IIT JAM 2023 MS Question Paper PDF

Time Allowed :1 Hour	Maximum Marks :100	Total Questions :60

General Instructions

Read the following instructions very carefully and strictly follow them:

1. Please check that this question paper contains 60 questions.
2. Please write down the Serial Number of the question in the answer- book at the given place before attempting it.
3. This Question Paper has 60 questions. All questions are compulsory.
4. Adhere to the prescribed word limit while answering the questions.

1. Let $M = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$**. If a non-zero vector** $X = (x, y, z)^T \in \mathbb{R}^3$ **satisfies** $M^6 X = X$**, then a subspace of** $\mathbb{R}^3$ **that contains the vector** X **is:**

(1) $\{(x, y, z)^T \in \mathbb{R}^3 : x = 0,\ y + z = 0\}$
(2) $\{(x, y, z)^T \in \mathbb{R}^3 : y = 0,\ x + z = 0\}$
(3) $\{(x, y, z)^T \in \mathbb{R}^3 : z = 0,\ x + y = 0\}$
(4) $\{(x, y, z)^T \in \mathbb{R}^3 : x = 0,\ y - z = 0\}$

---

2. Let $M = M_1 M_2$**, where** M_1 **and** M_2 **are two** 3×3 **distinct matrices. Consider the following two statements:**

(I) The rows of M are linear combinations of rows of M_2.

(II) The columns of M are linear combinations of columns of M_1.

Then:

(1) Only (I) is TRUE
(2) Only (II) is TRUE
(3) Both (I) and (II) are TRUE
(4) Neither (I) nor (II) is TRUE

---

3. Let $X \sim F_{6,2}$ **and** $Y \sim F_{2,6}$**. If** $P(X \leq 2) = \frac{216}{343}$ **and** $P(Y \leq \frac{1}{2}) = \alpha$**, then** 686α **equals:**

(1) 246
(2) 254
(3) 260
(4) 264

---

4. Let $Y \sim F_{4,2}$. Then $P(Y \leq 2)$ equals:

(1) 0.60
(2) 0.62
(3) 0.64
(4) 0.66

---

5. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables each having $U(0,1)$ distribution. Let Y be a random variable having distribution function G. Suppose that

$$\lim_{n\to\infty} P\left(\frac{X_1 + X_2 + \cdots + X_n}{4} \leq x\right) = G(x), \quad \forall x \in \mathbb{R}.$$

Then, $Var(Y)$ equals:

(1) $\frac{1}{12}$

(2) $\frac{1}{32}$

(3) $\frac{1}{48}$

(4) $\frac{1}{64}$

---

6. Let X_1, X_2, X_3 be a random sample from an $N(\theta, 1)$ distribution, where $\theta \in \mathbb{R}$ is an unknown parameter. Then, which one of the following conditional expectations does NOT depend on θ?

(1) $E(X_1 + X_2 - X_3 \mid X_1 + X_2)$
(2) $E(X_1 + X_2 - X_3 \mid X_2 + X_3)$
(3) $E(X_1 + X_2 - X_3 \mid X_1 - X_3)$
(4) $E(X_1 + X_2 - X_3 \mid X_1 + X_2 + X_3)$

---

7. For the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x,y) = 2x^2 - xy - 3y^2 - 3x + 7y$, the point (1,1) is:

(1) a point of local maximum
(2) a point of local minimum

(3) a saddle point
(4) NOT a critical point

---

8. Let E_1, E_2, E_3 be three events such that $P(E_1 \cap E_2) = \frac{1}{4},\ P(E_1 \cap E_3) = P(E_2 \cap E_3) = \frac{1}{5}$, and $P(E_1 \cap E_2 \cap E_3) = \frac{1}{6}$. Then, among the events E_1, E_2, E_3, the probability that at least two events occur equals:

(1) $\frac{17}{60}$

(2) $\frac{23}{60}$

(3) $\frac{19}{60}$

(4) $\frac{29}{60}$

---

9. Let X be a continuous random variable such that $P(X \geq 0) = 1$ and $Var(X) < \infty$. Then, $E(X^2)$ is:

(1) $2\int_0^\infty x^2 P(X > x)\, dx$

(2) $\int_0^\infty x^2 P(X > x)\, dx$

(3) $2\int_0^\infty x P(X > x)\, dx$

(4) $\int_0^\infty x P(X > x)\, dx$

---

10. Let X be a random variable having probability density function

$$f(x;\theta) = \begin{cases} (3-\theta)x^{2-\theta}, & 0 < x < 1, \\ 0, & \text{otherwise}, \end{cases}$$

where $\theta \in \{0, 1\}$. For testing the null hypothesis $H_0 : \theta = 0$ against $H_1 : \theta = 1$ at the significance level $\alpha = 0.125$, the power of the most powerful test equals:

(1) 0.15
(2) 0.25
(3) 0.35
(4) 0.45

---

11. Let X_1, X_2 be i.i.d. random variables having the common probability density

function

$$f(x) = \begin{cases} e^{-x}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Define $X_{(1)} = \min(X_1, X_2)$ and $X_{(2)} = \max(X_1, X_2)$. Then, which one of the following statements is FALSE?

(1) $\dfrac{2X_{(1)}}{X_{(2)} - X_{(1)}} \sim F_{2,2}$

(2) $2(X_{(2)} - X_{(1)}) \sim \chi^2_2$

(3) $E(X_{(1)}) = \dfrac{1}{2}$

(4) $P(3X_{(1)} < X_{(2)}) = \dfrac{1}{3}$

---

12. Let X and Y be random variables such that $X \sim N(1, 2)$ and $P(Y = \frac{X}{2} + 1) = 1$. Let $\alpha = \mathbf{Cov}(X, Y)$, $\beta = E(Y)$, and $\gamma = \mathbf{Var}(Y)$. Then, the value of $\alpha + 2\beta + 4\gamma$ equals:

(1) 5
(2) 6
(3) 7
(4) 8

---

13. A point (a, b) is chosen at random from the rectangular region $[0, 2] \times [0, 4]$. The probability that the area of the region

$$R = \{(x, y) \in \mathbb{R}^2 : bx + ay \leq ab,\ x, y \geq 0\}$$

is less than 2 equals:

(1) $\frac{1+\ln 2}{4}$
(2) $\frac{1+\ln 2}{2}$
(3) $\frac{2+\ln 2}{4}$
(4) $\frac{1+2\ln 2}{4}$

---

14. Let $X_1, X_2, \ldots$ be independent random variables such that $P(X_i = i) = \frac{1}{4}$ and $P(X_i = 2i) = \frac{3}{4}$, for $i = 1, 2, \ldots$. For some real constants c_1, c_2, suppose that

$$\frac{c_1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i}{i} + c_2\sqrt{n} \xrightarrow{d} Z \sim N(0, 1), \text{ as } n \to \infty.$$

Then, the value of $\sqrt{3}(3c_1 + c_2)$ equals:

(1) 2
(2) 3
(3) 4
(4) 5

---

15. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables such that $P(X_1 = 0) = P(X_1 = 1) = P(X_1 = 2) = \frac{1}{3}$. Let $S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $T_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2$. Suppose that

$$\alpha_1 = \lim_{n\to\infty} P\left(\left|S_n - \frac{1}{2}\right| < \frac{3}{4}\right), \quad \alpha_2 = \lim_{n\to\infty} P\left(\left|S_n - \frac{1}{3}\right| < 1\right),$$
$$\alpha_3 = \lim_{n\to\infty} P\left(\left|T_n - \frac{1}{3}\right| < \frac{3}{2}\right), \quad \alpha_4 = \lim_{n\to\infty} P\left(\left|T_n - \frac{2}{3}\right| < \frac{1}{2}\right).$$

Then, the value of $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4$ equals:

(1) 6
(2) 5
(3) 4
(4) 3

---

16. For $x \in \mathbb{R}$, the curve $y = x^2$ intersects the curve $y = x\sin x + \cos x$ at exactly n points. Then, n equals:

(1) 1
(2) 2
(3) 4
(4) 8

---

17. Let (X, Y) be a random vector having the joint pdf

$$f(x,y) = \begin{cases} \alpha|x|, & x^2 \le y \le 2x^2,\ -1 \le x \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

where α is a positive constant. Then, $P(X > Y)$ equals:

(1) $\frac{5}{48}$
(2) $\frac{7}{48}$
(3) $\frac{5}{24}$

(4) $\dfrac{7}{24}$

---

18. Let X_1, X_2, X_3, X_4 be a random sample of size 4 from $N(\theta, 1)$, where $\theta \in \mathbb{R}$. Let $\bar{X} = \frac{1}{4}\sum_{i=1}^{4} X_i$, $g(\theta) = \theta^2 + 2\theta$, and $L(\theta)$ be the Cramér–Rao lower bound on the variance of unbiased estimators of $g(\theta)$. Then, which one of the following statements is FALSE?

(1) $L(\theta) = (1+\theta)^2$
(2) $\bar{X} + e^{\bar{X}}$ is a sufficient statistic for θ
(3) $(1+\bar{X})^2$ is the UMVUE of $g(\theta)$
(4) $Var((1+\bar{X})^2) \geq \frac{(1+\theta)^2}{2}$

---

19. Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with pdf

$$f(x;\mu) = \begin{cases} \dfrac{1}{2}e^{-\frac{x-2\mu}{2}}, & x > 2\mu, \\ 0, & \text{otherwise,} \end{cases}$$

where $-\infty < \mu < \infty$. For estimating μ, consider estimators

$$T_1 = \frac{\bar{X} - 2}{2}, \quad T_2 = \frac{nX_{(1)} - 2}{2n},$$

where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $X_{(1)} = \min(X_1, X_2, \ldots, X_n)$. Which one of the following statements is TRUE?

(1) T_1 is consistent but T_2 is NOT consistent
(2) T_2 is consistent but T_1 is NOT consistent
(3) Both T_1 and T_2 are consistent
(4) Neither T_1 nor T_2 is consistent

---

20. Let $X_1, X_2, \ldots, X_n$ be a random sample from $U(\theta + \frac{\sigma}{\sqrt{3}}, \theta + \sqrt{3}\sigma)$, where $\theta \in \mathbb{R}$ and $\sigma > 0$ are unknown. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2}$. Let $\hat{\theta}$ and $\hat{\sigma}$ be the method of moments estimators of θ and σ, respectively. Which one of the following statements is FALSE?

(1) $\hat{\theta} + \sqrt{3}\hat{\sigma} = \sqrt{3}\bar{X} - 3S$
(2) $2\sqrt{3}\hat{\sigma} + \hat{\theta} = \bar{X} - 4\sqrt{3}S$
(3) $\sqrt{3}\hat{\sigma} + \hat{\theta} = \bar{X} + \sqrt{3}S$
(4) $\hat{\sigma} - \sqrt{3}\hat{\theta} = 9S - \sqrt{3}\bar{X}$

21. Let (X, Y, Z) be a random vector having the joint pdf

$$f(x,y,z) = \begin{cases} \dfrac{1}{2xy}, & 0 < z < y < x < 1, \\ \dfrac{1}{2xz^2}, & 0 < z < x < y < 2x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Then, which one of the following statements is FALSE?

(1) $P(Z < Y < X) = \frac{1}{2}$
(2) $P(X < Y < Z) = 0$
(3) $E(\min(X, Y)) = \frac{1}{4}$
(4) $Var(Y \mid X = \frac{1}{2}) = \frac{1}{12}$

---

22. Let X be a random variable such that its moment generating function exists near 0, and

$$E(X^n) = (-1)^n \frac{2}{5} + \frac{2^{n+1}}{5} + \frac{1}{5}, \quad n = 1, 2, 3, \ldots$$

Then, $P(|X - \frac{1}{2}| > 1)$ equals:

(1) $\frac{1}{5}$

(2) $\frac{2}{5}$

(3) $\frac{3}{5}$

(4) $\frac{4}{5}$

---

23. Let X be a random variable with pmf $p(x)$, positive for non-negative integers, satisfying

$$p(x+1) = \frac{\ln 3}{x+1} p(x), \quad x = 0, 1, 2, \ldots$$

Then, $Var(X)$ equals:

(1) $\ln 3$
(2) $\ln 6$
(3) $\ln 9$
(4) $\ln 18$

---

24. Let $\{a_n\}_{n\geq 1}$ be a sequence such that $a_1 = 1$ and $4a_{n+1} = \sqrt{45 + 16a_n}$, for $n = 1, 2, \ldots$. Then, which one of the following statements is TRUE?

(1) $\{a_n\}$ is monotonically increasing and converges to $\frac{17}{8}$
(2) $\{a_n\}$ is monotonically increasing and converges to $\frac{9}{4}$
(3) $\{a_n\}$ is bounded above by $\frac{17}{8}$
(4) $\sum_{n=1}^{\infty} a_n$ is convergent

---

25. Let the series S and T be defined by

$$S = \sum_{n=0}^{\infty} \frac{2 \cdot 5 \cdot 8 \cdots (3n+2)}{1 \cdot 5 \cdot 9 \cdots (4n+1)}, \quad T = \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-n^2}.$$

Then, which one of the following statements is TRUE?

(1) S is convergent and T is divergent
(2) S is divergent and T is convergent
(3) Both S and T are convergent
(4) Both S and T are divergent

---

26. The volume of the region

$$R = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 4,\ 0 \leq z \leq 4 - y\}$$

is:

(1) $16\pi - 16$
(2) 16π
(3) 8π
(4) $16\pi + 4$

---

27. For real constants α and β, suppose that the system of linear equations

$$x + 2y + 3z = 6, \quad x + y + \alpha z = 3, \quad 2y + z = \beta$$

has infinitely many solutions. Then, the value of $4\alpha + 3\beta$ equals:

(1) 18
(2) 23
(3) 28
(4) 32

28. Let x_1, x_2, x_3, x_4 be observed values of a random sample from $N(\theta, \sigma^2)$, where $\theta \in \mathbb{R}, \sigma > 0$. Suppose that

$$\bar{x} = 3.6, \quad \frac{1}{3}\sum_{i=1}^{4}(x_i - \bar{x})^2 = 20.25.$$

For testing $H_0 : \theta = 0$ against $H_1 : \theta \neq 0$, the p-value of the likelihood ratio test equals:

(1) 0.712
(2) 0.208
(3) 0.104
(4) 0.052

---

29. Let X and Y be jointly distributed random variables such that for every fixed $\lambda > 0$, the conditional distribution of $X|Y = \lambda$ is Poisson with mean λ. If $Y \sim$ Gamma$(2, \frac{1}{2})$, then the value of $P(X = 0) + P(X = 1)$ equals:

(1) $\frac{7}{27}$
(2) $\frac{20}{27}$
(3) $\frac{8}{27}$
(4) $\frac{16}{27}$

---

30. Among all points on the sphere $x^2 + y^2 + z^2 = 24$, the point (α, β, γ) closest to the point $(1, 2, -1)$ satisfies what value of $\alpha + \beta + \gamma$?

(1) 4
(2) -4
(3) 2
(4) -2

---

31. Let M be a 3×3 real matrix. If $P = M + M^T$ and $Q = M - M^T$, then which of the following statements is/are always TRUE?

(1) $\det(P^2Q^3) = 0$
(2) $\text{trace}(Q + Q^2) = 0$
(3) $X^TQ^2X = 0, \; \forall X \in \mathbb{R}^3$
(4) $X^TPX = 2X^TMX, \; \forall X \in \mathbb{R}^3$

32. Let X_1, X_2, X_3 be i.i.d. random variables, each following $N(0,1)$. Then, which of the following statements is/are TRUE?

(1) $\dfrac{\sqrt{2}(X_1 - X_2)}{\sqrt{(X_1+X_2)^2 + 2X_3^2}} \sim t_1$

(2) $\dfrac{(X_1+X_2)^2}{(X_1-X_2)^2 + 2X_3^2} \sim F_{1,2}$

(3) $E\left(\dfrac{X_1}{X_2^2 + X_3^2}\right) = 0$

(4) $P(X_1 < X_2 + X_3) = \frac{1}{3}$

---

33. Let $x_1, \ldots, x_{10}$ be a random sample from $N(\theta, \sigma^2)$. If $\bar{x} = 0$, $s = 2$, then using Student's t-distribution with 9 degrees of freedom, the 90% confidence interval for θ is:

(1) $(-0.8746, \infty)$
(2) $(-0.8746, 0.8746)$
(3) $(-1.1587, 1.1587)$
(4) $(-\infty, 0.8746)$

---

34. Let (X_1, X_2) have pmf

$$f(x_1, x_2) = \begin{cases} \dfrac{c}{x_1! x_2! (12 - x_1 - x_2)!}, & x_1, x_2 \in \{0, \ldots, 12\}, x_1 + x_2 \leq 12, \\ 0, & \text{otherwise.} \end{cases}$$

Then, which of the following statements is/are TRUE?

(1) $E(X_1 + X_2) = 8$
(2) $Var(X_1 + X_2) = \frac{8}{3}$
(3) $Cov(X_1, X_2) = -\frac{5}{3}$
(4) $Var(X_1 + 2X_2) = 8$

---

35. Let P be a 3×3 matrix with eigenvalues 1, 1, and 2. Let $(1, -1, 2)^T$ be the only linearly independent eigenvector corresponding to eigenvalue 1. If adjoint of $2P$ is Q, then which of the following statements is/are TRUE?

(1) trace$(Q) = 20$
(2) $\det(Q) = 64$

(3) $(2, -2, 4)^T$ is an eigenvector of Q
(4) $Q^3 = 20Q^2 - 124Q + 256I_3$

---

36. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \dfrac{xy(x+y)}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Then, which of the following statements is/are TRUE?

(1) f is continuous on $\mathbb{R} \times \mathbb{R}$
(2) The partial derivative of f w.r.t. y exists at $(0, 0)$ and is 0
(3) The partial derivative of f w.r.t. x is continuous on $\mathbb{R} \times \mathbb{R}$
(4) f is NOT differentiable at $(0, 0)$

---

37. Let X, Y be i.i.d. $N(0, 1)$. Let $U = \frac{X}{Y}$ and $Z = |U|$. Then, which of the following statements is/are TRUE?

(1) U has a Cauchy distribution
(2) $E(Z^p) < \infty$, for some $p \geq 1$
(3) $E(e^{tZ})$ does not exist for all $t \in (-\infty, 0)$
(4) $Z^2 \sim F_{1,1}$

---

38. Which of the following are TRUE?

$$\int_0^1 \int_0^1 e^{\max(x^2, y^2)}\, dx\, dy, \qquad \int_0^1 \int_0^1 e^{\min(x^2, y^2)}\, dx\, dy$$

are two given integrals.

(1) $\int_0^1 \int_0^1 e^{\max(x^2,y^2)} dx\, dy = e - 1$
(2) $\int_0^1 \int_0^1 e^{\min(x^2,y^2)} dx\, dy = \int_0^1 e^{t^2} dt - (e - 1)$
(3) $\int_0^1 \int_0^1 e^{\max(x^2,y^2)} dx\, dy = 2 \int_0^1 \int_0^y e^{y^2} dx\, dy$
(4) $\int_0^1 \int_0^1 e^{\min(x^2,y^2)} dx\, dy = 2 \int_0^1 \int_y^1 e^{x^2} dx\, dy$

---

39. Let X be a random variable with pdf

$$f(x) = \begin{cases} \dfrac{5}{x^6}, & x > 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, which of the following statements is/are TRUE?

(1) The coefficient of variation is $\dfrac{4}{\sqrt{15}}$
(2) The first quartile is $\left(\dfrac{4}{3}\right)^{1/5}$
(3) The median is $(2)^{1/5}$
(4) The upper bound by Chebyshev's inequality for $P(X \geq \frac{5}{2})$ is $\frac{1}{15}$

---

40. Given 10 data points (x_i, y_i), the regression lines of Y on X and X on Y are $2y - x = 8$ and $y - x = -3$, respectively. Let $\bar{x} = \frac{1}{10}\sum x_i$ and $\bar{y} = \frac{1}{10}\sum y_i$. Then, which of the following statements is/are TRUE?

(1) $\sum x_i = 140$
(2) $\sum y_i = 110$
(3) $\dfrac{\sum (x_i - \bar{x}) y_i}{\sqrt{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}} = -\dfrac{1}{\sqrt{2}}$
(4) $\dfrac{\sum (x_i - \bar{x})^2}{\sum (y_i - \bar{y})^2} = 2$

---

41. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 - x$. Let $g : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $g(x) = 0$ has exactly three distinct roots in (0,1). Let $h(x) = f(x)g(x)$, and $h''(x)$ be the second derivative of h. If n is the number of roots of $h''(x) = 0$ in (0,1), find the minimum possible value of n.

---

42. Let $X_1, X_2, \ldots$ be i.i.d. with pdf $f(x) = \frac{x^2 e^{-x}}{2}, x \geq 0$. For real constants β, γ, k, suppose

$$\lim_{n \to \infty} P\left(\frac{1}{n}\sum_{i=1}^{n} X_i \leq x\right) = \begin{cases} 0, & x < \beta, \\ kx, & \beta \leq x \leq \gamma, \\ k\gamma, & x > \gamma. \end{cases}$$

Find the value of $2\beta + 3\gamma + 6k$.

---

43. Let α, β be real constants such that

$$\lim_{x \to 0^+} \frac{\int_0^x \frac{\alpha t^2}{1+t^4} dt}{\beta x - \sin x} = 1.$$

Find the value of $\alpha + \beta$.

44. Let $X_1, \ldots, X_{10}$ be a random sample from $N(0, \sigma^2)$. For some real constant c, let

$$Y = \frac{c}{10} \sum_{i=1}^{10} |X_i|$$

be an unbiased estimator of σ. Find c (rounded to two decimal places).

---

45. Let X have pdf

$$f(x) = \begin{cases} \frac{x}{2}, & 0 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Then, find $Var\left(\ln \frac{2}{X}\right)$.

---

46. Let X_1, X_2, X_3 be i.i.d. random variables each following $N(2, 4)$. If $P(2X_1 - 3X_2 + 6X_3 > 17) = 1 - \Phi(\beta)$, then find β.

---

47. Let a discrete random variable X have pmf $P(X = n) = \dfrac{k}{(n-1)^n}$, $n = 2, 3, \ldots$. If $P(X \geq 17 \mid X \geq 5)$ is required, find its value.

---

48. Let

$$S_n = \sum_{k=1}^{n} \frac{1 + k2^k}{4^{k-1}}, \quad n = 1, 2, \ldots$$

Find $\lim_{n \to \infty} S_n$ (round off to two decimal places).

---

49. A box contains 80% white, 15% blue, 5% red balls. Among them, white, blue, and red balls have defect rates $\alpha\%, 6\%, 9\%$ respectively. If $P(\text{white} \mid \text{defective}) = 0.4$, find α.

---

50. Let X_1, X_2 be from pdf $f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}, x > 0$. To test $H_0 : \theta = 1$ vs $H_1 : \theta \neq 1$, consider test statistic $W = \frac{X_1 + X_2}{2}$. If $X_1 = 0.25, X_2 = 0.75$, find the p-value (round

off to two decimals).

---

51. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 \sin(x-1) + xe^{(x-1)}$. Then, find

$$\lim_{n\to\infty} n\left(f\left(1+\frac{1}{n}\right)+f\left(1+\frac{2}{n}\right)+\cdots+f\left(1+\frac{10}{n}\right)-10\right).$$

---

52. Let (X_1, X_2) follow a bivariate normal distribution with $E(X_1) = E(X_2) = 1$, $Var(X_1) = 1$, $Var(X_2) = 4$, $Cov(X_1, X_2) = 1$. Find $Var(X_1 + X_2 \mid X_1 = \frac{1}{2})$.

---

53. If $\displaystyle\int_0^{\infty} 2^{-x^2} dx = \alpha\sqrt{\pi}$, find α (round to two decimals).

---

54. Let $x_1 = 2.1, x_2 = 4.2, x_3 = 5.8, x_4 = 3.9$ be a sample from pdf $f(x;\theta) = \frac{x}{\theta^2} e^{-x^2/(2\theta)}$, $x > 0$. Find the MLE of $Var(X_1)$.

---

55. Let $X_i \sim$ Geometric(θ) with pmf $f(x;\theta) = \theta(1-\theta)^x, x = 0, 1, 2, \ldots$. If $\hat{\theta}$ is the UMVUE of θ, then find $156\,\hat{\theta} = ?$ given sample $x_1 = 2, x_2 = 5, x_3 = 4$.

---

56. Let $X_1, X_2, \ldots, X_5$ be i.i.d. Bin$(1, \frac{1}{2})$ random variables. Define $K = X_1 + X_2 + \cdots + X_5$ and

$$U = \begin{cases} 0, & K = 0, \\ X_1 + X_2 + \cdots + X_K, & K = 1, 2, \ldots, 5. \end{cases}$$

Find $E(U)$.

---

57. Let $X_1 \sim$ Gamma$(1, 4), X_2 \sim$ Gamma$(2, 2), X_3 \sim$ Gamma$(3, 4)$ be independent. If $Y = X_1 + 2X_2 + X_3$, find $E\left[\left(\frac{Y}{4}\right)^4\right]$.

---

58. Let $X_1, X_2 \sim U(0, \theta)$ i.i.d., with $\theta > 0$. For testing $H_0 : \theta \in (0, 1] \cup [2, \infty)$ vs $H_1 : \theta \in (1, 2)$, consider the critical region

$$R = \{(x_1, x_2) : \tfrac{5}{4} < \max(x_1, x_2) < \tfrac{7}{4}\}.$$

Find the size of the test (probability of Type-I error).

---

59. Let $X_1, \ldots, X_5 \sim$ Bin$(1, \theta)$. For $H_0 : \theta \leq 0.5$ vs $H_1 : \theta > 0.5$, define

$$T_1 : \text{Reject } H_0 \text{ if } \sum X_i = 5, \quad T_2 : \text{Reject } H_0 \text{ if } \sum X_i \geq 3.$$

If $\theta = \frac{2}{3}$, find $\beta_1 + \beta_2$ where β_i = Type-II error for T_i.

---

60. Let $X_1 \sim N(2, 1)$, $X_2 \sim N(-1, 4)$, $X_3 \sim N(0, 1)$ be independent. Find the probability that exactly two of them are less than 1 (round off to two decimals).
