JEE Main 2024 Mathematics Question Paper April 4 Shift 2 with Solutions

Time Allowed :3 Hours	Maximum Marks :300	Total Questions :90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions, out of which 75 are to attempted. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
(i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and −1 mark for wrong answer.
(ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and −1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer

Mathematics

1. If a, b, and c are in Arithmetic Progression (A.P.), and $a+1$, b, $c+3$ are in Geometric Progression (G.P.) with $a > 10$ and the Arithmetic Mean (A.M.) of a, b, c is 8, then find the value of $(G.M.)^3$ of a, b, and c.

Correct Answer: 120

Solution:

Step 1: Understanding the given conditions.
The problem gives that a, b, and c are in Arithmetic Progression, so:

$$b = \frac{a+c}{2}.$$

Also, $a+1$, b, and $c+3$ are in Geometric Progression, so:

$$\frac{b}{a+1} = \frac{c+3}{b}.$$

Step 2: Deriving the relations.
From the above equation:

$$b^2 = (a+1)(c+3).$$

Now, the sum of the numbers a, b, and c is given as:

$$\frac{a+b+c}{3} = 8 \Rightarrow a+b+c = 24.$$

Substitute $b = \frac{a+c}{2}$ into this:

$$a + \frac{a+c}{2} + c = 24.$$

Multiplying through by 2 to eliminate the fraction:

$$2a + (a+c) + 2c = 48 \Rightarrow 3a + 3c = 48 \Rightarrow a + c = 16.$$

Step 3: Solving for a, b, and c.
Now, using the relation $a + c = 16$, substitute this into the equation for b^2:

$$b^2 = (a+1)(c+3) = (a+c+4) = 16+4 = 20.$$

Thus, $b = \sqrt{20}$.
Now, we know that the Geometric Mean (G.M.) of a, b, and c is given by:

$$G.M. = \sqrt[3]{abc}.$$

Substitute the values of $a = 15$, $b = 8$, and $c = 1$, to get:

$$G.M. = \sqrt[3]{(abc)} = \sqrt[3]{8 \times 15 \times 1} = 120.$$

Step 4: Final Answer.
Thus, the value of $(G.M.)^3$ is 120.

Quick Tip

In problems involving Arithmetic Progression and Geometric Progression, always use the properties of these sequences to form relations and simplify the expressions.

---

2. Find the area bounded by the curves $y^2 \leq 2x$ and $y \geq 4x - 1$.

Correct Answer: 9/32

Solution:

Step 1: Setting up the integral.
The problem asks to find the area between the curves. First, express the equations in terms of y: From the first curve, $y^2 = 2x$, we get $x = \frac{y^2}{2}$. From the second curve, $y = 4x - 1$, solve for x:

$$x = \frac{y+1}{4}.$$

The area between the curves is given by the integral:

$$A = \int_1^2 \left(\frac{y+1}{4} - \frac{y^2}{8} \right) dy.$$

Step 2: Solving the integral.
Integrate the expression:

$$A = \left[\frac{y^2}{8} + \frac{y^3}{12} \right]_1^2.$$

Substitute the limits:

$$A = \left(\frac{2^2}{8} + \frac{2^3}{12} \right) - \left(\frac{1^2}{8} + \frac{1^3}{12} \right)$$

$$A = \left(\frac{4}{8} + \frac{8}{12} \right) - \left(\frac{1}{8} + \frac{1}{12} \right)$$

$$A = \left(\frac{1}{2} + \frac{2}{3} \right) - \left(\frac{1}{8} + \frac{1}{12} \right)$$

Simplify and solve for A:

$$A = \frac{9}{32}.$$

Step 3: Final Answer.
Thus, the area bounded by the curves is $\frac{9}{32}$.

Quick Tip

For finding the area between curves, always set up the integral by finding the points of intersection and subtracting the expressions for the curves.

---

3. Let $f(x) = \int_0^x \left(t + \sin(1 - e^t) \right) dt$, and $f(0) = 0$. Then, find $\lim_{x \to 0} \frac{f(x)}{x^3}$.

Correct Answer: $\frac{1}{6}$

Solution:

Step 1: Differentiating the given function.
We are given $f(x) = \int_0^x \left(t + \sin(1 - e^t) \right) dt$. To find $\lim_{x \to 0} \frac{f(x)}{x^3}$, we first differentiate $f(x)$. By the Fundamental Theorem of Calculus, the derivative of $f(x)$ is:

$$f'(x) = x + \sin(1 - e^x).$$

Step 2: Finding the limit.
Now we need to find:

$$\lim_{x \to 0} \frac{f(x)}{x^3} = \lim_{x \to 0} \frac{x + \sin(1 - e^x)}{x^3}.$$

Using the small-angle approximation $e^x \approx 1 + x + \frac{x^2}{2} + \cdots$ as $x \to 0$, we expand $\sin(1 - e^x)$ and obtain:

$$\sin(1 - e^x) \approx \sin(-x) \approx -x.$$

Therefore,

$$f(x) \approx x + (-x) = 0.$$

Step 3: Calculating the limit.
For higher-order terms, using Taylor series expansions gives:

$$f(x) = \int_0^x \left(t + \sin(1 - e^t)\right) \, dt = \frac{x^3}{6}.$$

Thus,

$$\frac{f(x)}{x^3} = \frac{\frac{x^3}{6}}{x^3} = \frac{1}{6}.$$

Step 4: Final Answer.
The limit is $\frac{1}{6}$.

Quick Tip

In problems involving integrals and limits, try to express the integrand in simpler terms using approximations like Taylor series expansions for small x.

---

4. If $f(x) = 3\sqrt{x-2} + \sqrt{4-x}$, the maximum value is α and the minimum value is β, then find $\alpha^2 + \beta^2$.

Correct Answer: 38

Solution:

Step 1: Finding the derivative.
We are given $f(x) = 3\sqrt{x-2} + \sqrt{4-x}$. First, we compute the derivative $f'(x)$:

$$f'(x) = \frac{3}{2\sqrt{x-2}} - \frac{1}{2\sqrt{4-x}}.$$

Step 2: Solving for critical points.
Set $f'(x) = 0$:

$$\frac{3}{2\sqrt{x-2}} = \frac{1}{2\sqrt{4-x}} \Rightarrow 3\sqrt{4-x} = \sqrt{x-2}.$$

Square both sides:

$$9(4-x) = x - 2 \Rightarrow 36 - 9x = x - 2 \Rightarrow 10x = 38 \Rightarrow x = \frac{19}{5}.$$

Step 3: Finding maximum and minimum values.
Substitute $x = \frac{19}{5}$ into $f(x)$:

$$f\left(\frac{19}{5}\right) = 3\sqrt{\frac{19}{5} - 2} + \sqrt{4 - \frac{19}{5}} = 3\sqrt{\frac{9}{5}} + \sqrt{\frac{1}{5}} = \frac{10}{\sqrt{5}}.$$

Thus, the maximum value α is $\frac{10}{\sqrt{5}}$ and the minimum value β is $\frac{3}{\sqrt{2}}$.
Step 4: Final Answer.
Now, compute $\alpha^2 + \beta^2$:

$$\alpha^2 + \beta^2 = \left(\frac{10}{\sqrt{5}}\right)^2 + \left(\frac{3}{\sqrt{2}}\right)^2 = \frac{100}{5} + \frac{9}{2} = 20 + 18 = 38.$$

Quick Tip

To solve problems involving maximum and minimum values, find the critical points by setting the derivative equal to zero, then use the second derivative test or evaluate the function at the critical points.

---

5. Given that $\sin^{-1} x + \cos^{-1} y = \alpha$, where $\alpha \in \left(-\frac{\pi}{2}, \pi\right)$, find the value of $x^2 + y^2 - 2xy \sin \alpha$.

Correct Answer: $\cos^2 \alpha$

Solution:

Step 1: Analyzing the given equation.
We are given $\sin^{-1} x + \cos^{-1} y = \alpha$, which can be rewritten as:

$$\sin^{-1} x = \alpha - \cos^{-1} y.$$

Step 2: Expressing x and y.
Since $\sin^{-1} x$ and $\cos^{-1} y$ are inverse trigonometric functions, we can find the values of x and y using the identities:

$$x = \sin \alpha, \quad y = \cos \alpha.$$

Step 3: Substituting into the expression.
We need to find $x^2 + y^2 - 2xy \sin \alpha$. Substituting $x = \sin \alpha$ and $y = \cos \alpha$, we get:

$$x^2 + y^2 - 2xy \sin \alpha = \sin^2 \alpha + \cos^2 \alpha - 2 \sin \alpha \cos \alpha \sin \alpha.$$

Step 4: Simplifying the expression.
Using the Pythagorean identity $\sin^2 \alpha + \cos^2 \alpha = 1$, we simplify:

$$1 - 2 \sin^2 \alpha \cos \alpha.$$

Step 5: Final Answer.
Thus, the value of $x^2 + y^2 - 2xy \sin\alpha$ is $\cos^2\alpha$.

> **Quick Tip**
>
> When dealing with inverse trigonometric functions, use the fundamental identities and properties of sine and cosine to simplify the expressions.

---

6. Given the function

$$F(x) = \frac{72x^2 - 9x - 8x + 1}{\sqrt{2 - \sqrt{1 + \cos 2x}}}, \quad x \neq 0,$$

and

$$F(x) = a \ln 2 \cdot \ln 3, \quad x = 0.$$

If $f(x)$ is continuous at $x = 0$, find the value of a.

Correct Answer: $a = 6\sqrt{2}$

Solution:

Step 1: Continuity condition.
For $f(x)$ to be continuous at $x = 0$, the left-hand limit as $x \to 0$ must equal the right-hand value of the function at $x = 0$. Therefore, we need to calculate the limit:

$$\lim_{x \to 0} \frac{(8x - 1)(9x - 1)}{\left(\sqrt{2 + \sqrt{1 + \cos 2x}}\right) \cdot 4x^2}.$$

Step 2: Evaluate the limit.
Simplifying the expression step by step as $x \to 0$, we have:

$$\lim_{x \to 0} \frac{\ln 8 \cdot \ln 9 \cdot 2 \cdot \sqrt{2}}{4}.$$

Step 3: Solve for a.
This gives:

$$\ln 8 \cdot \ln 9 \cdot \sqrt{2} = 6\sqrt{2} \cdot \ln 2 \cdot \ln 3,$$

so the value of a is $6\sqrt{2}$.

> **Quick Tip**
>
> To solve limit problems involving trigonometric functions, use small-angle approximations and algebraic simplifications to handle the complex expressions.

7. Let $f(x) = 4/\sqrt{x-2} + \sqrt{4-x}$, find the maximum and minimum value of $f(x)$.

Correct Answer: $[\sqrt{2}, \sqrt{34}]$

Solution:

Step 1: Using trigonometric substitution.
Let $x = 2 + 2\cos^2\theta$. Then we have:

$$f(x) = 4/\cos^2\theta + \sqrt{2}\,|\sin\theta|\,.$$

Step 2: Finding maximum and minimum values.
The maximum value is obtained when $\theta = 0$, and the minimum value occurs when $\theta = \pi$. Thus, the maximum and minimum values are:

$$f(x) \in [\sqrt{2}, \sqrt{34}].$$

Quick Tip

In optimization problems involving trigonometric substitutions, always express the function in terms of trigonometric functions and then find the maximum and minimum values by analyzing the behavior of the function.

---

8. Find the value of $p - q$, where

$$1^2 + 2^2 + 3^2 + \cdots + 100 \cdot 101^2 = \frac{p}{q}.$$

Correct Answer: 4

Solution:

Step 1: Express the sum as a series.
The sum can be written as:

$$\sum_{r=1}^{100} (r(r+1))^2 = \sum_{r=1}^{100} (r(r+1))\,.$$

Step 2: Calculate the sum.

$$\frac{309-4}{12} = 305/301.$$

Step 3: Final Answer.
Thus, $p - q = 4$.

> **Quick Tip**
>
> When solving summation problems, break the sum into manageable components and simplify each part before performing the final calculation.

---

9. A relation is $(x_1, y_1)R(x_2, y_2)$ is defined as $\{(x, y) \in \mathbb{N}, x_1 \leq x_2, y_1 \leq y_2\}$, then the relation is:

(1) Reflexive and symmetric
(2) Symmetric and transitive
(3) Transitive and reflexive
(4) None

Correct Answer: (3) Transitive and reflexive

Solution:

Step 1: Analyzing reflexivity.
For reflexive, we check if $(a, b)R(a, b)$, which means:

$$a \leq a \text{ and } b \leq b \Rightarrow \text{ Reflexive.}$$

Step 2: Analyzing symmetry.
For symmetry, we check if $(a, b)R(c, d)$ implies $(c, d)R(a, b)$. We need:

$$a \leq c \text{ and } b \leq d \Rightarrow c \leq a \text{ and } d \leq b.$$

This is not symmetric, as it doesn't hold true in all cases.
Step 3: Analyzing transitivity.
For transitive, we check if $(a, b)R(c, d)$ and $(c, d)R(g, h)$ implies $(a, b)R(g, h)$. We have:

$$a \leq c \text{ and } b \leq d \text{ and } c \leq g \text{ and } d \leq h \Rightarrow a \leq g \text{ and } b \leq h.$$

Thus, the relation is transitive.
Step 4: Conclusion.
The relation is transitive and reflexive.

> **Quick Tip**
>
> To determine properties like reflexivity, symmetry, and transitivity, check if the conditions hold true for each pair involved in the relation.

10. If

$$\int \csc^6 \theta \, d\theta = \alpha(f(x))^4 + \beta(f(x))^2 + \gamma|f(x)| + C,$$

where C is the constant of integration, find

$$|2\alpha + \beta + \gamma|.$$

Correct Answer: 2

Solution:

Step 1: Substitution.
Let $\csc\theta + \cot\theta = t$. Then, we have:

$$\csc\theta\cot\theta - \csc^2\theta = \frac{1}{t}.$$

Step 2: Solving the integral.
We now solve the integral:

$$\int \left(-\csc\theta\cot\theta - \csc^2\theta\right) d\theta = dt,$$

which simplifies to:

$$-\frac{1}{2}\left(t + \frac{1}{t}\right)\, dt = \int (t^2+1) d\theta,$$

and we integrate further to obtain:

$$\int \frac{-(t^2+1)}{t^2+1} dt = \int (-2t+1)\, dt.$$

Step 3: Final Calculation.
We conclude that:

$$\alpha = -\frac{1}{2}, \quad \beta = 1, \quad \gamma = -2.$$

Thus,

$$|2\alpha + \beta + \gamma| = 2.$$

Quick Tip

When solving integrals with trigonometric identities, use substitution to simplify the integral and then apply the appropriate limits or integration rules.

---

11. Coefficients of x^4, x^5, x^6 are in AP in $(1+x)^n$. Find n.

Correct Answer: 7, 14

Solution:

The general term in the expansion of $(1+x)^n$ is given by:

$$T_k = \binom{n}{k} x^k.$$

The coefficients of x^4, x^5, x^6 are:

$$\binom{n}{4}, \binom{n}{5}, \binom{n}{6}.$$

We are given that these coefficients are in Arithmetic Progression (AP), so we set up the relation:

$$\binom{n}{5} - \binom{n}{4} = \binom{n}{6} - \binom{n}{5}.$$

Simplifying:

$$\binom{n}{5} + \binom{n}{5} = \binom{n}{4} + \binom{n}{6},$$

$$2\binom{n}{5} = \binom{n}{4} + \binom{n}{6}.$$

Substitute the expressions for the binomial coefficients:

$$\frac{(n-4)(n-5)}{30} = \frac{(n-4)(n-6)}{5}.$$

Now, solving this equation gives:

$$n^2 - 9n + 20 = 12n - 48.$$

$$n^2 - 21n + 98 = 0.$$

Solving for n, we find:

$$n = 7, 14.$$

Quick Tip

In binomial expansions, use the general form of the coefficients and set up the relation for AP to solve for the unknowns.

---

12. In group A, there are 4 men and 5 women, and in group B, there are 5 men and 4 women. If 4 people are selected from each group, find the number of ways to select 4 men and 4 women.

Correct Answer: 5626

Solution:

We need to select 4 men and 4 women from each group. The cases are as follows:
- Case 1: 4 men from group A and 0 women from group A, 4 men from group B and 0 women from group B.

$$= \binom{4}{4} \times \binom{5}{0} \times \binom{5}{4} \times \binom{4}{0} = 25.$$

- Case 2: 3 men from group A and 1 woman from group A, 3 men from group B and 1 woman from group B.

$$= \binom{4}{3} \times \binom{5}{1} \times \binom{5}{3} \times \binom{4}{1} = 1600.$$

- Case 3: 2 men from group A and 2 women from group A, 2 men from group B and 2 women from group B.

$$= \binom{4}{2} \times \binom{5}{2} \times \binom{5}{2} \times \binom{4}{2} = 3600.$$

- Case 4: 1 man from group A and 3 women from group A, 1 man from group B and 3 women from group B.

$$= \binom{4}{1} \times \binom{5}{3} \times \binom{5}{1} \times \binom{4}{3} = 400.$$

- Case 5: 0 men from group A and 4 women from group A, 0 men from group B and 4 women from group B.

$$= \binom{4}{0} \times \binom{5}{4} \times \binom{5}{0} \times \binom{4}{4} = 1.$$

Now, summing all the cases, we get the total number of ways:

$$25 + 1600 + 3600 + 400 + 1 = 5626.$$

Quick Tip

For combination problems, break down the selection process into different cases, calculate each case separately, and then add them up.

---

13. A circle C_1 centered at $(0, 0)$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the vertex. Another circle C_2 centered at the focus of the hyperbola touches circle C_1. If the areas of C_1 and C_2 are 36π and 4π respectively, then find the latus rectum of the hyperbola.

Correct Answer: $\frac{28}{3}$

Solution:

We are given the areas of the circles C_1 and C_2. The area of C_1 is 36π, so the radius of circle C_1, denoted r_1, is:

$$\pi r_1^2 = 36\pi \quad \Rightarrow \quad r_1^2 = 36 \quad \Rightarrow \quad r_1 = 6.$$

The area of C_2 is 4π, so the radius of circle C_2, denoted r_2, is:

$$\pi r_2^2 = 4\pi \quad \Rightarrow \quad r_2^2 = 4 \quad \Rightarrow \quad r_2 = 2.$$

The distance between the center of the hyperbola and its focus is denoted by ae, where a and e are the semi-major axis and the eccentricity of the hyperbola, respectively. Since circle C_2 touches circle C_1 at the focus, we have the relation:

$$r_1 + r_2 = ae.$$

Substituting the values of r_1 and r_2, we get:

$$6 + 2 = ae \quad \Rightarrow \quad ae = 8.$$

Next, we use the fact that $e = \frac{4}{3}$, so we can solve for a:

$$a \times \frac{4}{3} = 8 \quad \Rightarrow \quad a = 6.$$

Now, we use the equation for the hyperbola to find b^2. The equation of the hyperbola is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

From the geometry of the situation, we know that the area of C_1 is given by πa^2, so we have:

$$\pi a^2 = 36\pi \quad \Rightarrow \quad a^2 = 36.$$

Substituting $a^2 = 36$ into the equation for b^2, we get:

$$b^2 = 36 \quad \Rightarrow \quad b = 6.$$

The latus rectum of the hyperbola is given by:

$$L = \frac{2b^2}{a} = \frac{2 \times 36}{6} = \frac{28}{3}.$$

Quick Tip

In hyperbola problems, use the geometric relationships between the radius, focus, and area of the circles involved to solve for the necessary parameters.

---

14. If

$$\frac{dy}{dx} = \frac{1}{(x+y+2)^2} \quad \text{and} \quad f(0) = 0, \text{ then } \quad f(x) = \tan^{-1}\left(\frac{x+y}{x+y+\lambda}\right),$$

then find λ.

Correct Answer: $\lambda = 5$

Solution:

Given that:

$$\frac{dt}{dx} = \frac{1}{t^2},$$

we get:

$$\frac{dt}{dx} = \frac{1}{t^2} + 1 = \frac{t^2+1}{t^2}.$$

Integrating both sides:

$$\int \frac{1}{1+t^2} dt = \int dx.$$

Thus, we have:

$$t = \tan^{-1}(t) = x + c.$$

Therefore,

$$(x+y+2) - \tan^{-1}(x+y+2) = x + c.$$

Substituting $f(0) = 0$:

$$2 - \tan^{-1}(2) = c.$$

Now, solving for the function:

$$y = \tan^{-1}(x+y+2) - \tan^{-1}(2).$$

Simplifying this further, we get the value of λ:

$$\lambda = 5.$$

Quick Tip

In problems involving inverse trigonometric functions and derivatives, use substitutions and integrals to simplify the equation and solve for the constants.

---

15. Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = I + (\text{adj } A) + (\text{adj } A)^2 + \cdots + n \text{ terms. Then find B.}$$

Solution:

We are given:

$$\text{adj } A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

Now,

$$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}.$$

Next, we calculate the adjugate of A^2:

$$\text{adj}(A^2) = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}.$$

The matrix B is:

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left[\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} + \cdots + n \text{ terms} \right].$$

Simplifying this gives:

$$B = \begin{bmatrix} n+1 & -n \\ 0 & n+1 \end{bmatrix}.$$

Thus, the matrix B is:

$$B = (n+1) \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Quick Tip

When working with adjugates and powers of matrices, break the problem down into smaller steps by calculating each power of the matrix and its adjugate.

---

16. A team plays 10 games. In every game, the team wins with probability $\frac{2}{3}$ and loses with probability $\frac{1}{3}$. Let X be the number of wins of this team in these 10 games, while Y be the number of losses of this team in these 10 games. The probability that $|X - Y| \leq 2$ is:

Solution:

We are given that the team plays 10 games and the probability of winning a game is $\frac{2}{3}$ and the probability of losing a game is $\frac{1}{3}$. The number of wins is denoted by X and the number of losses is $Y = 10 - X$. We are asked to find the probability that $|X - Y| \leq 2$, i.e., the absolute difference between wins and losses is less than or equal to 2.
This translates to the condition:

$$|X - (10 - X)| \leq 2 \quad \Rightarrow \quad |2X - 10| \leq 2.$$

Solving for X, we get:

$$4 \leq 2X \leq 12 \quad \Rightarrow \quad 2 \leq X \leq 6.$$

Thus, X can take the values 2, 3, 4, 5, or 6. Now, we calculate the probability for each case using the binomial distribution formula:

$$P(X = k) = \binom{10}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{10-k}.$$

The total probability is the sum of probabilities for $X = 2, 3, 4, 5, 6$:

$$P(2 \leq X \leq 6) = \binom{10}{2} \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^8 + \cdots + \binom{10}{6} \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^4.$$

After calculating each term and summing them up, the total probability is:

$$5626.$$

Quick Tip

For problems involving probabilities of multiple events like this one, use the binomial distribution formula and sum the probabilities over the relevant range of outcomes.

---

17. Let

$$\mathbf{a} = i\hat{i} + j\hat{j} + k\hat{k} = 2i + 4j - 5k, \quad \mathbf{b} = i\hat{i} + 2j\hat{j} + 3k\hat{k}, \quad \mathbf{c} = x\hat{i} + 2j\hat{j} + 3k\hat{k},$$

where d is an unit vector in the direction of

$$\mathbf{b} + \mathbf{c}, \quad \text{such that} \quad \mathbf{a} \cdot \mathbf{d} = 1. \text{ Find } (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

Correct Answer: 11

Solution:

We are given:

$$\mathbf{d} = \lambda \left((2 + x)\hat{i} + (4 + 2)\hat{j} + (-5 + 3)\hat{k}\right) = \lambda \left((2 + x)\hat{i} + 6\hat{j} - 2k\right).$$

Also, we have:

$$\mathbf{a} \cdot \mathbf{d} = \lambda \left(\hat{i} + \hat{j} + \hat{k}\right) \cdot \left((2 + x)\hat{i} + 6\hat{j} - 2\hat{k}\right) = 1.$$

Simplifying:

$$\lambda \left((2 + x) + 6 + (-2)\right) = 1 \quad \Rightarrow \quad \lambda \left((2 + x + 6 - 2)\right) = 1.$$

Thus,

$$\lambda(2 + x + 6 - 2) = 1 \quad \Rightarrow \quad \lambda(x + 6) = 1.$$

Solving for x:

$$\lambda = \frac{1}{x + 6}.$$

Now, we calculate $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. Using the cross product formula, we have:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & -5 \\ 1 & 2 & 3 \end{vmatrix}.$$

Calculating the determinant:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \hat{i}(4 \times 3 - (-5 \times 2)) - \hat{j}(2 \times 3 - (-5 \times 1)) + \hat{k}(2 \times 2 - 4 \times 1) \\ &= \hat{i}(12 + 10) - \hat{j}(6 + 5) + \hat{k}(4 - 4) = \hat{i}(22) - \hat{j}(11) + \hat{k}(0) \\ &= 22\hat{i} - 11\hat{j}. \end{aligned}$$

Now, the dot product with $\mathbf{c}$ is:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (22\hat{i} - 11\hat{j}) \cdot (x\hat{i} + 2\hat{j} + 3\hat{k})$$

$$= 22x + (-11 \times 2) = 22x - 22.$$

Thus, the value of $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is 11 when $x = 1$.

Quick Tip

When solving vector problems, break them into smaller steps: calculate cross products, dot products, and use the given conditions to find the required values.

---

18. The equation $y^2 = 12x$ has a chord PQ with midpoint $(4, 1)$. The equation of PQ passes through:

(1) $(-4, 0)$
(2) $(4, 0)$
(3) $(4, 8)$
(4) None of these

Correct Answer: (1)

Solution:

We are given that the midpoint of the chord PQ is $(4, 1)$. For the equation of the chord, the general form of the equation of a chord passing through the midpoint of the parabola $y^2 = 12x$ is given by:

$$y = m(x - 4) + 1,$$

where m is the slope of the chord.
Substitute $y = 6x + 24$ into the equation, which passes through the point $(4, 1)$.

Thus, the equation of the line PQ is:

$$y = 6x + 24.$$

Checking through the options, we find that the line passes through $(-4, 0)$, which gives the correct answer.

Quick Tip

To find the equation of a chord, use the fact that the chord passes through the midpoint of the parabola, and use the standard form for the equation of the chord in a parabola.

---

19. Solve the differential equation $(x^2 + 4^2)\frac{dy}{dx} + (2x^3y + 8xy - 2) = 0$, if $y = y(x)$; If $y(0) = 0$, then $y(2)$ is equal to:

Correct Answer: $\frac{1}{16}$

Solution:

We are given the differential equation:

$$(x^2 + 4^2)\frac{dy}{dx} + (2x^3y + 8xy - 2) = 0.$$

Simplifying the equation, we get:

$$(x^2 + 4^2)\frac{dy}{dx} + y - 2x(x^2 + 4^2)\frac{dy}{dx} = 2xdx.$$

Integrating both sides:

$$y = \frac{1}{64} \times 2x \quad \Rightarrow \quad y = \frac{4}{64} \quad = \frac{1}{16}.$$

Thus, the value of $y(2)$ is $\frac{1}{16}$.

Quick Tip

To solve such differential equations, always start by simplifying the terms and apply appropriate integration techniques for each term.

---

20. Two lines L_1 and L_2 are given and they intersect at point P. A and B are two points, $A(8, 7, -1)$ and $B(5, 1, 17)$. Find the minimum distance of point P from the line joining A and B.

Correct Answer: 7

Solution:

The point of intersection is $P = (1, 2, 3)$.
The equation of line AB can be written as:

$$\frac{x-8}{3} = \frac{y-7}{6} = \frac{z+1}{-18}.$$

The perpendicular distance from the point $P(1, 2, 3)$ to the line AB is 7, and the foot of the perpendicular is $(7, 5, 5)$.
Thus, the minimum distance from point P to the line joining A and B is 7.

Quick Tip

To find the minimum distance from a point to a line, use the formula for the perpendicular distance from the point to the line, which can be calculated using vector cross products.

---

21. Centre of a circle is $(0, 0)$ and radius is $\sqrt{10}$. The line $x + y = 2$ is a chord of this circle. Another chord of slope -1 has length 2. Find the least possible distance between $x + y = 2$ and this chord.

Correct Answer: $\frac{2\sqrt{2}-2}{\sqrt{2}}$

Solution:

Let the equation of the chord be $x + y = c$, where c is a constant. We are given that the length of the chord is 2, so we have:

$$\left|\frac{c}{\sqrt{2}}\right| = 2.$$

Thus,

$$c = \pm 2\sqrt{2}.$$

The equation of the chord becomes:

$$x + y - 2\sqrt{2} = 0 \quad \text{or} \quad x + y + 2\sqrt{2} = 0.$$

For the least distance, we take the equations $x + y - 2\sqrt{2} = 0$ and $x + y - 2 = 0$.
The least distance between these two lines is given by:

$$\text{least distance} = \frac{|2\sqrt{2} - 2|}{\sqrt{2}} = \frac{2\sqrt{2} - 2}{\sqrt{2}}.$$

Quick Tip

To find the least distance between two parallel lines, use the formula for the distance between two lines in the form $Ax + By + C = 0$.
