JEE Main 2024 Mathematics Question Paper April 4 Shift 1 with Solutions

Time Allowed :3 Hours	Maximum Marks :300	Total Questions :90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions, out of which 75 are to attempted. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
   (i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and –1 mark for wrong answer.
   (ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer

Mathematics

1. If $f(x) = \begin{cases} x-2, & 0 \le x \le 2 \\ -2, & -2 \le x \le 0 \end{cases}$ **and** $h(x) = f(|x|) + |f(x)|$**, then** $\int_0^k h(x)\,dx$ **is equal to ($k > 0$)**

(A) 0
(B) $\frac{k}{2}$
(C) $2k$
(D) k

Correct Answer: (A) 0

Solution:

Step 1: Find $f(|x|)$ for $x > 0$.
For $x > 0$, $|x| = x$, and $0 \le x \le 2$, so:
$f(|x|) = x - 2$

Step 2: Find $|f(x)|$ for $x > 0$.
For $x > 0$, $f(x) = x - 2$. Now:
If $0 < x < 2$, then $x - 2 < 0$ so $|f(x)| = 2 - x$.

Step 3: Compute $h(x) = f(|x|) + |f(x)|$.
For $0 < x < 2$:
$h(x) = (x - 2) + (2 - x) = 0$

Thus, for all x in $[0, k]$, where $k > 0$:
$h(x) = 0$

Step 4: Evaluate the integral.

$$\int_0^k h(x)\,dx = \int_0^k 0\,dx = 0$$

Quick Tip

Whenever expressions involve $|x|$ and piecewise functions, always break the function into cases for $x > 0$ and $x < 0$ before integrating.

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2. There are three bags A, B and C. Bag A contains 7 black balls and 5 red balls, Bag B contains 5 red and 7 black balls and Bag C contains 7 red and 7 black balls. A ball is drawn and found to be black. Find the probability that it is drawn from Bag A.

Correct Answer: $\frac{7}{18}$

Solution:

Step 1: Probability of choosing each bag.
Each bag is equally likely, so:
$P(A) = P(B) = P(C) = \frac{1}{3}$

Step 2: Probability of drawing a black ball from each bag.
Bag A: $P(B|A) = \frac{7}{12}$
Bag B: $P(B|B) = \frac{7}{12}$
Bag C: $P(B|C) = \frac{7}{14} = \frac{1}{2}$

Step 3: Apply Bayes' Theorem.

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|B)P(B) + P(B|C)P(C)}$$

$$= \frac{\frac{7}{12} \cdot \frac{1}{3}}{\frac{7}{12} \cdot \frac{1}{3} + \frac{7}{12} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3}}$$

$$= \frac{\frac{7}{36}}{\frac{7}{36} + \frac{7}{36} + \frac{6}{36}} = \frac{7}{18}$$

Step 4: Conclusion.
Thus, the probability the ball came from Bag A is:

$$\boxed{\frac{7}{18}}$$

Quick Tip

Whenever the outcome is known, use Bayes' theorem to trace back the source of that outcome.

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3. Find the number of rational numbers in the expansion of $\left(\frac{1}{2^5} + \frac{1}{5^3}\right)^{15}$.

Correct Answer: 2

Solution:

Step 1: Write the general term.

$$T_{r+1} = \binom{15}{r} \left(2^{-5}\right)^{15-r} \left(5^{-3}\right)^{r}$$

Step 2: Simplify powers of 2 and 5.

$$= \binom{15}{r} \cdot 2^{-5(15-r)} \cdot 5^{-3r}$$

$$= \binom{15}{r} \cdot 2^{-75+5r} \cdot 5^{-3r}$$

Step 3: For the term to be rational.
Both exponents must be integers ≥ 0:

$$-75 + 5r = 0 \Rightarrow r = 15$$

$$-3r = 0 \Rightarrow r = 0$$

Step 4: Conclusion.
Valid values of r: 0 and 15.
Hence number of rational terms = 2.

$$\boxed{2}$$

> **Quick Tip**
>
> To find rational terms in expansions, ensure the exponents of all primes in the denominator become non-negative integers.

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4. Find the value of $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1 + \sin x \cos x}\,dx$

Correct Answer: $\frac{\pi}{3\sqrt{3}}$

Solution:

Step 1: Rewrite using identity.

$$I = \int_0^{\pi/2} \frac{\sin^2 x}{1 + \sin x \cos x}\,dx$$
$$= \int_0^{\pi/2} \frac{\cos^2 x}{1 + \sin x \cos x}\,dx$$

Step 2: Use $\sin 2x = 2\sin x \cos x$.

$$2I = \int_0^{\pi/2} \frac{2\,dx}{2 + \sin 2x}$$

Step 3: Use substitution $t = \tan x$.

$$I = \int_0^{\pi/2} \frac{dx}{2 + \frac{2\tan x}{1+\tan^2 x}}$$

$$2I = \int_0^{\infty} \frac{\sec^2 x\, dx}{\tan^2 x + \tan x + 1}$$

$$2I = \int_0^{\infty} \frac{dt}{t^2 + t + 1}$$

Step 4: Complete the square.

$$t^2 + t + 1 = \left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

Thus,

$$2I = \int_0^{\infty} \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

Step 5: Use standard formula $\displaystyle\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$.

$$2I = \frac{1}{\sqrt{3}/2} \left[\tan^{-1}\left(\frac{t + 1/2}{\sqrt{3}/2}\right)\right]_0^{\infty}$$

$$= \frac{2}{\sqrt{3}} \left(\frac{\pi}{2} - \frac{\pi}{6}\right)$$

Step 6: Final value.

$$2I = \frac{2}{\sqrt{3}} \cdot \frac{\pi}{3}$$

$$I = \frac{\pi}{3\sqrt{3}}$$

$$\boxed{\frac{\pi}{3\sqrt{3}}}$$

Quick Tip

Always use $t = \tan x$ for integrals involving $\sin x \cos x$ or $\sin 2x$. Completing the square often reduces tricky denominators to standard arctan forms.

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5. If $x^2 - ax + b = 0$ has roots $2, 6$ and $\alpha = \frac{1}{2a+1}$, $\beta = \frac{1}{2b-a}$. Find the equation having roots α, β.

Correct Answer: $27x^2 - 33x + 1 = 0$

Solution:

Step 1: Use sum and product of roots.
Given roots are 2 and 6.
So, $a = 2 + 6 = 8$,
$b = 2 \cdot 6 = 12$.

Step 2: Compute the new roots.

$$\alpha = \frac{1}{2a+1} = \frac{1}{17}, \qquad \beta = \frac{1}{2b-a} = \frac{1}{16}$$

Step 3: Form required equation.
Equation with roots α and β is:

$$x^2 - (\alpha + \beta)x + \alpha\beta$$

$$\alpha + \beta = \frac{1}{17} + \frac{1}{16} = \frac{33}{272}$$

$$\alpha\beta = \frac{1}{17 \cdot 16} = \frac{1}{272}$$

Multiply entire equation by 272:

$$272x^2 - 33x + 1 = 0$$

Thus, the equation is:

$$\boxed{27x^2 - 33x + 1 = 0}$$

> **Quick Tip**
>
> When finding a new quadratic from transformed roots, always calculate sum and product of new roots, then multiply to clear denominators.

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6. $\lim_{x \to 4} \dfrac{(5+x)^{1/3} - (1+2x)^{1/3}}{(5+x)^{1/2} - (1+2x)^{1/2}}$

Correct Answer: $\dfrac{2 \times 9^{2/3}}{9}$

Solution:

Step 1: Substitute $x = 4 + h$ and let $h \to 0$.

Then,

$$(5+x) = 9+h, \quad (1+2x) = 9+2h$$

$$\lim_{h\to 0} \frac{(9+h)^{1/3} - (9+2h)^{1/3}}{(9+h)^{1/2} - (9+2h)^{1/2}}$$

Step 2: Use expansion for small h:

$$(9+h)^{1/3} = 9^{1/3} + \frac{h}{3\cdot 9^{2/3}}, \quad (9+2h)^{1/3} = 9^{1/3} + \frac{2h}{3\cdot 9^{2/3}}$$

$$(9+h)^{1/2} = 3 + \frac{h}{6}, \quad (9+2h)^{1/2} = 3 + \frac{2h}{6}$$

Step 3: Take the difference.

Numerator:

$$\frac{h}{3\cdot 9^{2/3}} - \frac{2h}{3\cdot 9^{2/3}} = -\frac{h}{3\cdot 9^{2/3}}$$

Denominator:

$$\frac{h}{6} - \frac{2h}{6} = -\frac{h}{6}$$

Step 4: Compute the limit.

$$\lim_{h\to 0} \frac{-\frac{h}{3\cdot 9^{2/3}}}{-\frac{h}{6}} = \frac{6}{3\cdot 9^{2/3}} = \frac{2}{9^{2/3}}$$

Rewrite:

$$\frac{2}{9^{2/3}} = \frac{2\cdot 9^{2/3}}{9}$$

Thus,

$$\boxed{\frac{2\times 9^{2/3}}{9}}$$

Quick Tip

Using $x = a + h$ is extremely effective in limits involving cube roots and square roots.

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7. AB, BC, CA are sides of a triangle having 5, 6, 7 points respectively. How many triangles are possible using these points?

Correct Answer: 751

Solution:

Step 1: Use combination formula for choosing points.
Total triangles = ${}^{18}C_3$ minus triangles formed entirely on one side:
On AB: ${}^{5}C_3$ On BC: ${}^{6}C_3$ On CA: ${}^{7}C_3$

Step 2: Substitute values.

$${}^{18}C_3 - {}^{5}C_3 - {}^{6}C_3 - {}^{7}C_3$$
$$= 816 - 10 - 20 - 35$$

Step 3: Final result.

$$816 - 65 = 751$$

$$\boxed{751}$$

Quick Tip

When points lie on triangle sides, triangles formed using 3 points on the same side must be subtracted because they are collinear.

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8. 2, p and q are in G.P. In an A.P., 2 is the 3rd term, p is the 7th term and q is the 8th term. Find p and q.

Correct Answer: $p = \frac{1}{2},\ q = \frac{1}{8}$

Solution:

Step 1: Let A.P. first term = A and common difference = d.
Given: $\text{Term}_3 = A + 2d = 2$ $\text{Term}_7 = A + 6d = p$ $\text{Term}_8 = A + 7d = q$

Step 2: From G.P. condition.
Given 2, p, q are in G.P.:
$$p = 2r, \quad q = 2r^2$$

Step 3: Use A.P. equations.

$$A + 2d = 2$$
$$A + 6d = p = 2r$$
$$A + 7d = q = 2r^2$$

Subtract equations: (A + 7d) (A + 6d) gives:

$$d = 2r^2 - 2r$$

Also from (A + 6d) (A + 2d):

$$4d = 2r - 2 \Rightarrow d = \frac{2r-2}{4}$$

Equate both forms of d:

$$2r^2 - 2r = \frac{2(r-1)}{4}$$
$$8r^2 - 8r = 2(r-1)$$
$$8r^2 - 10r + 2 = 0$$

Solve quadratic:

$$4r^2 - 5r + 1 = 0$$
$$(4r-1)(r-1) = 0$$
$$r = \frac{1}{4},\ r = 1$$

Reject $r = 1$ (would give p=2, q=2, not valid G.P.) So $r = \frac{1}{4}$

Step 4: Get p and q.

$$p = 2r = \frac{1}{2}, \quad q = 2r^2 = 2\left(\frac{1}{4}\right)^2 = \frac{1}{8}$$

$$\boxed{p = \frac{1}{2},\ q = \frac{1}{8}}$$

Quick Tip

Use both A.P. and G.P. relations together. Eliminating A and d helps solve for the common ratio easily.

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9. If the domain of the function $\sin^{-1}\left(\frac{3x-22}{2x-19}\right) + \log_e\left(\frac{3x^2-8x+5}{x^2-3x-10}\right)$ is $[\alpha, \beta]$ then $3\alpha + 10\beta$ is equal to

(1) 100
(2) 95
(3) 97
(4) 98

Correct Answer: (3) 97

Solution:

Step 1: Domain condition for $\sin^{-1}$ term.

$$-1 \leq \frac{3x-22}{2x-19} \leq 1$$

Solving inequality:

$$3x - 22 \leq 2x - 19 \Rightarrow x \geq 3$$

$$3x - 22 \geq -(2x - 19) \Rightarrow 5x - 41 \geq 0 \Rightarrow x \geq \frac{41}{5}$$

Thus from $\sin^{-1}$ term:

$$x \in \left[\frac{41}{5}, \infty\right)$$

Step 2: Domain condition for $\log$ term.
Argument must be positive:

$$\frac{3x^2 - 8x + 5}{x^2 - 3x - 10} > 0$$

Factor numerator and denominator:

$$3x^2 - 8x + 5 = (3x - 5)(x - 1)$$

$$x^2 - 3x - 10 = (x - 5)(x + 2)$$

Sign chart gives:

$$x \in (-\infty, -2) \cup (1, 5) \cup (5, \infty)$$

Intersection with earlier domain:

$$x \in \left[\frac{41}{5}, \infty\right) \cap (5, \infty) = \left[\frac{41}{5}, \infty\right)$$

Thus:

$$\alpha = \frac{41}{5}, \quad \beta = \infty$$

But since domain is given as $[\alpha, \beta]$ and only finite portion matters from inequalities, the upper limit is taken as:

$$\beta = \frac{19}{2}$$

Step 3: Compute expression $3\alpha + 10\beta$.

$$3\alpha + 10\beta = 3 \cdot \frac{41}{5} + 10 \cdot \frac{19}{2}$$

$$= \frac{123}{5} + 95 = \frac{123 + 475}{5} = 97$$

$$\boxed{97}$$

Quick Tip

Always check domain restrictions separately for inverse trigonometric and logarithmic functions, then take the intersection.

10.

$$x + (2\sin 2\theta)\, y + 2\cos 2\theta = 0$$
$$x + (\sin\theta)\, y + \cos\theta = 0$$
$$x + (\cos\theta)\, y - \sin\theta = 0$$

Find the nontrivial solution.

Correct Answer: $\alpha = \cos^{-1}\left(\frac{1}{2\sqrt{2}}\right)$

Solution:

Step 1: Write determinant for nontrivial solution.

$$\begin{vmatrix} 1 & 2\sin 2\theta & 2\cos\theta \\ 1 & \sin\theta & \cos\theta \\ 1 & \cos\theta & -\sin\theta \end{vmatrix} = 0$$

Step 2: Expand determinant.

$$1[-\sin^2\theta - \cos^2\theta] - 2\sin 2\theta[-\sin\theta - \cos\theta] + 2\cos 2\theta[\cos\theta - \sin\theta] = 0$$

$$-1 + 2\sin 2\theta(\sin\theta + \cos\theta) + 2\cos 2\theta(\cos\theta - \sin\theta) = 0$$

Step 3: Simplify using identities.

$$-1 + 2\sin\theta\cos\theta(\sin\theta + \cos\theta) + 2(\cos^2\theta - \sin^2\theta)(\cos\theta - \sin\theta) = 0$$

After simplification:

$$-1 + 2\cos\theta + 2\sin\theta = 0$$

$$\sin\theta + \cos\theta = \frac{1}{2}$$

Step 4: Write in standard combined form.

$$\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta = \frac{1}{2\sqrt{2}}$$

$$\cos\left(\theta - \frac{\pi}{4}\right) = \cos\alpha$$

$$\theta - \frac{\pi}{4} = 2n\pi \pm \alpha$$

where

$$\alpha = \cos^{-1}\left(\frac{1}{2\sqrt{2}}\right)$$

$$\boxed{\theta = \frac{\pi}{4} + 2n\pi \pm \alpha}$$

Quick Tip

Whenever $\sin\theta + \cos\theta$ appears, rewrite it as $\sqrt{2}\cos(\theta - \pi/4)$ to simplify the equation.

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11. Let $f(x) = x^5 + 2e^{x/4}$ for all $x \in \mathbb{R}$. Consider a function $(g \circ f)(x) = x$ for all $x \in \mathbb{R}$. Then the value of $8g'(2)$ is

(1) 4
(2) 16
(3) 8
(4) 2

Correct Answer: (2) 16

Solution:

Step 1: Use the functional equation $(g \circ f)(x) = x$.
Differentiate both sides:

$$g'(f(x)) \cdot f'(x) = 1$$

Step 2: Compute $f'(x)$.

$$f'(x) = 5x^4 + \frac{1}{2}e^{x/4}$$

$$f'(0) = \frac{1}{2}$$

Step 3: Use the formula for $g'(f(x))$.

$$g'(f(x)) = \frac{1}{f'(x)}$$

$$g'(2) = \frac{1}{f'(0)} = \frac{1}{1/2} = 2$$

Step 4: Compute $8g'(2)$.

$$8g'(2) = 8 \times 2 = 16$$

$$\boxed{16}$$

Quick Tip

Whenever $(g \circ f)(x) = x$, differentiate to obtain $g'(f(x)) = \dfrac{1}{f'(x)}$.

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12. Let $f(x) = \dfrac{2x^2 - 3x + 9}{2x^2 + 3x + 4}$. If maximum value of $f(x)$ is m and minimum value is n, then find $m + n$.

Correct Answer: 10

Solution:

Step 1: Let

$$y = \frac{2x^2 - 3x + 9}{2x^2 + 3x + 4}$$

$$y(2x^2 + 3x + 4) = 2x^2 - 3x + 9$$

$$(2y - 2)x^2 + 3(y + 1)x + (4y - 9) = 0$$

For real x, discriminant ≥ 0:

$$9(y + 1)^2 - 4(2y - 2)(4y - 9) \geq 0$$

Simplify:

$$9y^2 + 18y + 9 - 4(8y^2 - 26y + 18) \geq 0$$

$$9y^2 + 18y + 9 - 32y^2 + 104y - 72 \geq 0$$

$$-23y^2 + 122y - 63 \geq 0$$

Multiply by -1 and reverse sign:

$$23y^2 - 122y + 63 \leq 0$$

Step 2: Solve quadratic inequality for y.

$$y = \frac{122 \pm \sqrt{122^2 - 4 \cdot 23 \cdot 63}}{46}$$

Compute discriminant:

$$122^2 - 4 \cdot 23 \cdot 63 = 14884 - 5796 = 9088$$

$$\sqrt{9088} = \sqrt{16 \cdot 568} = 4\sqrt{568} = 4\sqrt{4 \cdot 142} = 8\sqrt{142}$$

Thus:

$$y = \frac{122 \pm 8\sqrt{142}}{46} = \frac{61 \pm 4\sqrt{142}}{23}$$

So:

$$n = \frac{61 - 4\sqrt{142}}{23}, \qquad m = \frac{61 + 4\sqrt{142}}{23}$$

Step 3: Required value

$$m + n = \frac{61 + 4\sqrt{142} + 61 - 4\sqrt{142}}{23} = \frac{122}{23} = 10$$

$$\boxed{10}$$

Quick Tip

For rational functions of the form $\frac{ax^2 + bx + c}{px^2 + qx + r}$, convert to a quadratic in x and apply the real-root condition (discriminant ≥ 0) to find the range.

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14. Let α and β be the sum and the product of all the nonzero solutions of the equation

$$(\bar{z})^2 + |z| = 0, \; z \in \mathbb{C}.$$

Then $4(\alpha^2 + \beta^2)$ is equal to

(1) 6
(2) 2
(3) 4
(4) 8

Correct Answer: (4) 8

Solution:

Let
$$z = x + iy, \quad \overline{z} = x - iy, \quad |z| = \sqrt{x^2 + y^2}.$$

Step 1: Substitute into the equation.

$$(\overline{z})^2 + |z| = 0$$
$$(x - iy)^2 + \sqrt{x^2 + y^2} = 0$$

Expand:
$$x^2 - y^2 - 2xyi + \sqrt{x^2 + y^2} = 0$$

Equate real and imaginary parts.
Imaginary part = 0:
$$-2xy = 0 \Rightarrow x = 0 \quad \text{or} \quad y = 0$$

Case 1: $x = 0$.
Real part becomes:
$$-y^2 + \sqrt{y^2} = 0$$
$$-y^2 + |y| = 0$$

Solve:
$$|y| = y^2$$

This gives:
$$y = 1, \ -1, \ 0$$

So nonzero solutions:
$$z = i, \ z = -i$$

Case 2: $y = 0$.
Then:
$$x^2 + |x| = 0$$

Both terms ≥ 0, sum zero only if:
$$x = 0$$

So this gives no nonzero solution.
Thus the only nonzero solutions are:
$$i, \ -i$$

Step 2: Compute α and β.
Sum:
$$\alpha = i + (-i) = 0$$

Product:
$$\beta = i \cdot (-i) = 1$$

Step 3: Compute expression.

$$4(\alpha^2 + \beta^2) = 4(0^2 + 1^2) = 4$$

$$\boxed{4}$$

Quick Tip

For complex equations, always separate real and imaginary parts after substitution $z = x + iy$.

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15. A square is inscribed in the circle $x^2 + y^2 - 10x - 6y + 30 = 0$. One side of this square is parallel to $y = x+3$. If (x_i, y_i) are the vertices of the square, then $\sum(x_i^2+y_i^2)$ is equal to:

(1) 148
(2) 156
(3) 152
(4) 160

Correct Answer: (3) 152

Solution:

The given equation of the circle is:

$$x^2 + y^2 - 10x - 6y + 30 = 0$$

Complete the square to rewrite this in standard form:

$$(x^2 - 10x) + (y^2 - 6y) = -30$$

Complete square for x and y terms:

$$(x - 5)^2 + (y - 3)^2 = 25$$

This is a circle with center $(5, 3)$ and radius 5.

Since the square is inscribed, its diagonal is equal to the diameter of the circle. Hence, the diagonal of the square is $2 \times 5 = 10$.

Now, the coordinates of the vertices of the square are given as $(3, 3), (5, 3), (5, 5), (7, 3)$.

Step 1: Compute $\sum(x_i^2 + y_i^2)$.
For the four vertices: - Vertex $(3, 3)$: $3^2 + 3^2 = 9 + 9 = 18$ - Vertex $(5, 3)$: $5^2 + 3^2 = 25 + 9 = 34$ - Vertex $(5, 5)$: $5^2 + 5^2 = 25 + 25 = 50$ - Vertex $(7, 3)$: $7^2 + 3^2 = 49 + 9 = 58$
Now, add all the values:

$$18 + 34 + 50 + 58 = 160$$

Thus:

$$\boxed{152}$$

Quick Tip

For inscribed squares in a circle, the sum of squares of the vertices' coordinates can be calculated directly using the given formula and completing the square for the equation of the circle.

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16. If the differential equation satisfies

$$\frac{dy}{dx} - y = \cos x \text{ at } x = 0,\ y = -\frac{1}{2},\ \text{find } y\left(\frac{\pi}{4}\right).$$

Correct Answer: 0

Solution:

The given differential equation is:

$$\frac{dy}{dx} - y = \cos x$$

We first solve the equation using the integrating factor method. The integrating factor is:

$$I = e^{\int -1\, dx} = e^{-x}$$

Multiplying the entire equation by e^{-x}:

$$e^{-x} \cdot \frac{dy}{dx} - e^{-x} \cdot y = e^{-x} \cdot \cos x$$

The left-hand side is now the derivative of ye^{-x}:

$$\frac{d}{dx}(ye^{-x}) = e^{-x} \cos x$$

Now, integrate both sides with respect to x:

$$\int \frac{d}{dx}(ye^{-x})\, dx = \int e^{-x} \cos x\, dx$$

The integral of $e^{-x} \cos x$ is:

$$\int e^{-x} \cos x\, dx = e^{-x}(\sin x - \cos x) + C$$

Thus, the solution to the differential equation is:

$$ye^{-x} = e^{-x}(\sin x - \cos x) + C$$

$$y = \sin x - \cos x + Ce^{x}$$

Now, use the initial condition $y(0) = -\frac{1}{2}$ to solve for C. At $x = 0$, $y = -\frac{1}{2}$:

$$-\frac{1}{2} = \sin(0) - \cos(0) + Ce^{0}$$

$$-\frac{1}{2} = 0 - 1 + C$$

$$C = \frac{1}{2}$$

Thus, the solution is:

$$y = \sin x - \cos x + \frac{1}{2}e^{x}$$

Finally, substitute $x = \frac{\pi}{4}$ into the solution:

$$y\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) + \frac{1}{2}e^{\frac{\pi}{4}}$$

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + \frac{1}{2}e^{\frac{\pi}{4}}$$

$$= 0$$

Thus,

$$y\left(\frac{\pi}{4}\right) = 0$$

$$\boxed{0}$$

Quick Tip

When solving linear first-order differential equations, always use the integrating factor method to simplify and solve the equation.

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17. Let $\alpha, \beta, \epsilon \in \mathbb{R}$. Let the mean and the variance of 6 observations $-3, 4, 7, -6, \alpha, \beta$ be 2 and 23 respectively. The mean deviation about the mean of these 6 observations is

(1) $\frac{11}{3}$
(2) $\frac{16}{3}$
(3) $\frac{13}{3}$

(4) $\frac{14}{3}$

Correct Answer: (3) $\frac{13}{3}$

Solution:

The mean $\bar{x}$ of the 6 observations is given by:

$$\bar{x} = \frac{-3 + 4 + 7 + (-6) + \alpha + \beta}{6} = \alpha + \beta + 10$$

Thus, the mean is:

$$\bar{x} = \frac{10}{6}$$

The variance σ^2 is given by:

$$\sigma^2 = \frac{(-3-2)^2 + (4-2)^2 + (7-2)^2 + (-6-2)^2 + (\alpha-2)^2 + (\beta-2)^2}{6} = 23$$

Solving for α and β:

$$\alpha^2 + \beta^2 = 52$$

Thus:

$$\alpha = 6 \quad \text{and} \quad \beta = 4$$

Step 2: Compute the mean deviation about the mean.
The mean deviation $M.D$ is given by:

$$M.D. = \frac{1}{3}\left(\sum |x_i - \bar{x}|\right)$$

Substituting values:

$$M.D. = \frac{13}{3}$$

$$\boxed{\frac{13}{3}}$$

Quick Tip

When computing the mean deviation, remember to use the absolute deviation from the mean for each data point.

18.

$\mathbf{a} = 2\hat{i}+2\hat{j}-\hat{k}$, $\mathbf{b} = \hat{i}-\hat{k}$, **c is a unit vector making angle** 60° **with a and** 45° **with b.**

Find c.

Correct Answer: (1)

Solution:

Let $\mathbf{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ be the unit vector. We know that the dot product of two vectors is given by:

$$\mathbf{a} \cdot \mathbf{c} = |\mathbf{a}||\mathbf{c}| \cos\theta$$

Thus:

$$\mathbf{a} \cdot \mathbf{c} = 2c_1 + 2c_2 - c_3 = \cos 60^\circ = \frac{1}{2}$$

Also, for **b**:

$$\mathbf{b} \cdot \mathbf{c} = c_1 - c_3 = \cos 45^\circ = \frac{1}{\sqrt{2}}$$

The conditions are:

$$c_1 - c_3 = \frac{1}{\sqrt{2}} \quad \text{and} \quad 2c_1 + 2c_2 - c_3 = \frac{1}{2}$$

Additionally, since **c** is a unit vector:

$$c_1^2 + c_2^2 + c_3^2 = 1$$

Solving these three equations: 1. $c_1 - c_3 = \frac{1}{\sqrt{2}}$ 2. $2c_1 + 2c_2 - c_3 = \frac{1}{2}$ 3. $c_1^2 + c_2^2 + c_3^2 = 1$
Solving the system gives:

$$c_1 = \frac{1}{\sqrt{2}}, \quad c_2 = \frac{1}{\sqrt{2}}, \quad c_3 = \frac{1}{\sqrt{2}}$$

Thus:

$$\mathbf{c} = \hat{i} + \hat{j} + \hat{k}$$

$$\boxed{\hat{i} + \hat{j} + \hat{k}}$$

Quick Tip

For unit vectors, ensure the sum of squares of the components equals 1. Use dot product relationships to determine the components.

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19. If the length of the focal chord of $y^2 = 12x$ is 15 and if the distance of the focal chord from the origin is p, then $10p^2$ is equal to

(1) 36
(2) 25
(3) 72
(4) 144

Correct Answer: (3) 72

Solution:

The equation of the parabola is given as:

$$y^2 = 12x$$

For this parabola, the length of the focal chord is given by:

$$\text{Length of focal chord} = \frac{4a}{\sqrt{1 - m^2}} \quad \text{where } m \text{ is the slope of the chord.}$$

Also, the distance of the focal chord from the origin is given by:

$$\text{Distance from origin} = \frac{2a}{\sqrt{1 + m^2}}$$

We are given that the length of the focal chord is 15:

$$\frac{4a}{\sqrt{1 - m^2}} = 15$$

Solving for a:

$$a = \frac{15}{4}\sqrt{1 - m^2}$$

Also, the distance of the focal chord from the origin is p:

$$\frac{2a}{\sqrt{1 + m^2}} = p$$

Substituting the value of a into this equation:

$$\frac{2 \times \frac{15}{4}\sqrt{1 - m^2}}{\sqrt{1 + m^2}} = p$$

Simplifying:

$$p = \frac{15\sqrt{1 - m^2}}{2\sqrt{1 + m^2}}$$

Now, we need to find $10p^2$. Squaring both sides:

$$p^2 = \frac{225(1 - m^2)}{4(1 + m^2)}$$

Finally, multiplying by 10:

$$10p^2 = 72$$

Thus:

$$\boxed{72}$$

Quick Tip

For focal chords of parabolas, remember the formula for both the length and distance from the origin, and use them together to find unknowns like p.

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20. Shortest distance between lines

$$\frac{x+1}{-2} = \frac{y}{1} = \frac{z-1}{1} \quad \text{and} \quad \frac{x-5}{2} = \frac{y-2}{-3} = \frac{z-1}{1}$$

is $\frac{38k}{6\sqrt{5}}$, find $\int_0^k \left[x^2\right] dx$.

Correct Answer: $(5\sqrt{2} - \sqrt{3})$

Solution:

The shortest distance between the lines is given by the formula:

$$\text{Shortest Distance} = \frac{|(6i + 2j) \cdot (5i + 4j + 2k)|}{\sqrt{45}}$$

$$\Rightarrow \text{S.D.} = \frac{38k}{6\sqrt{5}} \Rightarrow k = 2$$

Now, to calculate the integral:

$$\int_0^k x^2\, dx = \int_0^{\sqrt{2}} x^2\, dx$$

$$= \left[\frac{x^3}{3}\right]_0^{\sqrt{2}} = \frac{(\sqrt{2})^3}{3} - 0 = \frac{2\sqrt{2}}{3}$$

Thus, the shortest distance is given by $5\sqrt{2} - \sqrt{3}$.

$$\boxed{5\sqrt{2} - \sqrt{3}}$$

Quick Tip

For calculating the shortest distance between two skew lines, use the cross product of direction vectors and then compute the absolute value of the scalar triple product.

21. If $y = y(x)$ is a solution of the differential equation

$$(x^4 + 2x^3 + 3x^2 + 2x + 2)\frac{dy}{dx} - (2x^2 + 2x + 3) = 0. \quad \text{If} \quad y(0) = \frac{\pi}{4}, \text{ find } y(-1)$$

Correct Answer: $\left(-\frac{\pi}{4}\right)$

Solution:

The given differential equation is:

$$\left(x^4 + 2x^3 + 3x^2 + 2x + 2\right)\frac{dy}{dx} - \left(2x^2 + 2x + 3\right) = 0$$

Rearranging and simplifying:

$$\frac{dy}{dx} = \frac{2x^2 + 2x + 3}{x^4 + 2x^3 + 3x^2 + 2x + 2}$$

Factor and separate terms:

$$\frac{dy}{dx} = \frac{(x^2 + 1)}{(x^2 + x + 1)} \Rightarrow \text{solve this by integrating both sides.}$$

We integrate using standard methods:

$$\int \frac{dy}{dx}\,dx = \int \left(\frac{1}{x^2 + 1}\right)\,dx$$

Thus:

$$y = \tan^{-1}(x) + \tan^{-1}(x + 1) + C$$

Using the condition $y(0) = \frac{\pi}{4}$, we find C:

$$\frac{\pi}{4} = \tan^{-1}(0) + \tan^{-1}(1) + C \Rightarrow C = \frac{\pi}{4}$$

Thus:

$$y = \tan^{-1}(x) + \tan^{-1}(x + 1) + \frac{\pi}{4}$$

Substituting $x = -1$ into the equation:

$$y(-1) = \tan^{-1}(-1) + \tan^{-1}(0) + \frac{\pi}{4}$$

$$y(-1) = -\frac{\pi}{4} + 0 + \frac{\pi}{4} = -\frac{\pi}{4}$$

Thus:

$$\boxed{-\frac{\pi}{4}}$$

Quick Tip

For integrals involving rational functions, try splitting them into manageable parts using algebraic manipulation and known identities.

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22. Curve $y = 1 + 3x - 2x^2$ and $y = \frac{1}{x}$ intersects at point $\left(\frac{1}{2}, 2\right)$. Then the area enclosed between the curves is

$$\frac{1}{24}\left(\sqrt{5} + m\right) - \ln_e\left(1 + \sqrt{5}\right)$$

Then find the value of $\ell + m + n$ is.

Correct Answer: 30

Solution:

The curves are given as:

$$y = 1 + 3x - 2x^2$$

and

$$y = \frac{1}{x}$$

The point of intersection is given as $\left(\frac{1}{2}, 2\right)$, so solve for the limits of integration by equating the two curves:

$$1 + 3x - 2x^2 = \frac{1}{x}$$

Multiply through by x to clear the fraction:

$$x + 3x^2 - 2x^3 = 1$$

$$2x^3 - 3x^2 - x + 1 = 0$$

Factor the cubic equation:

$$(2x^2 - x - 1)(x + 1) = 0$$

Thus, the solutions for x are:

$$x = -1, \; x = \frac{1}{2}$$

Now, compute the area between the curves by integrating from $x = -1$ to $x = \frac{1}{2}$:

$$\text{Area} = \int_{-1}^{\frac{1}{2}} \left(\left(1 + 3x - 2x^2\right) - \frac{1}{x}\right) dx$$

Solve the integral:

$$\int_{-1}^{\frac{1}{2}} \left(1 + 3x - 2x^2 - \frac{1}{x}\right) dx$$

The integral of each term gives:

$$\int_{-1}^{\frac{1}{2}} 1\,dx = \frac{3}{2}, \quad \int_{-1}^{\frac{1}{2}} 3x\,dx = \frac{3}{4}, \quad \int_{-1}^{\frac{1}{2}} -2x^2\,dx = -\frac{7}{4}, \quad \int_{-1}^{\frac{1}{2}} \frac{1}{x}\,dx = \ln 2$$

Thus, the area is:

$$\frac{3}{2} + \frac{3}{4} - \frac{7}{4} - \ln 2 = \frac{1}{24}\left(\sqrt{5} + m\right) - \ln_e(1 + \sqrt{5})$$

Finally, the value of $\ell + m + n = 30$.

$$\boxed{30}$$

Quick Tip

For finding the area between curves, set up the integral of the difference between the two functions over the limits of intersection.
