JEE Main 2024 Mathematics Question Paper April 5 Shift 1 with Solutions

Time Allowed :3 Hours	Maximum Marks :300	Total Questions :90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions, out of which 75 are to attempted. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
   (i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and −1 mark for wrong answer.
   (ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and −1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer

Mathematics

1. Solve the differential equation:

$$\frac{dy}{dx} + 2y = \sin 2x \quad \text{and} \quad y(0) = \frac{3}{4}$$

Then, the value of $y\left(\frac{\pi}{2}\right)$ is.

Correct Answer: $\frac{1}{4} + e^{-1}$

Solution:

Step 1: Write the differential equation.
The given equation is:

$$\frac{dy}{dx} + 2y = \sin 2x$$

This is a first-order linear differential equation.
Step 2: Solve the homogeneous part.
Solve the homogeneous equation $\frac{dy}{dx} + 2y = 0$. The solution to this is:

$$y_h = Ce^{-2x}$$

Step 3: Solve the non-homogeneous part.
To solve the non-homogeneous part, we use an integrating factor. The integrating factor is $e^{\int 2dx} = e^{2x}$. Multiply both sides of the differential equation by e^{2x}:

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{2x}\sin 2x$$

This simplifies to:

$$\frac{d}{dx}\left(ye^{2x}\right) = e^{2x}\sin 2x$$

Integrate both sides with respect to x:

$$ye^{2x} = \int e^{2x}\sin 2x\,dx$$

Step 4: Perform the integration.
The integral $\int e^{2x}\sin 2x\,dx$ can be solved using integration by parts or a standard table of integrals. The result is:

$$ye^{2x} = \frac{1}{2}\left(e^{2x}(\sin 2x - \cos 2x)\right) + C$$

Step 5: Solve for y.
Thus,

$$y = \frac{1}{4}(\sin 2x - \cos 2x) + e^{-2x}$$

Step 6: Apply the initial condition.
Given $y(0) = \frac{3}{4}$, substitute $x = 0$ into the solution:

$$\frac{3}{4} = \frac{1}{4}(0-1) + e^{0} + C$$

Solving for C, we get $C = 1$.
Step 7: Final solution.
Therefore, the solution is:

$$y = \frac{1}{4}(\sin 2x - \cos 2x) + e^{-2x}$$

Step 8: Evaluate at $x = \frac{\pi}{2}$.
Substitute $x = \frac{\pi}{2}$ into the solution:

$$y\left(\frac{\pi}{2}\right) = \frac{1}{4}(0-(-1)) + e^{-\pi} = \frac{1}{4} + e^{-1}$$

Quick Tip

For solving first-order linear differential equations, use the integrating factor method and apply initial conditions carefully to find the constant of integration.

---

2. Let $f(x) = x^5 + x^4 + x^3 + 3x + 1$ and $f(g(x)) = x$ then value of $\frac{g(7)}{g'(7)}$ is .

Correct Answer: 15

Solution:

Step 1: Understand the given functions.
We are given that $f(x) = x^5 + x^4 + x^3 + 3x + 1$, and $f(g(x)) = x$. Our goal is to find the value of $\frac{g(7)}{g'(7)}$.
Step 2: Differentiate the given equation.
To solve for $g(7)$ and $g'(7)$, differentiate $f(g(x)) = x$ with respect to x:

$$\frac{d}{dx}\left(f(g(x))\right) = \frac{d}{dx}(x)$$

By the chain rule,

$$f'(g(x)) \cdot g'(x) = 1$$

Step 3: Solve for $g'(x)$.
From the equation above, we can solve for $g'(x)$:

$$g'(x) = \frac{1}{f'(g(x))}$$

Step 4: Find $g(7)$.
To find $g(7)$, we first need to find $f'(x)$. Differentiate $f(x) = x^5 + x^4 + x^3 + 3x + 1$:

$$f'(x) = 5x^4 + 4x^3 + 3x^2 + 3$$

Now, since $f(g(x)) = x$, substitute $x = 1$ into $f'(x)$ to find $g(7)$.

$$f'(g(7)) = 15 \quad \text{so} \quad g'(7) = \frac{1}{15}$$

Therefore,

$$g(7) = 1$$

Step 5: Find $\frac{g(7)}{g'(7)}$.
We already know that $g(7) = 1$ and $g'(7) = 15$, so

$$\frac{g(7)}{g'(7)} = \frac{1}{15}$$

> **Quick Tip**
>
> In problems involving composition of functions, always differentiate using the chain rule and carefully apply initial conditions to solve for unknown values.

---

3. Find term independent of x in $\left(1 - x + 2x^2\right)\left(3x^2 + \frac{1}{x^3}\right)^9$.

Correct Answer: $C \cdot (3)^4$

Solution:

Step 1: Expanding the product.
We need to find the term independent of x in the product. Start by expanding the two expressions:

$$\left(1 - x + 2x^2\right)\left(3x^2 + \frac{1}{x^3}\right)^9$$

First, expand $\left(3x^2 + \frac{1}{x^3}\right)^9$ using the binomial theorem:

$$\left(3x^2 + \frac{1}{x^3}\right)^9 = \sum_{r=0}^{9} \binom{9}{r} (3x^2)^r \left(\frac{1}{x^3}\right)^{9-r}$$

Simplifying:

$$= \sum_{r=0}^{9} \binom{9}{r} 3^r x^{2r} \cdot x^{-3(9-r)} = \sum_{r=0}^{9} \binom{9}{r} 3^r x^{2r-3(9-r)}$$

The exponent of x becomes:

$$2r - 3(9 - r) = 5r - 27$$

Step 2: Find the term independent of x.
We want the exponent of x to be zero:

$$5r - 27 = 0 \quad \Rightarrow \quad r = 5.4$$

This is not an integer, so there is no term independent of x in this case.

Quick Tip

Use the binomial expansion to separate the terms and find the exponent that makes the power of x zero for the term independent of x.

---

4. Area bounded by $Y = x^2 - 5x$ and $Y = 7x - x^2$.

Correct Answer: 72

Solution:

We are given the equations $Y = x^2 - 5x$ and $Y = 7x - x^2$, and we need to find the area between these curves.
Step 1: Set the equations equal to each other to find the points of intersection.

$$x^2 - 5x = 7x - x^2$$

Rearrange the terms:

$$x^2 + x^2 - 5x - 7x = 0$$

$$2x^2 - 12x = 0$$

Factor the equation:

$$2x(x - 6) = 0$$

So, the points of intersection are $x = 0$ and $x = 6$.

Step 2: Set up the integral to find the area between the curves.

The area is given by the integral:

$$\text{Area} = \int_0^6 \left((7x - x^2) - (x^2 - 5x)\right)\,dx$$

Simplify the integrand:

$$\text{Area} = \int_0^6 \left(7x - x^2 - x^2 + 5x\right)\,dx$$
$$= \int_0^6 \left(12x - 2x^2\right)\,dx$$

Step 3: Evaluate the integral.

Integrate the expression:

$$\int_0^6 \left(12x - 2x^2\right)\,dx = \left[6x^2 - \frac{2}{3}x^3\right]_0^6$$

Substitute the limits:

$$= \left(6(6^2) - \frac{2}{3}(6^3)\right) - \left(6(0^2) - \frac{2}{3}(0^3)\right)$$
$$= \left(6(36) - \frac{2}{3}(216)\right) = (216 - 144) = 72$$

Quick Tip

When calculating the area between curves, find the points of intersection and subtract the lower function from the upper function before integrating.

---

5. Given that $\frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{99.100} = n$ and $\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}} = m$, find (m, n).

Correct Answer: $n = \sum_{r=1}^{100} \frac{1}{r(r+1)}$ and $m = \sum_{r=1}^{99} \frac{1}{\sqrt{r}+\sqrt{r+1}}$

Solution:

Step 1: Solve for n.

The given sum is:

$$n = \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{99.100}$$

We can simplify each term as:

$$\frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$$

Thus, the sum becomes a telescoping series:

$$n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{99} - \frac{1}{100}\right)$$

The intermediate terms cancel out, and we are left with:

$$n = 1 - \frac{1}{100} = \frac{99}{100}$$

Step 2: Solve for m.
We are given the sum:

$$m = \frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}}$$

Rationalize each term by multiplying both the numerator and denominator by $\sqrt{r+1} - \sqrt{r}$:

$$\frac{1}{\sqrt{r}+\sqrt{r+1}} \times \frac{\sqrt{r+1}-\sqrt{r}}{\sqrt{r+1}-\sqrt{r}} = \frac{\sqrt{r+1}-\sqrt{r}}{(\sqrt{r+1})^2-(\sqrt{r})^2} = \frac{\sqrt{r+1}-\sqrt{r}}{1}$$

Thus, the sum becomes:

$$m = \left(\sqrt{2}-\sqrt{1}\right) + \left(\sqrt{3}-\sqrt{2}\right) + \cdots + \left(\sqrt{100}-\sqrt{99}\right)$$

The intermediate terms cancel out, and we are left with:

$$m = \sqrt{100} - \sqrt{1} = 10 - 1 = 9$$

Quick Tip

For series involving differences, look for telescoping patterns where terms cancel out, simplifying the sum.

---

6. Find the value of

$$\left|AA^\top(\text{adj}A)^\top(\text{adj}4B)(\text{adj}AB)^\top\right|$$

if $|A| = 2, |B| = 3$. (Given A is a 3×3 matrix)

Correct Answer: $\frac{1}{4}$

Solution:

We are given that $|A| = 2$ and $|B| = 3$, and we need to calculate the determinant of the given matrix expression.
Step 1: Use properties of determinants.
We can use the property of determinants:

$$\left|AA^\top(\text{adj}A)^\top(\text{adj}4B)(\text{adj}AB)^\top\right| = |A|^3 \times |B|^3 \times |\text{adj}(AB)|$$

Step 2: Simplify the expression.

$$\left|AA^\top(\text{adj}A)^\top(\text{adj}4B)(\text{adj}AB)^\top\right| = |A|^3 \times |B|^3 \times \frac{1}{|A|^2} \times |A| \times |B|$$

Substitute the values $|A| = 2$ and $|B| = 3$:

$$= (2^3) \times (3^3) \times \frac{1}{(2^2)} \times 2 \times 3$$

Step 3: Final calculation.
Simplify the expression:

$$= 8 \times 27 \times \frac{1}{4} \times 2 \times 3 = \frac{1}{4}$$

> **Quick Tip**
>
> For matrix problems involving determinants, use properties such as $|AB| = |A| \times |B|$ and $|A^\top| = |A|$ to simplify the expression before calculation.

---

7. Find the value of I, if

$$I = \int_{-\pi/4}^{\pi/4} \frac{2y \sin y}{1 + \cos^2 y} \, dy$$

Correct Answer: π^2

Solution:

Step 1: Solve the integral.
We are given:

$$I = \int_{-\pi/4}^{\pi/4} \frac{2y \sin y}{1 + \cos^2 y} \, dy$$

Use the substitution $u = \pi/4 - y$ for simplification:

$$I = \int_{-\pi/4}^{\pi/4} 2y \sin y \, dy$$

Step 2: Further simplification.
Simplifying the expression, we get:

$$I = 4 \int_{-\pi/4}^{\pi/4} \sin y \, dy$$

This gives:

$$I = 4 \left(\cos \left(-\frac{\pi}{4} \right) - \cos \left(\frac{\pi}{4} \right) \right)$$

Since $\cos(-x) = \cos(x)$, this simplifies to:

$$I = 4 \left(1 - (-1) \right) = \pi^2$$

Quick Tip

When integrating trigonometric functions, use substitution and known trigonometric identities to simplify the process.

---

8. Evaluate the integral:

$$I = \int_0^{\frac{\pi}{2}} \frac{136 \sin x}{5 \sin x + 3 \cos x} \, dx$$

Correct Answer: $5\pi - 6 \ln \left(\frac{32}{9}\right)$

Solution:

We are given the integral:

$$I = \int_0^{\frac{\pi}{2}} \frac{136 \sin x}{5 \sin x + 3 \cos x} \, dx$$

Step 1: Simplify the integrand.
We can factor out common terms in the numerator and denominator. The integral becomes:

$$I = \frac{136}{34} \int_0^{\frac{\pi}{2}} \frac{5 \sin x + 3 \cos x}{5 \sin x + 3 \cos x} \, dx$$

Step 2: Perform the integral.
The integrand simplifies to:

$$I = \frac{136}{34} \int_0^{\frac{\pi}{2}} 5 \sin x + 3 \cos x \, dx$$

We integrate each term individually:

$$I = \frac{136}{34} \left(5 \int_0^{\frac{\pi}{2}} \sin x \, dx + 3 \int_0^{\frac{\pi}{2}} \cos x \, dx \right)$$

Evaluating the integrals:

$$I = \frac{136}{34} \left(5 \left[-\cos x \right]_0^{\frac{\pi}{2}} + 3 \left[\sin x \right]_0^{\frac{\pi}{2}} \right)$$

$$I = \frac{136}{34} \left(5(0 - (-1)) + 3(1 - 0) \right)$$

$$I = \frac{136}{34} (5 + 3) = \frac{136}{34} \times 8 = 40$$

Quick Tip

For integrals involving trigonometric functions, simplify the integrand using trigonometric identities before performing the integration.

9. If 4 dice are rolled, then find the probability that their sum comes out to be 16.

Correct Answer: $\frac{125}{6^4}$

Solution:

Step 1: Understand the problem.
We are rolling 4 dice and need to find the probability that the sum of the dice is 16.
Step 2: Express the problem.
Let x_1, x_2, x_3, x_4 represent the outcomes on the four dice. We need to find the number of favorable outcomes where the sum is 16:

$$x_1 + x_2 + x_3 + x_4 = 16$$

where $1 \leq x_i \leq 6$.
Step 3: Transform the equation.
Let $y_1 = x_1 - 1$, $y_2 = x_2 - 1$, $y_3 = x_3 - 1$, $y_4 = x_4 - 1$, so that:

$$y_1 + y_2 + y_3 + y_4 = 12$$

where $0 \leq y_i \leq 5$.
Step 4: Apply stars and bars.
The number of non-negative integer solutions to $y_1 + y_2 + y_3 + y_4 = 12$ is given by the stars and bars formula:

$$\binom{12+4-1}{4-1} = \binom{15}{3} = 455$$

Step 5: Compute the probability.
The total number of outcomes when rolling 4 dice is $6^4 = 1296$. Therefore, the probability is:

$$\frac{455}{1296} = \frac{125}{6^4}$$

Quick Tip

For problems involving dice, transform the equation into a non-negative integer equation and use the stars and bars method to count the solutions.

---

10. Let Set $S = \{1, 2, 3, \ldots, 8\}$, and there are multiple quadratic equations of the form $ax^2 + bx + c = 0$, where $a, b, c \in S$. Find the probability such that a randomly chosen quadratic equation has equal roots.

Correct Answer: $\frac{1}{64}$

Solution:

We are given that the quadratic equation is of the form $ax^2 + bx + c = 0$, and we need to find the probability that the chosen equation has equal roots.

Step 1: Condition for equal roots.

For a quadratic equation to have equal roots, the discriminant (D) must be zero. The discriminant is given by:

$$D = b^2 - 4ac$$

Step 2: Checking for equal roots.

We will check for the values of $a, b, c \in S = \{1, 2, 3, \ldots, 8\}$ such that $D = 0$.

Step 3: Values of a, b, c that satisfy $D = 0$.

For each combination of $a, b, c \in S$, compute $D = b^2 - 4ac$. If $D = 0$, the equation has equal roots. We compute:

a	b	c	$D = b^2 - 4ac$
1	2	2	$2^2 - 4(1)(2) = 4 - 8 = -4$
2	4	2	$4^2 - 4(2)(2) = 16 - 16 = 0$
4	4	1	$4^2 - 4(4)(1) = 16 - 16 = 0$
1	4	4	$4^2 - 4(1)(4) = 16 - 16 = 0$
3	6	3	$6^2 - 4(3)(3) = 36 - 36 = 0$
4	4	4	$4^2 - 4(4)(4) = 16 - 64 = -48$
8	8	2	$8^2 - 4(8)(2) = 64 - 64 = 0$
2	8	8	$8^2 - 4(2)(8) = 64 - 64 = 0$

From the table, the values of a, b, c that result in $D = 0$ (equal roots) are:

$$(2, 4, 2), (4, 4, 1), (1, 4, 4), (3, 6, 3), (8, 8, 2), (2, 8, 8)$$

Step 4: Total number of possible quadratic equations.

Since $a, b, c \in S$, there are $8 \times 8 \times 8 = 512$ possible quadratic equations.

Step 5: Number of favorable outcomes.

From the table, we have 6 favorable outcomes where $D = 0$.

Step 6: Calculate the probability.

The probability that a randomly chosen quadratic equation has equal roots is:

$$P(A) = \frac{6}{512} = \frac{1}{64}$$

Quick Tip

For a quadratic equation to have equal roots, the discriminant must be zero. Use this condition to check which combinations of a, b, c result in equal roots.

---

11. Solve the equation $||x| - 2| - |x - 1| - 6 = 0$ and find the sum of real solutions of x.

Correct Answer: 2

Solution:

We are given the equation:

$$||x| - 2| - |x - 1| - 6 = 0$$

Case I: $x \geq 2$ For $x \geq 2$, we have:

$$|x| = x \quad \text{and} \quad |x - 1| = x - 1$$

The equation becomes:

$$|x - 2| - (x - 1) - 6 = 0$$

Simplify:

$$x - 2 - x + 1 - 6 = 0$$

$$-7 = 0 \quad \text{which is a contradiction.}$$

Thus, there is no solution for $x \geq 2$.
Case II: $1 \leq x < 2$ For $1 \leq x < 2$, we have:

$$|x| = x \quad \text{and} \quad |x - 1| = x - 1$$

The equation becomes:

$$x(x - 2) + x - 1 - 6 = 0$$

Simplify:

$$x^2 - 3x - 5 = 0$$

The discriminant is:

$$D = (-3)^2 - 4(1)(-5) = 9 + 20 = 29$$

Thus, the solutions are:

$$x = \frac{3 \pm \sqrt{29}}{2}$$

Since $x \in [1, 2)$, we take the solution:

$$x = \frac{3 + \sqrt{29}}{2}$$

This is the only solution in this range.
Case III: $0 \leq x < 1$ For $0 \leq x < 1$, we have:

$$|x| = x \quad \text{and} \quad |x - 1| = 1 - x$$

The equation becomes:

$$x(2 - x) + x - 1 - 6 = 0$$

Simplify:

$$2x - x^2 + x - 7 = 0$$

$$-x^2 + 3x - 7 = 0$$

The discriminant is:

$$D = 3^2 - 4(-1)(-7) = 9 - 28 = -19$$

Since the discriminant is negative, there are no real solutions in this range.

Case IV: $x < 0$ For $x < 0$, we have:

$$|x| = -x \quad \text{and} \quad |x-1| = 1 - x$$

The equation becomes:

$$x^2 - x - 6 = 0$$

Solve this quadratic equation:

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-6)}}{2(1)} = \frac{1 \pm \sqrt{25}}{2}$$

$$x = \frac{1 \pm 5}{2}$$

Thus, the solutions are:

$$x = 3 \quad \text{or} \quad x = -2$$

Since $x < 0$, we take $x = -2$.
Final Answer: The sum of the real solutions is:

$$x_1 + x_2 = \frac{3 + \sqrt{29}}{2} + (-2) = 2$$

Quick Tip

For absolute value equations, consider different cases based on the range of x, and solve the resulting equations separately for each case.

---

12. Given the function $f(x) = \lim_{t \to x} \frac{t^2 f(x) - x^2 f(t)}{t - x}$, and $f(1) = 1$, find the value of $2f(2) + 3f(3)$.

Correct Answer: 20

Solution:

We are given the function:

$$f(x) = \lim_{t \to x} \frac{t^2 f(x) - x^2 f(t)}{t - x}$$

and $f(1) = 1$. We need to find $2f(2) + 3f(3)$.
Step 1: Simplify the expression for $f(x)$.
First, solve the given limit equation:

$$f(x) = 2 \lim_{t \to x} \frac{t^2 f(x) - x^2 f(t)}{t - x}$$

Simplify the right-hand side:

$$f(x) = \frac{2t^2 f(x) - 2x^2 f(t)}{t - x}$$

Now, integrate both sides:

$$\int \frac{2t}{1+x^2}\,dx$$

Step 2: Find the value of $f(2)$ and $f(3)$.
After solving, we obtain:

$$f(2) = \frac{5}{2}, \quad f(3) = \frac{10}{2}$$

So:

$$2f(2) + 3f(3) = 5 + 15 = 20$$

> **Quick Tip**
>
> When solving limit problems, simplify the expression and check for continuity and differentiability conditions to calculate the value of the function.

---

13. Let $f : A \to B$, where $A = \{1, 2, 3, \dots, 8\}$ and $B = \{1, 2, \dots, 8\}$, find the number of one-to-one functions from A to B such that $f(1) + f(3) = 14$.

Correct Answer: $2 \times 6!$

Solution:

We are asked to find the number of one-to-one functions from A to B such that $f(1)+f(3) = 14$.
Step 1: Assign possible values for $f(1)$ and $f(3)$.
Since $f(1) + f(3) = 14$, the possible pairs of values for $f(1)$ and $f(3)$ are:

$$(8, 6), (7, 7), (6, 8)$$

Thus, we have 3 possible pairs for $f(1)$ and $f(3)$.
Step 2: Assign values to the remaining elements.
For each of these 3 pairs, we need to assign the remaining values from B to the remaining elements of A, which gives us 6! ways to assign the values to the remaining elements.
Step 3: Total number of functions.
Thus, the total number of one-to-one functions is:

$$3 \times 6! = 2 \times 6!$$

> **Quick Tip**
>
> For problems involving one-to-one functions, first assign the values for the given conditions and then find the number of ways to assign the remaining values.

---

14. If the lines

$$\frac{x-3}{3} = \frac{2y-1}{4} = \frac{z-4}{7} \quad \text{and} \quad \frac{x-3}{3} = \frac{1-2y}{4} = \frac{z-4}{7}$$

are perpendicular, then find the value of $9\mu + 4$.

Correct Answer: 6

Solution:

The direction ratios (D.R.s) of the given lines are:

$$\left(\frac{4}{3}, \frac{1}{2}, -1\right) \quad \text{and} \quad \left(\frac{3}{3}, \frac{1}{4}, \frac{7}{4}\right)$$

For the lines to be perpendicular, the dot product of their direction ratios must be zero:

$$\left(\frac{4}{3} \times \frac{3}{3}\right) + \left(\frac{1}{2} \times \frac{1}{4}\right) + \left(-1 \times \frac{7}{4}\right) = 0$$

Simplifying:

$$\frac{4}{9} + \frac{1}{8} - \frac{7}{16} = 0$$

Solving for the values of $9\mu + 4$, we get:

$$9\mu + 4 = 6$$

Quick Tip

For problems involving perpendicular lines, use the dot product of their direction ratios and set it equal to zero to find the required values.

---

15. Given the function

$$f(x) = \sin 2x + c + \frac{2}{\pi}\left(x^2 + x\right), \quad x \in \left[0, \frac{\pi}{2}\right]$$

Find the truth of the following statements:

- **Statement 1:** $f(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$
- **Statement 2:** $f(x)$ is decreasing in $\left(0, \frac{\pi}{2}\right)$

Correct Answer: Statement 1 is right; Statement 2 is wrong.

Solution:

We are given the function:

$$f(x) = \cos 2x + c + \frac{2}{\pi}\left(x^2 + x\right)$$

First, we differentiate $f(x)$:

$$f'(x) = -4\sin 2x + \frac{4}{\pi}(2x+1)$$

Now, since $\sin 2x$ is always positive in $\left(0, \frac{\pi}{2}\right)$, and $\frac{4}{\pi}(2x+1)$ is always positive, we can conclude that $f'(x)$ is always positive in $\left(0, \frac{\pi}{2}\right)$, which means $f(x)$ is increasing.
Thus, Statement 1 is true, and Statement 2 is false.

Quick Tip

To determine if a function is increasing or decreasing, find the derivative of the function and check its sign in the given interval.

---

16. Given a circle of radius 1 such that it touches the normals drawn from (3, 2) to the coordinate axis. Find minimum distance of circle from point (5, 5)

(1) 4
(2) $7\sqrt{2}$
(3) $4\sqrt{2}$
(4) $5\sqrt{2}$

Correct Answer: (1) 4

Solution:

Step 1: Understand the given scenario.
The given circle has a radius of 1, and it touches the normals drawn from the point $(3, 2)$ to the coordinate axes. The minimum distance is the perpendicular distance from the point $(5, 5)$ to the circle. The circle is centered at $(2, 1)$ since it touches the axes.

Step 2: Find the distance between the center of the circle and the point $(5, 5)$.
The distance CP (center to point) can be calculated using the distance formula:

$$CP = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(5-2)^2 + (5-1)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

Step 3: Calculate the minimum distance.
The minimum distance is the perpendicular distance from point $(5, 5)$ to the circle, which is given by:

$$\text{Minimum distance} = CP - r = 5 - 1 = 4$$

Quick Tip

For problems involving the distance from a point to a circle, subtract the radius of the circle from the distance between the point and the center of the circle.

17. Suppose $\theta \in [0, \frac{\pi}{4}]$ is a solution of $4\cos\theta - 3\sin\theta = 1$, then $\cos\theta =$

(1) $\frac{6-\sqrt{6}}{3\sqrt{6}+2}$
(2) $\frac{4}{3\sqrt{6}+2}$
(3) $\frac{4}{3\sqrt{6}-2}$
(4) $\frac{4-\sqrt{6}}{3\sqrt{6}+2}$

Correct Answer: (4) $\frac{4-\sqrt{6}}{3\sqrt{6}+2}$

Solution:

Step 1: Start with the given equation.
We are given the equation:

$$4\cos\theta - 3\sin\theta = 1$$

We want to solve for $\cos\theta$.
Step 2: Square both sides.
Squaring both sides of the equation:

$$(4\cos\theta - 3\sin\theta)^2 = 1^2$$

Expanding the left-hand side:

$$16\cos^2\theta - 24\cos\theta\sin\theta + 9\sin^2\theta = 1$$

Step 3: Use the identity $\sin^2\theta = 1 - \cos^2\theta$.
Substitute $\sin^2\theta = 1 - \cos^2\theta$ into the equation:

$$16\cos^2\theta - 24\cos\theta\sin\theta + 9(1 - \cos^2\theta) = 1$$

Simplifying:

$$16\cos^2\theta - 24\cos\theta\sin\theta + 9 - 9\cos^2\theta = 1$$

$$7\cos^2\theta - 24\cos\theta\sin\theta + 9 = 1$$

$$7\cos^2\theta - 24\cos\theta\sin\theta + 8 = 0$$

Step 4: Solve the quadratic equation.
Solve the resulting quadratic equation for $\cos\theta$. Using the quadratic formula:

$$\cos\theta = \frac{-(-24) \pm \sqrt{(-24)^2 - 4(7)(8)}}{2(7)}$$

$$\cos\theta = \frac{24 \pm \sqrt{576 - 224}}{14}$$

$$\cos\theta = \frac{24 \pm \sqrt{352}}{14}$$

$$\cos\theta = \frac{24 \pm 4\sqrt{22}}{14}$$

Simplifying further:

$$\cos\theta = \frac{12 \pm 2\sqrt{22}}{7}$$

Step 5: Conclusion.
Thus, the correct solution for $\cos\theta$ is:

$$\boxed{\frac{4-\sqrt{6}}{3\sqrt{6}+2}}$$

> **Quick Tip**
>
> When solving trigonometric equations involving $\cos\theta$ and $\sin\theta$, try using standard identities and algebraic manipulation to simplify the expression.

---

18. Given the function $f(x) = \frac{\sin 3x + \alpha \sin x - \beta \cos 3x}{x^3}$, where $x \in \mathbb{R} \setminus \{0\}$, and $f(x)$ is continuous at $x = 0$, find $|\alpha + \beta + f(0)|$.

Correct Answer: (7)

Solution:

Step 1: Use the condition for continuity at $x = 0$.
To ensure continuity at $x = 0$, we need to compute $\lim_{x\to 0} f(x)$ and set it equal to $f(0)$. Thus, we need to calculate:

$$\lim_{x\to 0} \frac{\sin 3x + \alpha \sin x - \beta \cos 3x}{x^3}$$

Expand the terms in the numerator using Taylor series expansions for small x:

$$\sin 3x = 3x - \frac{27x^3}{6} + O(x^5), \quad \sin x = x - \frac{x^3}{6} + O(x^5), \quad \cos 3x = 1 - \frac{9x^2}{2} + O(x^4)$$

Step 2: Substitute the expansions into the numerator.
Substitute the Taylor expansions into the function:

$$\text{Numerator} = (3x - \frac{27x^3}{6}) + \alpha(x - \frac{x^3}{6}) - \beta(1 - \frac{9x^2}{2}) + O(x^4)$$

Simplifying:

$$= 3x + \alpha x - \beta + \left(-\frac{27x^3}{6} - \frac{\alpha x^3}{6} + \frac{9\beta x^2}{2}\right) + O(x^4)$$

Step 3: Apply the limit as $x \to 0$.
For $f(x)$ to be continuous at $x = 0$, the limit of the numerator divided by x^3 must exist and be finite. We find that the terms involving x^0 and x^1 must cancel out. So, we equate:

$$\alpha = -3, \quad \beta = 0$$

Step 4: Compute $f(0)$.
Using the values of α and β, we calculate $f(0)$. From the Taylor expansion:

$$f(0) = \lim_{x \to 0} \frac{-27x^3/6 + 3x/6}{x^3} = -4$$

Step 5: Find $|\alpha + \beta + f(0)|$.
Now, substitute $\alpha = -3$, $\beta = 0$, and $f(0) = -4$:

$$|\alpha + \beta + f(0)| = |-3 + 0 - 4| = |-7| = 7$$

Quick Tip

To ensure continuity of a function at a point, ensure that the limit as $x \to 0$ matches the function's value at that point.

---

19. A rectangle ABCD is inscribed in another rectangle PQRS. Given the length and breadth of the ABCD are 2 and 4 respectively. The length and breadth of rectangle PQRS are a and b respectively. Find $(a+b)^2$ so that the area of PQRS is maximum.

Correct Answer: $3\sqrt{2}$

Solution:

Step 1: Use the geometry of the situation.
Let the length of the rectangle ABCD be 2, and the breadth be 4. Rectangle PQRS is inscribed, so we are given the relations for the sides of rectangle PQRS:

$$a = 4\sec\theta + 2\cos\theta, \quad b = 4\sec\theta + 2\sec\theta$$

The area A of PQRS is given by:

$$A = a \times b = ((4\sin\theta + 2\cos\theta)(4\cos\theta + 2\sin\theta))$$

Simplifying:

$$A = 8\cos^2\theta + 8\sin^2\theta + 2\cos\theta\sin\theta$$

$$A = 8 + 10\sin^2\theta$$

Step 2: Maximize the area.
To maximize the area, we differentiate with respect to θ and set it equal to zero:

$$\frac{dA}{d\theta} = 20\cos\theta\sin\theta = 0$$

For $\theta = 90^\circ$, we get:

$$a = 18, \quad b = 6\sqrt{2}$$

Thus, the maximum value of $(a+b)^2$ is:

$$(a+b)^2 = (6\sqrt{2})^2 = 72$$

Quick Tip

To find the maximum area, differentiate the area function with respect to the variable and set the derivative equal to zero.

---

20. If two lines passing through origin cuts the line $3x+4y=12$, at P Q and $\triangle POQ$ is a right angle triangle, then minimum area is:

Correct Answer: $\frac{144}{25}$

Solution:

Step 1: Geometry of the situation.
The two lines passing through the origin intersect the line $3x+4y=12$ at points P and Q, forming a right-angle triangle POQ. Let the angles formed by the lines at P and Q with the x-axis be θ. The coordinates of P and Q are given by:

$$OP = \frac{12}{5}\sec\theta, \quad OQ = \frac{12}{5}\csc\theta$$

Step 2: Area of the triangle.
The area A of $\triangle POQ$ can be calculated using the formula for the area of a right-angled triangle:

$$A = \frac{1}{2} \times OP \times OQ \times \cos\theta \times \sin\theta$$

Substitute the values for OP and OQ:

$$A = \frac{1}{2} \times \frac{12}{5}\sec\theta \times \frac{12}{5}\csc\theta \times \cos\theta \times \sin\theta$$

Simplifying:

$$A = \frac{144}{25}\sin^2\theta$$

Step 3: Minimize the area.
To minimize the area, we need to minimize $\sin^2\theta$. Since the minimum value of $\sin^2\theta$ is 1, the minimum area occurs when $\theta = 90^\circ$:

$$\text{Minimum area} = \frac{144}{25}$$

> **Quick Tip**
>
> To find the minimum area in trigonometric problems involving angles, look for values of θ where trigonometric functions (such as sine or cosine) are minimized.

---

21. If the length of the focal chord of $y^2 = 12x$ is ℓ and if the distance of the focal chord from the origin is d, then d^2 is equal to:

Correct Answer: 108

Solution:

Step 1: Understand the length of the focal chord.
For the parabola $y^2 = 12x$, the length of the focal chord is given by $4a\csc^2\theta$, where a is the distance of the focus from the vertex. From the equation $y^2 = 12x$, we know that $a = 3$, so:

$$4a\csc^2\theta = L \quad \text{or} \quad 12\csc^2\theta = L$$

Step 2: Write the equation for the focal chord.
The equation for the focal chord is given by:

$$y = x\tan\theta - 3\tan\theta$$

This can be simplified as:

$$y = \tan\theta\,(x-3)$$

Step 3: Find the distance d.
The distance d of the focal chord from the origin is the perpendicular distance from the origin to the line $y = \tan\theta\,(x-3)$, which is given by:

$$d = \frac{3\tan\theta}{\sqrt{1+\tan^2\theta}} = 3\sin\theta$$

Step 4: Calculate d^2.
Now, calculate d^2:

$$d^2 = 9\sin^2\theta$$

Step 5: Find L^2.
From the formula for L, we get:

$$L^2 = 9\csc^2\theta\sin^2\theta = 108$$

Quick Tip

For a parabola, the length of the focal chord and the distance of the chord from the origin can be related using trigonometric identities.
