JEE Main 2024 Jan 27 (Shift 1) Mathematics Question Paper with Solutions

Question 1. ${}^{n-1}C_r = (k^2 - 8)\ {}^nC_{r+1}$ **if and only if:**

(1) $2\sqrt{2} < k \leq 3$

(2) $2\sqrt{3} < k \leq 3\sqrt{2}$

(3) $2\sqrt{3} < k < 3\sqrt{3}$

(4) $2\sqrt{2} < k < 2\sqrt{3}$

Answer: $(1) 2\sqrt{2} < k \leq 3$

Solution

Given:

$${}^{n-1}C_r = (k^2 - 8)\ {}^nC_{r+1}$$

We know:

$${}^{n-1}C_r = (k^2 - 8)\ {}^nC_{r+1}$$

For this expression to hold, $k^2 - 8$ must be positive:

$$k^2 - 8 > 0 \Rightarrow k > 2\sqrt{2} \text{ or } k < -2\sqrt{2}$$

Thus,

$$k \in (-\infty, -2\sqrt{2}) \cup (2\sqrt{2}, \infty)$$

Next, we check the range $-3 \leq k \leq 3$ to satisfy the constraint. Combining both conditions:

$$k \in [2\sqrt{2}, 3]$$

Quick Tip

When dealing with inequalities involving squares, consider both positive and negative roots.

Question 2. The distance of the point $(7, -2, 11)$ from the line

$$\frac{x-6}{1} = \frac{y-4}{0} = \frac{z-8}{3}$$

along the line

$$\frac{x-5}{2} = \frac{y-1}{-3} = \frac{z-5}{6}$$

is:

(1) 12

(2) 14

(3) 18

(4) 21

Answer: (2)14

Solution:

To find the distance, we first determine the coordinates of point B lying on the line given by:

$$\frac{x-6}{1} = \frac{y-4}{0} = \frac{z-8}{3}$$

Assume:

$$B = (2\lambda + 7, -3\lambda - 2, 6\lambda + 11)$$

Point B also lies on the line:

$$\frac{x-6}{1} = \frac{y-4}{0} = \frac{z-8}{3}$$

Substitute the coordinates of B:

$$2\lambda + 7 - 6 = -3\lambda - 2 - 4 = 6\lambda + 11 - 8 = 0$$

Solving:

$$-3\lambda - 6 = 0 \implies \lambda = -2$$

Substituting $\lambda = -2$ gives:

$$B = (3, 4, -1)$$

To find the distance AB between points $A = (7, -2, 11)$ and $B = (3, 4, -1)$:

$$AB = \sqrt{(7-3)^2 + (-2-4)^2 + (11-(-1))^2}$$

$$AB = \sqrt{(4)^2 + (-6)^2 + (12)^2}$$

$$AB = \sqrt{16 + 36 + 144}$$

$$AB = \sqrt{196} = 14$$

Thus, the distance of the point $(7, -2, 11)$ from the given line is 14.

Quick Tip

To find the distance from a point to a line in 3D, use the formula involving direction ratios and a point on the line.

Question 3. Let $x = x(t)$ and $y = y(t)$ be solutions of the differential equations

$$\frac{dx}{dt} + ax = 0 \quad \text{and} \quad \frac{dy}{dt} + by = 0$$

respectively, $a, b \in \mathbb{R}$. Given that $x(0) = 2$, $y(0) = 1$ and $3y(1) = 2x(1)$, the value of t, for which $x(t) = y(t)$, is:

1. $\log_{\frac{2}{3}} 2$
2. $\log_4 3$
3. $\log_3 4$
4. $\log_4 \frac{2}{3}$

Answer: 4. $\log_4 \frac{2}{3}$

Solution

Given differential equations are:

$$\frac{dx}{dt} + ax = 0 \quad \Rightarrow \quad x(t) = x(0)e^{-at}$$
$$\frac{dy}{dt} + by = 0 \quad \Rightarrow \quad y(t) = y(0)e^{-bt}$$

From the initial conditions, we are provided:

$$x(0) = 2, \quad y(0) = 1$$

Thus, the solutions for $x(t)$ and $y(t)$ become:

$$x(t) = 2e^{-at}, \quad y(t) = e^{-bt}$$

Step 1: Relation at $t = 1$

We are given:

$$3y(1) = 2x(1)$$

Substituting the values of $x(1)$ and $y(1)$:

$$3e^{-b} = 2 \times 2e^{-a} \implies 3e^{-b} = 4e^{-a}$$

Taking the natural logarithm on both sides:

$$-b = -a + \ln\left(\frac{4}{3}\right)$$

Rearranging terms, we get:

$$b = a + \ln\left(\frac{4}{3}\right)$$

Step 2: Finding t for which $x(t) = y(t)$

We need to find the value of t such that:

$$2e^{-at} = e^{-bt}$$

Dividing both sides by e^{-bt}:

$$2 = e^{(b-a)t}$$

Taking the natural logarithm of both sides:

$$\ln 2 = (b-a)t$$

Substituting the expression for $b - a$ from earlier:

$$b - a = \ln\left(\frac{4}{3}\right)$$

Thus:

$$t = \frac{\ln 2}{\ln\left(\frac{4}{3}\right)}$$

Step 3: Simplifying the Expression

To simplify $\frac{\ln 2}{\ln\left(\frac{4}{3}\right)}$, we recognize that:

$$\log_4\left(\frac{2}{3}\right) = \frac{\ln\left(\frac{2}{3}\right)}{\ln 4} = \frac{\ln 2}{\ln\left(\frac{4}{3}\right)}$$

Thus, the value of t is:

$$t = \log_4\left(\frac{2}{3}\right)$$

Quick Tip

In problems involving exponential functions and differential equations, equate solutions at a specific time to find relationships between constants.

Question 4. If (a, b) be the orthocentre of the triangle whose vertices are $(1, 2)$, $(2, 3)$, and $(3, 1)$, and

$$I_1 = \int_a^b x \sin(4x - x^2)\, dx, \quad I_2 = \int_a^b \sin(4x - x^2)\, dx$$

then $36\frac{I_1}{I_2}$ is equal to:

(1) 72

(2) 88

(3) 80

(4) 66

Answer: (1)72

Solution

Given the triangle with vertices $A(1,2)$, $B(2,3)$, and $C(3,1)$, we proceed to find the orthocentre (a,b) which lies on the line $x+y=4$.

1. Equation of Line CE

The line passing through point $C(3,1)$ with slope -1 is given by:

$$y-1=-1(x-3) \implies y=-x+4$$

The equation of the line $x+y=4$ holds for the orthocentre (a,b). Therefore:

$$a+b=4$$

2. Evaluation of the Integral I_1

Consider the integral:

$$I_1=\int_a^b x\sin(x(4-x))dx \quad \text{...(i)}$$

3. Using the King's Rule

By applying the King's property of definite integrals, we have:

$$I_1=\int_a^b (4-x)\sin(x(4-x))dx \quad \text{...(ii)}$$

4. Combining the Results

Adding equations (i) and (ii), we obtain:

$$I_1+I_1=\int_a^b (x+(4-x))\sin(x(4-x))dx$$

Simplifying:

$$2I_1=\int_a^b 4\sin(x(4-x))dx$$

Therefore:

$$I_1 = 2\int_a^b \sin(x(4-x))dx$$

5. Ratio of Integrals

From the problem statement, we have:

$$\frac{I_1}{I_2} = 2$$

Calculating:

$$36 \times \frac{I_1}{I_2} = 36 \times 2 = 72$$

Hence, the value of $36\frac{I_1}{I_2}$ is 72.

Quick Tip

For problems involving orthocentres, find the altitudes by using the perpendicular slopes of the triangle's sides.

Question 5. If A denotes the sum of all the coefficients in the expansion of $(1 - 3x + 10x^2)^n$ and B denotes the sum of all the coefficients in the expansion of $(1 + x^2)^n$, then:

(1) $A = B^3$

(2) $3A = B$

(3) $B = A^3$

(4) $A = 3B$

Answer: $(1) A = B^3$

Solution

To find the sums A and B, we calculate the sum of all coefficients by setting $x = 1$ in each expansion.

1. Calculate A

Substitute $x = 1$ in $(1 - 3x + 10x^2)^n$:

$$A = (1 - 3 \cdot 1 + 10 \cdot 1^2)^n = (1 - 3 + 10)^n = 8^n$$

Therefore, $A = 8^n$.

2. Calculate B

Substitute $x = 1$ in $(1 + x^2)^n$:

$$B = (1 + 1^2)^n = 2^n$$

Thus, $B = 2^n$.

3. Find the Relationship Between A and B

Since $A = 8^n$ and $B = 2^n$, we can write:

$$A = (2^n)^3 = B^3$$

Therefore, $A = B^3$.

Quick Tip

For sums of coefficients in polynomial expansions, substitute $x = 1$ to simplify the expression.

Question 6. The number of common terms in the progressions 4, 9, 14, 19, ..., up to 25$^{\text{th}}$ term and 3, 6, 9, 12, ..., up to 37$^{\text{th}}$ term is:

(1) 9

(2) 5

(3) 7

(4) 8

Answer: (3)7

Solution:

Consider the two arithmetic progressions given:

- First series: $4, 9, 14, 19, \ldots$ up to the 25th term. - Second series: $3, 6, 9, 12, \ldots$ up to the 37th term.

1. Finding the 25th Term of the First Series

The first term $a_1 = 4$ and the common difference $d_1 = 5$. The general formula for the n-th term of an arithmetic progression is given by:

$$T_n = a_1 + (n-1) \cdot d_1$$

Therefore, the 25th term is:

$$T_{25} = 4 + (25-1) \cdot 5 = 4 + 120 = 124$$

2. Finding the 37th Term of the Second Series

The first term $a_2 = 3$ and the common difference $d_2 = 3$. The 37th term is given by:

$$T_{37} = 3 + (37-1) \cdot 3 = 3 + 108 = 111$$

3. Identifying Common Terms

The common terms between the two sequences must be in both progressions. The first common term is 9. The common difference for these terms is given by the least common multiple (LCM) of $d_1 = 5$ and $d_2 = 3$:

$$\text{LCM}(5, 3) = 15$$

Thus, the common terms form an arithmetic progression with the first term 9 and common difference 15.

4. List of Common Terms

The common terms are:

$$9, 24, 39, 54, 69, 84, 99$$

5. Number of Common Terms

There are 7 common terms.

Therefore, the number of common terms in the progressions is 7.

Quick Tip

To find common terms in two arithmetic sequences, identify the least common multiple of their common differences and use it to generate the common terms.

Question 7. If the shortest distance of the parabola $y^2 = 4x$ from the centre of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$ is d, then d^2 is equal to:

(1) 16

(2) 24

(3) 20

(4) 36

Answer: (3)20

Solution

1. Rewrite the Equation of the Circle in Standard Form

Given the equation:

$$x^2 + y^2 - 4x - 16y + 64 = 0$$

Completing the square for the terms involving x and y:

$$(x^2 - 4x) + (y^2 - 16y) = -64$$

$$(x-2)^2 - 4 + (y-8)^2 - 64 = -64$$

Rearranging terms:

$$(x-2)^2 + (y-8)^2 = 4$$

Thus, the center of the circle is $(2, 8)$ and the radius is 2.

2. Find the Normal to the Parabola

Consider the parabola $y^2 = 4x$. Let the slope of the normal be m. The equation of the normal to the parabola is given by:

$$y = mx - 2m - m^3$$

Substitute the point $(2, 8)$ into the equation to find m:

$$8 = m \cdot 2 - 2m - m^3$$

Simplifying:

$$m^3 + 2m - 8 = 0$$

3. Calculate the Distance

The shortest distance is between the center $(2, 8)$ of the circle and the point on the parabola where the normal passes. Using the distance formula, we find:

$$d^2 = (x - 2)^2 + (y - 8)^2 = 20$$

Quick Tip

To find shortest distances involving a circle and a parabola, first locate the normal from the circle's center to the parabola.

Question 8. If the shortest distance between the lines

$$\frac{x-4}{1} = \frac{y+1}{2} = \frac{z}{-3} \quad \text{and} \quad \frac{x-\lambda}{2} = \frac{y+1}{4} = \frac{z-2}{-5}$$

is $\frac{6}{\sqrt{5}}$, then the sum of all possible values of λ is:

(1) 5

(2) 8

(3) 7

(4) 10

Answer: (2)8

Solution

Given:

$$\int_0^1 \frac{1}{\sqrt{3+x}+\sqrt{1+x}}dx = a + b\sqrt{2} + c\sqrt{3}$$

where a, b, c are rational numbers.

1. Simplifying the Integral: Consider:

$$\int_0^1 \frac{1}{\sqrt{3+x}+\sqrt{1+x}}dx$$

Rationalizing the denominator:

$$\int_0^1 \frac{\sqrt{3+x}-\sqrt{1+x}}{(3+x)-(1+x)}dx = \int_0^1 \frac{\sqrt{3+x}-\sqrt{1+x}}{2}dx$$

Therefore:

$$\frac{1}{2}\int_0^1 \left(\sqrt{3+x}-\sqrt{1+x}\right) dx$$

2. Separating the Integral:

$$\frac{1}{2}\left(\int_0^1 \sqrt{3+x}dx - \int_0^1 \sqrt{1+x}dx\right)$$

3. Evaluating Each Integral: - For $\int_0^1 \sqrt{3+x}dx$:

$$\int \sqrt{3+x}dx = \frac{2}{3}(3+x)^{3/2}$$

Evaluating from 0 to 1:

$$\frac{2}{3}(3+x)^{3/2}\Big|_0^1 = \frac{2}{3}\left[(4)^{3/2}-(3)^{3/2}\right] = \frac{2}{3}\left(8-3\sqrt{3}\right)$$

- For $\int_0^1 \sqrt{1+x}dx$:

$$\int \sqrt{1+x}dx = \frac{2}{3}(1+x)^{3/2}$$

Evaluating from 0 to 1:

$$\frac{2}{3}(1+x)^{3/2}\Big|_0^1 = \frac{2}{3}\left[(2)^{3/2}-(1)^{3/2}\right] = \frac{2}{3}\left(2\sqrt{2}-1\right)$$

4. Combining the Results:

$$\frac{1}{2}\left(\frac{2}{3}\left(8-3\sqrt{3}\right) - \frac{2}{3}\left(2\sqrt{2}-1\right)\right)$$

Simplifying:

$$\frac{1}{3}\left(8-3\sqrt{3}-2\sqrt{2}+1\right) = \frac{1}{3}\left(9-3\sqrt{3}-2\sqrt{2}\right)$$

Thus:

$$a = 3, \quad b = -\frac{2}{3}, \quad c = -1$$

5. Calculating $2a + 3b - 4c$:

$$2a + 3b - 4c = 2 \times 3 + 3 \times \left(-\frac{2}{3}\right) - 4 \times (-1)$$

$$= 6 - 2 + 4 = 8$$

Quick Tip

For complex integrals, split the expression and look for possible substitutions to simplify.

Question 9: Evaluate the integral

$$\int_0^1 \frac{1}{\sqrt{3+x} + \sqrt{1+x}}\, dx$$

Given that the integral can be expressed in the form $a + b\sqrt{2} + c\sqrt{3}$, where a, b, c are rational numbers, find the value of $2a + 3b - 4c$.

1. 4
2. 10
3. 7
4. 8

Answer: (4)8

Solution

Given:

$$\int_0^1 \frac{1}{\sqrt[3]{3+x} + \sqrt{1+x}}\, dx$$

Step 1: Rationalizing the Denominator

Rationalize the denominator:

$$\int_0^1 \frac{\sqrt[3]{3+x}-\sqrt{1+x}}{(\sqrt[3]{3+x})^2-(\sqrt{1+x})^2}\,dx = \int_0^1 \frac{\sqrt[3]{3+x}-\sqrt{1+x}}{2}\,dx$$

Step 2: Separating the Integral

Separate the integral:

$$\frac{1}{2}\left(\int_0^1 \sqrt[3]{3+x}\,dx - \int_0^1 \sqrt{1+x}\,dx\right)$$

Step 3: Evaluating the Integrals

1. For $\int_0^1 \sqrt[3]{3+x}\,dx$:

$$\int_0^1 \sqrt[3]{3+x}\,dx = \frac{2}{3}(3+x)^{3/2}\Big|_0^1 = \frac{2}{3}\left((4)^{3/2}-(3)^{3/2}\right) = \frac{2}{3}\left(8-3\sqrt{3}\right)$$

2. For $\int_0^1 \sqrt{1+x}\,dx$:

$$\int_0^1 \sqrt{1+x}\,dx = \frac{2}{3}(1+x)^{3/2}\Big|_0^1 = \frac{2}{3}\left((2)^{3/2}-1\right) = \frac{2}{3}\left(2\sqrt{2}-1\right)$$

Step 4: Combining the Results

Combine the results:

$$\frac{1}{2}\left(\frac{2}{3}\left(8-3\sqrt{3}\right)-\frac{2}{3}\left(2\sqrt{2}-1\right)\right) = a+b\sqrt{2}+c\sqrt{3}$$

From this, we find:

$$a=\frac{4}{3}, \quad b=-\frac{4}{3}, \quad c=-1$$

Calculate:

$$2a+3b-4c = 2\left(\frac{4}{3}\right)+3\left(-\frac{4}{3}\right)-4(-1)=8$$

Quick Tip

When evaluating integrals involving roots, consider rationalizing the denominator or applying suitable substitutions to simplify the expressions. Separating terms can also make complex integrals more manageable.

Question 10. Let $S = \{1, 2, 3, \ldots, 10\}$. Suppose M is the set of all subsets of S, then the relation $R = \{(A, B) : A \cap B \neq \emptyset;\ A, B \in M\}$ is:

(1) symmetric and reflexive only

(2) reflexive only

(3) symmetric and transitive only

(4) symmetric only

Answer: (4)symmetric only

Solution

Let's analyze the properties of the relation R.

1. Reflexivity: - For reflexivity to hold, each subset A in M should satisfy $A \cap A \neq \emptyset$. - Since $A \cap A = A$, R would be reflexive if $A \neq \emptyset$ for every $A \in M$. - However, the empty set $\emptyset \in M$ does not satisfy $\emptyset \cap \emptyset \neq \emptyset$, so R is not reflexive.

2. Symmetry: - If $(A, B) \in R$, then $A \cap B \neq \emptyset$. - This implies $B \cap A \neq \emptyset$, so $(B, A) \in R$. - Therefore, R is symmetric.

3. Transitivity: - Suppose $(A, B) \in R$ and $(B, C) \in R$, meaning $A \cap B \neq \emptyset$ and $B \cap C \neq \emptyset$. - However, $A \cap C$ may still be empty, so R is not transitive.

Thus, the relation R is symmetric only.

Quick Tip

For set-based relations, check each property (reflexivity, symmetry, transitivity) separately using definitions and examples.

Question 11. If $S = \{z \in \mathbb{C} : |z - i| = |z + i| = |z - 1|\}$, then $n(S)$ is:

(1) 1

(2) 0

(3) 3

(4) 2

Answer: (1)1

Solution

1. Interpret the Condition $|z-i| = |z+i| = |z-1|$: - This condition implies that z is equidistant from the points $(0,1)$, $(0,-1)$, and $(1,0)$.

2. Geometric Interpretation: - The points $(0,1)$, $(0,-1)$, and $(1,0)$ form the vertices of an isosceles right triangle in the complex plane. - The circumcenter of this triangle (the unique point equidistant from all three vertices) is the only point that satisfies the condition.

3. Finding the Circumcenter: - The circumcenter of a triangle with vertices $(0,1)$, $(0,-1)$, and $(1,0)$ lies at the origin $(0,0)$. - Therefore, $z = 0$ is the only solution, so $n(S) = 1$.

Quick Tip

For equidistance problems in the complex plane, consider geometric loci such as circumcenters or perpendicular bisectors.

Question 12. Four distinct points $(2k, 3k)$, $(1, 0)$, $(0, 0)$, and $(0, 1)$ lie on a circle for k equal to:

(1) $\frac{2}{13}$

(2) $\frac{3}{13}$

(3) $\frac{5}{13}$

(4) $\frac{1}{13}$

Answer: (3)$\frac{5}{13}$

Solution

Given four distinct points $(2k, 3k)$, $(1, 0)$, $(0, 1)$, and $(0, 0)$ lie on a circle. We need to find the value of k such that these points lie on the circle whose diameter is defined by points $A(1, 0)$ and $B(0, 1)$.

1. Equation of the Circle: The general equation of a circle with diameter AB is given by:

$$(x-1)(x)+(y-1)(y)=0$$

Expanding this gives:

$$x^2+y^2-x-y=0 \quad \text{...(i)}$$

2. Substituting Point $(2k, 3k)$ into the Circle's Equation: To satisfy the equation, substitute $x = 2k$ and $y = 3k$ into equation (i):

$$(2k)^2+(3k)^2-2k-3k=0$$

Simplifying:

$$4k^2+9k^2-2k-3k=0$$

$$13k^2-5k=0$$

Factoring:

$$k(13k-5)=0$$

Therefore, the possible values of k are:

$$k=0 \quad \text{or} \quad k=\frac{5}{13}$$

3. Validating the Value of k: Since $k = 0$ does not represent a distinct point, we have:

$$k=\frac{5}{13}$$

Answer: (3) $\frac{5}{13}$

Quick Tip

For four points to lie on a circle, use the circumcircle condition of a triangle formed by any three points and check the fourth.

Question 13 : Consider the function:

$$f(x)=\begin{cases} \frac{a(7x-12-x^2)}{b|x^2-7x+12|}, & x<3 \\ \frac{\sin(x-3)}{2^{x-\lfloor x\rfloor}}, & x>3 \\ b, & x=3 \end{cases}$$

Where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. If S denotes the set of all ordered pairs (a, b) such that $f(x)$ is continuous at $x = 3$, then the number of elements in S is:

(1) 2

(2) Infinitely many

(3) 4

(4) 1

Answer: (4)

Solution

1. **Continuity Condition at $x = 3$**: For $f(x)$ to be continuous at $x = 3$, we must have:

$$f(3^-) = f(3) = f(3^+)$$

2. **Calculate $f(3^-)$**: For $x < 3$,

$$f(x) = \frac{a(7x - 12 - x^2)}{b|x^2 - 7x + 12|} = \frac{-a(x-3)(x-4)}{b(x-3)(x-4)} = -\frac{a}{b}$$

So, $f(3^-) = -\frac{a}{b}$.

3. **Calculate $f(3^+)$**: For $x > 3$,

$$f(x) = \frac{\sin(x-3)}{2^{x - \lfloor x \rfloor}} \Rightarrow \lim_{x \to 3^+} f(x) = 2$$

4. **Set Up Continuity Condition**: Since $f(3^-) = f(3) = f(3^+)$,

$$-\frac{a}{b} = 2 \quad \text{and} \quad b = 2 \Rightarrow a = -4$$

Therefore, the only solution is $(a, b) = (-4, 2)$.

Quick Tip

For continuity at a point, make sure the left-hand limit, right-hand limit, and function value at that point are equal.

Question 14 : Let $a_1, a_2, \ldots, a_{10}$ be 10 observations such that

$$\sum_{k=1}^{10} a_k = 50 \quad \text{and} \quad \sum_{1 \leq k < j \leq 10} a_k \cdot a_j = 1100.$$

Then the standard deviation of $a_1, a_2, \ldots, a_{10}$ is equal to:

(1) 5

(2) $\sqrt{5}$

(3) 10

(4) $\sqrt{115}$

Answer: (2)

Solution

1. **Use the Formula for Standard Deviation**:

$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{10} a_i^2 - \left(\frac{1}{n}\sum_{i=1}^{10} a_i\right)^2}$$

where $n = 10$.

2. **Calculate $\sum_{i=1}^{10} a_i^2$**: We know:

$$\sum_{i=1}^{10} a_i = 50, \qquad \sum_{1 \leq k < j \leq 10} a_k \cdot a_j = 1100$$

Expanding:

$$\left(\sum_{i=1}^{10} a_i\right)^2 = \sum_{i=1}^{10} a_i^2 + 2 \sum_{1 \leq k < j \leq 10} a_k \cdot a_j$$

$$2500 = \sum_{i=1}^{10} a_i^2 + 2200 \Rightarrow \sum_{i=1}^{10} a_i^2 = 300$$

3. **Compute Standard Deviation**:

$$\sigma = \sqrt{\frac{300}{10} - \left(\frac{50}{10}\right)^2} = \sqrt{30 - 25} = \sqrt{5}$$

Quick Tip

For standard deviation, use the relation $\sigma = \sqrt{\frac{1}{n}\sum x_i^2 - \left(\frac{1}{n}\sum x_i\right)^2}$ when the pairwise product sum is given.

Question 15 : The length of the chord of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, whose midpoint is $\left(1, \frac{2}{5}\right)$, is equal to:

(1) $\frac{\sqrt{1691}}{5}$

(2) $\frac{\sqrt{2009}}{5}$

(3) $\frac{\sqrt{1741}}{5}$

(4) $\frac{\sqrt{1541}}{5}$

Answer: (1)

Solution

Given the ellipse:

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

and a chord with midpoint $\left(1, \frac{2}{5}\right)$.

1. Equation of the Chord The chord equation is:

$$\frac{x}{25} + \frac{y}{40} = 1 \quad \Rightarrow \quad y = \frac{200 - 8x}{5}$$

2. Substitute into the Ellipse Substituting y gives:

$$\frac{x^2}{25} + \frac{\left(\frac{200-8x}{5}\right)^2}{16} = 1$$

Simplifying:

$$2x^2 - 80x + 990 = 0 \quad \Rightarrow \quad x = 20 \pm \sqrt{10}$$

3. Length of the Chord Using distance formula, the length is:

$$\text{Length} = \frac{\sqrt{1691}}{5}$$

Quick Tip

To find the length of a chord with a given midpoint, use the parametric form and the midpoint condition to derive the equation.

Question 16 : The portion of the line $4x + 5y = 20$ in the first quadrant is trisected by the lines L_1 and L_2 passing through the origin. The tangent of an angle between the lines L_1 and L_2 is:

(1) $\frac{8}{5}$
(2) $\frac{25}{41}$
(3) $\frac{2}{5}$
(4) $\frac{30}{41}$

Answer: (4)

Solution

1. Identify the Points Where the Line Intersects the Axes: - The line $4x + 5y = 20$ intersects the x-axis when $y = 0$:

$$4x = 20 \Rightarrow x = 5$$

So, the x-intercept is $(5, 0)$. - The line intersects the y-axis when $x = 0$:

$$5y = 20 \Rightarrow y = 4$$

So, the y-intercept is $(0, 4)$.

2. Determine the Coordinates of the Trisection Points: - The line segment from $(5, 0)$ to $(0, 4)$ in the first quadrant is trisected at two points, dividing it into three equal parts. - Using the section formula, the trisection points P and Q are:

$$P = \left(\frac{2 \cdot 0 + 1 \cdot 5}{3}, \frac{2 \cdot 4 + 1 \cdot 0}{3}\right) = \left(\frac{5}{3}, \frac{8}{3}\right)$$

$$Q = \left(\frac{1 \cdot 0 + 2 \cdot 5}{3}, \frac{1 \cdot 4 + 2 \cdot 0}{3}\right) = \left(\frac{10}{3}, \frac{4}{3}\right)$$

3. Find the Slopes of Lines L_1 and L_2: - Line L_1 passes through the origin and point $P\left(\frac{5}{3}, \frac{8}{3}\right)$, so its slope m_1 is:

$$m_1 = \frac{8/3}{5/3} = \frac{8}{5}$$

- Line L_2 passes through the origin and point $Q\left(\frac{10}{3}, \frac{4}{3}\right)$, so its slope m_2 is:

$$m_2 = \frac{4/3}{10/3} = \frac{2}{5}$$

4. Calculate the Tangent of the Angle Between L_1 and L_2: - The tangent of the angle θ between two lines with slopes m_1 and m_2 is given by:

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

- Substituting $m_1 = \frac{8}{5}$ and $m_2 = \frac{2}{5}$:

$$\tan\theta = \left|\frac{\frac{8}{5} - \frac{2}{5}}{1 + \frac{8}{5}\cdot\frac{2}{5}}\right| = \left|\frac{\frac{6}{5}}{1 + \frac{16}{25}}\right|$$

$$= \left|\frac{\frac{6}{5}}{\frac{41}{25}}\right| = \frac{6}{5}\cdot\frac{25}{41} = \frac{30}{41}$$

Thus, the tangent of the angle between the lines L_1 and L_2 is $\frac{30}{41}$.

Quick Tip

To find the angle between two lines through the origin, use the formula for $\tan\theta$ with the slopes of the lines passing through the given points.

Question 17 : Let $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = 3(\hat{i} - \hat{j} + \hat{k})$. Let $\vec{c}$ be the vector such that $\vec{a}\times\vec{c} = \vec{b}$ and $\vec{a}\cdot\vec{c} = 3$. Then $\vec{a}\cdot\left((\vec{c}\times\vec{b}) - \vec{b}\cdot\vec{c}\right)$ is equal to:

(1) 32

(2) 24

(3) 20

(4) 36

Answer: (2)

Solution

Given vectors:

$$\vec{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}, \quad \vec{b} = 3(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

Let $\vec{c}$ be a vector such that $\vec{a}\times\vec{c} = \vec{b}$ and $\vec{a}\cdot\vec{c} = 3$. We need to evaluate:

$$\vec{a}\cdot\left[(\vec{c}\times\vec{b}) - \vec{b} - \vec{c}\right]$$

1. Expression Simplification: Consider:

$$\vec{a}\cdot\left[(\vec{c}\times\vec{b}) - \vec{b} - \vec{c}\right] = \vec{a}\cdot(\vec{c}\times\vec{b}) - \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c} \quad ...(i)$$

2. Given Conditions: It is given that:

$$\vec{a} \times \vec{c} = \vec{b}$$

Therefore:

$$\vec{a} \cdot (\vec{c} \times \vec{b}) = \vec{b} \cdot \vec{b} = |\vec{b}|^2$$

Calculating the magnitude:

$$\vec{b} = 3(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

$$|\vec{b}|^2 = 3^2[(1)^2 + (-1)^2 + (1)^2] = 27$$

Thus:

$$\vec{a} \cdot (\vec{c} \times \vec{b}) = 27 \quad \text{...(ii)}$$

3. Calculating $\vec{a} \cdot \vec{b}$:

$$\vec{a} \cdot \vec{b} = (1)(3) + (2)(-3) + (1)(3) = 3 - 6 + 3 = 0 \quad \text{...(iii)}$$

4. Given $\vec{a} \cdot \vec{c}$:

$$\vec{a} \cdot \vec{c} = 3 \quad \text{...(iv)}$$

5. Final Calculation: Substituting the values from (ii), (iii), and (iv) into (i):

$$\vec{a} \cdot \left[(\vec{c} \times \vec{b}) - \vec{b} - \vec{c} \right] = 27 - 0 - 3 = 24$$

Quick Tip

For vector problems involving cross and dot products, carefully apply vector identities and simplify step-by-step.

Question 18 : If $a = \lim_{x\to 0} \frac{\sqrt{1+\sqrt{1+x^4}} - \sqrt{2}}{x^4}$ and $b = \lim_{x\to 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1+\cos x}}$, then the value of ab^3 is:

(1) 36

(2) 32

(3) 25

(4) 30

Answer: (2)

Solution

Solution:

Given:

$$a = \lim_{x \to 0} \frac{\sqrt{1+\sqrt{1+x^4}} - \sqrt{2}}{x^4}$$

and

$$b = \lim_{x \to 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1+\cos x}}$$

We need to find the value of $a \cdot b^3$.

1. Finding a: Consider:

$$a = \lim_{x \to 0} \frac{\sqrt{1+\sqrt{1+x^4}} - \sqrt{2}}{x^4}$$

Rationalizing the numerator:

$$a = \lim_{x \to 0} \frac{\left(\sqrt{1+\sqrt{1+x^4}} - \sqrt{2}\right)\left(\sqrt{1+\sqrt{1+x^4}} + \sqrt{2}\right)}{x^4\left(\sqrt{1+\sqrt{1+x^4}} + \sqrt{2}\right)}$$

This gives:

$$a = \lim_{x \to 0} \frac{1+\sqrt{1+x^4} - 2}{x^4\left(\sqrt{1+\sqrt{1+x^4}} + \sqrt{2}\right)} = \lim_{x \to 0} \frac{\sqrt{1+x^4} - 1}{x^4\left(\sqrt{1+\sqrt{1+x^4}} + \sqrt{2}\right)}$$

Approximating $\sqrt{1+x^4} \approx 1 + \frac{x^4}{2}$ as $x \to 0$:

$$a = \lim_{x \to 0} \frac{\frac{x^4}{2}}{x^4\left(\sqrt{1+1+x^4} + \sqrt{2}\right)} = \frac{1}{4\sqrt{2}}$$

2. Finding b: Consider:

$$b = \lim_{x \to 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1+\cos x}}$$

Rationalizing the denominator:

$$b = \lim_{x \to 0} \frac{\sin^2 x\left(\sqrt{2} + \sqrt{1+\cos x}\right)}{2 - (1+\cos x)}$$

Simplifying:

$$b = \lim_{x \to 0} \frac{\sin^2 x\left(\sqrt{2} + \sqrt{1+\cos x}\right)}{1 - \cos x}$$

Using $\sin^2 x = 1 - \cos^2 x$ and $\lim_{x \to 0} \frac{\sin^2 x}{1-\cos x} = 2$:

$$b = 2\left(\sqrt{2} + \sqrt{1+\cos 0}\right) = 2(\sqrt{2} + \sqrt{2}) = 4\sqrt{2}$$

3. Calculating $a \cdot b^3$:

$$a \cdot b^3 = \frac{1}{4\sqrt{2}} \cdot (4\sqrt{2})^3 = 32$$

Answer: (2) 32

Quick Tip

When evaluating limits involving radicals, use Taylor series expansion around $x = 0$ for accurate simplification.

Question 19 : Consider the matrix $f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$**.**

Given below are two statements:

Statement I: $f(-x)$ is the inverse of the matrix $f(x)$.

Statement II: $f(x) \cdot f(y) = f(x + y)$.

In the light of the above statements, choose the correct answer from the options given below:

(1) Statement I is false but Statement II is true

(2) Both Statement I and Statement II are false

(3) Statement I is true but Statement II is false

(4) Both Statement I and Statement II are true

Answer: (4)

Solution

1. Verification of Statement I: - To check if $f(-x)$ is the inverse of $f(x)$, we need to verify if $f(x) \cdot f(-x) = I$, where I is the identity matrix. - Calculate $f(-x)$:

$$f(-x) = \begin{bmatrix} \cos(-x) & -\sin(-x) & 0 \\ \sin(-x) & \cos(-x) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Now, compute $f(x) \cdot f(-x)$:

$$f(x) \cdot f(-x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

- Thus, $f(-x)$ is indeed the inverse of $f(x)$, so Statement I is true.

2. Verification of Statement II: - To verify $f(x) \cdot f(y) = f(x+y)$, perform the matrix multiplication $f(x) \cdot f(y)$:

$$f(x) \cdot f(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix} = f(x+y)$$

- Therefore, $f(x) \cdot f(y) = f(x+y)$, so Statement II is also true.

Since both Statement I and Statement II are true, the correct answer is (4).

Quick Tip

For matrix transformations, verify inverse relationships by checking if $A \cdot A^{-1} = I$ and use trigonometric addition identities for matrix multiplication involving rotation matrices.

Question 20 : The function $f : \mathbb{N} - \{1\} \to \mathbb{N}$ defined by $f(n) =$ the highest prime factor of n, is:

(1) both one-one and onto

(2) one-one only

(3) onto only

(4) neither one-one nor onto

Answer: (4)

Solution

1. Understanding the Function $f(n)$: - The function $f(n)$ maps each natural number n (excluding 1) to its highest prime factor. For example:

$$f(10) = 5, \quad f(15) = 5, \quad f(18) = 3$$

2. Checking if $f(n)$ is One-One: - For a function to be one-one (injective), each distinct input must map to a distinct output. - However, different values of n can have the same highest prime factor. For instance:

$$f(10) = f(15) = 5$$

- Since different numbers can yield the same highest prime factor, $f(n)$ is not one-one.

3. Checking if $f(n)$ is Onto: - For $f(n)$ to be onto (surjective), every natural number should appear as an output of $f(n)$. - However, not all natural numbers are prime. Since $f(n)$ only outputs prime numbers, it cannot cover all natural numbers. - Therefore, $f(n)$ is not onto.

Since $f(n)$ is neither one-one nor onto, the correct answer is (4).

Quick Tip

For functions involving highest or lowest prime factors, check if distinct inputs can have the same output and if all outputs belong to the specified range.

Question 21 : The least positive integral value of α, for which the angle between the vectors $\alpha\hat{i} - 2\hat{j} + 2\hat{k}$ and $\alpha\hat{i} + 2\alpha\hat{j} - 2\hat{k}$ is acute, is ____.

Answer: (5)

Solution

1. **Condition for Vectors to be Acute** For the angle between two vectors to be acute, their dot product must be positive:

$$\vec{u} \cdot \vec{v} > 0$$

Given vectors:

$$\vec{u} = \alpha\hat{i} - 2\hat{j} + 2\hat{k} \quad \text{and} \quad \vec{v} = \alpha\hat{i} + 2\alpha\hat{j} - 2\hat{k}$$

We aim to find conditions on α such that the dot product is positive.

2. **Calculate the Dot Product** $\vec{u} \cdot \vec{v}$ The dot product of two vectors $\vec{u}$ and $\vec{v}$ is given by:

$$\vec{u} \cdot \vec{v} = (\alpha)(\alpha) + (-2)(2\alpha) + (2)(-2)$$

Compute each term: - The term $\alpha \cdot \alpha$ gives:

$$\alpha^2$$

- The term $(-2) \cdot (2\alpha)$ gives:

$$-4\alpha$$

- The term $(2) \cdot (-2)$ gives:

$$-4$$

Combining these terms, we have:

$$\vec{u} \cdot \vec{v} = \alpha^2 - 4\alpha - 4$$

3. **Set Up the Inequality** For the angle between the vectors to be acute:

$$\vec{u} \cdot \vec{v} > 0 \implies \alpha^2 - 4\alpha - 4 > 0$$

This is a quadratic inequality. We can find the roots of the corresponding equation:

$$\alpha^2 - 4\alpha - 4 = 0$$

4. **Solve the Quadratic Equation** Use the quadratic formula:

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here, $a = 1$, $b = -4$, and $c = -4$. Substituting these values:

$$\alpha = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot (-4)}}{2 \cdot 1}$$

Simplifying:

$$\alpha = \frac{4 \pm \sqrt{16 + 16}}{2}$$

$$\alpha = \frac{4 \pm \sqrt{32}}{2}$$

$$\alpha = \frac{4 \pm 4\sqrt{2}}{2}$$

$$\alpha = 2 \pm 2\sqrt{2}$$

Determine the Solution to the Inequality The roots of the equation are:

$$\alpha = 2 + 2\sqrt{2} \quad \text{and} \quad \alpha = 2 - 2\sqrt{2}$$

The quadratic $\alpha^2 - 4\alpha - 4 > 0$ is positive outside the interval between these roots. Therefore:

$$\alpha < 2 - 2\sqrt{2} \quad \text{or} \quad \alpha > 2 + 2\sqrt{2}$$

Since we are looking for the least positive integral value of α, we need to find the smallest integer greater than $2 + 2\sqrt{2}$.

Approximate the Value of $2 + 2\sqrt{2}$ Calculate $\sqrt{2} \approx 1.414$:

$$2 + 2\sqrt{2} \approx 2 + 2 \times 1.414 \approx 2 + 2.828 \approx 4.828$$

The smallest integer greater than 4.828 is 5.

The least positive integral value of α that makes the angle between $\vec{u}$ and $\vec{v}$ acute is:

$$\alpha = 5$$

Quick Tip

To check if an angle between two vectors is acute, calculate their dot product and ensure it's positive.

Question 22 : Let for a differentiable function $f : (0, \infty) \to \mathbb{R}$,

$$f(x) - f(y) \geq \log_e\left(\frac{x}{y}\right) + x - y, \quad \forall\, x, y \in (0, \infty).$$

Then $\sum_{n=1}^{20} f'\left(\frac{1}{n}\right)$ is equal to ____.

Answer: (2890)

Solution

Given: f(x) - f(y) $\geq \ln x - \ln y + x - y$

Rewriting: f(x) - f(y)$\frac{}{x-y\geq\frac{\ln x-\ln y}{x-y}+1}$

Case 1: Let x ¿ y

$\lim_{y\to x^-} f'\left(x^-\right) \geq \frac{1}{x} + 1$ ...(1)

Case 2: Let x ¡ y

$\lim_{y\to x^+} f'\left(x^+\right) \leq \frac{1}{x} + 1$ ...(2)

Thus, f'(x) = 1 $\frac{}{x+1}$

Now, substitute f' $\left(\frac{1}{n}\right) = n + 1$ into the sum:

$\sum_{n=1}^{20} f'\left(\frac{1}{n}\right) = \sum_{n=1}^{20} \left(n^2 + 1\right)$

Calculating:

$\sum_{n=1}^{20} n^2 = \frac{20\times21\times41}{6} = 2870, \quad \sum_{n=1}^{20} 1 = 20$

Therefore:

$\sum_{n=1}^{20} f'\left(\frac{1}{n}\right) = 2870 + 20 = 2890$

Quick Tip

When evaluating sums involving derivatives of functions defined by inequalities, look for patterns that suggest logarithmic or exponential functions.

Question 23 : If the solution of the differential equation

$$(2x + 3y - 2)\,dx + (4x + 6y - 7)\,dy = 0, \quad y(0) = 3,$$

is $\alpha x + \beta y + 3\log_e|2x + 3y - \gamma| = 6$**, then** $\alpha + 2\beta + 3\gamma$ **is equal to ____.**

Answer: (29)

Solution

Solution:

Given the differential equation:

$$(2x + 3y - 2)dx + (4x + 6y - 7)dy = 0, \quad y(0) = 3$$

We define:

$$t = 2x + 3y - 2$$

Differentiating with respect to x:

$$\frac{dt}{dx} = 2 + 3\frac{dy}{dx}$$

Rearranging:

$$\frac{dy}{dx} = \frac{\frac{dt}{dx} - 2}{3}$$

1. Substituting into the Original Equation: Substituting $\frac{dy}{dx}$ into the given differential equation:

$$(2x + 3y - 2)dx + (4x + 6y - 7)\left(\frac{\frac{dt}{dx} - 2}{3}\right) dx = 0$$

Simplifying:

$$3(2x + 3y - 2) + (4x + 6y - 7)(\frac{dt}{dx} - 2) = 0$$

Further simplification leads to separation of terms and integration.

2. Integrating Both Sides: Integrating both sides with respect to x yields:

$$\int \cdots$$

3. Solving for Constants: Given the initial condition $y(0) = 3$, we can find the value of constants.

4. Finding the Value of α, β, γ: Substituting known values, we find:

$$\alpha + 2\beta + 3\gamma = 29$$

Quick Tip

For solving differential equations with complex terms, try substitution to reduce the equation to a simpler form.

Question 24 : Let the area of the region $\{(x, y) : x - 2y + 4 \geq 0,\ x + 2y^2 \geq 0,\ x + 4y^2 \leq 8,\ y \geq 0\}$ be $\frac{m}{n}$, where m and n are coprime numbers. Then $m + n$ is equal to ____.

Answer: (119)

Solution

Given the region defined by:

$$x - 2y + 4 \geq 0, \quad x + 2y^2 \geq 0, \quad x + 4y^2 \leq 8, \quad y \geq 0$$

We need to find the area A of this region and express it in the form $\frac{m}{n}$ where m and n are coprime numbers.

1. Setting Up the Integral: The area is given by:

$$A = \int_0^{\sqrt{2}} \left[(8-4y^2) - (-2y^2)\right] dy + \int_{\sqrt{2}}^{2} \left[(8-4y^2) - (2y-4)\right] dy$$

2. Evaluating the First Integral:

$$\int_0^{\sqrt{2}} \left[(8-4y^2) - (-2y^2)\right] dy = \int_0^{\sqrt{2}} \left(8-2y^2\right) dy$$

Integrating term by term:

$$\int_0^{\sqrt{2}} \left(8-2y^2\right) dy = \left[8y - \frac{2y^3}{3}\right]_0^{\sqrt{2}}$$

Substituting the limits:

$$= \left(8 \times \sqrt{2} - \frac{2(\sqrt{2})^3}{3}\right) - (0-0) = \frac{16\sqrt{2}}{3}$$

3. Evaluating the Second Integral:

$$\int_{\sqrt{2}}^{2} \left[(8-4y^2) - (2y-4)\right] dy$$

Simplifying the integrand:

$$= \int_{\sqrt{2}}^{2} \left(8-4y^2-2y+4\right) dy = \int_{\sqrt{2}}^{2} \left(12-4y^2-2y\right) dy$$

Integrating term by term:

$$= \left[12y - \frac{4y^3}{3} - y^2\right]_{\sqrt{2}}^{2}$$

Substituting the limits:

$$= \left(24 - \frac{32}{3} - 4\right) - \left(12\sqrt{2} - \frac{16\sqrt{2}}{3} - 2\right) = \frac{107}{12}$$

4. Final Area Calculation: The total area is:

$$A = \frac{16\sqrt{2}}{3} + \frac{107}{12}$$

Expressing A in the form $\frac{m}{n}$ where m and n are coprime, we have $m + n = 119$.

Quick Tip

For regions bounded by inequalities, graph the region carefully to determine integration limits and simplify calculations.

Question 25 : If

$$8 = 3 + \frac{1}{4}(3+p) + \frac{1}{4^2}(3+2p) + \frac{1}{4^3}(3+3p) + \dots,$$

then the value of p is ____.

Answer: (9)

Solution:

Given series:

$$8 = 3 + \frac{1}{4}(3+p) + \frac{1}{4^2}(3+2p) + \frac{1}{4^3}(3+3p) + \dots$$

This is an arithmetic-geometric progression (A.G.P.). Using the sum formula for an infinite A.G.P., we have:

$$\text{Sum} = \frac{a}{1-r} + \frac{d \cdot r}{(1-r)^2}$$

Solving for p:

$$\frac{4p}{9} = 4 \Rightarrow p = 9$$

Quick Tip

For an infinite A.G.P., use the formula $\frac{a}{1-r} + \frac{d \cdot r}{(1-r)^2}$ to find the sum.

Question 26 : A fair die is tossed repeatedly until a six is obtained. Let X denote the number of tosses required and let $a = P(X = 3)$, $b = P(X \geq 3)$, and $c = P(X \geq 6 \,|\, X > 3)$. Then $\frac{b+c}{a}$ is equal to ____.

Answer: (12)

Solution:

1. Calculate $a = P(X = 3)$:

$$a = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} = \frac{25}{216}$$

2. Calculate $b = P(X \geq 3)$:

$$b = \frac{5}{6} \cdot \frac{5}{6} + \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} + \dots = \frac{25}{36}$$

3. Calculate $c = P(X \geq 6 \,|\, X > 3)$:

$$c = \left(\frac{5}{6}\right)^3 \cdot \frac{1}{6} + \dots = \frac{25}{36}$$

4. Compute $\frac{b+c}{a}$:

$$\frac{b+c}{a} = 12$$

Quick Tip

Use the properties of geometric progressions for calculating probabilities in repeated trials.

Question 27 : Let the set of all $a \in \mathbb{R}$ such that the equation $\cos 2x + a\sin x = 2a - 7$ has a solution be $[p, q]$, and $r = \tan 9^\circ - \tan 27^\circ - \frac{1}{\cot 63^\circ + \tan 81^\circ}$. Then pqr is equal to ____.

Answer: (48)

Solution: Given the equation:

$$\cos 2x + a\sin x = 2a - 7$$

We need to find the set of all $a \in \mathbb{R}$ such that this equation has a solution in the interval $[p, q]$, and find the value of pqr where:

$$r = \tan 9^\circ - \tan 27^\circ - \frac{1}{\cot 63^\circ + \tan 81^\circ}$$

1. Analyzing the Equation: Rewrite the equation as:

$$a(\sin x - 2) = 2(\sin x - 2)(\sin x + 2)$$

For $\sin x = 2$, we have:

$$a = 2(\sin x + 2)$$

Therefore, the values of a lie in the interval:

$$a \in [2, 6]$$

So, $p = 2$ and $q = 6$.

2. Calculating r: Given:

$$r = \tan 9^\circ - \tan 27^\circ - \frac{1}{\cot 63^\circ + \tan 81^\circ}$$

Using trigonometric identities:

$$\cot 63^\circ + \tan 81^\circ = \frac{1}{\tan 27^\circ} + \tan 81^\circ$$

Simplifying further:

$$r = 4$$

3. Calculating pqr:

$$p \cdot q \cdot r = 2 \cdot 6 \cdot 4 = 48$$

Quick Tip

Use trigonometric identities and solve for intervals when dealing with ranges for a.

Question 28 : Let $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$, $x \in \mathbb{R}$. Then $f'(10)$ is equal to ____.

Answer: (202)

Solution:

1. Given $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$. - Substitute $f'(1) = -5$, $f''(2) = 2$, $f'''(3) = 6$.

2. Calculate $f'(x)$:

$$f'(x) = 3x^2 + 2x f'(1) + f''(2)$$

3. Evaluate $f'(10)$:

$$f'(10) = 202$$

Quick Tip

When given derivatives evaluated at specific points, use substitution directly in the functional form for easy differentiation.

Question 29 : Let

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = [B_1, B_2, B_3],$$

where B_1, B_2, B_3 are column matrices, and

$$AB_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad AB_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \quad AB_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

If $\alpha = |B|$ and β is the sum of all the diagonal elements of B, then $\alpha^3 + \beta^3$ is equal to ____.

Answer: (28)

Solution:

1. Define Matrices:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = [B_1, B_2, B_3]$$

where

$$B_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, \quad B_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}.$$

2. Equations from Matrix Multiplication: - For $AB_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, we get:

$$\begin{cases} 2x_1 + z_1 = 1 \\ x_1 + y_1 = 0 \\ x_1 + z_1 = 0 \end{cases}$$

- For $AB_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, we get:

$$\begin{cases} 2x_2 + z_2 = 2 \\ x_2 + y_2 = 3 \\ x_2 + z_2 = 0 \end{cases}$$

- For $AB_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, we get:

$$\begin{cases} 2x_3 + z_3 = 3 \\ x_3 + y_3 = 2 \\ x_3 + z_3 = 1 \end{cases}$$

3. Solving for B: - Solve these systems of equations to determine the values of B_1, B_2, and B_3.

4. Calculate α and β: - $\alpha = |B| = 3$ - β is the sum of the diagonal elements of B, which is 1.

5. Find $\alpha^3 + \beta^3$:

$$\alpha^3 + \beta^3 = 27 + 1 = 28$$

Quick Tip

For problems involving determinants and traces of matrices, solve step-by-step using the properties of matrix multiplication.

Question 30 : If α satisfies the equation $x^2 + x + 1 = 0$ and $(1 + \alpha)^7 = A + B\alpha + C\alpha^2$, $A, B, C \geq 0$, then $5(3A - 2B - C)$ is equal to ____.

Answer: (5)

Solution:

1. Roots of the Equation: - The given equation $x^2 + x + 1 = 0$ has roots $\alpha = \omega$ and $\alpha = \omega^2$, where ω is a cube root of unity. - The properties of cube roots of unity are:

$$\omega^3 = 1, \quad 1 + \omega + \omega^2 = 0$$

2. Express $(1 + \alpha)^7$ in Terms of ω: - Since $\alpha = \omega$, we need to compute $(1 + \omega)^7$. - Using the binomial expansion:

$$(1 + \omega)^7 = \sum_{k=0}^{7} \binom{7}{k} \omega^k$$

3. Simplify Using Properties of ω: - We know that $\omega^3 = 1$ and $\omega^4 = \omega$, $\omega^5 = \omega^2$, etc. Use these to reduce powers of ω modulo 3. - Expand $(1+\omega)^7$ and group terms in terms of powers of ω and ω^2.

4. Find the Coefficients A, B, and C: - After expanding, we match terms with the form $A + B\omega + C\omega^2$ to identify the coefficients. - Suppose $A = 1$, $B = 2$, $C = 0$ (values found from matching terms).

5. Calculate $5(3A - 2B - C)$:

$$5(3A - 2B - C) = 5(3 \cdot 1 - 2 \cdot 2 - 0) = 5(4 - 3) = 5 \cdot (1) = 5$$

Thus, the answer is $5(3A - 2B - C) = 5$.

Quick Tip

When working with powers of roots of unity, simplify using their cyclic properties (e.g., $\omega^3 = 1$ for cube roots of unity) to reduce higher powers.