IIT JAM Mathematical Statistics - 2025 Question Paper

Time Allowed :3 Hours	**Maximum Marks :**100	**Total Questions :**60

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The examination is of 3 hours duration. There are a total of 60 questions carrying 100 marks. The entire paper is divided into three sections, A, B and C. All sections are compulsory. Questions in each section are of different types.
2. Section A contains a total of 30 Multiple Choice Questions (MCQ). Each MCQ type question has four choices out of which only one choice is the correct answer. Questions Q.1 – Q.30 belong to this section and carry a total of 50 marks. Q.1 – Q.10 carry 1 mark each and Questions Q.11 – Q.30 carry 2 marks each.
3. Section B contains a total of 10 Multiple Select Questions (MSQ). Each MSQ type question is similar to MCQ but with a difference that there will be one or more than one choices that are correct out of the four given choices. The candidate gets full credit if he/she selects all the correct answers only and no wrong answers. Questions Q.31 – Q.40 belong to this section and carry 2 marks each with a total of 20 marks.
4. Section C contains a total of 20 Numerical Answer Type (NAT) questions. For these NAT type questions, the answer is a real number which needs to be entered using the virtual keyboard on the monitor. No choices will be shown for these type of questions. Questions Q.41 – Q.60 belong to this section and carry a total of 30 marks. Q.41 – Q.50 carry 1 mark each and Questions Q.51 – Q.60 carry 2 marks each.
5. In all sections, questions not attempted will result in zero marks. In Section A (MCQ), wrong answer will result in NEGATIVE marks. For all 1-mark questions, 1/3 marks will be deducted for each wrong answer. For all 2-mark questions, 2/3 marks will be deducted for each wrong answer. In Section B (MSQ), there is NO NEGATIVE and NO PARTIAL marking provisions. There is NO NEGATIVE marking in Section C (NAT) as well.
6. Only Virtual Scientific Calculator is allowed. Charts, graph sheets, tables, cellular phone or other electronic gadgets are NOT allowed in the examination hall.
7. A Scribble Pad will be provided for rough work.

Section - A

1. Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be sequences given by

$$a_n = \left\lfloor \frac{n^2}{n+1} \right\rfloor \quad \text{and} \quad b_n = \frac{n^2}{n+1} - a_n \,.$$

Then

(A) $\{a_n\}_{n\geq 1}$ converges and $\{b_n\}_{n\geq 1}$ diverges
(B) $\{a_n\}_{n\geq 1}$ diverges and $\{b_n\}_{n\geq 1}$ converges
(C) Both $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ diverge
(D) Both $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ converge

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2. Let a, b, c be real numbers with $b \neq c$. Define the matrix

$$M = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$$

Then the number of characteristic roots of M that are real is

(A) 3
(B) 2
(C) 1
(D) 0

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3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous odd function that is not identically zero. Further, suppose that f is a periodic function. Define

$$g(x) = \int_0^x f(t)\, dt\,.$$

Then

(A) g is odd and not periodic
(B) g is odd and periodic
(C) g is even and not periodic
(D) g is even and periodic

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4. Suppose $Z_1, Z_2, \ldots, Z_{128}$ are i.i.d. Bin(1, 0.5) random variables. Define

$$\mathbf{X} = (Z_1, Z_2, \ldots, Z_{64})^T \quad \text{and} \quad \mathbf{Y} = (Z_{65}, Z_{66}, \ldots, Z_{128})^T\,.$$

Then the value of Var($\mathbf{X}^T\mathbf{Y}$) is

(A) 4
(B) 8
(C) 12

(D) 16

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5. Let X_1, X_2, X_3 be i.i.d. Bin(1, θ) random variables. Consider the problem of testing the null hypothesis $H_0 : \theta = \frac{1}{2}$ against the alternative hypothesis $H_1 : \theta = \frac{1}{4}$ based on X_1, X_2, X_3. Then the power of the most powerful test of size 0.125 is

(A) 0
(B) $\frac{1}{64}$
(C) $\frac{27}{64}$
(D) $\frac{7}{8}$

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6. Suppose X is a Poisson(λ) random variable. Define $Y = (-1)^X$. Then the expected value of Y is

(A) $-\lambda e^{-2\lambda}$
(B) $-e^{-2\lambda}$
(C) $\lambda e^{-2\lambda}$
(D) λ

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7. Let $\{Y_n\}_{n\geq 1}$ be a sequence of i.i.d. Bin(1, p) random variables, where $0 < p < 1$ is an unknown parameter. Let $\hat{p}_n$ be the maximum likelihood estimator of p based on $Y_1, Y_2, \ldots, Y_n$. It is claimed that:

$$\frac{\hat{p}_n - p}{\sqrt{\frac{p(1-p)}{n}}} \xrightarrow{d} N(0,1) \quad \text{as } n \to \infty \tag{I}$$

$$\frac{\hat{p}_n - p}{\sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}}} \xrightarrow{d} N(0,1) \quad \text{as } n \to \infty \tag{II}$$

Which of the following statements is correct?

(A) (I) is correct and (II) is incorrect
(B) (I) is incorrect and (II) is correct
(C) Both (I) and (II) are correct
(D) Both (I) and (II) are incorrect

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8. Let X be a continuous random variable with probability density function $f(x)$.

Consider the problem of testing the null hypothesis

$$H_0 : f(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

against the alternative hypothesis

$$H_1 : f(x) = \begin{cases} 2x & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then the power of the most powerful size α test, where $0 < \alpha < 1$, based on a single sample, is

(A) $\alpha(1-\alpha)$
(B) $\alpha(2-\alpha)$
(C) $1-\alpha$
(D) α

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9. Suppose $X \sim N(0,4)$ and $Y \sim N(0,9)$ are independent random variables. Then the value of $P(9X^2 + 4Y^2 < 6)$ is

(A) $1 - e^{-1/4}$
(B) $1 - e^{-1/12}$
(C) $1 - e^{-1/6}$
(D) $1 - e^{-1/9}$

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10. Let X be a single sample from a continuous distribution with probability density function

$$f(x;\theta) = \begin{cases} \frac{2(\theta - x)}{\theta^2} & \text{if } 0 < x < \theta, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is an unknown parameter. For $0 < \alpha < 0.05$, a $100(1-\alpha)\%$ confidence interval for θ based on X is

(A) $\left[\frac{X}{1-\sqrt{\alpha/2}}, \frac{X}{1-\sqrt{1-\alpha/2}}\right]$
(B) $\left(\frac{X}{1-\sqrt{\alpha}}, \frac{X}{1-\sqrt{1-\alpha}}\right)$
(C) $\left(\left(1-\sqrt{1-\frac{\alpha}{2}}\right)X, \left(1-\sqrt{\frac{\alpha}{2}}\right)X\right)$
(D) $\left(\frac{\alpha}{2}X, \left(1-\frac{\alpha}{2}\right)X\right)$

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11. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x + \pi \cos x$. Then the number of solutions of the equation $f(x) = 0$ is

(A) 1
(B) 2
(C) 3
(D) 4

---

12. A fair die is thrown three times independently. The probability that 4 is the maximum value that appears among these throws is equal to

(A) $\frac{8}{27}$
(B) $\frac{1}{216}$
(C) $\frac{37}{216}$
(D) $\frac{1}{2}$

---

13. Let $\mathcal{A}$ be an $n \times n$ matrix. Which of the following statements is NOT necessarily true?

(A) If $\text{rank}(\mathcal{A}^5) = \text{rank}(\mathcal{A}^6)$, then $\text{rank}(\mathcal{A}^6) = \text{rank}(\mathcal{A}^7)$
(B) If $\text{rank}(\mathcal{A}) = n$, then it is possible to obtain a singular matrix by suitably changing a single entry of $\mathcal{A}$
(C) If $\text{rank}(\mathcal{A}) = n$, then $\text{rank}(\mathcal{A} + \mathcal{A}^T) \geq \frac{n}{2}$
(D) If $\text{rank}(\mathcal{A}) < n$, then it is possible to obtain a nonsingular matrix by suitably changing $n - \text{rank}(\mathcal{A})$ entries of $\mathcal{A}$

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14. Let V be a subspace of $\mathbb{R}^{10}$. Suppose $\mathcal{A}$ is a 10×10 matrix with real entries. Let $\mathcal{A}^k(V) = \{\mathcal{A}^k\mathbf{x} : \mathbf{x} \in V\}$ for $k \geq 1$ and $\mathcal{A}(V) = \mathcal{A}^1(V)$. Which one of the following statements is NOT true?

(A) If $\mathcal{A}$ is nonsingular, then $\dim(V) = \dim(\mathcal{A}(V))$ necessarily holds
(B) It is possible that $\mathcal{A}$ is singular and $\dim(V) = \dim(\mathcal{A}(V))$
(C) If $\text{rank}(\mathcal{A}) = 8$, then $\dim(\mathcal{A}(V)) \geq \dim(V) - 2$ necessarily holds
(D) If $\dim(V) = \dim(\mathcal{A}(V)) = \dim(\mathcal{A}^2(V)) = \cdots = \dim(\mathcal{A}^5(V))$, then $\dim(\mathcal{A}^6(V)) = \dim(V)$ necessarily holds

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15. A function $f : (0, 1) \to \mathbb{R}$ is said to have property $\mathcal{I}$ if, for any $0 < x_1 < x_2 < 1$ and for any c between $f(x_1)$ and $f(x_2)$, there exists $y \in [x_1, x_2]$ such that $f(y) = c$.

Consider the following statements:

(I) If $g : (0,1) \to \mathbb{R}$ satisfies property $\mathcal{I}$, then g is necessarily continuous.

(II) If $h : (0,1) \to \mathbb{R}$ is differentiable, then h' necessarily satisfies property $\mathcal{I}$.

Then

(A) (I) is correct and (II) is incorrect
(B) (I) is incorrect and (II) is correct
(C) Both (I) and (II) are correct
(D) Both (I) and (II) are incorrect

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16. Suppose f is a polynomial of degree n with real coefficients, and $\mathcal{A}$ is an $n \times n$ matrix with real entries satisfying $f(\mathcal{A}) = 0$. Consider the following statements:

(I) If $f(0) \neq 0$, then $\mathcal{A}$ is necessarily nonsingular.

(II) If $f(0) = 0$, then $\mathcal{A}$ is necessarily singular.

Then

(A) (I) is correct and (II) is incorrect
(B) (I) is incorrect and (II) is correct
(C) Both (I) and (II) are correct
(D) Both (I) and (II) are incorrect

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17. Let X_1 and X_2 be i.i.d. $N(0, \sigma^2)$ random variables. Define $Z_1 = X_1 + X_2$ and $Z_2 = X_1 - X_2$. Then which one of the following statements is NOT correct?

(A) Z_1 and Z_2 are independently distributed
(B) Z_1 and Z_2 are identically distributed
(C) $P\left(\left|\frac{Z_1}{Z_2}\right| < 1\right) = 0.5$
(D) $\frac{Z_1}{Z_2}$ and $Z_1^2 + Z_2^2$ are independently distributed

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18. Consider a circle C with unit radius and center at $A = (0,0)$. Let $B = (1,0)$. Suppose $\Theta \sim U(0,\pi)$ and $D = (\cos\Theta, \sin\Theta)$. Note that the angle $\angle DAB = \Theta$. Then the expected area of the triangle ABD is

(A) $\frac{1}{\pi}$
(B) $\frac{2}{\pi}$
(C) $\frac{1}{2\pi}$

(D) 1

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19. Suppose $Y \sim U(0,1)$ and the conditional distribution of X given $Y = y$ is Bin(6, y), for $0 < y < 1$. Then the probability that $(X+1)$ is an even number is

(A) $\frac{3}{7}$
(B) $\frac{1}{2}$
(C) $\frac{4}{7}$
(D) $\frac{5}{14}$

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20. Let X be an Exp(λ) random variable. Suppose $Y = \min\{X, 2\}$. Let F_X and F_Y denote the distribution functions of X and Y respectively. Then which of the following statements is true?

(A) F_Y is a continuous function
(B) $F_Y(y)$ is discontinuous at $y = 2$
(C) $F_Y(t) \leq F_X(t)$ for all $t \in \mathbb{R}$
(D) $E(Y) > E(X)$

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21. Let $X_1, X_2, \ldots, X_n$ be i.i.d. $N(0, \sigma^2)$ random variables. Suppose c is such that

$$E\left[c\sqrt{\sum_{i=1}^{n} X_i^2}\right] = \sigma.$$

Then the value of c is

(A) $\sqrt{\frac{9\pi}{128}}$
(B) $\sqrt{\frac{9\pi}{64}}$
(C) $\sqrt{\frac{9}{128\pi}}$
(D) $\sqrt{\frac{3}{64\pi}}$

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22. Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution with probability density function

$$f(x;\theta) = \frac{1}{2\theta}e^{-|x|/\theta}, \quad x \in \mathbb{R},$$

where $\theta > 0$ is an unknown parameter. The critical region for the uniformly most powerful test for testing the null hypothesis $H_0 : \theta = 2$ against the alternative hypothesis $H_1 : \theta > 2$ at level α, where $0 < \alpha < 1$, is

(A) $\left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : 2\sum_{i=1}^n |x_i| < \chi^2_{2n,1-\alpha}\right\}$
(B) $\left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : 2\sum_{i=1}^n |x_i| > \chi^2_{2n,\alpha}\right\}$
(C) $\left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n |x_i| < \chi^2_{2n,1-\alpha}\right\}$
(D) $\left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n |x_i| > \chi^2_{2n,\alpha}\right\}$

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23. Let (X, Y) have the $N_2(0, 0, 1, 1, 0.25)$ distribution. Then the correlation coefficient between e^X and e^{2Y} is

(A) $\frac{e^3 - e^{5/2}}{(e^5(e-1)(e^4-1))^{1/2}}$
(B) $\frac{e^3 - e^{5/2}}{(e^4(e-1)(e^8-1))^{1/2}}$
(C) $\frac{e^2 - e^{5/2}}{(e^5(e-1)(e^4-1))^{1/2}}$
(D) $\frac{e^{3/2} - e^{5/2}}{(e^5(e^2-1)(e^4-1))^{1/2}}$

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24. Let $\{X_k\}_{k\geq 1}$ be a sequence of i.i.d. U(-1,1) random variables. Suppose

$$Y_n = \sqrt{3n}\frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i^4}.$$

Then $\{Y_n\}_{n\geq 1}$ converges in distribution as $n \to \infty$ to a

(A) $N(0, 1)$ random variable
(B) random variable degenerate at 0
(C) $N(0, 25)$ random variable
(D) $N(0, 0.04)$ random variable

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25. If $(X, Y) \sim N_2(0, 0, 1, 1, 0.5)$, then the value of $E[e^{-XY}]$ is

(A) $\frac{2}{\sqrt{5}}$
(B) $\frac{2}{\sqrt{3}}$
(C) $\frac{1}{\sqrt{2}}$
(D) $\frac{1}{2}$

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26. Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be a random sample from a $N_2(0, 0, 1, 1, \rho)$ distribution, where ρ is an unknown parameter. Which of the following statements is NOT correct?

(A) $\left(\sum_{i=1}^n X_i^2, \sum_{i=1}^n Y_i^2, \sum_{i=1}^n X_i Y_i\right)$ is a sufficient statistic for ρ
(B) $\left(\sum_{i=1}^n X_i^2, \sum_{i=1}^n Y_i^2, \sum_{i=1}^n X_i Y_i\right)$ is not a minimal sufficient statistic for ρ
(C) $\sum_{i=1}^n X_i^2$ is an ancillary statistic
(D) $\left(\sum_{i=1}^n X_i^2, \sum_{i=1}^n Y_i^2\right)$ is an ancillary statistic

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27. Let $(Y_1, Y_2, Y_3) \in \{0, 1, \ldots, n\}^3$ be a discrete random vector having joint probability mass function

$$P(Y_1 = y_1, Y_2 = y_2, Y_3 = y_3) = \begin{cases} \frac{n!}{y_1! y_2! y_3!} p^{y_1} (2p)^{y_2} (1 - 3p)^{y_3} & \text{if } y_1 + y_2 + y_3 = n, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 \leq p \leq \frac{1}{3}$ is an unknown parameter. Assume the convention $0^0 = 1$. The maximum likelihood estimator of p is denoted by $\hat{p}$. Which of the following statements is correct?

(A) $E(\hat{p}) > p$
(B) $\hat{p}$ is an unbiased estimator of p, but not the uniformly minimum variance unbiased estimator of p
(C) $\hat{p}$ is the uniformly minimum variance unbiased estimator of p
(D) $E(\hat{p}) < p$

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28. Let $X_1, X_2, \ldots, X_n$ be i.i.d. $N(0, 1)$ random variables, where $n > 3$. If

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \quad \text{and} \quad S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2,$$

then Var$(\frac{\bar{X}}{S})$ is equal to

(A) $\frac{n-3}{n(n-1)}$
(B) $\frac{n-1}{n(n-3)}$
(C) $\frac{n-1}{n(n-2)}$
(D) $\frac{n-2}{n(n-1)}$

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29. Suppose (X_i, Y_i), $i = 1, 2, \ldots, 200$, are i.i.d. random vectors each having joint

probability density function

$$f(x, y) = \begin{cases} \frac{1}{25\pi} & \text{if } x^2 + y^2 \leq 25, \\ 0 & \text{otherwise.} \end{cases}$$

Let M be the cardinality of the set $\{i \in \{1, 2, \ldots, 200\} : X_i^2 + Y_i^2 \leq 0.25\}$. Then $P(M \geq 1)$ is closest to

(A) $\Phi(0.5)$
(B) $1 - e^{-1}$
(C) $\Phi(3)$
(D) $1 - e^{-2}$

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30. Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables, where $X_n \sim$ Bin(n, p_n) with $p_n \in (0, 1)$. Which of the following conditions implies that $X_n \xrightarrow{d} 0$ as $n \to \infty$?

(A) $\lim_{n\to\infty} p_n = 0$
(B) $\lim_{n\to\infty} P(X_n = k) = 0$ for each $k \in \mathbb{N}$
(C) $\lim_{n\to\infty} E(X_n) = 0$
(D) $\sup_{n\geq 1} \text{Var}(X_n) < \infty$

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31. Let $\{x_n\}_{n\geq 1}$ be a sequence given by

$$x_n = \frac{2}{3}\left(x_{n-1} + \frac{2}{x_{n-1}}\right), \quad \textbf{for } n \geq 2,$$

with $x_1 = -10$. Then which of the following statement(s) is/are correct?

(A) $\{x_n\}_{n\geq 1}$ converges
(B) $\{x_n\}_{n\geq 1}$ diverges
(C) $x_{2025} - x_{2024}$ is positive
(D) $x_{2025} - x_{2024}$ is negative

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32. Let $a, b \in \mathbb{R}$. Consider the system of linear equations

$$x + y + 3z = 5,$$

$$ax - y + 4z = 11,$$

$$2x + by + z = 3.$$

Then which of the following statements is/are correct?

(A) There are finitely many pairs (a, b) such that the system has a unique solution
(B) There are finitely many pairs (a, b) such that the system has no solution
(C) There are finitely many pairs (a, b) such that the system has infinitely many solutions
(D) If $a = b = 1$, the system has no solution

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33. Suppose $f : (0, \infty) \to (0, \infty)$ is continuously differentiable. Assume further that $\lim_{x\to\infty} f(x) = 0$. Which of the following statements is/are necessarily true?

(A) $\lim_{x\to\infty} f'(x)$ exists and is equal to 0
(B) $\limsup_{x\to\infty} f'(x) = 0$
(C) $\liminf_{x\to\infty} f'(x) = 0$
(D) $\liminf_{x\to\infty} |f'(x)| = 0$

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34. The joint moment generating function of (X, Y) is given by

$$M_{X,Y}(s,t) = \left(\frac{1}{4} + \frac{1}{2}e^s + \frac{1}{4}e^t\right)^2, \quad (s,t) \in \mathbb{R}^2.$$

Then which of the following statements is/are correct?

(A) $E(X) = 1$
(B) $E(Y^2) = \frac{3}{8}$
(C) $\mathrm{Cov}(X, Y) = -\frac{1}{4}$
(D) $\mathrm{Var}(X) = \frac{1}{2}$

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35. The joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} c(x+y) & \textbf{if } 0 \le x, y \le 1, \\ 0 & \textbf{otherwise}, \end{cases}$$

for some constant c. Which of the following statements is/are correct?

(A) $c = 1$
(B) X and Y are independent
(C) The probability density function of X is $g(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$
(D) $X + Y$ has a probability density function

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36. Let $X_1, X_2, \ldots, X_{30}$ be a random sample from a $N(\mu, \sigma^2)$ population. Suppose $P = \frac{1}{10}\sum_{i=1}^{10} X_i$ and $Q = \frac{1}{9}\sum_{i=1}^{10}(X_i - P)^2$. Then which of the following statements is/are correct?

(A) $\frac{X_{11}+P-X_{12}-X_{20}}{\sqrt{Q}} \sim \sqrt{\frac{31}{10}}t_9$

(B) $\frac{P-X_{15}}{\sqrt{9Q+(X_{18}-\mu)^2}} \sim \sqrt{\frac{11}{10}}t_{10}$

(C) $\frac{(X_{12}-X_{20})^2}{Q} \sim 2F_{1,9}$

(D) $\frac{P-X_{14}}{\sqrt{Q}} \sim \sqrt{\frac{11}{10}}t_9$

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37. Let $X_1, X_2, \ldots, X_n$, where $n > 1$, be a random sample from a $N(\theta, \theta)$ distribution, where $\theta > 0$ is an unknown parameter. Suppose $T_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - T_n)^2$. Then which of the following statements is/are correct?

(A) $T_n S_n^2$ is a consistent estimator for θ^2

(B) $T_n^2 - S_n^2$ is a consistent estimator for θ^2

(C) $(\sum X_i, \sum X_i^2)$ is a complete statistic

(D) $\sum X_i^2$ is a complete sufficient statistic for θ

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38. Let $X_1, X_2, \ldots, X_n$, where $n > 1$, be a random sample from a continuous distribution with probability density function

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1} & \textbf{if } 0 < x < 1, \\ 0 & \textbf{otherwise}, \end{cases}$$

where $\theta > 0$ is an unknown parameter. Then which of the following statistics is/are sufficient for θ?

(A) $(X_1, X_2, \ldots, X_n)$

(B) $(X_{(1)}, X_{(2)}, \ldots, X_{(n)})$, where $X_{(r)}$ is the r^{th} order statistic, $r = 1, \ldots, n$

(C) $\sum_{i=1}^{n} X_i$

(D) $\prod_{i=1}^{n} X_i$

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39. A simple linear regression model $Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, with $x_i = (-1)^i$ for $i = 1, 2, \ldots, 20$, is fitted. The random error variables ϵ_i are uncorrelated with mean 0 and finite variance $\sigma^2 > 0$. Let $\hat{\beta}_0$ and $\hat{\beta}_1$ be the least squares estimators of β_0 and β_1 respectively. Let $\hat{Y}_i$ be the fitted value of the i^{th} response variable Y_i for $i = 1, \ldots, 20$. Which of the following statements is/are correct?

(A) $\text{Cov}(\hat{\beta}_0, \hat{\beta}_1) = 0$

(B) $\text{Var}(\hat{\beta}_0) = \text{Var}(\hat{\beta}_1)$

(C) $\text{Var}(\hat{\beta}_0) = \text{Cov}(\hat{Y}_i, \hat{\beta}_0)$ for all $i = 1, \ldots, 20$
(D) $\text{Var}(\hat{\beta}_1) = \text{Cov}(\hat{Y}_i, \hat{\beta}_1)$ for all $i = 1, \ldots, 20$

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40. Let $Y_1, Y_2, \ldots, Y_n$ be i.i.d. discrete random variables from a population with probability mass function

$$P(Y = y; \theta) = \begin{cases} \theta(1-\theta)^y & \textbf{if } y \in \mathbb{N} \cup \{0\}, \\ 0 & \textbf{otherwise}, \end{cases}$$

where $0 < \theta < 1$ is an unknown parameter. Assume the convention $0^0 = 1$. If $\hat{\theta}$ is the method of moments estimator of θ, then which of the following statements is/are correct?
(A) $\hat{\theta}$ is also the maximum likelihood estimator of θ
(B) $\hat{\theta}$ is an unbiased estimator of θ
(C) $\hat{\theta}$ is a consistent estimator of θ
(D) $1/\hat{\theta}$ is an unbiased estimator of $1/\theta$

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41. Let A be a 7×7 real matrix with $\text{rank}(A) = 1$. Suppose the trace of A^2 is 2025. Let the characteristic polynomial of A be written as $\sum_{n=0}^{7} a_n x^n$. Then $\sum_{n=0}^{7} |a_n|$ is ____________.
(answer in integer)

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42. The radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$ is equal to ____________.
(round off to 2 decimal places)

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43. Let $f(x) = x \sin\left(\frac{\pi}{2x}\right), x > 0$. Then

$$\lim_{h \to 0} \frac{1}{\pi^2 h^2} [3f(1) - 2f(1+h) - f(1-2h)]$$

is equal to ____________.
(round off to 2 decimal places)

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44. Let $\{X_i\}_{i \geq 1}$ be a sequence of i.i.d. random variables with $E(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2 < \infty$. Suppose c is a constant that does not depend on n such that

$$\frac{c}{n} \sum_{i=1}^{n} (X_{2i} - X_{2i-1})^2$$

is a consistent estimator of σ^2. Then c is equal to ____________.
(round off to 2 decimal places)

45. Let $X_1, X_2, \ldots, X_{10}$ be i.i.d. $\mathrm{U}(0,\theta)$ random variables, where $\theta > 0$ is unknown. For testing the null hypothesis $H_0 : \theta = 1$ against the alternative hypothesis $H_1 : \theta = 0.9$, consider a test that rejects H_0 if

$$X_{(10)} = \max\{X_1, X_2, \ldots, X_{10}\} < 0.8.$$

Then the probability of type I error of the test is equal to ___________.
(round off to 2 decimal places)

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46. Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_{100}, Y_{100})$ be i.i.d. discrete random vectors each having joint probability mass function

$$P(X = x, Y = y) = \frac{e^{-(1+x)\lambda}((1+x)\lambda)^y}{y!} p^x (1-p)^{1-x}, \quad x \in \{0,1\}, y \in \mathbb{N} \cup \{0\},$$

where $\lambda > 0$ and $0 < p < 1$ are unknown parameters. If the observed values of $\sum_{i=1}^{100} X_i$ and $\sum_{i=1}^{100} Y_i$ are 54 and 521 respectively, the maximum likelihood estimate of λ is equal to ___________.
(round off to 2 decimal places)

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47. Let X and Y be i.i.d. random variables with probability density function

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $Z = \min\{X, Y\}$, then $E(Z)$ is equal to ___________.
(answer in integer)

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48. The joint probability density function of the random vector (X, Y, Z) is given by

$$f(x,y,z) = \begin{cases} xy & \text{if } 0 < z < y < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then the value of $P(X > 5Y)$ is equal to ___________.
(round off to 2 decimal places)

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49. Suppose U_1 and U_2 are i.i.d. U(0,1) random variables. Further, let X be a Bin(2, 0.5) random variable that is independent of (U_1, U_2). Then

$$36\mathbb{P}(U_1 + U_2 > X)$$

is equal to ________________ (answer in integer)

50. A drawer contains 5 pairs of shoes of different sizes. Assume that all 10 shoes are distinguishable. A person selects 5 shoes from the drawer at random. Then the probability that there are exactly 2 complete pairs of shoes among these 5 shoes is equal to ________________ (round off to 2 decimal places)

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51. Let $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 12\}$ be a circle in the plane. Let (a, b) be the point on C which minimizes the distance to the point (1,2). Then $b - a$ is ________________ (round off to 2 decimal places)

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52. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3 + 2x^2 - 15x$ and $g(x) = x$ respectively. Let x_0 be the smallest strictly positive number such that $f(x_0) = 0$. Then the area of the region enclosed by the graphs of f and g between the lines $x = 0$ and $x = x_0$ is ________________

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53. Let V be the volume of the region

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + \frac{z^2}{4} \leq 1 \text{ and } |z| \leq 1\}.$$

Then $\frac{V}{\pi}$ is equal to ________________

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54. Suppose $X_1, X_2, \ldots, X_{10}, Y_1, Y_2, \ldots, Y_{10}$ are independent random variables, where $X_i \sim N(0, \sigma^2)$ and $Y_i \sim N(0, 3\sigma^2)$ for $i = 1, 2, \ldots, 10$. The observables are $D_1, \ldots, D_{10}$, where D_i denotes the Euclidean distance between the points $(X_i, Y_i, 0)$ and $(0, 0, 5)$ for $i = 1, 2, \ldots, 10$. If the observed value of $\sum_{i=1}^{10} D_i^2$ is equal to 1050, then the method of moments estimate of σ^2 is equal to ________________ (answer in integer)

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55. Consider a sequence of independent Bernoulli trials with success probability $p = \frac{1}{7}$. Then the expected number of trials required to get two consecutive successes for the first time is equal to ________________

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56. Let X be a real valued random variable with $E(X) = 1$, $E(X^2) = 4$, $E(X^4) = 16$. Then $E(X^3)$ is equal to ________________

57. Let X_1, X_2, X_3, X_4 be a random sample from a continuous distribution with probability density function

$$f(x;\theta) = \begin{cases} 2\theta^2 x^{-3} & \textbf{if } \theta < x < \infty, \\ 0 & \textbf{otherwise,} \end{cases}$$

where $\theta > 0$ is an unknown parameter. It is known that $X_{(1)} = \min\{X_1, X_2, X_3, X_4\}$ is a complete sufficient statistic for θ. If the observed values are $x_1 = 15, x_2 = 11, x_3 = 10, x_4 = 17$, the uniformly minimum variance unbiased estimate of θ^2 is equal to ________________

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58. Let X be a random variable with probability density function

$$f(x) = \begin{cases} 3x^2 & \textbf{if } 0 < x < 1, \\ 0 & \textbf{otherwise.} \end{cases}$$

Let δ denote the conditional expectation of X given that $X \leq \frac{1}{2}$. Then the value of 80δ is equal to ________________

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59. Let $X_1, X_2, \ldots, X_7$ be i.i.d. continuous random variables with median θ. If $X_{(1)} < X_{(2)} < \cdots < X_{(7)}$ are the corresponding order statistics, then $\mathbb{P}(X_{(2)} > \theta)$ is equal to ________________ **(round off to 3 decimal places).**

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60. Suppose (X, Y) has the $N_2(3, 0, 4, 1, 0.5)$ distribution. Then $4\text{Cov}(X + Y, Y^3)$ is equal to ________________
