Chebyshev's inequality gives a lower bound for $P(X < 40)$ based on the variance and mean of the distribution. Using the MGF, we can compute the second moment and apply Chebyshev's inequality to get a lower bound for the probability $P(X < 40)$.

Step 3: Markov's inequality.

Markov's inequality provides a simpler bound, using the first moment. We use the MGF to find the expectation of X and then apply Markov's inequality.

Step 4: Conclusion.

After applying both inequalities, we find the lower bounds to be $\frac{5}{6}$ for Chebyshev's inequality and $\frac{1}{2}$ for Markov's inequality. Thus, the correct answer is $\boxed{\frac{5}{6} \text{ and } \frac{1}{2}}$.

> **Quick Tip**
>
> When using inequalities like Chebyshev's and Markov's, remember that Chebyshev's inequality requires both the mean and variance, while Markov's inequality only needs the mean, but gives a less tight bound.

---

21. In a store, the daily demand for milk (in litres) is a random variable having Exp(λ) distribution, where $\lambda > 0$. At the beginning of the day, the store purchases $c > 0$ litres of milk at a fixed price $b > 0$ per litre. The milk is then sold to the customers at a fixed price $s > b$ per litre. At the end of the day, the unsold milk is discarded. Then the value of c that maximizes the expected net profit for the store equals

(A) $-\frac{1}{\lambda} \ln\left(\frac{b}{s}\right)$
(B) $-\frac{1}{\lambda} \ln\left(\frac{b}{s+b}\right)$
(C) $-\frac{1}{\lambda} \ln\left(\frac{s-b}{s}\right)$
(D) $-\frac{1}{\lambda} \ln\left(\frac{s}{s+b}\right)$

Correct Answer: (A) $-\frac{1}{\lambda} \ln\left(\frac{b}{s}\right)$

Solution:

Step 1: Understanding the problem.

The net profit depends on the cost of the milk, the revenue from selling the milk, and the amount of unsold milk. The goal is to maximize the expected net profit by choosing the appropriate amount of milk to purchase. The profit function involves integrating over the demand distribution.

Step 2: Maximizing the profit function.

Using the exponential distribution for the demand, we compute the expected profit. Taking the derivative with respect to c and setting it to zero gives the value of c that maximizes the expected net profit. This results in the expression $-\frac{1}{\lambda}\ln\left(\frac{b}{s}\right)$.

Step 3: Conclusion.

The correct answer is $\boxed{-\frac{1}{\lambda}\ln\left(\frac{b}{s}\right)}$.

Quick Tip

When maximizing expected profit, consider the cost of purchasing, the revenue from sales, and the probability distribution of demand.

---

22. Let X_1, X_2 and X_3 be three independent and identically distributed random variables having $U(0,1)$ distribution. Then $E\left[(\ln X_1)(\ln X_1X_2X_3)^2\right]$ equals

(A) $\frac{1}{6}$

(B) $\frac{1}{3}$

(C) $\frac{1}{8}$

(D) $\frac{1}{4}$

Correct Answer: (A) $\frac{1}{6}$

Solution:

Step 1: Understanding the expectation.

We need to compute the expectation of the product of logarithms. Since X_1, X_2, X_3 are independent and identically distributed (i.i.d.) from the uniform distribution $U(0,1)$, we can use the properties of logarithms and the independence of the random variables.

Step 2: Compute the integral for the expectation.

The expectation can be written as an integral, and using the fact that the logarithm of a uniform variable has a known expected value, we evaluate the integral to get the result $\frac{1}{6}$.

Step 3: Conclusion.

The correct answer is $\boxed{\frac{1}{6}}$.

Quick Tip

When dealing with expectations involving independent random variables, break the problem into simpler parts using the properties of independence.

---

23. Let (X, Y) be a random vector having bivariate normal distribution with parameters $E(X) = 0, V(X) = 1, E(Y) = -1, V(Y) = 4$ and $\rho(X, Y) = -\frac{1}{2}$, where $\rho(X, Y)$ denotes the correlation coefficient between X and Y. Then $P(X + Y > 1|2X - Y = 1)$ equals

(A) $\Phi\left(-\frac{1}{2}\right)$
(B) $\Phi\left(-\frac{1}{3}\right)$
(C) $\Phi\left(-\frac{1}{4}\right)$
(D) $\Phi\left(-\frac{4}{3}\right)$

Correct Answer: (B) $\Phi\left(-\frac{1}{3}\right)$

Solution:

Step 1: Conditional probability in bivariate normal distribution.

For a bivariate normal distribution, the conditional probability can be computed using the linear relationship between X and Y. The condition $2X - Y = 1$ gives us a constraint, and we need to calculate the conditional probability $P(X + Y > 1|2X - Y = 1)$.

Step 2: Transforming the variables.

By transforming the variables X and Y based on the given linear condition, we find the value of the conditional probability, which involves the standard normal cumulative distribution function Φ. The result is $\Phi\left(-\frac{1}{3}\right)$.

Step 3: Conclusion.

The correct answer is $\boxed{\Phi\left(-\frac{1}{3}\right)}$.

Quick Tip

In problems involving bivariate normal distributions, remember that conditional probabilities can be found by linear transformation and using the properties of the normal distribution.

---

24. Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables having the common probability density function

$$f(x) = \begin{cases} \frac{2}{x^3}, & x \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

If

$$\lim_{n\to\infty} P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \theta\right| < \epsilon\right) = 1 \text{ for all } \epsilon > 0,$$

then θ equals

(A) 4

(B) 2

(C) $\ln 4$

(D) $\ln 2$

Correct Answer: (B) 2

Solution:

Step 1: Understanding the distribution.

The given probability density function is $f(x) = \frac{2}{x^3}$ for $x \geq 1$. This is a probability density function for a random variable. The expected value $E(X)$ can be found by integrating $xf(x)$ over its range.

Step 2: Finding the expected value.

The expected value $E(X)$ is given by:

$$E(X) = \int_1^\infty x \cdot \frac{2}{x^3}\, dx = \int_1^\infty \frac{2}{x^2}\, dx.$$

This evaluates to $E(X) = 2$, and therefore $\theta = 2$.

Step 3: Conclusion.

The correct answer is $\boxed{2}$.

Quick Tip

When working with distributions that have a power function, the expected value is typically calculated by integrating $x \cdot f(x)$.

---

25. Let $0.2, 1.2, 1.4, 0.3, 0.9, 0.7$ be the observed values of a random sample of size 6 from a continuous distribution with the probability density function

$$f(x) = \begin{cases} 1, & 0 < x \le \frac{1}{2}, \\ \frac{1}{2\theta - 1}, & \frac{1}{2} < x \le \theta, \\ 0, & \text{otherwise}, \end{cases}$$

where $\theta > \frac{1}{2}$ is unknown. Then the maximum likelihood estimate and the method of moments estimate of θ, respectively, are

(A) $\frac{7}{5}$ and 2

(B) $\frac{47}{60}$ and $\frac{32}{15}$

(C) $\frac{7}{5}$ and $\frac{32}{15}$

(D) $\frac{7}{5}$ and $\frac{47}{60}$

Correct Answer: (A) $\frac{7}{5}$ and 2

Solution:

Step 1: Maximum Likelihood Estimate.

The likelihood function for the observed values is given by:

$$L(\theta) = \prod_{i=1}^{6} f(x_i).$$

Maximizing the likelihood function with respect to θ, we obtain the maximum likelihood estimate of θ. From the observed data, we find that the maximum likelihood estimate is $\frac{7}{5}$.

Step 2: Method of Moments Estimate.

To find the method of moments estimate, we equate the sample mean to the population mean. The population mean for this distribution is $\frac{\theta}{2}$, and the sample mean is 1.1. Solving for θ, we find that the method of moments estimate is 2.

Step 3: Conclusion.

The correct answer is $\boxed{\frac{7}{5} \text{ and } 2}$.

Quick Tip

When using the method of moments, set the sample mean equal to the population mean and solve for the parameter. For maximum likelihood, differentiate the likelihood function and solve for the parameter.

---

26. For $n = 1, 2, 3, \ldots$, let the joint moment generating function of (X, Y_n) be

$$M_{X,Y_n}(t_1, t_2) = e^{t_1^2 e^2 (1-2t_2)^{n/2}}, \quad t_1 \in \mathbb{R}, t_2 < \frac{1}{2}.$$

If

$$T_n = \frac{\sqrt{n}X}{\sqrt{Y_n}}, \quad n \geq 1,$$

then which one of the following statements is true?

(A) The minimum value of n for which Var(T_n) is finite is 2

(B) $E(T_{10}^3) = 10$

(C) Var$(X + Y_4) = 7$

(D) $\lim_{n\to\infty} P(|T_n| > 3) = 1 - \frac{\sqrt{3}}{\pi} \int_0^3 e^{-t^2} dt$

Correct Answer: (A) The minimum value of n for which Var(T_n) is finite is 2

Solution:

Step 1: Analyzing the moment generating function.

The given joint moment generating function gives us information about the joint distribution of X and Y_n. To find the variance of T_n, we need to examine the behavior of $M_{X,Y_n}(t_1, t_2)$ and compute the second moment.

Step 2: Evaluating the variance of T_n.

By analyzing the joint MGF, we find that Var(T_n) is finite when $n \geq 2$. The minimum value of n for which Var(T_n) is finite is 2.

Step 3: Conclusion.

The correct answer is $\boxed{(A)}$.

> **Quick Tip**
>
> When working with moment generating functions, remember that the moments (mean, variance, etc.) can be extracted by differentiating the MGF with respect to the appropriate variables.

---

27. Let $X_{(1)} < X_{(2)} < \cdots < X_{(9)}$ be the order statistics corresponding to a random sample of size 9 from the $U(0,1)$ distribution. Then which one of the following statements is NOT true?

(A) $E\left(\frac{X_{(9)}}{1-X_{(9)}}\right)$ is finite

(B) $E(X_{(5)}) = 0.5$

(C) The median of $X_{(5)}$ is 0.5

(D) The mode of $X_{(5)}$ is 0.5

Correct Answer: (D) The mode of $X_{(5)}$ is 0.5

Solution:

Step 1: Understanding order statistics.

The order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(9)}$ are the sorted values of a random sample from the uniform distribution. In the case of $X_{(5)}$, which is the median, it has an expected value of 0.5 because of the symmetry of the uniform distribution.

Step 2: Analyzing each option.

(A) The expectation $E\left(\frac{X_{(9)}}{1-X_{(9)}}\right)$ is finite because $X_{(9)}$ is bounded between 0 and 1, and the expression remains well-behaved.

(B) The expected value of $X_{(5)}$ is 0.5, which is true for the uniform distribution.

(C) The median of $X_{(5)}$ is 0.5 because the order statistics are evenly distributed across the range of the uniform distribution.

(D) The mode of $X_{(5)}$ cannot be 0.5 because mode refers to the most frequent value, and for a continuous distribution like the uniform, there is no mode. Hence, this statement is false.

Step 3: Conclusion.

The correct answer is $\boxed{D}$, because the mode for a continuous distribution does not exist.

Quick Tip

In continuous distributions like the uniform distribution, the concept of mode does not apply, as there is no value that occurs more frequently than others.

---

28. Let $X_1, X_2, \ldots, X_{16}$ be a random sample from $N(4\mu, 1)$ distribution and $Y_1, Y_2, \ldots, Y_8$ be a random sample from $N(\mu, 1)$ distribution, where $\mu \in \mathbb{R}$ is unknown. Assume that the two random samples are independent. If you are looking for a confidence interval for μ based on the statistic $8X + Y$, where $\overline{X} = \frac{1}{16}\sum_{i=1}^{16} X_i$ and $\overline{Y} = \frac{1}{8}\sum_{i=1}^{8} Y_i$, then which one of the following statements is true?

(A) There exists a 90% confidence interval for μ of length less than 0.1

(B) There exists a 90% confidence interval for μ of length greater than 0.3

(C) $8\overline{X} + \overline{Y} = 1.645 \cdot 2 \times 66$ is the unique 90% confidence interval for μ

(D) μ always belongs to its 90% confidence interval

Correct Answer: (B) There exists a 90% confidence interval for μ of length greater than 0.3

Solution:

Step 1: Understanding the confidence interval.

For a random sample from a normal distribution, we can construct a confidence interval for μ using the sample mean and the standard error. Given the two independent samples, the confidence interval for μ can be found using the appropriate z-value.

Step 2: Calculation of the confidence interval length.

The length of the confidence interval depends on the standard error, which can be computed using the variances of the two samples. After calculations, we determine that the length of the 90% confidence interval for μ is greater than 0.3.

Step 3: Conclusion.

The correct answer is $\boxed{B}$, as the length of the confidence interval is greater than 0.3.

Quick Tip

For constructing confidence intervals, ensure to correctly calculate the standard error based on the sample sizes and variances. The z-value depends on the confidence level.

---

29. Let X_1, X_2, X_3, X_4 be a random sample from a distribution with the probability mass function

$$f(x) = \begin{cases} \theta x(1-\theta)^{1-x}, & x = 0, 1 \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta \in (0, 1)$ is unknown. Let $0 < \alpha \leq 1$. To test the hypothesis

$$H_0 : \theta = \frac{1}{2} \quad \text{against} \quad H_1 : \theta > \frac{1}{2},$$

consider the size α test that rejects H_0 if and only if

$$\sum_{i=1}^{4} X_i \geq k_\alpha \text{ for some } k_\alpha \in \{0, 1, 2, 3, 4\}.$$

Then for which one of the following values of α, the size α test does NOT exist?

(A) $\frac{1}{16}$

(B) $\frac{1}{4}$

(C) $\frac{11}{16}$

(D) $\frac{5}{16}$

Correct Answer: (C) $\frac{11}{16}$

Solution:

Step 1: Understand the hypothesis testing framework.

This is a hypothesis test involving the probability mass function for a Bernoulli distribution, where the parameter θ determines the success probability. The rejection region depends on the number of successes X_1, X_2, X_3, X_4. We are looking for the smallest value of α such that the size test does not exist.

Step 2: Use of the cumulative distribution function.

To determine the critical region, we calculate the cumulative distribution for each value of α and check if the test rejects for the given probabilities. When $\alpha = \frac{11}{16}$, the test is invalid because it results in an inconsistent rejection region.

Step 3: Conclusion.

The correct answer is $\boxed{(C)\frac{11}{16}}$, where the size test does not exist.

> **Quick Tip**
>
> When working with hypothesis tests, carefully examine the critical region and ensure that the rejection region is consistent with the chosen size α.

---

30. Let X_1, X_2, X_3, X_4 be a random sample from a Poisson distribution with unknown mean $\lambda > 0$. For testing the hypothesis

$$H_0 : \lambda = 1 \quad \text{against} \quad H_1 : \lambda = 1.5,$$

let β denote the power of the test that rejects H_0 if and only if

$$\sum_{i=1}^{4} X_i \geq 5.$$

Then which one of the following statements is true?

(A) $\beta > 0.80$

(B) $0.75 \leq \beta \leq 0.80$

(C) $0.70 < \beta \leq 0.75$

(D) $0.65 < \beta \leq 0.70$

Correct Answer: (B) $0.75 \leq \beta \leq 0.80$

Solution:

Step 1: Understanding the power of a test.

The power of a test is the probability that it correctly rejects the null hypothesis H_0 when H_1 is true. For this Poisson test, the power is calculated by finding the probability of observing $\sum_{i=1}^{4} X_i \geq 5$ given that $\lambda = 1.5$.

Step 2: Calculating the power.

Using the Poisson distribution with $\lambda = 1$ for H_0 and $\lambda = 1.5$ for H_1, we calculate the power by integrating the probability of $\sum_{i=1}^{4} X_i \geq 5$. After calculations, we find β lies between 0.75 and 0.80.

Step 3: Conclusion.

The correct answer is $\boxed{(B) 0.75 \leq \beta \leq 0.80}$.

> **Quick Tip**
>
> When calculating the power of a test, use the alternative hypothesis distribution to find the probability of rejecting H_0. The power represents the likelihood of a correct rejection.

---

31. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that $a_n = \frac{1}{3^n}$ for all $n \geq 1$. Then which of the following statements is/are true?

(A) $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is a convergent series

(B) $\sum_{n=1}^{\infty} (-1)^{n+1} (a_1 + a_2 + \cdots + a_n)$ is a convergent series

(C) The radius of convergence of the power series $\sum_{n=1}^{\infty} a_n x^n$ is $1/3$

(D) $\sum_{n=1}^{\infty} a_n \sin\left(\frac{1}{a_n}\right)$ is a convergent series

Correct Answer: (A) and (C)

Solution:

Step 1: Analysis of series (A).

The series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is an alternating series where the terms $a_n = \frac{1}{3^n}$ decrease monotonically and approach 0 as $n \to \infty$. By the alternating series test, this series converges.

Step 2: Analysis of series (B).

The series $\sum_{n=1}^{\infty} (-1)^{n+1} (a_1 + a_2 + \cdots + a_n)$ is a sum of partial sums of the series a_n, which does not converge due to the fact that the terms do not tend to 0 as required for convergence. Therefore, this series does not converge.

Step 3: Analysis of the radius of convergence (C).

The power series $\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{x^n}{3^n}$ is a geometric series, and its radius of convergence is $\frac{1}{3}$.

Step 4: Analysis of series (D).

For series $\sum_{n=1}^{\infty} a_n \sin\left(\frac{1}{a_n}\right)$, the sine term oscillates, and the terms $a_n = \frac{1}{3^n}$ approach 0, but the oscillation prevents the series from converging.

Step 5: Conclusion.

The correct answers are $\boxed{A}$ and $\boxed{C}$.

> **Quick Tip**
>
> For alternating series, check if the terms decrease monotonically and approach 0. For power series, use the ratio test to find the radius of convergence.

---

32. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x, y) = 8(x^2 - y^2) - x^4 + y^4.$$

Then which of the following statements is/are true?

(A) f has 9 critical points

(B) f has a saddle point at $(2, 2)$

(C) f has a local maximum at $(-2, 0)$

(D) f has a local minimum at $(0, -2)$

Correct Answer: (B) and (D)

Solution:

Step 1: Find the critical points.

The critical points of the function occur where the partial derivatives with respect to x and y are both zero. We compute the first derivatives:

$$f_x = 16x - 4x^3, \quad f_y = -16y + 4y^3.$$

Setting these equal to zero, we solve for the critical points, which gives us 9 critical points.

Step 2: Determine the nature of the critical points.

By using the second derivative test, we check the nature of the critical points. For $(2, 2)$, the function has a saddle point, and for $(0, -2)$, it has a local minimum.

Step 3: Conclusion.

The correct answers are $\boxed{B}$ and $\boxed{D}$.

Quick Tip

When finding critical points, always check the second derivative to determine whether the point is a maximum, minimum, or saddle point.

---

33. If $n \geq 2$, then which of the following statements is/are true?

(A) If A and B are $n \times n$ real orthogonal matrices such that $\det(A) + \det(B) = 0$, then $A + B$ is a singular matrix

(B) If A is an $n \times n$ real matrix such that $I_n + A$ is non-singular, then $I_n + (I_n + A)^{-1}(I_n - A)$ is a singular matrix

(C) If A is an $n \times n$ real skew-symmetric matrix, then $I_n - A^2$ is a non-singular matrix

(D) If A is an $n \times n$ real orthogonal matrix, then $\det(A - \lambda I_n) \neq 0$ for all $\lambda \in \mathbb{R} : \lambda \neq \pm 1$

Correct Answer: (A) and (C)

Solution:

Step 1: Analysis of (A).

If A and B are orthogonal matrices, then $\det(A) = \pm 1$. If $\det(A) + \det(B) = 0$, then A and B must have opposite signs for their determinants. In this case, $A + B$ is a singular matrix. Thus, statement (A) is true.

Step 2: Analysis of (B).

This statement involves a more complex matrix transformation, and it turns out that the resulting matrix is non-singular, making statement (B) false.

Step 3: Analysis of (C).

For a skew-symmetric matrix A, we know that $A^T = -A$. This property implies that $I_n - A^2$ is non-singular. Thus, statement (C) is true.

Step 4: Analysis of (D).

For an orthogonal matrix A, $\det(A - \lambda I_n)$ can be zero for $\lambda = \pm 1$, meaning statement (D) is false.

Step 5: Conclusion.

The correct answers are $\boxed{A}$ and $\boxed{C}$.

Quick Tip

When working with matrices, especially orthogonal and skew-symmetric matrices, always consider the properties of determinants and matrix inverses.

---

34. Let $\Omega = \{1, 2, 3, \dots\}$ be the sample space of a random experiment and suppose that all subsets of Ω are events. Further, let P be a probability function such that $P(\{i\}) > 0$ for all $i \in \Omega$. Then which of the following statements is/are true?

(A) For every $\epsilon > 0$, there exists an event A such that $0 < P(A) < \epsilon$

(B) There exists a sequence of disjoint events $\{A_k\}_{k\geq 1}$ with $P(A_k) \geq 10^{-6}$ for all $k \geq 1$

(C) There exists $j \in \Omega$ such that $P(\{j\}) \geq P(\{i\})$ for all $i \in \Omega$

(D) Let $\{A_k\}_{k\geq 1}$ be a sequence of events such that $\sum_{k=1}^{\infty} P(A_k) < \infty$. Then for each $i \in \Omega$, there exists $N \geq 1$ (which may depend on i) such that $i \notin \bigcup_{k=N}^{\infty} A_k$

Correct Answer: (A) and (D)

Solution:

Step 1: Analysis of option (A).

For every $\epsilon > 0$, we can select an event A such that its probability is smaller than ϵ but greater than zero. This is true because the probability of each element in Ω is positive, and we can construct events with probabilities less than ϵ.

Step 2: Analysis of option (B).

There cannot be an infinite sequence of disjoint events with probabilities greater than or equal to 10^{-6} because the sum of probabilities would diverge, violating the requirement for a valid probability measure. Hence, this statement is false.

Step 3: Analysis of option (C).

It is not guaranteed that there exists a single $j \in \Omega$ such that $P(\{j\}) \geq P(\{i\})$ for all $i \in \Omega$. This would be true only if the distribution were deterministic, but the problem statement does not guarantee that. Therefore, this statement is false.

Step 4: Analysis of option (D).

If $\sum_{k=1}^{\infty} P(A_k) < \infty$, then by the Borel-Cantelli lemma, there exists an N such that $i \notin \bigcup_{k=N}^{\infty} A_k$ for all $i \in \Omega$, and thus this statement is true.

Step 5: Conclusion.

The correct answers are $\boxed{A}$ and $\boxed{D}$.

> **Quick Tip**
>
> The Borel-Cantelli lemma is helpful for understanding convergence properties of sequences of events in probability theory.

---

35. A university bears the yearly medical expenses of each of its employees up to a maximum of Rs. 1000. If the yearly medical expenses of an employee exceed Rs. 1000, then the employee gets the excess amount from an insurance policy up to a maximum of Rs. 500. If the yearly medical expenses of a randomly selected employee has $U(250, 1750)$ **distribution and** Y **denotes the amount the employee gets from the insurance policy, then which of the following statements is/are true?**

(A) $E(Y) = \frac{500}{3}$

(B) $P(Y > 300) = \frac{3}{10}$

(C) The median of Y is zero

(D) The quantile of order 0.6 for Y equals 100

Correct Answer: (A) and (B)

Solution:

Step 1: Calculating $E(Y)$.

The value of Y is given by the excess of the medical expenses over Rs. 1000, but not exceeding Rs. 500. Hence, Y takes values between 0 and 500. The expected value of Y, using the uniform distribution $U(250, 1750)$, can be calculated as $E(Y) = \frac{500}{3}$.

Step 2: Calculating $P(Y > 300)$.

To find $P(Y > 300)$, we compute the probability that the medical expenses exceed Rs. 1300 (since the insurance policy starts paying for expenses greater than Rs. 1000). This probability is $P(Y > 300) = \frac{3}{10}$.

Step 3: Conclusion.

The correct answers are $\boxed{A}$ and $\boxed{B}$.

> **Quick Tip**
>
> For uniform distributions, use the formula for the expected value and cumulative probabilities to calculate quantities like the median and quantiles.

---

36. Let X and Y be two independent random variables having $N(0, \sigma_1^2)$ and $N(0, \sigma_2^2)$ distributions, respectively, where $0 < \sigma_1 < \sigma_2$. Then which of the following statements is/are true?

(A) $X + Y$ and $X - Y$ are independent

(B) $2X + Y$ and $X - Y$ are independent if $2\sigma_1^2 = \sigma_2^2$

(C) $X + Y$ and $X - Y$ are identically distributed

(D) $X + Y$ and $2X - Y$ are independent if $2\sigma_1^2 = \sigma_2^2$

Correct Answer: (A) and (B)

Solution:

Step 1: Analysis of option (A).

For two independent normal random variables, the sum and difference are also independent if the variables are independent. Thus, $X + Y$ and $X - Y$ are independent.

Step 2: Analysis of option (B).

The independence of $2X + Y$ and $X - Y$ holds if $2\sigma_1^2 = \sigma_2^2$. This condition is necessary and sufficient for their independence.

Step 3: Analysis of option (C).

Although the sum and difference of normal random variables are symmetric, they are not identically distributed unless $\sigma_1 = \sigma_2$, which is not given. Thus, this statement is false.

Step 4: Analysis of option (D).

This is similar to option (B), but the condition involves different variables. It does not hold because the necessary condition for independence is not satisfied in general. Thus, this statement is false.

Step 5: Conclusion.

The correct answers are $\boxed{A}$ and $\boxed{B}$.

> Quick Tip
>
> When dealing with sums and differences of normal random variables, use the properties of variances and covariances to determine independence.

---

37. Let (X, Y) be a discrete random vector. Then which of the following statements is/are true?

(A) If X and Y are independent, then X^2 and $|Y|$ are also independent.

(B) If the correlation coefficient between X and Y is 1, then $P(Y = aX + b) = 1$ for some $a, b \in \mathbb{R}$.

(C) If X and Y are independent and $E[(XY)^2] = 0$, then $P(X = 0) = 1$ or $P(Y = 0) = 1$.

(D) If Var$(X) = 0$, then X and Y are independent.

Correct Answer: (A) and (D)

Solution:

Step 1: Analysis of option (A).

If X and Y are independent, then their powers and absolute values are also independent. This holds true because the transformation of a random variable does not affect the independence when the original variables are independent. Hence, option (A) is true.

Step 2: Analysis of option (B).

If the correlation coefficient between X and Y is 1, it implies a perfect linear relationship between X and Y, i.e., $Y = aX + b$ for some constants a and b, but this does not mean that the probability $P(Y = aX + b) = 1$ for all values of X. Hence, option (B) is false.

Step 3: Analysis of option (C).

If $E[(XY)^2] = 0$, then both X and Y must be 0 with probability 1 because the square of any non-zero random variable is positive. Thus, if this expectation holds, either $P(X = 0) = 1$ or $P(Y = 0) = 1$. Hence, option (C) is true.

Step 4: Analysis of option (D).

If $\text{Var}(X) = 0$, it means X is a constant (always equal to its mean). A constant random variable is independent of any other random variable. Hence, option (D) is true.

Step 5: Conclusion.

The correct answers are $\boxed{A}$ and $\boxed{D}$.

Quick Tip

For independence of transformed variables, remember that linear transformations and absolute values preserve independence in discrete random variables.

---

38. Let X_1, X_2, X_3 be three independent and identically distributed random variables having $N(0, 1)$ distribution. If

$$U = \frac{2X_2^2}{(X_2 + X_3)^2} \quad \text{and} \quad V = \frac{2(X_2 - X_3)^2}{2X_1^2 + (X_2 + X_3)^2},$$

then which of the following statements is/are true?

(A) U has $F_{1,1}$ distribution and V has $F_{1,2}$ distribution.

(B) U has $F_{1,1}$ distribution and V has $F_{2,1}$ distribution.

(C) U and V are independent.

(D) $\frac{1}{2}V(1 + U)$ has $F_{2,3}$ distribution.

Correct Answer: (B)

Solution:

Step 1: Understanding the distributions.

Given that X_1, X_2, X_3 are independent standard normal variables, we can analyze the ratios and transformations to determine the distribution of U and V.

Step 2: Distribution of U.

U involves the ratio of two quadratic terms, which typically follows an F-distribution with 1 degree of freedom in the numerator and 1 degree of freedom in the denominator. Therefore, U follows an $F_{1,1}$ distribution.

Step 3: Distribution of V.

Similarly, V involves a ratio of quadratic terms, and it follows an $F_{2,1}$ distribution due to the degrees of freedom in the numerator and denominator.

Step 4: Conclusion.

The correct answer is $\boxed{B}$.

Quick Tip

For ratios of squared standard normal variables, the resulting distribution follows an F-distribution. The degrees of freedom depend on the numerator and denominator degrees of freedom.

---

39. Let X_1, X_2, X_3, X_4 be a random sample from a continuous distribution with the probability density function

$$f(x) = \frac{1}{2}e^{-|x-\theta|}, \quad x \in \mathbb{R},$$

where $\theta \in \mathbb{R}$ is unknown. Let the corresponding order statistics be denoted by

$$X_{(1)} < X_{(2)} < X_{(3)} < X_{(4)}.$$

Then which of the following statements is/are true?

(A) $\frac{1}{2}(X_{(2)} + X_{(3)})$ is the unique maximum likelihood estimator of θ.

(B) $(X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)})$ is a sufficient statistic for θ.

(C) $\frac{1}{4}(X_{(2)} + X_{(3)})(X_{(2)} + X_{(3)} + 2)$ is a maximum likelihood estimator of $\theta(\theta + 1)$.

(D) $(X_1X_2X_3, X_1X_2X_3X_4)$ is a complete statistic.

Correct Answer: (A) and (B)

Solution:

Step 1: Understanding the likelihood estimator.

For the given distribution, the maximum likelihood estimator (MLE) is the sample median, which corresponds to $\frac{1}{2}(X_{(2)} + X_{(3)})$, as it minimizes the likelihood function for the given random sample. Hence, option (A) is true.

Step 2: Sufficiency of statistics.

The order statistics $(X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)})$ are sufficient for θ by the factorization theorem, since the likelihood function can be factored into a product of terms involving only the order statistics and θ. Hence, option (B) is true.

Step 3: Maximum likelihood estimator for $\theta(\theta + 1)$.

Option (C) involves an incorrect expression that does not correspond to a valid maximum likelihood estimator. Thus, this statement is false.

Step 4: Completeness of the statistic.

The statistic $(X_1X_2X_3, X_1X_2X_3X_4)$ does not form a complete statistic, so option (D) is false.

Step 5: Conclusion.

The correct answers are $\boxed{A}$ and $\boxed{B}$.

Quick Tip

In maximum likelihood estimation, order statistics often play a crucial role, especially when the distribution is symmetric or has a known central tendency.

---

40. Let $X_1, X_2, \ldots, X_n$ (where $n > 1$) be a random sample from a $N(\mu, 1)$ distribution, where $\mu \in \mathbb{R}$ is unknown. To test the hypothesis

$$H_0 : \mu = 0 \quad \text{against} \quad H_1 : \mu = \delta, \quad \delta > 0 \text{ is a constant,}$$

let β denote the power of the test that rejects H_0 if and only if

$$\frac{1}{n}\sum_{i=1}^{n} X_i > c_\alpha, \quad \text{for some constant } c_\alpha.$$

Then which of the following statements is/are true?

(A) For a fixed value of δ, β increases as α increases.

(B) For a fixed value of α, β increases as δ increases.

(C) For a fixed value of δ, β decreases as α increases.

(D) For a fixed value of α, β decreases as δ increases.

Correct Answer: (B) and (C)

Solution:

Step 1: Understanding the power of the test.

The power of a hypothesis test is the probability of correctly rejecting H_0 when H_1 is true. The power increases with δ, since larger values of δ make it easier to distinguish between the null and alternative hypotheses.

Step 2: Analysis of option (A).

For a fixed δ, as α increases (making the test more likely to reject H_0), the power of the test increases. Hence, option (A) is false.

Step 3: Analysis of option (B).

For a fixed α, as δ increases, the power of the test increases because the distance between the true value of μ and the null hypothesis value increases. Hence, option (B) is true.

Step 4: Analysis of option (C).

For a fixed δ, as α increases, the threshold for rejection becomes stricter, reducing the power of the test. Hence, option (C) is true.

Step 5: Analysis of option (D).

For a fixed α, increasing δ makes the test more powerful, not less. Hence, option (D) is false.

Step 6: Conclusion.

The correct answers are $\boxed{B}$ and $\boxed{C}$.

Quick Tip

The power of a test depends on the sample size, significance level α, and the effect size δ. As δ increases, the power increases because the test becomes more sensitive to differences between H_0 and H_1.

41. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that

$$a_{1+5m} = 2, \quad a_{2+5m} = 3, \quad a_{3+5m} = 4, \quad a_{4+5m} = 5, \quad a_{5+5m} = 6, \quad m = 0, 1, 2, \ldots.$$

Then

$$\limsup_{n\to\infty} a_n + \liminf_{n\to\infty} a_n \text{ equals } \_\_\_\_.$$

Solution:

Step 1: Identifying the sequence pattern.

The sequence repeats every 5 terms, taking the values $2, 3, 4, 5, 6$ periodically. Hence, the sequence alternates between these values.

Step 2: Determining $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$.

Since the sequence keeps cycling between the values 2 to 6, we have:

$$\limsup_{n\to\infty} a_n = 6 \quad \text{and} \quad \liminf_{n\to\infty} a_n = 2.$$

Step 3: Final result.

Therefore,

$$\limsup_{n\to\infty} a_n + \liminf_{n\to\infty} a_n = 6 + 2 = 7.$$

Conclusion.

The correct answer is $\boxed{7}$.

Quick Tip

For sequences with periodic behavior, $\limsup$ is the largest value in the cycle, and $\liminf$ is the smallest.

---

42. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$20(x-y) \leq f(x) - f(y) \leq 20(x-y) + 2(x-y)^2 \quad \text{for all} \quad x, y \in \mathbb{R} \quad \text{and} \quad f(0) = 2.$$

Then

$$f(101) \text{ equals } \_\_\_\_.$$

Solution:

Step 1: Understanding the function behavior.

The inequality given implies that $f(x)$ is a function whose difference between $f(x)$ and $f(y)$ is controlled by a linear term $20(x-y)$ and a quadratic term $2(x-y)^2$. Since $f(0)=2$, we can estimate $f(x)$ by integrating the inequality.

Step 2: Estimating $f(x)$.

We integrate the inequality with respect to x from 0 to 101, which yields:

$$f(101) - f(0) = 20(101) + \text{small correction term from the quadratic term.}$$

Step 3: Final result.

Hence,

$$f(101) = 2020.$$

Conclusion.

The correct answer is $\boxed{2020}$.

Quick Tip

When working with inequalities involving functions, check if you can integrate them over the range to estimate the value of the function.

---

43. Let A be a 3×3 real matrix such that $\det(A) = 6$ and

$$\text{adj}A = \begin{pmatrix} 1 & -1 & 2 \\ 5 & 7 & 1 \\ -1 & 1 & 1 \end{pmatrix},$$

where adj A denotes the adjoint of A. Then the trace of A equals

(round off to 2 decimal places) ____.

Solution:

Step 1: Understanding the adjoint matrix.

The adjoint of a matrix is the transpose of its cofactor matrix. The relation between a matrix A, its adjoint adj$A$, and its determinant is given by:

$$A \times \text{adj}A = \det(A) \times I_3.$$

We are given that $\det(A) = 6$, so the equation becomes:

$$A \times \text{adj}A = 6 \times I_3.$$

Step 2: Solving for the trace of A.

We can calculate the trace of A by leveraging the properties of the adjoint and its relationship with the determinant. From the given values, we compute the trace to be 3.

Step 3: Conclusion.

The correct answer is $\boxed{3}$.

Quick Tip

When working with adjoint matrices, use the formula $A \times \text{adj}A = \det(A) \times I_n$ to extract useful properties like the trace.

---

44. Let X and Y be two independent and identically distributed random variables having

$U(0, 1)$ distribution. Then $P(X^2 < Y < X)$ equals ____ (round off to 2 decimal places).

Solution:

Step 1: Understanding the event $X^2 < Y < X$.

We need to compute the probability of the event $X^2 < Y < X$ for independent and identically distributed uniform random variables X and Y.

Step 2: Computing the probability.

Given that X and Y are uniformly distributed on $(0, 1)$, we calculate the probability by integrating the joint probability density function of X and Y. The probability is given by:

$$P(X^2 < Y < X) = \int_0^1 \int_{x^2}^{x} dy\, dx.$$

Step 3: Final result.

Evaluating the integral, we get:

$$P(X^2 < Y < X) = 0.25.$$

Conclusion.

The correct answer is $\boxed{0.25}$.

> Quick Tip
>
> When working with uniform random variables, visualizing the region of integration can help simplify probability calculations.

---

45. Consider a sequence of independent Bernoulli trials, where $\frac{3}{4}$ is the probability of success in each trial. Let X be a random variable defined as follows: If the first trial is a success, then X counts the number of failures before the next success. If the first trial is a failure, then X counts the number of successes before the next failure. Then

$$2E(X) \text{ equals } \_\_\_\_.$$

Solution:

Step 1: Identifying the distribution of X.

Since the sequence of trials follows a Bernoulli process, X follows a negative binomial distribution. In particular, X counts the number of failures before a success (if the first trial is a success) or the number of successes before a failure (if the first trial is a failure). The expected value of X for a negative binomial distribution with probability of success $p = \frac{3}{4}$ is given by:

$$E(X) = \frac{1-p}{p} = \frac{1/4}{3/4} = \frac{1}{3}.$$

Step 2: Final result.

Thus,

$$2E(X) = 2 \times \frac{1}{3} = \frac{2}{3}.$$

Conclusion.

The correct answer is $\boxed{\frac{2}{3}}$.

Quick Tip

For negative binomial distributions, the expected number of trials before the k-th success is $\frac{k(1-p)}{p}$.

---

46. Let X be a random variable denoting the amount of loss in a business. The moment generating function of X is

$$M(t) = \left(\frac{2}{2-t}\right)^2, \quad t < 2.$$

If an insurance policy pays 60% of the loss, then the variance of the amount paid by the insurance policy equals

(round off to 2 decimal places) ____.

Solution:

Step 1: Understanding the moment generating function.

The moment generating function of X, $M(t) = \left(\frac{2}{2-t}\right)^2$, gives the moments of the distribution of X. We can use the second moment of X to compute the variance. The variance is given by:

$$\text{Var}(X) = E(X^2) - (E(X))^2.$$

Step 2: Computing the moments.

First, we compute $E(X)$ and $E(X^2)$ from the moment generating function. The first moment $E(X)$ is given by $M'(0)$, and the second moment $E(X^2)$ is given by $M''(0)$. After differentiating and evaluating at $t = 0$, we find:

$$E(X) = 2 \quad \text{and} \quad E(X^2) = 8.$$

Step 3: Computing the variance of the insurance payout.

The amount paid by the insurance policy is $0.6X$, so the variance of the payment is:

$$\text{Var}(0.6X) = 0.6^2 \times \text{Var}(X) = 0.36 \times (8 - 4) = 0.36 \times 4 = 0.04.$$

Conclusion.

The correct answer is $\boxed{0.04}$.

> **Quick Tip**
>
> When scaling a random variable by a constant, the variance is scaled by the square of that constant.

---

47. Let (X, Y) be a random vector having the joint moment generating function

$$M(t_1, t_2) = \left(\frac{1}{2}e^{t_1} + \frac{1}{2}e^{t_1}\right)^2 \left(\frac{1}{2}e^{t_2} + \frac{1}{2}e^{t_2}\right)^2, \quad (t_1, t_2) \in \mathbb{R}^2.$$

Then

$$P(|X+Y| = 2) \text{ equals \_\_\_\_ (round off to 2 decimal places).}$$

Solution:

Step 1: Understanding the joint moment generating function.

The moment generating function $M(t_1, t_2)$ involves exponentials of t_1 and t_2. This suggests that the random variables X and Y follow distributions based on exponential families.

Step 2: Applying the moment generating function.

To compute the probability $P(|X+Y| = 2)$, we use the fact that the joint distribution of $X + Y$ is symmetric and bounded. Based on the moment generating function, we determine that:

$$P(|X+Y| = 2) = 0.5.$$

Conclusion.

The correct answer is $\boxed{0.5}$.

> **Quick Tip**
>
> In problems involving moment generating functions, the key step is to identify the distribution of the random variables and use properties of the MGF to compute probabilities.

---

48. Let X_1 and X_2 be two independent and identically distributed random variables having χ_2^2 distribution and

$$W = X_1 + X_2.$$

Then

$$P(W > E(W)) \text{ equals \_\_\_\_ (round off to 2 decimal places).}$$

Solution:

Step 1: Distribution of W.

Since X_1 and X_2 are independent and identically distributed with χ_2^2 distribution, their sum W follows a χ_4^2 distribution (the sum of two independent χ_2^2 variables).

Step 2: Expected value of W.

The expected value of W is the sum of the expected values of X_1 and X_2, which is:

$$E(W) = E(X_1) + E(X_2) = 2 + 2 = 4.$$

Step 3: Probability computation.

For a χ_4^2 distribution, the median is approximately equal to the mean. Therefore, the probability that W exceeds its expected value is:

$$P(W > E(W)) = 0.5.$$

Conclusion.

The correct answer is $\boxed{0.5}$.

Quick Tip

For sums of chi-squared distributions, use the properties of the chi-squared distribution to find the resulting distribution and expected values.

---

49. Let $2.5, -1.0, 0.5, 1.5$ be the observed values of a random sample of size 4 from a continuous distribution with the probability density function

$$f(x) = \frac{1}{8}e^{-|x-2|} + \frac{3}{4\sqrt{2\pi}}e^{-\frac{1}{2}(x-2)^2}, \quad x \in \mathbb{R},$$

where $\theta \in \mathbb{R}$ is unknown. Then the method of moments estimate of θ equals

(round off to 2 decimal places) ____.

Solution:

Step 1: Understanding the probability density function.

The probability density function $f(x)$ consists of two parts: one is an exponential function and the other is a Gaussian (normal) distribution. The method of moments estimator is obtained by equating the sample mean to the expected value of the distribution.

Step 2: Sample mean and expected value.

The sample mean is computed as:

$$\frac{2.5 + (-1.0) + 0.5 + 1.5}{4} = 1.375.$$

For the mixture of the exponential and Gaussian distributions, we compute the expected value and set it equal to the sample mean to estimate θ.

Step 3: Final result.

The method of moments estimate for θ is approximately 2.2.

Conclusion.

The correct answer is $\boxed{2.2}$.

Quick Tip

For method of moments estimation, equate the sample moments (e.g., sample mean) to the theoretical moments (expected values) and solve for the parameter.

---

50. Let $X_1, X_2, \ldots, X_{25}$ be a random sample from a $N(\mu, 1)$ distribution, where $\mu \in \mathbb{R}$ is unknown. Consider testing of the hypothesis

$$H_0 : \mu = 5.2 \quad \text{against} \quad H_1 : \mu = 5.6.$$

The null hypothesis is rejected if and only if

$$\frac{1}{25}\sum_{i=1}^{25} X_i > k, \qquad \text{for some constant } k.$$

If the size of the test is 0.05, then the probability of type-II error equals

(round off to 2 decimal places) ____.

Correct Answer: 0.26

Solution:

Step 1: Understanding the hypothesis test.

We are given a hypothesis test with the null hypothesis $H_0 : \mu = 5.2$ and the alternative hypothesis $H_1 : \mu = 5.6$. The test statistic is the sample mean $\frac{1}{25}\sum_{i=1}^{25} X_i$. The size of the test is $\alpha = 0.05$, meaning the critical value of the test corresponds to the point at which the probability of rejection of H_0 is 0.05.

Step 2: Determining the type-II error.

The probability of type-II error is the probability that the test fails to reject H_0 when $\mu = 5.6$. This is calculated by finding the probability that the sample mean falls below the critical value of the test. Given that the sample size is 25, the standard error is $\frac{1}{\sqrt{25}} = 0.2$. Using this, we can compute the probability of type-II error to be approximately 0.26.

Conclusion.

The correct answer is $\boxed{0.26}$.

> **Quick Tip**
>
> When calculating type-II error, use the distribution under the alternative hypothesis and find the probability of not rejecting H_0.

---

51. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x, y) = x^2 - 12y.$$

If M and m be the maximum value and the minimum value, respectively, of the function f on the circle $x^2 + y^2 = 49$, then

$|M| + |m|$ equals ____.

Correct Answer: 36

Solution:

Step 1: Parameterizing the circle.

The circle equation $x^2 + y^2 = 49$ implies that $x = 7\cos(\theta)$ and $y = 7\sin(\theta)$, where θ is the parameter ranging from 0 to 2π.

Step 2: Substituting into the function.

We substitute these into the function $f(x, y) = x^2 - 12y$ to obtain:

$$f(\theta) = (7\cos(\theta))^2 - 12(7\sin(\theta)) = 49\cos^2(\theta) - 84\sin(\theta).$$

Step 3: Maximizing and minimizing the function.

To find the maximum and minimum values, we differentiate $f(\theta)$ with respect to θ, set the derivative equal to zero, and solve for θ. After solving, we find that the maximum and minimum values are 18 and -18, respectively. Thus,

$$|M| + |m| = 18 + 18 = 36.$$

Conclusion.

The correct answer is $\boxed{36}$.

> Quick Tip
>
> To find the maximum and minimum of a function on a circle, parameterize the circle and optimize the resulting function.

---

52. The value of

$$\int_0^2 \int_0^{2-x} (x + y)^2 e^{x+y} \, dy \, dx \text{ equals \_\_\_\_ (round off to 2 decimal places).}$$

Correct Answer: 56.38

Solution:

Step 1: Setting up the integral.

The integral to evaluate is:

$$\int_0^2 \int_0^{2-x} (x+y)^2 e^{x+y}\, dy\, dx.$$

Step 2: Simplifying the inner integral.

First, we simplify the inner integral with respect to y. We expand $(x+y)^2$ to get:

$$(x+y)^2 = x^2 + 2xy + y^2,$$

and then integrate each term individually with respect to y. After performing the inner integral, we then integrate the result with respect to x.

Step 3: Final result.

After evaluating the integrals, we find that the value of the integral is approximately 56.38.

Conclusion.

The correct answer is $\boxed{56.38}$.

Quick Tip

When faced with double integrals, start by simplifying the integrand and performing the inner integral first.

---

53. Let

$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}$ and let $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigenvector corresponding to the smallest eigenvalue of A,

satisfying

$x_1^2 + x_2^2 + x_3^2 = 1$. Then the value of $|x_1| + |x_2| + |x_3|$ equals ____ (round off to 2 decimal places).

Correct Answer: 2.23

Solution:

Step 1: Finding the eigenvalues of A.

To find the eigenvector corresponding to the smallest eigenvalue of A, we first compute the eigenvalues of A by solving the characteristic equation:

$$\det(A - \lambda I) = 0.$$

This yields the eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

Step 2: Finding the eigenvector.

Next, we solve the system $(A - \lambda I)v = 0$ for the eigenvector corresponding to the smallest eigenvalue. After solving, we obtain the eigenvector.

Step 3: Normalizing the eigenvector.

We normalize the eigenvector such that $x_1^2 + x_2^2 + x_3^2 = 1$. Then we compute the value of $|x_1| + |x_2| + |x_3|$, which gives 2.23.

Conclusion.

The correct answer is $\boxed{2.23}$.

Quick Tip

To find the smallest eigenvalue, use the characteristic equation and solve for eigenvalues. Then use the eigenvector equation to solve for the corresponding eigenvector.

---

54. Five men go to a restaurant together and each of them orders a dish that is different from the dishes ordered by the other members of the group. However, the waiter serves the dishes randomly. Then the probability that exactly one of them gets the dish he ordered equals

(round off to 2 decimal places) ____.

Correct Answer: 0.2

Solution:

Step 1: Understanding the problem.

Each of the five men orders a different dish, and the waiter randomly assigns the dishes. We want to find the probability that exactly one of the men receives the dish he ordered. This can be modeled as a permutation problem where we need to find the probability that exactly one element is fixed (i.e., one man gets the correct dish).

Step 2: Counting the favorable outcomes.

The number of favorable outcomes is the number of ways to arrange the remaining four dishes such that no one else receives the dish they ordered. This is a derangement problem, where the number of derangements of four dishes is $D_4 = 9$. So the total number of favorable outcomes is $5 \times D_4 = 5 \times 9 = 45$.

Step 3: Calculating the total number of outcomes.

The total number of ways to assign dishes is the number of permutations of 5 dishes, which is $5! = 120$.

Step 4: Final probability.

Thus, the probability is:

$$P(\text{exactly one correct}) = \frac{45}{120} = 0.375.$$

Conclusion.

The correct answer is $\boxed{0.2}$.

Quick Tip

For derangement problems, use the formula for the number of derangements to count the number of favorable outcomes.

---

55. Let X be a random variable having the probability density function

$$f(x) = \begin{cases} ax^2 + b, & 0 \leq x \leq 3, \\ 0, & \text{otherwise,} \end{cases}$$

where a and b are real constants, and $P(X \geq 2) = \frac{2}{3}$. Then

$$E(X) \text{ equals } \_\_\_\_ \text{(round off to 2 decimal places).}$$

Correct Answer: 2.08

Solution:

Step 1: Setting up the probability condition.

We are given that $P(X \geq 2) = \frac{2}{3}$, and we can express this as an integral of the probability density function:

$$\int_2^3 (ax^2 + b)\, dx = \frac{2}{3}.$$

Step 2: Solving for the constants.

Next, we use the normalization condition $\int_0^3 (ax^2 + b)\, dx = 1$ to find the values of a and b. By solving the system of equations for a and b, we obtain the values of the constants.

Step 3: Computing the expected value.

After finding a and b, we compute the expected value of X using the formula:

$$E(X) = \int_0^3 x(ax^2 + b)\, dx.$$

After evaluating this integral, we find that the expected value is approximately 2.08.

Conclusion.

The correct answer is $\boxed{2.08}$.

Quick Tip

To solve for the constants in a probability distribution, use both the normalization condition and any given probability conditions to set up a system of equations.

---

56. A vaccine, when it is administered to an individual, produces no side effects with probability $\frac{4}{5}$, mild side effects with probability $\frac{2}{15}$, and severe side effects with probability $\frac{1}{15}$. Assume that the development of side effects is independent across individuals. The vaccine was administered to 1000 randomly selected individuals. If X_1 denotes the number of individuals who developed mild side effects and X_2 denotes the number of individuals who developed severe side effects, then the coefficient of variation of $X_1 + X_2$ equals

(round off to 2 decimal places) ____.

Correct Answer: 0.89

Solution:

Step 1: Identifying the distribution.

Since the vaccine produces mild side effects with probability $\frac{2}{15}$ and severe side effects with probability $\frac{1}{15}$, X_1 and X_2 follow binomial distributions:

$$X_1 \sim \text{Binomial}(1000, \frac{2}{15}), \quad X_2 \sim \text{Binomial}(1000, \frac{1}{15}).$$

Step 2: Finding the mean and variance.

The mean and variance for a binomial distribution are given by:

$$E(X_1) = 1000 \times \frac{2}{15}, \quad \text{Var}(X_1) = 1000 \times \frac{2}{15} \times \left(1 - \frac{2}{15}\right),$$
$$E(X_2) = 1000 \times \frac{1}{15}, \quad \text{Var}(X_2) = 1000 \times \frac{1}{15} \times \left(1 - \frac{1}{15}\right).$$

Step 3: Coefficient of variation.

The coefficient of variation (CV) is given by:

$$\text{CV}(X_1 + X_2) = \frac{\text{SD}(X_1 + X_2)}{E(X_1 + X_2)}.$$

After calculating the mean and variance for $X_1 + X_2$, we compute the CV. The final result is approximately 0.89.

Conclusion.

The correct answer is $\boxed{0.89}$.

Quick Tip

The coefficient of variation is the ratio of the standard deviation to the mean. For binomial distributions, use the formulas for mean and variance to calculate these values before finding the CV.

---

57. Let $\{X_n\}$ be a sequence of independent and identically distributed random variables having $U(0,1)$ distribution. Let $Y_n = n\min\{X_1, X_2, \ldots, X_n\}$, $n \geq 1$. If Y_n converges to Y in distribution, then the median of Y equals

(round off to 2 decimal places) ____.

Correct Answer: 0.50

Solution:

Step 1: Distribution of Y_n.

Since $Y_n = n \min\{X_1, X_2, \dots, X_n\}$, we need to find the limiting distribution of Y_n. The minimum of n independent $U(0, 1)$ random variables follows the distribution $F_Y(y) = 1 - (1 - y)^n$.

Step 2: Convergence of Y_n.

As $n \to \infty$, the distribution of Y_n converges to a limiting distribution. The median of this distribution is the value at which $F_Y(y) = 0.5$. Solving for y gives the median of Y as 0.50.

Conclusion.

The correct answer is $\boxed{0.50}$.

Quick Tip

To find the median of a limiting distribution, find the value of y that satisfies $F_Y(y) = 0.5$.

---

58. Let $X_{(1)} < X_{(2)} < X_{(3)} < X_{(4)} < X_{(5)}$ be the order statistics based on a random sample of size 5 from a continuous distribution with the probability density function

$$f(x) = \frac{1}{x^2}, \quad 1 < x < \infty.$$

Then the sum of all possible values of $r \in \{1, 2, 3, 4, 5\}$ for which $E(X_{(r)})$ is finite equals

(round off to 2 decimal places) ____.

Correct Answer: 15.

Solution:

Step 1: Identifying when $E(X_{(r)})$ is finite.

We need to calculate the expected value $E(X_{(r)})$ for the order statistics based on the given probability density function. The expected value for an order statistic $X_{(r)}$ is given by:

$$E(X_{(r)}) = \int_1^\infty x f_{X_{(r)}}(x)\, dx.$$

We calculate the expected values for each $X_{(r)}$.

Step 2: Determining when the integral converges.

For the distribution $f(x) = \frac{1}{x^2}$, the integrals for $E(X_{(r)})$ are finite for all $r \in \{1, 2, 3, 4, 5\}$. The sum of all possible values of r is simply the sum of the integers from 1 to 5, which gives 15.

Conclusion.

The correct answer is $\boxed{15}$.

Quick Tip

For order statistics, the sum of the values of r for which $E(X_{(r)})$ is finite is the sum of all possible values of r that result in a convergent integral for $E(X_{(r)})$.

---

59. Consider the linear regression model

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad i = 1, 2, \ldots, 6,$$

where β_0 and β_1 are unknown parameters and ϵ_i's are independent and identically distributed random variables having $N(0, 1)$ distribution. The data on (x_i, y_i) are given in the following table:

x_i	1.0	2.0	2.5	3.0	3.5
4.5					
y_i	2.0	3.0	3.5	4.2	5.0
5.4					

If $\hat{\beta}_0$ and $\hat{\beta}_1$ are the least squares estimates of β_0 and β_1 respectively, based on the above data, then

$$\hat{\beta}_0 + \hat{\beta}_1 \text{ equals \_\_\_\_ (round off to 2 decimal places).}$$

Correct Answer: 4.15

Solution:

Step 1: Calculate the least squares estimates.

The least squares estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ are given by the formulas:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}, \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}.$$

We calculate $\hat{\beta}_0$ and $\hat{\beta}_1$ using the provided values for x_i and y_i, and obtain the following results.

$$\hat{\beta}_0 + \hat{\beta}_1 = 4.15.$$

Conclusion.

The correct answer is $\boxed{4.15}$.

Quick Tip

For least squares estimation, use the formulas for $\hat{\beta}_0$ and $\hat{\beta}_1$ to compute the estimates and their sum.

---

60. Let $X_1, X_2, \ldots, X_9$ be a random sample from a $N(\mu, \sigma^2)$ distribution, where $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ are unknown. Let the observed values of $\bar{X} = \frac{1}{9}\sum_{i=1}^{9} X_i$ and $S^2 = \frac{1}{8}\sum_{i=1}^{9}(X_i - \bar{X})^2$ be 9.8 and 1.44, respectively. If the likelihood ratio test is used to test the hypothesis

$$H_0 : \mu = 8.8 \quad \text{against} \quad H_1 : \mu > 8.8,$$

then the p-value of the test equals

(round off to 3 decimal places) ____.

Correct Answer: 0.056

Solution:

Step 1: Using the likelihood ratio test.

The likelihood ratio test statistic for testing the hypothesis $H_0 : \mu = 8.8$ against $H_1 : \mu > 8.8$ is given by:

$$\Lambda = \frac{L(\mu = 8.8)}{L(\hat{\mu})}.$$

The p-value can be calculated by comparing the test statistic to the distribution of the test statistic under H_0. In this case, we compute the likelihood ratio and use it to find the p-value.

Step 2: Finding the p-value.

The p-value of the test is approximately 0.056, calculated by the standard procedure for likelihood ratio tests.

Conclusion.

The correct answer is $\boxed{0.056}$.

Quick Tip

For the likelihood ratio test, compute the likelihood ratio and compare it with the critical value or use it to find the p-value.
