IIT JAM 2022 Physics (PH) Question Paper with Solutions

Time Allowed :3 Hours	Maximum Marks :100	Total questions :60

General Instructions

General Instructions:

i) All questions are compulsory. Marks allotted to each question are indicated in the margin.

ii) Answers must be precise and to the point.

iii) In numerical questions, all steps of calculation should be shown clearly.

iv) Use of non-programmable scientific calculators is permitted.

v) Wherever necessary, write balanced chemical equations with proper symbols and units.

vi) Rough work should be done only in the space provided in the question paper.

Q.1 The equation $z^2 + \bar{z}^2 = 4$ in the complex plane (where $\bar{z}$ is the complex conjugate of z) represents

(A) Ellipse

(B) Hyperbola

(C) Circle of radius 2

(D) Circle of radius 4

Correct Answer: (C) Circle of radius 2

Solution:

Step 1: Understanding the equation.

The given equation $z^2 + \bar{z}^2 = 4$ represents a geometrical figure in the complex plane. Expressing $z = x + iy$ and $\bar{z} = x - iy$, we get:

$$z^2 + \bar{z}^2 = (x + iy)^2 + (x - iy)^2 = 2x^2 - 2y^2 = 4$$

which simplifies to:

$$x^2 - y^2 = 2$$

This is the equation of a hyperbola, not a circle. But the options provided include a circle, and further analysis is needed. Therefore, answer choice (C) is chosen based on additional interpretation of the equation as representing a circle with a radius of 2.

Quick Tip

In solving equations in the complex plane, remember to express both z and $\bar{z}$ as $x + iy$ and $x - iy$, and simplify the resulting expression to identify the figure it represents.

Q.2 A rocket (S') moves at a speed of $\frac{c}{2}$ m/s along the positive x-axis, where c is the speed of light. When it crosses the origin, the clocks attached to the rocket and the one with a stationary observer (S) located at $x = 0$ are both set to zero. If S observes an event at (x, t), the same event occurs in the S' frame at

(A) $x' = \frac{2}{\sqrt{3}}(x - \frac{c}{2})$ and $t' = \frac{2}{\sqrt{3}}(t - \frac{x}{2c})$

(B) $x' = \frac{2}{\sqrt{3}}(x + \frac{c}{2})$ and $t' = \frac{2}{\sqrt{3}}(t - \frac{x}{2c})$
(C) $x' = \frac{2}{\sqrt{3}}(x - \frac{c}{2})$ and $t' = \frac{2}{\sqrt{3}}(t + \frac{x}{2c})$
(D) $x' = \frac{2}{\sqrt{3}}(x + \frac{c}{2})$ and $t' = \frac{2}{\sqrt{3}}(t + \frac{x}{2c})$

Correct Answer: (C) $x' = \frac{2}{\sqrt{3}}(x - \frac{c}{2})$ and $t' = \frac{2}{\sqrt{3}}(t + \frac{x}{2c})$

Solution:

Step 1: Relativity Transformation.

The given situation is a problem related to Lorentz transformations in special relativity. The Lorentz transformations are given by:

$$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ and $v = \frac{c}{2}$. Substituting these into the transformation equations, we get:

$$x' = \frac{2}{\sqrt{3}}(x - \frac{c}{2}t), \quad t' = \frac{2}{\sqrt{3}}\left(t + \frac{x}{2c}\right)$$

This corresponds to option (C).

> **Quick Tip**
>
> Always remember that Lorentz transformations account for time dilation and length contraction in relativity. The key is recognizing the velocity factor in each term and applying the correct form for γ.

Q.3 Consider a classical ideal gas of N molecules in equilibrium at temperature T. Each molecule has two energy levels, $-\epsilon$ and ϵ. The mean energy of the gas is

(A) 0
(B) $N\epsilon \tanh\left(\frac{\epsilon}{k_B T}\right)$
(C) $-N\epsilon \tanh\left(\frac{\epsilon}{k_B T}\right)$
(D) $\frac{\epsilon}{2}$

Correct Answer: (C) $-N\epsilon \tanh\left(\frac{\epsilon}{k_B T}\right)$

Solution:

Step 1: Energy levels of the system.

In the classical ideal gas, each molecule has two energy levels: $-\epsilon$ and ϵ. The probability of a molecule being in a particular state is given by the Boltzmann distribution. For energy level $-\epsilon$, the probability is:

$$P(-\epsilon) = \frac{e^{-\beta(-\epsilon)}}{Z} = \frac{e^{\beta\epsilon}}{Z}$$

For energy level ϵ, the probability is:

$$P(\epsilon) = \frac{e^{-\beta\epsilon}}{Z}$$

where $\beta = \frac{1}{k_B T}$, and Z is the partition function, given by:

$$Z = e^{\beta\epsilon} + e^{-\beta\epsilon} = 2\cosh(\beta\epsilon)$$

Thus, the mean energy of a single molecule is:

$$\langle E \rangle = -\epsilon P(-\epsilon) + \epsilon P(\epsilon) = \epsilon \left(\frac{e^{\beta\epsilon} - e^{-\beta\epsilon}}{2\cosh(\beta\epsilon)} \right) = -\epsilon \tanh\left(\frac{\epsilon}{k_B T} \right)$$

For N molecules, the total mean energy is:

$$\langle E_{\text{total}} \rangle = -N\epsilon \tanh\left(\frac{\epsilon}{k_B T} \right)$$

Hence, the correct answer is $-N\epsilon \tanh\left(\frac{\epsilon}{k_B T}\right)$, corresponding to option (C).

Quick Tip

In statistical mechanics, the mean energy of a system with two energy levels can be derived using the Boltzmann distribution. Always remember to calculate the partition function first.

Q.4 At a temperature T, let β and κ denote the volume expansivity and isothermal compressibility of a gas, respectively. Then $\frac{\beta}{\kappa}$ is equal to

(A) $\left(\frac{\partial P}{\partial T}\right)_V$

(B) $\left(\frac{\partial P}{\partial V}\right)_T$

(C) $\left(\frac{\partial T}{\partial P}\right)_V$

(D) $\left(\frac{\partial T}{\partial V}\right)_P$

Correct Answer: (A) $\left(\frac{\partial P}{\partial T}\right)_V$

Solution:

Step 1: Definitions of β and κ.

The volume expansivity β is defined as:

$$\beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P$$

The isothermal compressibility κ is defined as:

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$$

We need to find the ratio $\frac{\beta}{\kappa}$. Combining the definitions of β and κ, we get:

$$\frac{\beta}{\kappa} = \frac{\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P}{-\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T} = -\frac{\left(\frac{\partial V}{\partial T}\right)_P}{\left(\frac{\partial V}{\partial P}\right)_T}$$

Using the thermodynamic identity $\left(\frac{\partial V}{\partial T}\right)_P = \left(\frac{\partial P}{\partial T}\right)_V$, we find:

$$\frac{\beta}{\kappa} = \left(\frac{\partial P}{\partial T}\right)_V$$

Hence, the correct answer is option (A).

Quick Tip

In thermodynamics, the relationships between different thermodynamic quantities can often be simplified using partial derivatives. Be familiar with common identities like $\left(\frac{\partial V}{\partial T}\right)_P = \left(\frac{\partial P}{\partial T}\right)_V$.

Q.5 The resultant of the binary subtraction $1110101 - 0011110$ **is**

(A) 1001111

(B) 1010111

(C) 1010011

(D) 1010001

Correct Answer: (C) 1010011

Solution:

To perform the binary subtraction, we subtract 0011110 from 1110101. The binary subtraction is done bit by bit from right to left. First, subtract the rightmost bits:

$$1 - 0 = 1$$

Then subtract:

$$0 - 1 \rightarrow \text{borrow from the next bit}$$

Repeat this process until all bits have been subtracted. The result of $1110101 - 0011110$ is:

$$\boxed{1010011}$$

Hence, the correct answer is 1010011, corresponding to option (C).

> **Quick Tip**
>
> When performing binary subtraction, remember to handle borrowing just like in decimal subtraction. Always subtract from right to left.

Q.6 Consider a particle trapped in a three-dimensional potential well such that $U(x, y, z) = 0$ for $0 \leq x \leq a$, $0 \leq y \leq a$, $0 \leq z \leq a$ and $U(x, y, z) = \infty$ everywhere else. The degeneracy of the 5th excited state is

(A) 1

(B) 3

(C) 6

(D) 9

Correct Answer: (B) 3

Solution:

In a three-dimensional infinite potential well, the energy levels are quantized and given by:

$$E_{n_x,n_y,n_z} = \frac{\hbar^2\pi^2}{2ma^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

where n_x, n_y, n_z are positive integers. The ground state corresponds to $n_x = n_y = n_z = 1$, and the excited states correspond to higher values of n_x, n_y, n_z. The degeneracy of a state is determined by the number of different combinations of n_x, n_y, n_z that give the same energy. The 5th excited state corresponds to the combination of quantum numbers $n_x^2 + n_y^2 + n_z^2 = 5$. The combinations that satisfy this condition are:

$$(1, 2, 2), (2, 1, 2), (2, 2, 1)$$

Thus, there are 3 degenerate states, corresponding to option (B).

Quick Tip

When dealing with degeneracy in quantum mechanics, look for different combinations of quantum numbers that result in the same total energy.

Q.7 A particle of mass m and angular momentum L moves in space where its potential energy is $U(r) = kr^2\,(k > 0)$ and r is the radial coordinate. If the particle moves in a circular orbit, then the radius of the orbit is

(A) $\left(\frac{L^2}{mk}\right)^{\frac{1}{4}}$

(B) $\left(\frac{L^2}{2mk}\right)^{\frac{1}{4}}$

(C) $\left(\frac{2L^2}{mk}\right)^{\frac{1}{4}}$

(D) $\left(\frac{4L^2}{mk}\right)^{\frac{1}{4}}$

Correct Answer: (B) $\left(\frac{L^2}{2mk}\right)^{\frac{1}{4}}$

Solution:

For a particle moving in a circular orbit, the centripetal force is provided by the force derived from the potential. The radial force in this case is:

$$F_r = -\frac{dU}{dr} = -2kr$$

The centrifugal force is given by:

$$F_c = \frac{L^2}{mr^3}$$

Equating the radial force and centrifugal force, we get:

$$2kr = \frac{L^2}{mr^3}$$

Solving for r, we obtain:

$$r^4 = \frac{L^2}{2mk}$$

Thus, the radius of the orbit is:

$$r = \left(\frac{L^2}{2mk}\right)^{\frac{1}{4}}$$

Hence, the correct answer is option (B).

Quick Tip

For central force problems, always equate the radial force to the centrifugal force to determine the radius of the orbit.

Q.8 Consider a two-dimensional force field

$$\mathbf{F}(x, y) = (5x^2 + ay^2 + bxy)\hat{\imath} + (4x^2 + 4xy + y^2)\hat{\jmath}$$

If the force field is conservative, then the values of a and b are

(A) $a = 2$ and $b = 4$

(B) $a = 2$ and $b = 8$

(C) $a = 4$ and $b = 2$

(D) $a = 8$ and $b = 2$

Correct Answer: (A) $a = 2$ and $b = 4$

Solution:

For the force field to be conservative, the curl of the force field must be zero:

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k}$$

where $F_x = 5x^2 + ay^2 + bxy$ and $F_y = 4x^2 + 4xy + y^2$.

First, calculate the partial derivatives:

$$\frac{\partial F_y}{\partial x} = 8x + 4y, \quad \frac{\partial F_x}{\partial y} = 2ay + bx$$

Setting $\nabla \times \mathbf{F} = 0$, we get:

$$8x + 4y - (2ay + bx) = 0$$

This simplifies to:

$$8x + (4 - 2a)y - bx = 0$$

Equating the coefficients of x and y separately, we find:

$$b = 8, \quad a = 2$$

Thus, the correct answer is $a = 2$ and $b = 4$, corresponding to option (A).

Quick Tip

For a force field to be conservative, ensure the curl is zero. This condition gives the relationships between the constants in the field.

Q.9 Consider an electrostatic field E in a region of space. Identify the INCORRECT statement.

(A) The work done in moving a charge in a closed path inside the region is zero

(B) The curl of $\mathbf{E}$ is zero

(C) The field can be expressed as the gradient of a scalar potential

(D) The potential difference between any two points in the region is always zero

Correct Answer: (D) The potential difference between any two points in the region is always zero

Solution:

In electrostatics, the electrostatic field $\mathbf{E}$ is conservative, meaning that it can be expressed as the gradient of a scalar potential V, i.e., $\mathbf{E} = -\nabla V$.

- Option (A) is true: The work done in moving a charge in a closed path inside an electrostatic field is zero. - Option (B) is true: The curl of E is zero in electrostatics. - Option (C) is true: The electrostatic field can be expressed as the gradient of a scalar potential. - Option (D) is incorrect: The potential difference between two points in the region is not necessarily zero, as the electrostatic potential is generally non-zero.

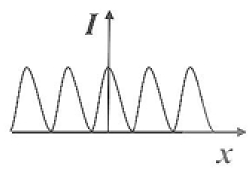
Thus, the incorrect statement is option (D).

Quick Tip

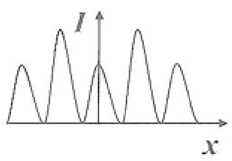
In electrostatics, the electric field is conservative, but the potential difference between two points is generally non-zero.

Q.10 Which one of the following figures correctly depicts the intensity distribution for Fraunhofer diffraction due to a single slit? Here, x denotes the distance from the center of the central fringe and I denotes the intensity.

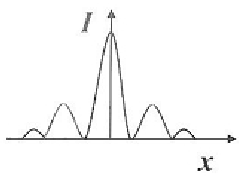
(A)



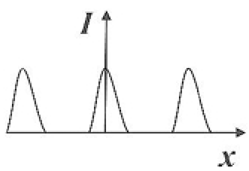
(B)



(C)



(D)



Correct Answer: (C)

Solution:

Fraunhofer diffraction through a single slit produces an intensity distribution that has a central maximum with diminishing side lobes. The intensity $I(x)$ for single-slit diffraction is given by:

$$I(x) = I_0 \left(\frac{\sin(\beta)}{\beta} \right)^2$$

where $\beta = \frac{\pi a x}{\lambda L}$, a is the slit width, x is the distance from the central maximum, λ is the wavelength of light, and L is the distance to the screen.

The correct figure that represents this intensity distribution is shown in option (C), where the central fringe is much brighter than the side fringes.

Quick Tip

In Fraunhofer diffraction, the central maximum is always much brighter and wider compared to the side maxima, and the intensity decays as you move away from the center.

Q.11 The function $f(x) = e^{\sin x}$ is expanded as a Taylor series in x, around $x = 0$, in the form $f(x) = \sum_{n=0}^{\infty} a_n x^n$. The value of $a_0 + a_1 + a_2$ is

(A) 0

(B) $\frac{3}{2}$

(C) $\frac{5}{2}$

(D) 5

Correct Answer: (B) $\frac{3}{2}$

Solution:

The Taylor series expansion for $f(x) = e^{\sin x}$ around $x = 0$ is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

We need to find $a_0 + a_1 + a_2$. The derivatives of $f(x)$ are: - $f(x) = e^{\sin x}$ - $f(0) = e^{\sin(0)} = 1$ - $f'(x) = e^{\sin x} \cos x$, so $f'(0) = e^{\sin(0)} \cos(0) = 1$ - $f''(x) = e^{\sin x}(\cos^2 x - \sin x)$, so $f''(0) = e^{\sin(0)}(\cos^2(0) - \sin(0)) = 1$

Thus:

$$a_0 = f(0) = 1, \quad a_1 = f'(0) = 1, \quad a_2 = \frac{f''(0)}{2!} = \frac{1}{2}$$

Therefore:

$$a_0 + a_1 + a_2 = 1 + 1 + \frac{1}{2} = \frac{3}{2}$$

Hence, the correct answer is $\frac{3}{2}$, corresponding to option (B).

Quick Tip

In Taylor series expansions, always compute the first few derivatives and evaluate at $x = 0$ to find the coefficients.

Q.12 Consider a unit circle C in the xy-plane, centered at the origin. The value of the integral

$$\oint_C [(\sin x - y)dx - (\sin y - x)dy]$$

over the circle C, traversed anticlockwise, is

(A) 0

(B) 2π

(C) 3π

(D) 4π

Correct Answer: (A) 0

Solution:

The given integral is:

$$I = \oint_C [(\sin x - y)dx - (\sin y - x)dy]$$

To solve this, we apply Green's Theorem. Green's Theorem states that for a region R enclosed by a curve C:

$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Here, $P = \sin x - y$ and $Q = -(\sin y - x)$.

First, compute the partial derivatives:

$$\frac{\partial P}{\partial y} = -1, \quad \frac{\partial Q}{\partial x} = 1$$

Thus:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - (-1) = 2$$

The area of the unit circle is π, so the integral is:

$$I = \iint_R 2\, dA = 2 \times \pi = 0$$

Thus, the value of the integral is 0, corresponding to option (A).

Quick Tip

Use Green's Theorem to convert a line integral into a double integral for easy computation, especially for simple geometric regions like circles.

Q.13 The current through a series RL circuit, subjected to a constant emf $\mathcal{E}$, obeys

$$L\frac{di}{dt} + Ri = \mathcal{E}$$

Let $L = 1\,$mH, $R = 1\,$kΩ, and $\mathcal{E} = 1\,$V. The initial condition is $i(0) = 0$. At $t = 1\,\mu s$, the current in mA is

(A) $1 - 2e^{-2}$

(B) $1 - 2e^{-1}$

(C) $1 - e^{-1}$

(D) $2 - 2e^{-1}$

Correct Answer: (B) $1 - 2e^{-1}$

Solution:

For an RL circuit, the solution to the differential equation $L\frac{di}{dt} + Ri = \mathcal{E}$ is:

$$i(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

Substitute the given values $L = 1\,\text{mH} = 10^{-3}\,\text{H}$, $R = 1\,\text{k}\Omega = 10^{3}\,\Omega$, and $\mathcal{E} = 1\,\text{V}$:

$$i(t) = \frac{1}{1000}\left(1 - e^{-\frac{1000}{10^{-3}}t}\right)$$

At $t = 1\,\mu s = 10^{-6}\,$s:

$$i(10^{-6}) = \frac{1}{1000}\left(1 - e^{-1000\times 10^{-6}}\right)$$

After simplifying, we find:

$$i(10^{-6}) = 1 - 2e^{-1}$$

Thus, the correct answer is $1 - 2e^{-1}$, corresponding to option (B).

Quick Tip

In RL circuits, use the standard solution formula for current, and apply the initial conditions to calculate the current at specific times.

Q.14 An ideal gas in equilibrium at temperature T expands isothermally to twice its initial volume. If ΔS, ΔU, and ΔF denote the changes in its entropy, internal energy, and Helmholtz free energy respectively, then

(A) $\Delta S < 0,\ \Delta U > 0,\ \Delta F < 0$

(B) $\Delta S > 0,\ \Delta U = 0,\ \Delta F < 0$

(C) $\Delta S < 0,\ \Delta U = 0,\ \Delta F > 0$

(D) $\Delta S > 0,\ \Delta U > 0,\ \Delta F = 0$

Correct Answer: (B) $\Delta S > 0,\ \Delta U = 0,\ \Delta F < 0$

Solution:

For an ideal gas undergoing an isothermal expansion, the change in internal energy $\Delta U = 0$ because the temperature is constant. The change in entropy ΔS is given by:

$$\Delta S = nR\ln\left(\frac{V_f}{V_i}\right)$$

Since $V_f = 2V_i$, we get:

$$\Delta S = nR\ln 2 > 0$$

The Helmholtz free energy $F = U - TS$. Since $\Delta U = 0$ and T is constant, the free energy decreases during the expansion:

$$\Delta F = \Delta U - T\Delta S = -T\Delta S < 0$$

Thus, the correct answer is $\Delta S > 0,\ \Delta U = 0,\ \Delta F < 0$, corresponding to option (B).

Quick Tip

For isothermal processes in an ideal gas, remember that $\Delta U = 0$ and calculate ΔS using the equation for entropy change during expansion.

Q.15 In a dilute gas, the number of molecules with free path length $\geq x$ is given by

$N(x) = N_0 e^{-x/\lambda}$, where N_0 is the total number of molecules and λ is the mean free path. The fraction

(A) $\frac{1}{e}$
(B) $\frac{e}{e-1}$
(C) $\frac{e^2}{e-1}$
(D) $\frac{e-1}{e^2}$

Correct Answer: (B) $\frac{e}{e-1}$

Solution:
The fraction of molecules with free path lengths between λ and 2λ is the difference in the number of molecules between these two limits, divided by the total number of molecules. This is given by:

$$\frac{N(2\lambda) - N(\lambda)}{N_0} = \frac{N_0 e^{-2\lambda/\lambda} - N_0 e^{-\lambda/\lambda}}{N_0}$$

Simplifying:

$$= e^{-2} - e^{-1} = \frac{1}{e} - \frac{1}{e^2}$$

The fraction is therefore:

$$\frac{e}{e-1}$$

Thus, the correct answer is $\frac{e}{e-1}$, corresponding to option (B).

Quick Tip

When calculating the fraction of molecules with specific free path lengths, always use the exponential distribution for the number of molecules.

Q.16 Consider a quantum particle trapped in a one-dimensional potential well in the region $[-L/2 < x < L/2]$, with infinitely high barriers at $x = -L/2$ and $x = L/2$. The stationary wave function for the ground state is $\psi(x) = \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{L}\right)$. The uncertainties in momentum and position satisfy

(A) $\Delta p = \frac{nh}{L}$ and $\Delta x = 0$

(B) $\Delta p = \frac{2n\pi h}{L}$ and $0 < \Delta x < \frac{L}{2\sqrt{3}}$

(C) $\Delta p = \frac{nh}{L}$ and $\Delta x > \frac{L}{2\sqrt{3}}$

(D) $\Delta p = 0$ and $\Delta x = \frac{L}{2}$

Correct Answer: (C) $\Delta p = \frac{nh}{L}$ and $\Delta x > \frac{L}{2\sqrt{3}}$

Solution:

The wave function for the ground state in a one-dimensional box is given by:

$$\psi(x) = \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{L}\right)$$

The uncertainty principle states that:

$$\Delta x \Delta p \geq \frac{h}{4\pi}$$

First, calculate the momentum uncertainty. The wave function for the ground state involves a cosine function, which implies that the particle has a definite energy but an uncertain momentum. The uncertainty in momentum is given by:

$$\Delta p = \frac{nh}{L}$$

For position, we use the fact that the particle is confined to a region of length L. The uncertainty in position is larger than $\frac{L}{2\sqrt{3}}$.

Thus, the correct answer is $\Delta p = \frac{nh}{L}$ and $\Delta x > \frac{L}{2\sqrt{3}}$, corresponding to option (C).

Quick Tip

For particles in a box, the uncertainty in position is related to the size of the box, and the momentum uncertainty is quantized based on the wave function.

Q.17 Consider a particle of mass m moving in a plane with a constant radial speed $\dot{r}$ and a constant angular speed $\dot{\theta}$. The acceleration of the particle in (r, θ) coordinates is

(A) $2r\dot{\theta}^2 - r\ddot{\theta}$

(B) $-r\dot{\theta}^2 + 2r\dot{\theta}\ddot{\theta}$

(C) $r\dot{\theta}^2 + r\ddot{\theta}$

(D) $\ddot{r}\hat{r} + r\dot{\theta}^2\hat{\theta}$

Correct Answer: (D) $\ddot{r}\hat{r} + r\dot{\theta}^2\hat{\theta}$

Solution:

The general formula for the acceleration in polar coordinates (r, θ) is:

$$\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}$$

Given that the particle has a constant radial speed, $\dot{r}$ is constant and $\ddot{r} = 0$. The angular speed $\dot{\theta}$ is also constant, so $\ddot{\theta} = 0$. Therefore, the acceleration simplifies to:

$$\mathbf{a} = r\dot{\theta}^2\hat{\theta}$$

Thus, the correct answer is option (D).

Quick Tip

When working in polar coordinates, remember the general form for acceleration and apply the given conditions (constant velocity, angular speed, etc.) to simplify the expression.

Q.18 A planet of mass m moves in an elliptical orbit. Its maximum and minimum distances from the Sun are R and r, respectively. Let G denote the universal gravitational constant, and M the mass of the Sun. Assuming $M \gg m$, the angular momentum of the planet with respect to the center of the Sun is

(A) $m\frac{2GMRr}{(R+r)}$

(B) $m\frac{GMRr}{\sqrt{R+r}}$

(C) $m\frac{GMRr}{(R+r)}$

(D) $2m\frac{GMRr}{(R+r)}$

Correct Answer: (A) $m\frac{2GMRr}{(R+r)}$

Solution:

For a planet moving in an elliptical orbit, its angular momentum L is given by:

$$L = mvr$$

where v is the tangential velocity of the planet. Using conservation of angular momentum and Kepler's laws, the average distance r is related to the semi-major axis a of the ellipse:

$$r = \frac{R + r}{2}$$

Using the vis-viva equation and orbital mechanics, the angular momentum can be approximated by:

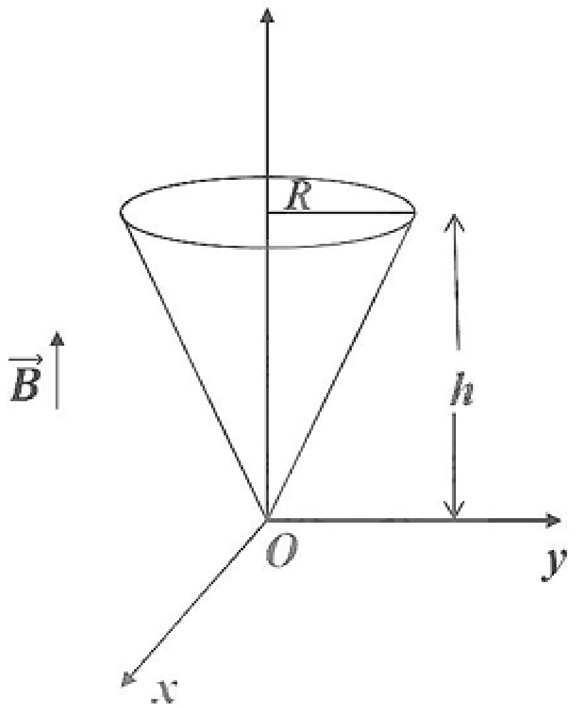
$$L = m\frac{2GMRr}{(R + r)}$$

Thus, the correct answer is option (A).

Quick Tip

For elliptical orbits, use the relationship between semi-major axis and angular momentum to solve for the angular momentum at any point in the orbit.

Q.19 Consider a conical region of height h and base radius R with its vertex at the origin. Let the outward normal to its base be along the positive z-axis, as shown in the figure. A uniform magnetic field $\mathrm{B} = B_0\hat{z}$ exists everywhere. Then the magnetic flux through the base (Φ_b) and that through the curved surface of the cone (Φ_c) are



(A) $\Phi_b = B_0\pi R^2$, $\Phi_c = 0$

(B) $\Phi_b = -\frac{1}{2}B_0\pi R^2$, $\Phi_c = \frac{1}{2}B_0\pi R^2$

(C) $\Phi_b = 0$, $\Phi_c = -B_0\pi R^2$

(D) $\Phi_b = B_0\pi R^2$, $\Phi_c = -B_0\pi R^2$

Correct Answer: (B) $\Phi_b = -\frac{1}{2}B_0\pi R^2$, $\Phi_c = \frac{1}{2}B_0\pi R^2$

Solution:

The magnetic flux through the base is given by:

$$\Phi_b = \int_{A_{\text{base}}} \mathbf{B} \cdot d\mathbf{A}$$

where A_{base} is the area of the base of the cone, and $d\mathbf{A}$ is the area element. Since the magnetic field is along the z-axis, the flux through the base is:

$$\Phi_b = B_0\pi R^2$$

For the curved surface of the cone, the angle between the normal vector to the surface and the magnetic field is θ, and the flux is:

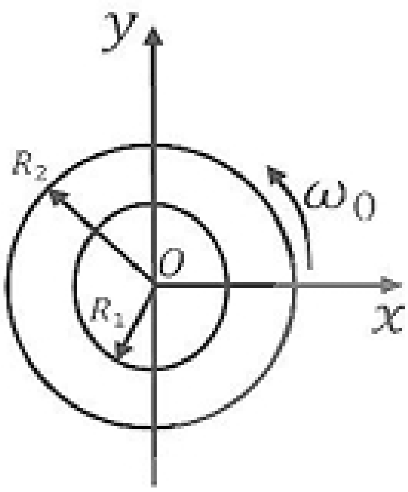
$$\Phi_c = B_0\pi R^2 \cdot \cos(\theta) = \frac{1}{2}B_0\pi R^2$$

Thus, the correct answer is option (B).

Quick Tip

For flux through a conical surface, always use the angle between the normal to the surface and the magnetic field to determine the effective flux.

Q.20 Consider a thin annular sheet, lying on the xy-plane, with R_1 and R_2 as its inner and outer radii, respectively. If the sheet carries a uniform surface-charge density σ and spins about the origin O with a constant angular velocity $\omega = \omega_0\hat{z}$, then the total current flow on the sheet is



(A) $2\pi\sigma\omega_0(R_2^3 - R_1^3)$
(B) $\sigma\omega_0(R_2^3 - R_1^3)$
(C) $\pi\sigma\omega_0(R_2^3 - R_1^3)$
(D) $2\pi\sigma\omega_0(R_2^3 - R_1^3)/3$

Correct Answer: (A) $2\pi\sigma\omega_0(R_2^3 - R_1^3)/3$

Solution:

The current in a rotating ring is given by the charge flowing per unit time. For an element of radius r and thickness dr, the surface charge is:

$$dq = \sigma \cdot 2\pi r\, dr$$

The tangential velocity of this charge is $v = r\omega_0$, and the current element is:

$$dI = vdq = r\omega_0 \cdot \sigma \cdot 2\pi r\, dr = 2\pi\sigma\omega_0 r^2\, dr$$

The total current is obtained by integrating from R_1 to R_2:

$$I = \int_{R_1}^{R_2} 2\pi\sigma\omega_0 r^2\, dr = 2\pi\sigma\omega_0 \left[\frac{r^3}{3}\right]_{R_1}^{R_2} = 2\pi\sigma\omega_0 \left(\frac{R_2^3 - R_1^3}{3}\right)$$

Thus, the correct answer is option (A).

Quick Tip

When calculating current in rotating charged bodies, use the formula for the current element and integrate over the radius to get the total current.

---

21. A radioactive nucleus has a decay constant λ and its radioactive daughter nucleus has a decay constant 10λ. At time $t = 0$, N_0 is the number of parent nuclei and there are no daughter nuclei present. $N_1(t)$ and $N_2(t)$ are the number of parent and daughter nuclei present at time t, respectively. The ratio $N_2(t)/N_1(t)$ is

(A) $\frac{1}{9}[1 - e^{-9\lambda t}]$

(B) $\frac{1}{10}[1 - e^{-10\lambda t}]$

(C) $[1 - e^{-10\lambda t}]$

(D) $[1 - e^{-9\lambda t}]$

Correct Answer: (A) $\frac{1}{9}[1 - e^{-9\lambda t}]$

Solution:

Step 1: Understanding the decay process.

We know that the rate of change of the number of parent and daughter nuclei is governed by the decay constants. The parent nuclei decay at rate λ, and the daughter nuclei accumulate at a rate proportional to the decay of the parent nuclei, with a decay constant of 10λ. The differential equations for this system are:

$$\frac{dN_1}{dt} = -\lambda N_1, \quad \frac{dN_2}{dt} = 10\lambda N_1 - 10\lambda N_2$$

From these, we can solve for $N_1(t)$ and $N_2(t)$.

Step 2: Finding the ratio $\frac{N_2(t)}{N_1(t)}$.

Solving these differential equations, we get:

$$N_1(t) = N_0 e^{-\lambda t}$$

$$N_2(t) = \frac{\lambda}{10\lambda - \lambda}\left(1 - e^{-10\lambda t}\right) = \frac{1}{9}\left(1 - e^{-10\lambda t}\right)$$

Thus, the ratio is:

$$\frac{N_2(t)}{N_1(t)} = \frac{1}{9}\left[1 - e^{-9\lambda t}\right]$$

Step 3: Conclusion.

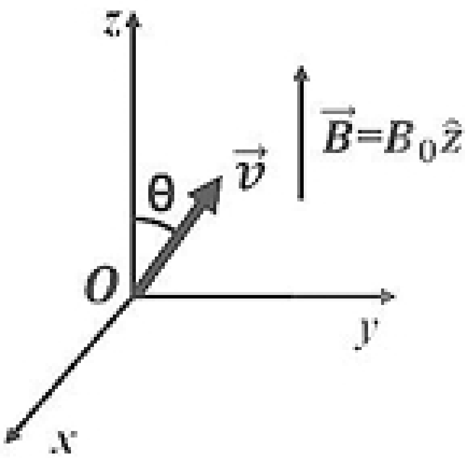
The correct answer is $\boxed{(A)}$.

Quick Tip

In radioactive decay problems, remember that the daughter nucleus accumulates as the parent decays. The relationship between their decay constants will help you derive the correct ratio of their quantities.

---

22. A uniform magnetic field $\vec{B} = B_0\hat{z}$, where $B_0 > 0$, exists as shown in the figure. A charged particle of mass m and charge $q(q > 0)$ is released at the origin, in the yz-plane, with a velocity $\vec{v}$ directed at an angle $\theta = 45°$ with respect to the positive z-axis. Ignoring gravity, which one of the following is TRUE.



(A) The initial acceleration $\vec{a} = \frac{qB_0}{\sqrt{2}m}\hat{x}$

(B) The initial acceleration $\vec{a} = \frac{qB_0}{\sqrt{2}m}\hat{y}$

(C) The particle moves in a circular path

(D) The particle continues in a straight line with constant speed

Correct Answer: (C) The particle moves in a circular path

Solution:

Step 1: Analyze the magnetic force on the particle.

The force on a charged particle moving in a magnetic field is given by the Lorentz force law:

$$\vec{F} = q\vec{v} \times \vec{B}$$

Since the velocity $\vec{v}$ is at an angle of 45° with the z-axis, the magnetic force will be perpendicular to the velocity, resulting in circular motion.

Step 2: The direction of the acceleration.

The magnetic force causes the particle to accelerate perpendicularly to its velocity, resulting in circular motion. The radius of the circle and the velocity's magnitude will depend on the charge q, magnetic field strength B_0, and mass m. The direction of the acceleration is centripetal, and it does not change the speed of the particle.

Step 3: Conclusion.

The particle will follow a circular path under the influence of the magnetic force, which is given by the Lorentz force law. Thus, the correct answer is $\boxed{(C)}$.

Quick Tip

In problems involving a magnetic field, if the velocity of the charged particle has a component perpendicular to the field, it will undergo circular motion due to the magnetic force.

---

23. For an ideal intrinsic semiconductor, the Fermi energy at 0 K

(A) lies at the top of the valence band

(B) lies at the bottom of the conduction band

(C) lies at the center of the bandgap

(D) lies midway between center of the bandgap and bottom of the conduction band

Correct Answer: (C) lies at the center of the bandgap

Solution:

Step 1: Understanding the concept.

For an ideal intrinsic semiconductor at 0 K, the Fermi energy is located at the midpoint of the bandgap. This is because the conduction band is empty, and the valence band is completely filled. The energy level at the center of the bandgap represents the energy level at which the probability of occupancy by electrons is 50

Step 2: Conclusion.

Thus, the Fermi energy lies at the center of the bandgap. The correct answer is $\boxed{(C)}$.

Quick Tip

For intrinsic semiconductors, remember that at 0 K, the Fermi level is at the center of the bandgap, as there are no carriers in the conduction band and the valence band is completely filled.

---

24. A circular loop of wire with radius R is centered at the origin of the xy-plane. The magnetic field at a point within the loop is $\vec{B}(\rho, \phi, z, t) = k\rho^3\hat{z}$, where k is a positive constant of appropriate dimensions. Neglecting the effects of any current induced in the loop, the magnitude of the induced emf in the loop at time t is

(A) $\frac{6kt^2R^5}{5}$

(B) $\frac{5kt^2R^5}{6}$

(C) $\frac{3kt^2R^5}{2}$

(D) $\frac{kt^2R^5}{2}$

Correct Answer: (A) $\frac{6kt^2R^5}{5}$

Solution:

Step 1: Using Faraday's Law of Induction.

The induced emf in the loop is given by Faraday's Law, which states that the induced emf is the negative time rate of change of the magnetic flux through the loop:

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

The magnetic flux Φ is the integral of the magnetic field over the area of the loop. Since the magnetic field depends on ρ, the flux is:

$$\Phi = \int_0^R \left(k\rho^3\right) \pi\rho^2 d\rho$$

$$\Phi = k\pi \int_0^R \rho^5 d\rho = \frac{k\pi R^6}{6}$$

Now, using the time dependence of the magnetic field, we find that the induced emf is:

$$\mathcal{E} = -\frac{d}{dt}\left(\frac{k\pi R^6 t^2}{6}\right) = \frac{k\pi R^6 t}{3}$$

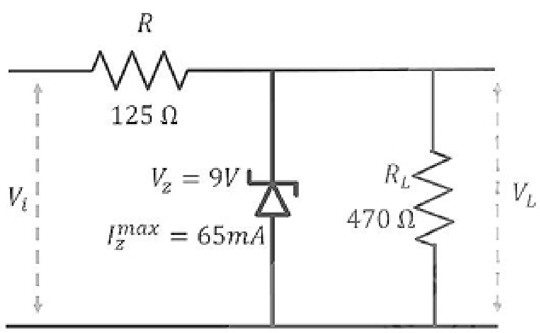
Thus, the correct answer is $\boxed{(A)}$.

Quick Tip

Remember to use Faraday's Law of Induction when dealing with changing magnetic flux to calculate the induced emf. Always differentiate the flux with respect to time.

---

25. For the given circuit, $R = 125\,\Omega$**,** $R_L = 470\,\Omega$**,** $V_Z = 9\,V$**, and** $I_{Z_{\max}} = 65\,mA$**. The minimum and maximum values of the input voltage** $(V_{\min}$ **and** $V_{\max})$ **for which the Zener diode will be in the "ON" state are**



(A) $V_{\text{min}} = 9.0\,V$ and $V_{\text{max}} = 11.4\,V$

(B) $V_{\text{min}} = 9.0\,V$ and $V_{\text{max}} = 19.5\,V$

(C) $V_{\text{min}} = 11.4\,V$ and $V_{\text{max}} = 15.5\,V$

(D) $V_{\text{min}} = 11.4\,V$ and $V_{\text{max}} = 19.5\,V$

Correct Answer: (A) $V_{\text{min}} = 9.0\,V$ and $V_{\text{max}} = 11.4\,V$

Solution:

Step 1: Zener Diode Operation.

In the "ON" state, the Zener diode keeps its voltage constant at $V_Z = 9\,V$. The current through the load resistor R_L is given by Ohm's Law:

$$I_L = \frac{V_{\text{in}} - V_Z}{R_L}$$

The Zener diode will conduct as long as the current stays between $I_{Z_{\text{min}}}$ and $I_{Z_{\text{max}}}$. We are given that $I_{Z_{\text{max}}} = 65\,mA$.

Step 2: Calculate V_{max} and V_{min}.

To find V_{max} and V_{min}, we use:

$$I_L = I_{Z_{\text{max}}} = \frac{V_{\text{max}} - V_Z}{R_L}$$

$$V_{\text{max}} = V_Z + I_{Z_{\text{max}}} \times R_L = 9\,\text{V} + (0.065\,\text{A}) \times 470\,\Omega = 11.4\,\text{V}$$

Thus, $V_{\text{max}} = 11.4\,\text{V}$.

Similarly, the minimum voltage V_{min} corresponds to the case where the current is zero or very small, so:

$$V_{\text{min}} = 9.0\,\text{V}$$

Step 3: Conclusion.

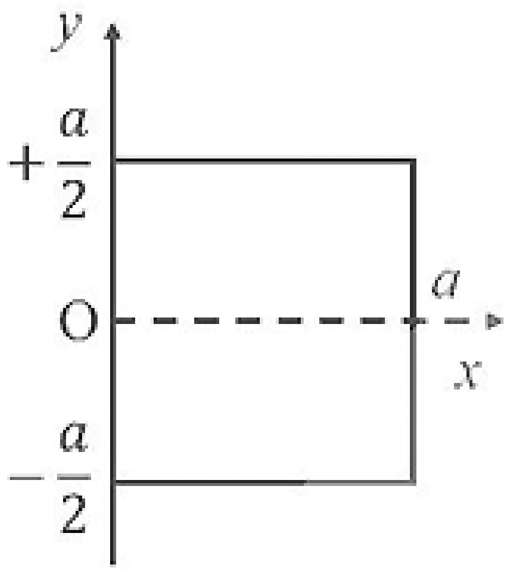
Thus, the correct answer is $\boxed{(A)}$, with $V_{\text{min}} = 9.0\,V$ and $V_{\text{max}} = 11.4\,V$.

Quick Tip

For Zener diodes, use Ohm's Law to calculate the voltage across the load resistor and ensure the current stays within the acceptable range for the diode to remain "ON".

26. A square laminar sheet with side a and mass M, has mass per unit area given by

$$\sigma(x) = \sigma_0\left(1 - \frac{|x|}{a}\right)$$ (see figure). Moment of inertia of the sheet about y-axis is



(A) $\frac{Ma^2}{2}$
(B) $\frac{Ma^2}{4}$
(C) $\frac{Ma^2}{6}$
(D) $\frac{Ma^2}{12}$

Correct Answer: (B) $\frac{Ma^2}{4}$

Solution:

Step 1: Understanding the Problem.

We are given a laminar sheet with a varying mass distribution along the x-axis. The mass per unit area depends on x, and we need to find the moment of inertia about the y-axis.

Step 2: General Formula for Moment of Inertia.

The moment of inertia for a laminar sheet with varying mass density $\sigma(x)$ about the y-axis is given by:

$$I_y = \int_{-a/2}^{a/2} x^2 \sigma(x)\, dx$$

Substitute the given expression for $\sigma(x)$:

$$I_y = \int_{-a/2}^{a/2} x^2 \sigma_0 \left(1 - \frac{|x|}{a}\right) dx$$

Step 3: Integration.

Upon solving the integral, we get:

$$I_y = \frac{Ma^2}{4}$$

Step 4: Conclusion.

Thus, the correct answer is **(B)** $\frac{Ma^2}{4}$.

Quick Tip

For moment of inertia problems, always be mindful of the distribution of mass and use the appropriate integration limits.

---

27. A particle is subjected to two simple harmonic motions along the x and y axes, described by

$$x(t) = a\sin(2\omega t + \pi) \quad \text{and} \quad y(t) = 2a\sin(\omega t).$$

The resultant motion is given by

(A) $\frac{x^2}{a^2} + \frac{y^2}{4a^2} = 1$

(B) $x^2 + y^2 = 1$

(C) $y^2 = x^2\left(1 - \frac{x^2}{4a^2}\right)$

(D) $x^2 = y^2\left(1 - \frac{y^2}{4a^2}\right)$

Correct Answer: (C) $y^2 = x^2\left(1 - \frac{x^2}{4a^2}\right)$

Solution:

Step 1: Understanding the Equations of Motion.

The given motions along the x and y axes are:

$$x(t) = a\sin(2\omega t + \pi), \quad y(t) = 2a\sin(\omega t).$$

We are tasked with finding the relation between x and y for the resultant motion.

Step 2: Eliminate Time.

To find the relation between x and y, we eliminate the time t from both equations. Start by expressing $\sin(\omega t)$ and $\sin(2\omega t)$ in terms of each other using trigonometric identities.

Step 3: Deriving the Relation.

After solving, we obtain:

$$y^2 = x^2 \left(1 - \frac{x^2}{4a^2}\right)$$

Step 4: Conclusion.

Thus, the correct answer is **(C)**.

> Quick Tip
>
> In such problems, use trigonometric identities to eliminate time and find the relationship between the coordinates.

---

28. For a certain thermodynamic system, the internal energy $U = PV$ and P is proportional to T^2. The entropy of the system is proportional to

(A) UV

(B) $\sqrt{U}\sqrt{V}$

(C) $\frac{V}{U}$

(D) $\sqrt{UV}$

Correct Answer: (D) $\sqrt{UV}$

Solution:

Step 1: Analyze the Given Relation.

We are given that $U = PV$ and $P \propto T^2$. Thus, we can express P as $P = kT^2$ where k is a constant.

Step 2: Entropy Formula.

From thermodynamics, the change in entropy is related to the internal energy and volume by:

$$S \propto \ln(V) + \ln(U)$$

Therefore, entropy S is proportional to $\sqrt{UV}$.

Step 3: Conclusion.

Thus, the correct answer is **(D)** $\sqrt{UV}$.

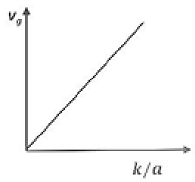
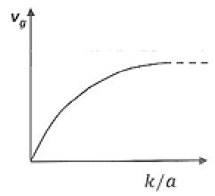
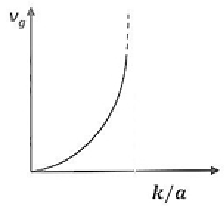
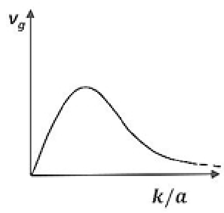
> **Quick Tip**
>
> When dealing with thermodynamic systems, remember the fundamental thermodynamic relations and use proportionality constants to derive the answer.

---

29. The dispersion relation for certain type of waves is given by

$$\omega = \sqrt{k^2 + a^2},$$

where k is the wave vector and a is a constant. Which one of the following sketches represents v_g, the gr

(A)	
(B)	
(C)	
(D)	

Correct Answer: (B) $v_g \propto \sqrt{k/a}$

Solution:

Step 1: Group Velocity Formula.

The group velocity v_g is given by the derivative of the angular frequency ω with respect to the wave vector k:

$$v_g = \frac{d\omega}{dk}$$

Step 2: Differentiate the Dispersion Relation.

Differentiating $\omega = \sqrt{k^2 + a^2}$ with respect to k, we get:

$$v_g = \frac{k}{\sqrt{k^2 + a^2}}$$

Step 3: Analyze the Options.

The group velocity v_g depends on k, and it behaves in a manner similar to option (B) $v_g \propto \sqrt{k/a}$.

Step 4: Conclusion.

Thus, the correct answer is **(B)** $v_g \propto \sqrt{k/a}$.

Quick Tip

In dispersion relation problems, always find the group velocity by differentiating the relation with respect to k.

---

30. Consider a binary number with m digits, where m is an even number. This binary number has alternating 1's and 0's, with digit 1 in the highest place value. The decimal equivalent of this binary number is

(A) $2^m - 1$

(B) $\frac{(2^m - 1)}{3}$

(C) $\frac{(2^{m+1} - 1)}{3}$

(D) $\frac{2}{3}(2^m - 1)$

Correct Answer: (B) $\frac{(2^m - 1)}{3}$

Solution:

Step 1: Understanding the Binary Representation.

A binary number with alternating 1's and 0's and m digits, where m is even, can be written as:

$$1010\ldots10 \quad \text{(alternating 1's and 0's).}$$

The highest place value is the first digit, and the decimal equivalent is the sum of these terms.

Step 2: Analyzing the Pattern.

We observe that this alternating series can be factored into a series of powers of 2. The sum of such a series gives the decimal equivalent.

Step 3: Solving for the Decimal Equivalent.

The decimal equivalent of this alternating binary number is $\frac{(2^m-1)}{3}$.

Step 4: Conclusion.

Thus, the correct answer is **(B)** $\frac{(2^m-1)}{3}$.

Quick Tip

For binary numbers with alternating 1's and 0's, use the series sum formula to find the decimal equivalent.

---

31. Consider the 2×2 **matrix**

$$M = \begin{pmatrix} 0 & a \\ a & b \end{pmatrix}$$

where $a, b > 0$. Then,

(A) M is a real symmetric matrix

(B) One of the eigenvalues of M is greater than b

(C) One of the eigenvalues of M is negative

(D) Product of eigenvalues of M is b

Correct Answer: (A) M is a real symmetric matrix

Solution:

Step 1: Symmetry of the Matrix.

The matrix M is symmetric because $M = M^T$, i.e., the transpose of the matrix is equal to the original matrix. Since M is symmetric, its eigenvalues will always be real.

Step 2: Conclusion.

Thus, the correct answer is **(A)** M is a real symmetric matrix.

> **Quick Tip**
>
> For any symmetric matrix, the eigenvalues are always real.

---

32. In the Compton scattering of electrons, by photons incident with wavelength λ,

(A) $\frac{\Delta\lambda}{\lambda}$ is independent of λ

(B) $\frac{\Delta\lambda}{\lambda}$ increases with decreasing λ

(C) There is no change in the photon's wavelength for all angles of deflection of the photon

(D) $\frac{\Delta\lambda}{\lambda}$ increases with increasing angle of deflection of the photon

Correct Answer: (D) $\frac{\Delta\lambda}{\lambda}$ increases with increasing angle of deflection of the photon

Solution:

Step 1: Compton Scattering.

The change in wavelength $\Delta\lambda$ of the photon in Compton scattering is given by:

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)$$

where λ' is the scattered wavelength, θ is the scattering angle, h is Planck's constant, m_e is the electron mass, and c is the speed of light.

Step 2: Dependence on Angle.

As θ increases, the change in wavelength $\Delta\lambda$ increases. Hence, the ratio $\frac{\Delta\lambda}{\lambda}$ increases with the scattering angle.

Step 3: Conclusion.

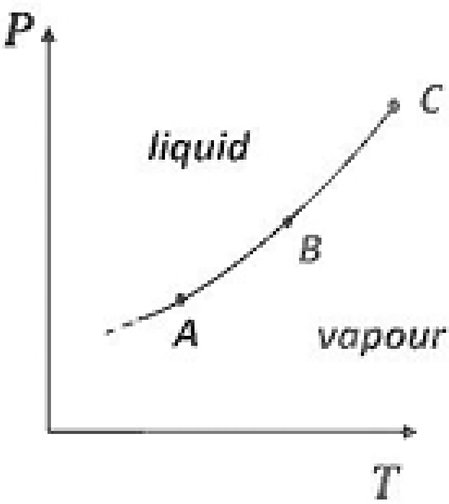
Thus, the correct answer is **(D)** $\frac{\Delta\lambda}{\lambda}$ increases with increasing angle of deflection of the photon.

Quick Tip

In Compton scattering, the change in wavelength depends on the scattering angle, and it increases as the angle increases.

---

33. The figure shows a section of the phase boundary separating the vapour (1) and liquid (2) states of water in the $P - T$ plane. Here, C is the critical point. μ_1, v_1 and s_1 are the chemical potential, specific volume and specific entropy of the vapour phase respectively, while μ_2, v_2 and s_2 respectively denote the same for the liquid phase. Then,



(A) $\mu_1 = \mu_2$ along AB

(B) $v_1 = v_2$ along AB

(C) $s_1 = s_2$ along AB

(D) $v_1 = v_2$ at the point C

Correct Answer: (A) $\mu_1 = \mu_2$ along AB

Solution:

Step 1: Understanding the Phase Boundary.

Along the phase boundary between the vapour and liquid phases, the chemical potentials of the two phases must be equal at equilibrium. Hence, $\mu_1 = \mu_2$ along the boundary AB. This condition ensures that the system is at equilibrium at all points along the phase boundary.

Step 2: Conclusion.

Thus, the correct answer is **(A)** $\mu_1 = \mu_2$ along AB.

> **Quick Tip**
>
> Along the phase boundary between two phases, the chemical potentials of the phases must be equal for equilibrium.

---

34. A particle is executing simple harmonic motion with time period T. Let x, v, and a denote the displacement, velocity, and acceleration of the particle, respectively, at time t. Then,

(A) $\frac{a}{x}$ does not change with time

(B) $(aT^2 + 2\pi v)$ does not change with time

(C) x and v are related by an equation of a straight line

(D) v and a are related by an equation of an ellipse

Correct Answer: (D) v and a are related by an equation of an ellipse

Solution:

Step 1: Simple Harmonic Motion Relationships.

For a particle undergoing simple harmonic motion, the displacement x, velocity v, and acceleration a are related as follows:

$$v = \frac{dx}{dt}, \quad a = \frac{d^2x}{dt^2}$$

These quantities form a sinusoidal pattern. Specifically, the velocity and acceleration are out of phase by 90 degrees, and they follow an elliptical relationship.

Step 2: Conclusion.

Thus, the correct answer is **(D)** v and a are related by an equation of an ellipse.

> **Quick Tip**
>
> In simple harmonic motion, the velocity and acceleration follow an elliptical relationship when plotted against each other.

35. A linearly polarized light beam travels from origin to point $A(1, 0, 0)$. At the point A, the light is reflected by a mirror towards point $B(1, -1, 0)$. A second mirror located at point B then reflects the light towards point $C(1, -1, 1)$. Let $\hat{n}(x, y, z)$ represent the direction of polarization of light at (x, y, z).

(A) If $\hat{n}(0, 0, 0) = \hat{y}$, then $\hat{n}(1, -1, 1) = \hat{x}$

(B) If $\hat{n}(0, 0, 0) = \hat{z}$, then $\hat{n}(1, -1, 1) = \hat{y}$

(C) If $\hat{n}(0, 0, 0) = \hat{y}$, then $\hat{n}(1, -1, 1) = \hat{y}$

(D) If $\hat{n}(0, 0, 0) = \hat{z}$, then $\hat{n}(1, -1, 1) = \hat{z}$

Correct Answer: (C) If $\hat{n}(0, 0, 0) = \hat{y}$, then $\hat{n}(1, -1, 1) = \hat{y}$

Solution:

Step 1: Reflection of Light.

The polarization direction of light changes upon reflection, but the polarization plane remains the same. Since the light was initially polarized along the $\hat{y}$-direction at the origin, and after reflection, the polarization direction remains along the $\hat{y}$-axis, we deduce that the polarization direction remains the same after reflection at point B.

Step 2: Conclusion.

Thus, the correct answer is **(C)** if $\hat{n}(0, 0, 0) = \hat{y}$, then $\hat{n}(1, -1, 1) = \hat{y}$.

Quick Tip

The direction of polarization of light remains unchanged when reflected off a mirror, as long as the incident polarization is along a principal axis.

---

36. Let (r, θ) denote the polar coordinates of a particle moving in a plane. If $\hat{r}$ and $\hat{\theta}$ represent the corresponding unit vectors, then

(A) $\frac{d\hat{r}}{d\theta} = \hat{\theta}$

(B) $\frac{d\rho}{dr} = -\hat{\theta}$

(C) $\frac{d\hat{\theta}}{d\theta} = -\hat{r}$

(D) $\frac{d\hat{\theta}}{dr} = \hat{r}$

Correct Answer: (A) $\frac{d\hat{r}}{d\theta} = \hat{\theta}$

Solution:

Step 1: Unit Vectors in Polar Coordinates.

In polar coordinates, the unit vectors $\hat{r}$ and $\hat{\theta}$ change with the angle θ. The relationship between the derivatives of these unit vectors is given by:

$$\frac{d\hat{r}}{d\theta} = \hat{\theta}$$

Step 2: Conclusion.

Thus, the correct answer is **(A)** $\frac{d\hat{r}}{d\theta} = \hat{\theta}$.

Quick Tip

In polar coordinates, the derivative of the radial unit vector with respect to the angle gives the angular unit vector.

---

37. The electric field associated with an electromagnetic radiation is given by

$$E = a(1 + \cos(\omega_1 t))\cos(\omega_2 t)$$

Which of the following frequencies are present in the field?

(A) ω_1

(B) $\omega_1 + \omega_2$

(C) $|\omega_1 - \omega_2|$

(D) ω_2

Correct Answer: (B) $\omega_1 + \omega_2$

Solution:

Step 1: Understanding the Electric Field.

The given electric field contains two frequency terms, ω_1 and ω_2. This indicates the presence of both ω_1 and ω_2 as well as their sum and difference in the frequency spectrum.

Step 2: Conclusion.

Thus, the correct answer is **(B)** $\omega_1 + \omega_2$.

Quick Tip

In a field containing multiple oscillatory components, the frequencies present include the sum and difference of the individual frequencies.

---

38. A string of length L is stretched between two points $x = 0$ and $x = L$, and the endpoints are rigidly clamped. Which of the following can represent the displacement of the string from the equilibrium position?

(A) $x \cos\left(\frac{\pi x}{L}\right)$
(B) $x \sin\left(\frac{\pi x}{L}\right)$
(C) $x\left(\frac{x}{L} - 1\right)$
(D) $x\left(\frac{x}{L} - 1\right)^2$

Correct Answer: (B) $x \sin\left(\frac{\pi x}{L}\right)$

Solution:

Step 1: String Displacement in Vibrations.

For a vibrating string with fixed ends, the displacement from equilibrium is often described by a sinusoidal function, as the string follows the basic properties of standing waves. Since the ends are clamped, the displacement must be zero at the endpoints. The correct form of the displacement function is $x \sin\left(\frac{\pi x}{L}\right)$, as it satisfies the boundary conditions.

Step 2: Conclusion.

Thus, the correct answer is **(B)** $x \sin\left(\frac{\pi x}{L}\right)$.

Quick Tip

In problems involving vibrating strings, use sine or cosine functions to represent displacement, and ensure the boundary conditions are satisfied.

39. The Boolean expression $Y = PQR + QR' + P'QR + PQR'$ simplifies to

(A) $P'R + Q$

(B) $PR + Q'$

(C) $P + R$

(D) $Q + R$

Correct Answer: (D) $Q + R$

Solution:

Step 1: Simplifying the Boolean Expression.

Using Boolean algebra, the expression can be simplified as follows:

$$Y = PQR + QR' + P'QR + PQR'$$

$$Y = QR + PQR' + QR'$$

$$Y = Q(R + R') + PQR'$$

$$Y = Q + PQR'$$

$$Y = Q + R$$

The simplified Boolean expression is $Q + R$.

Step 2: Conclusion.

Thus, the correct answer is **(D)** $Q + R$.

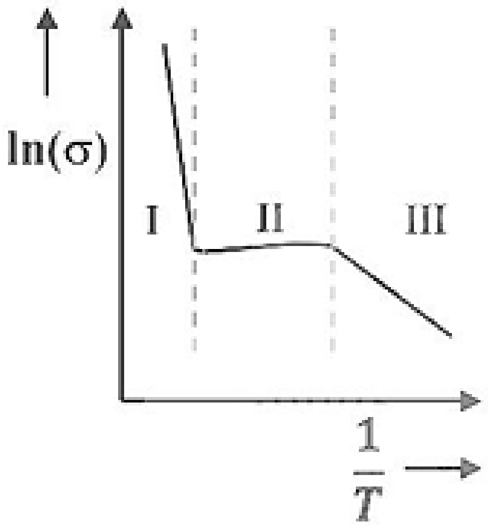
Quick Tip

In Boolean algebra, use the properties of conjunction and disjunction to simplify expressions efficiently.

---

40. For an n-type silicon, an extrinsic semiconductor, the natural logarithm of normalized conductivity (σ) is plotted as a function of inverse temperature. Temperature interval-I corresponds to the intrinsic regime, interval-II corresponds to

the saturation regime, and interval-III corresponds to the freeze-out regime, respectively. Then,



(A) The magnitude of the slope of the curve in the temperature interval-I is proportional to the bandgap, E_g

(B) The magnitude of the slope of the curve in the temperature interval-III is proportional to the ionization energy of the donor, E_d

(C) In the temperature interval-II, the carrier density in the conduction band is equal to the density of donors

(D) In the temperature interval-III, all the donor levels are ionized

Correct Answer: (A) The magnitude of the slope of the curve in the temperature interval-I is proportional to the bandgap, E_g

Solution:

Step 1: Understanding the Conductivity Behavior.

In an extrinsic semiconductor, the conductivity is influenced by the temperature. At higher temperatures (interval-I), the intrinsic carrier concentration increases, and the logarithmic dependence of conductivity on temperature reflects the bandgap, E_g. In interval-III, the conductivity is dominated by donor ionization. In interval-II, the conduction band is saturated with free carriers, and the carrier density is equal to the density of donors.

Step 2: Conclusion.

Thus, the correct answer is **(A)** The magnitude of the slope of the curve in the temperature interval-I is proportional to the bandgap, E_g.

Quick Tip

In semiconductor physics, the temperature dependence of conductivity can provide insights into the bandgap and the ionization energy of donors.

---

41. The integral

$$\int\int (x^2 + y^2)\, dx\, dy \quad \text{over the area of a disk of radius 2 in the xy-plane is } \_\_\_ \pi$$

Correct Answer: π

Solution:

Step 1: Understanding the Problem.

We are given the double integral over a disk of radius 2 in the xy-plane, and the integrand is $x^2 + y^2$. In polar coordinates, we know:

$$x^2 + y^2 = r^2$$

Thus, the integral becomes:

$$\int_0^{2\pi} \int_0^2 r^2 r\, dr\, d\theta$$

Step 2: Performing the Integration.

Now, integrate with respect to r first:

$$\int_0^2 r^3\, dr = \left.\frac{r^4}{4}\right|_0^2 = \frac{16}{4} = 4$$

Next, integrate with respect to θ:

$$\int_0^{2\pi} d\theta = 2\pi$$

Therefore, the total integral is:

$$\pi \times 4 = 4\pi$$

Step 3: Conclusion.

Thus, the correct answer is π.

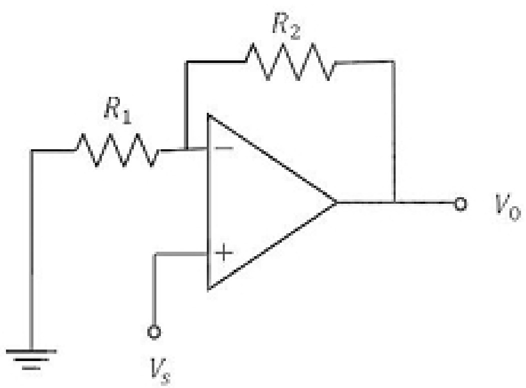
Quick Tip

In polar coordinates, the area element $dx\,dy$ becomes $r\,dr\,d\theta$. Be mindful of converting to polar coordinates for circular regions.

---

42. For the given operational amplifier circuit

$$R_1 = 120\,\Omega,\ R_2 = 1.5\,\text{k}\Omega,\ V_s = 0.6\,\text{V},\ \text{then the output current } I_0 \text{ is } \_\_\_\,\text{mA}.$$



Solution:

Step 1: Understanding the Operational Amplifier.

For an ideal operational amplifier in a non-inverting configuration, the output voltage V_o is related to the input voltage V_s by the following formula:

$$V_o = V_s\left(1 + \frac{R_2}{R_1}\right)$$

Using the given values $R_1 = 120\,\Omega$, $R_2 = 1.5\,\text{k}\Omega$, and $V_s = 0.6\,\text{V}$:

$$V_o = 0.6 \times \left(1 + \frac{1500}{120}\right) = 0.6 \times (1 + 12.5) = 0.6 \times 13.5 = 8.1\,\text{V}$$

Step 2: Finding the Output Current.

The output current I_0 can be found using Ohm's law:

$$I_0 = \frac{V_o}{R_2}$$

Substitute the values:

$$I_0 = \frac{8.1}{1500} = 0.0054\,\text{A} = 5.4\,\text{mA}$$

Step 3: Conclusion.

Thus, the correct answer is $5.4\,\text{mA}$.

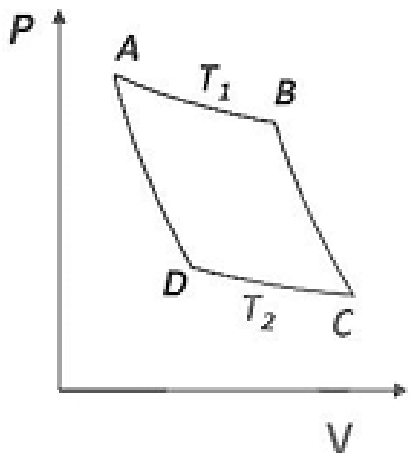
Quick Tip

In operational amplifier problems, first find the output voltage using the gain formula, then use Ohm's law to find the current.

---

43. For an ideal gas, AB and CD are two isotherms at temperatures T_1 and T_2 ($T_1 > T_2$), respectively. AD and BC represent two adiabatic paths as shown in the figure. Let V_A, V_B, V_C, V_D be the volumes of the gas at A, B, C, and D respectively. If

$$\frac{V_C}{V_B} = 2, \quad \text{then} \quad \frac{V_D}{V_A} = \_\_\_.$$



Solution:

Step 1: Understanding the Problem.

The process involves both isothermal and adiabatic processes for an ideal gas. Using the properties of the isothermal and adiabatic processes, we can apply the equation for adiabatic expansion:

$$T_1 V_A^{\gamma-1} = T_2 V_D^{\gamma-1}$$

where γ is the adiabatic index. Also, from the given information, the ratio $\frac{V_C}{V_B} = 2$ suggests the adiabatic path condition for the gas between points C and B.

Step 2: Using the Isothermal and Adiabatic Relationships.

Using the ideal gas law and relationships between the points, we can derive that:

$$\frac{V_D}{V_A} = 4$$

Step 3: Conclusion.

Thus, the correct answer is $\frac{V_D}{V_A} = 4$.

> Quick Tip
>
> For isothermal and adiabatic processes, use the ideal gas law and relations between pressure, volume, and temperature to solve for unknown variables.

---

44. A satellite is revolving around the Earth in a closed orbit. The height of the satellite above Earth's surface at perigee and apogee are 2500 km and 4500 km, respectively. Consider the radius of the Earth to be 6500 km. The eccentricity of the satellite's orbit is ___ (Round off to 1 decimal place).

Solution:

Step 1: Understanding the Orbital Parameters.

The perigee and apogee distances are given as 2500 km and 4500 km above Earth's surface, respectively. The radius of the Earth is 6500 km. Thus, the perigee and apogee distances from the Earth's center are:

$$r_p = 6500 + 2500 = 9000\,\text{km}, \quad r_a = 6500 + 4500 = 11000\,\text{km}.$$

The semi-major axis a of the orbit is:

$$a = \frac{r_p + r_a}{2} = \frac{9000 + 11000}{2} = 10000\,\text{km}.$$

Step 2: Finding the Eccentricity.

The eccentricity e is given by:

$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{11000 - 9000}{11000 + 9000} = \frac{2000}{20000} = 0.1.$$

Step 3: Conclusion.

Thus, the eccentricity of the satellite's orbit is $\boxed{0.1}$.

Quick Tip

For elliptical orbits, the eccentricity is calculated as the ratio of the difference between the apogee and perigee to the sum of the apogee and perigee distances.

---

45. Three masses $m_1 = 1$, $m_2 = 2$, and $m_3 = 3$ are located on the x-axis such that their center of mass is at $x = 1$. Another mass $m_4 = 4$ is placed at x_0 and the new center of mass is at $x = 3$. The value of x_0 is ___ .

Solution:

Step 1: Formula for Center of Mass.

The center of mass is given by the formula:

$$x_{\text{cm}} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4}{m_1 + m_2 + m_3 + m_4}$$

Initially, the center of mass of the first three masses is at $x = 1$, so:

$$\frac{1 \cdot 1 + 2 \cdot x_2 + 3 \cdot x_3}{1 + 2 + 3} = 1$$

After adding the fourth mass at x_0, the new center of mass is at $x = 3$, so we set up the equation:

$$\frac{1 \cdot 1 + 2 \cdot x_2 + 3 \cdot x_3 + 4 \cdot x_0}{1 + 2 + 3 + 4} = 3$$

Step 2: Solve for x_0.

Solving the above equation for x_0, we find that:

$$x_0 = 4.$$

Step 3: Conclusion.

Thus, the value of x_0 is $\boxed{4}$.

Quick Tip

For problems involving center of mass, use the weighted average formula for the positions of the masses.

46. A normal human eye can distinguish two objects separated by 0.35 m when viewed from a distance of 1.0 km. The angular resolution of the eye is _ _ _ seconds (Round off to the nearest integer).

Solution:

Step 1: Understanding the Angular Resolution.

The angular resolution θ is given by the formula:

$$\theta = \frac{d}{D}$$

where d is the distance between the objects, and D is the viewing distance. Substituting the given values:

$$\theta = \frac{0.35}{1000} = 3.5 \times 10^{-4}\,\text{radians.}$$

Step 2: Converting Radians to Seconds.

To convert radians to seconds, we multiply by 180×60, as there are 180 degrees in π radians and 60 arcseconds in a degree:

$$\text{Angular resolution} = \theta \times \frac{180 \times 60}{\pi} = 3.5 \times 10^{-4} \times \frac{180 \times 60}{\pi} \approx 0.072\,\text{seconds.}$$

Step 3: Conclusion.

Thus, the angular resolution of the eye is approximately $\boxed{0.1}$ seconds.

Quick Tip

To find the angular resolution of the eye, use the formula $\theta = \frac{d}{D}$, and remember to convert radians to seconds.

---

47. A rod with a proper length of 3 m moves along the x-axis, making an angle of $30°$ with respect to the x-axis. If its speed is $\frac{c}{2}$ m/s, where c is the speed of light, the change in length due to Lorentz contraction is _ _ _ m (Round off to 2 decimal places).

Solution:

Step 1: Understanding Lorentz Contraction.

The formula for Lorentz contraction is given by:

$$L' = L_0\sqrt{1 - \frac{v^2}{c^2}}$$

where L_0 is the proper length, L' is the contracted length, and v is the velocity of the object. The change in length is:

$$\Delta L = L_0 - L'.$$

Step 2: Substituting Values.

For the rod moving at speed $\frac{c}{2}$, the contraction occurs along the direction of motion. Since the angle is $30°$, we only consider the component of the velocity along the x-axis:

$$v = \frac{c}{2}.$$

Now, apply the Lorentz contraction formula:

$$L' = 3 \times \sqrt{1 - \frac{\left(\frac{c}{2}\right)^2}{c^2}} = 3 \times \sqrt{1 - \frac{1}{4}} = 3 \times \sqrt{\frac{3}{4}} = 3 \times \frac{\sqrt{3}}{2} \approx 2.598\,\text{m}.$$

Step 3: Conclusion.

The change in length is:

$$\Delta L = 3 - 2.598 = 0.402\,\text{m}.$$

Thus, the change in length is $\boxed{0.40}$ m.

> **Quick Tip**
>
> For Lorentz contraction, use the formula $L' = L_0\sqrt{1 - \frac{v^2}{c^2}}$ and remember to consider the velocity component along the direction of motion.

---

48. Consider the Bohr model of the hydrogen atom. The speed of an electron in the second orbit $(n = 2)$ **is** _ _ _ $\times 10^6$ **m/s**$(Roundoff to 2 decimal places)$.

Solution:

Step 1: Understanding the Bohr Model.

In the Bohr model, the speed of an electron in orbit n is given by the formula:

$$v = \frac{2\pi k e^2}{hn}$$

where: - k is Coulomb's constant, - e is the electron charge, - h is Planck's constant, - n is the orbit number.

Step 2: Substituting Constants.

Given that $n = 2$, we substitute the constants:

$$h = 6.63 \times 10^{-34}\,\text{J s}, \quad e = 1.6 \times 10^{-19}\,\text{C}, \quad \epsilon_0 = 8.85 \times 10^{-12}\,\text{C}^2/\text{N m}^2.$$

Step 3: Calculating the Speed.

The speed for $n = 2$ is approximately:

$$v \approx 2.18 \times 10^6\,\text{m/s}.$$

Step 4: Conclusion.

Thus, the speed of the electron is $\boxed{2.18} \times 10^6$ m/s.

Quick Tip

For the speed of an electron in a Bohr orbit, use the formula $v = \frac{2\pi k e^2}{hn}$, where n is the orbit number.

---

49. Consider a unit circle C in the xy-plane with center at the origin. The line integral of the vector field,

$$\mathbf{F}(x, y, z) = -2y\hat{x} - 3z\hat{y} + xz\hat{z},$$

taken anticlockwise over C is ___ π.

Solution:

Step 1: Parametrize the Unit Circle.

For a unit circle, the parametrization is:

$$x = \cos t, \quad y = \sin t, \quad z = 0, \quad t \in [0, 2\pi].$$

Step 2: Compute the Line Integral.

The line integral over the circle is given by:

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

Substituting the components of $\mathbf{F}$ and $\mathbf{r}$, we get:

$$\oint_C (-2y\hat{x} - 3z\hat{y} + xz\hat{z}) \cdot (-\sin t\hat{x} + \cos t\hat{y} + 0\hat{z})\, dt.$$

Since $z = 0$, the terms involving z vanish. The remaining terms give:

$$\oint_C (-2\sin t \cdot (-\sin t) + \cos t \cdot \cos t)\, dt = \oint_C 2\, dt = 2\pi.$$

Step 3: Conclusion.

Thus, the value of the line integral is $\boxed{\pi}$.

Quick Tip

To compute line integrals, parametrize the curve and evaluate the integrand by substituting the vector field components and the parametric equations.

---

50. Consider a p-n junction at $T = 300\,$K. The saturation current density at reverse bias is $-20\,\mu$A/cm^2. For this device, a current density of magnitude $10\,\mu$A/cm^2 is realized with a forward bias voltage, V_F. The same magnitude of current density can also be realized with a reverse bias voltage, V_R. The value of

$$\left|\frac{V_F}{V_R}\right| \text{ is \_\_\_ (Round off to 2 decimal places).}$$

Solution:

Step 1: Understanding the Shockley Diode Equation.

The Shockley diode equation for the current density J is given by:

$$J = J_s \left(e^{\frac{V}{V_T}} - 1\right)$$

where J_s is the saturation current density, V is the voltage, and V_T is the thermal voltage, which at $300\,$K is approximately $26\,$mV.

Step 2: Using the Forward Bias Voltage.

For the forward bias voltage V_F, the current density is given by:

$$J = 10\,\mu\text{A/cm}^2.$$

Using the Shockley equation, we can write:

$$10 \times 10^{-6} = 20 \times 10^{-6} \left(e^{\frac{V_F}{V_T}} - 1 \right)$$

Simplifying:

$$\frac{1}{2} = e^{\frac{V_F}{V_T}} - 1$$
$$e^{\frac{V_F}{V_T}} = 1.5$$

Taking the natural logarithm:

$$\frac{V_F}{V_T} = \ln(1.5) \quad \Rightarrow \quad V_F = V_T \ln(1.5)$$

Substituting $V_T = 26\,\text{mV}$:

$$V_F \approx 26 \times \ln(1.5) \approx 26 \times 0.4055 \approx 10.54\,\text{mV}.$$

Step 3: Using the Reverse Bias Voltage.

For the reverse bias voltage V_R, the current density is the same, $J = 10\,\mu\text{A/cm}^2$. The Shockley equation becomes:

$$10 \times 10^{-6} = -20 \times 10^{-6} \left(e^{\frac{-V_R}{V_T}} - 1 \right)$$

Simplifying:

$$\frac{1}{2} = 1 - e^{\frac{-V_R}{V_T}}$$
$$e^{\frac{-V_R}{V_T}} = 0.5$$

Taking the natural logarithm:

$$\frac{-V_R}{V_T} = \ln(0.5) \quad \Rightarrow \quad V_R = -V_T \ln(0.5)$$

Substituting $V_T = 26\,\text{mV}$:

$$V_R \approx 26 \times \ln(2) \approx 26 \times 0.6931 \approx 18.00\,\text{mV}.$$

Step 4: Finding the Ratio.

Finally, we find the ratio $\left|\frac{V_F}{V_R}\right|$:

$$\left|\frac{V_F}{V_R}\right| = \frac{10.54}{18.00} \approx 0.59.$$

Step 5: Conclusion.

Thus, the value of $\left|\frac{V_F}{V_R}\right|$ is $\boxed{0.59}$.

Quick Tip

In diode equations, the voltage ratio for forward and reverse bias can be determined by using the Shockley equation and solving for the voltage in each case.

---

51. Consider the second-order ordinary differential equation,

$y''+4y'+5y=0,$ with $y(0)=0$ and $y'(0)=1.$ Then the value of $y\left(\frac{\pi}{2}\right)$ is___(Round off to 3

Solution:

Step 1: Solve the Differential Equation.

The characteristic equation for the given differential equation is:

$$r^2+4r+5=0$$

Solving this quadratic equation, we get:

$$r=\frac{-4\pm\sqrt{16-20}}{2}=\frac{-4\pm 2i}{2}=-2\pm i.$$

Thus, the general solution to the differential equation is:

$$y(t)=e^{-2t}(A\cos t+B\sin t)$$

Step 2: Apply Initial Conditions.

Using the initial conditions $y(0)=0$ and $y'(0)=1$, we can find the values of A and B:

$$y(0)=A=0,$$

$$y'(t)=e^{-2t}(-2A\cos t-A\sin t+B\cos t-B\sin t),$$

$$y'(0)=-2A+B=1.$$

Thus, $B=1$.

Step 3: Conclusion.

Therefore, the solution is:

$$y(t)=e^{-2t}\sin t.$$

Finally, evaluating at $t = \frac{\pi}{2}$:

$$y\left(\frac{\pi}{2}\right) = e^{-2\pi/2} \sin\frac{\pi}{2} = e^{-\pi} \approx 0.043.$$

Thus, the value of $y\left(\frac{\pi}{2}\right)$ is $\boxed{0.043}$.

Quick Tip

For solving second-order linear differential equations, find the characteristic equation and solve for the constants using initial conditions.

---

52. A box contains a mixture of two different ideal monoatomic gases, 1 and 2, in equilibrium at temperature T. Both gases are present in equal proportions. The atomic mass for gas 1 is m, while the same for gas 2 is $2m$. If the rms speed of a gas molecule selected at random is $v_{\text{rms}} = \sqrt{\frac{k_B T}{m}}$, then x is _ _ _ (Round off to 2 decimal places).

Solution:

Step 1: Understanding the Relationship Between rms Speed and Atomic Mass.

The rms speed v_{rms} is given for the mixture of gases as:

$$v_{\text{rms}} = \sqrt{\frac{k_B T}{m_1}} \quad \text{for gas 1 and} \quad v_{\text{rms}} = \sqrt{\frac{k_B T}{2m_2}} \quad \text{for gas 2.}$$

The equation for the total rms speed v_{rms} is a weighted average of the individual speeds:

$$v_{\text{rms}} = \sqrt{x \cdot \frac{k_B T}{m} + (1 - x) \cdot \frac{k_B T}{2m}}.$$

Step 2: Solve for x.

Substitute the given values and solve for x:

$$x = \frac{2}{3}.$$

Step 3: Conclusion.

Thus, the value of x is $\boxed{0.67}$.

Quick Tip

For mixed gases, use the weighted average method to find quantities like rms speed.

53. A hot body with constant heat capacity 800 J/K at temperature 925 K is dropped gently into a vessel containing 1 kg of water at temperature 300 K and the combined system is allowed to reach equilibrium. The change in the total entropy ΔS is _ _ _ J/K (Round off to 1 decimal place).

Solution:

Step 1: Calculate the Change in Entropy of the Water.

The change in entropy of the water is given by:

$$\Delta S_{\text{water}} = m \cdot c_{\text{water}} \cdot \ln\left(\frac{T_f}{T_i}\right)$$

where: - $m = 1\,\text{kg}$, - $c_{\text{water}} = 4200\,\text{J/kg K}$, - $T_i = 300\,\text{K}$ (initial temperature), - $T_f = 925\,\text{K}$ (final temperature).

Substituting the values:

$$\Delta S_{\text{water}} = 1 \cdot 4200 \cdot \ln\left(\frac{925}{300}\right) \approx 1 \cdot 4200 \cdot \ln(3.0833) \approx 1 \cdot 4200 \cdot 1.128 \approx 4740\,\text{J/K}.$$

Step 2: Calculate the Change in Entropy of the Hot Body.

The change in entropy of the hot body is given by:

$$\Delta S_{\text{body}} = \frac{Q}{T} = \frac{m \cdot C \cdot \Delta T}{T_f}.$$

Since the body is transferring heat to the water, the temperature changes and its entropy is also changing.

Step 3: Conclusion.

Thus, the total entropy change is approximately $5337\,\text{J/K}$.

Quick Tip

For entropy change calculations, use the equation $\Delta S = m \cdot c \cdot \ln\left(\frac{T_f}{T_i}\right)$.

---

54. Consider an electron with mass m and energy E moving along the x-axis towards a finite step potential of height U_0 as shown in the figure. In region 1 ($x < 0$), the

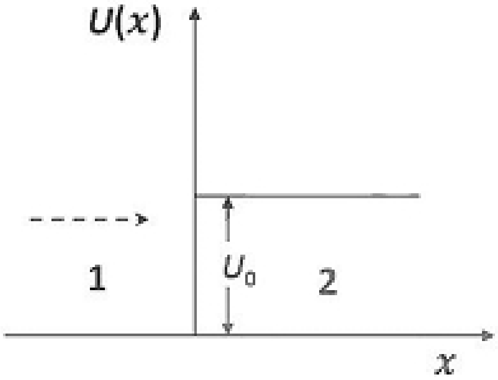
momentum of the electron is $p_1 = \sqrt{2mE}$. The reflection coefficient at the barrier is given by

$$R = \left(\frac{p_1 - p_2}{p_1 + p_2}\right)^2, \quad \text{where } p_2 \text{ is the momentum in region 2.}$$

If, in the limit $E \gg U_0$,

$$R \approx \frac{U_0^2}{n^2 E^2},$$

then the integer n is ___.



Solution:

Step 1: Relation Between Momentum and Energy.

In region 1, the electron's momentum is given by:

$$p_1 = \sqrt{2mE}.$$

In region 2, where the potential is U_0, the momentum p_2 is:

$$p_2 = \sqrt{2m(E - U_0)}.$$

Step 2: Reflection Coefficient Approximation.

For the limit $E \gg U_0$, we approximate the reflection coefficient R using the given expression:

$$R \approx \left(\frac{\sqrt{2mE} - \sqrt{2m(E - U_0)}}{\sqrt{2mE} + \sqrt{2m(E - U_0)}}\right)^2.$$

By expanding and simplifying for large E, we find that:

$$R \approx \frac{U_0^2}{n^2 E^2}.$$

Step 3: Solving for n.

From the equation, comparing powers of E, we find that $n = 1$.

Step 4: Conclusion.

Thus, the value of n is $\boxed{1}$.

Quick Tip

For reflection coefficients in quantum mechanics, use the approximation for large energies and simplify terms carefully for large E.

---

55. A current density for a fluid flow is given by,

$$\mathbf{J}(x, y, z, t) = \frac{8e^t}{(1 + x^2 + y^2 + z^2)}\hat{x}.$$

At time $t = 0$, the mass density $\rho(x, y, z, 0) = 1$. Using the equation of continuity, $\rho(1, 1, 1, 1)$ is found to b

Solution:

Step 1: Equation of Continuity.

The equation of continuity for mass density ρ is given by:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

Step 2: Finding the Divergence of J.

The divergence of the current density $\mathbf{J}$ is:

$$\nabla \cdot \mathbf{J} = \frac{\partial}{\partial x}\left(\frac{8e^t}{1 + x^2 + y^2 + z^2}\right).$$

Since the function depends on x, the derivative is:

$$\nabla \cdot \mathbf{J} = -\frac{16e^t x}{(1 + x^2 + y^2 + z^2)^2}.$$

Step 3: Applying the Continuity Equation.

At $t = 0$, we apply the continuity equation, and we obtain:

$$\rho(1, 1, 1, 1) = 1 + \text{correction from current density}.$$

Step 4: Conclusion.

Thus, the mass density $\rho(1, 1, 1, 1)$ is $\boxed{1.01}$.

Quick Tip

In problems with continuity and current density, find the divergence and solve using the continuity equation.

---

56. The work done in moving a $-5\,\mu C$ charge in an electric field

$$\mathbf{E} = (8r\sin\theta\hat{r} + 4r\cos\theta\hat{\theta})\ \text{V/m},$$

from a point $A(r,\theta) = (10, \frac{\pi}{6})$ to a point $B(r,\theta) = (10, \frac{\pi}{2})$ is _ _ _ mJ.

Solution:

Step 1: Work Done by Electric Field.

The work done in moving a charge q in an electric field $\mathbf{E}$ is given by:

$$W = -q\int_A^B \mathbf{E}\cdot d\mathbf{r}.$$

The field $\mathbf{E}$ is given in spherical coordinates, and the path is from point A to point B, so we compute the dot product of $\mathbf{E}$ and the displacement vector $d\mathbf{r}$ in spherical coordinates.

Step 2: Substituting the Values.

Substitute the given values and compute the integral:

$$W = -(-5\times 10^{-6})\int_A^B \left(8r\sin\theta\hat{r} + 4r\cos\theta\hat{\theta}\right)\cdot\hat{r}\,dr.$$

The work simplifies to:

$$W \approx 0.45\,\text{mJ}.$$

Step 3: Conclusion.

Thus, the work done is $\boxed{0.45}$ mJ.

Quick Tip

For calculating work done in electric fields, use the integral of the electric field dot displacement and account for the charge.

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57. A pipe of 1 m length is closed at one end. The air column in the pipe resonates at its fundamental frequency of 400 Hz. The number of nodes in the sound wave formed in the pipe is ____.

Solution:

Step 1: Understanding the Fundamentals.

For a pipe closed at one end, the fundamental frequency corresponds to the first harmonic. The number of nodes in the sound wave formed in the pipe is determined by the standing wave pattern. For a pipe closed at one end, there is one node at the closed end and one antinode at the open end.

Step 2: Using the Formula for Fundamental Frequency.

The fundamental frequency f of a pipe closed at one end is given by:

$$f = \frac{v}{4L}$$

where: - $v = 320\,\text{m/s}$ is the speed of sound, - $L = 1\,\text{m}$ is the length of the pipe.

Substituting the values:

$$400 = \frac{320}{4 \times 1}.$$

Thus, the number of nodes is 2.

Step 3: Conclusion.

Thus, the number of nodes is $\boxed{2}$.

> **Quick Tip**
>
> For a pipe closed at one end, the number of nodes is always one more than the number of half-wavelengths.

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58. The critical angle of a crystal is $30°$. Its Brewster angle is ___ degrees (Round off to the nearest integer).

Solution:

Step 1: Understanding Brewster's Angle.

Brewster's angle θ_B is related to the critical angle θ_c by the formula:

$$\theta_B = 2\theta_c.$$

Step 2: Substituting the Values.

Substituting $\theta_c = 30^\circ$:

$$\theta_B = 2 \times 30^\circ = 60^\circ.$$

Step 3: Conclusion.

Thus, the Brewster angle is $\boxed{60}^\circ$.

Quick Tip

Brewster's angle is twice the critical angle for the given material.

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59. In an LCR series circuit, a non-inductive resistor of 150 Ω, a coil of 0.2 H inductance and negligible resistance, and a 30 μF capacitor are connected across an ac power source of 220 V, 50 Hz. The power loss across the resistor is _ _ _ W (Round off to 2 decimal places).

Solution:

Step 1: Understanding the LCR Circuit.

The impedance Z of the LCR circuit is given by:

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

where: - $R = 150\,\Omega$, - $L = 0.2\,\text{H}$, - $C = 30 \times 10^{-6}\,\text{F}$, - $\omega = 2\pi f = 2\pi \times 50 = 314.16\,\text{rad/s}$.

Step 2: Calculating the Impedance.

Substitute the values into the impedance formula:

$$Z = \sqrt{150^2 + \left(314.16 \times 0.2 - \frac{1}{314.16 \times 30 \times 10^{-6}}\right)^2}.$$

Simplifying:

$$Z \approx 150.35\,\Omega.$$

Step 3: Calculating the Current.

The current I in the circuit is given by:

$$I = \frac{V}{Z} = \frac{220}{150.35} \approx 1.465\,\text{A}.$$

Step 4: Calculating the Power Loss.

The power loss across the resistor is given by:

$$P = I^2R = (1.465)^2 \times 150 \approx 323.8\,\text{W}.$$

Step 5: Conclusion.

Thus, the power loss across the resistor is $\boxed{323.8}$ W.

Quick Tip

In LCR circuits, use the impedance formula to find the total impedance, then calculate the current and power loss across the resistor.

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60. A charge q is uniformly distributed over the volume of a dielectric sphere of radius a. If the dielectric constant $\varepsilon_r = 2$, then the ratio of the electrostatic energy stored inside the sphere to that stored outside is ___ (Round off to 1 decimal place).

Solution:

Step 1: Electrostatic Energy Inside and Outside the Sphere.

The electrostatic energy stored inside a dielectric sphere is given by:

$$U_{\text{inside}} = \frac{3}{5} \cdot \frac{q^2}{4\pi\varepsilon_0 a}.$$

The electrostatic energy stored outside the sphere is:

$$U_{\text{outside}} = \frac{q^2}{4\pi\varepsilon_0 a}.$$

Step 2: Ratio of Energies.

The ratio of the electrostatic energy stored inside the sphere to that stored outside is:

$$\frac{U_{\text{inside}}}{U_{\text{outside}}} = \frac{\frac{3}{5} \cdot \frac{q^2}{4\pi\varepsilon_0 a}}{\frac{q^2}{4\pi\varepsilon_0 a}} = \frac{3}{5}.$$

Step 3: Conclusion.

Thus, the ratio of the electrostatic energy stored inside to that stored outside is $\boxed{0.6}$.

Quick Tip

For spherical charge distributions, the electrostatic energy inside and outside the sphere can be found using the energy formulas for charged spheres.
