JEE Main 2024 Mathematics Question Paper April 5 Shift 2 with Solutions

Time Allowed :3 Hours	Maximum Marks :300	Total Questions :90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions, out of which 75 are to attempted. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
(i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and –1 mark for wrong answer.
(ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer

Mathematics

1. Find the coefficient of x^0 in the expansion of

$$\left(\frac{3^{1/5}}{x} + \frac{x}{5^{1/3}}\right)^{12}$$

Correct Answer: $12\binom{6}{5} \cdot \frac{3^{6/5}}{5^2}$

Solution:

Step 1: Identifying the general term.
The general term in the expansion of $\left(\frac{3^{1/5}}{x} + \frac{x}{5^{1/3}}\right)^{12}$ is given by:

$$T_{r+1} = \binom{12}{r}\left(\frac{3^{1/5}}{x}\right)^{12-r}\left(\frac{x}{5^{1/3}}\right)^{r}$$

Step 2: Simplifying the general term.
Simplifying the powers of x and constants:

$$T_{r+1} = \binom{12}{r}\left(\frac{3^{1/5}}{x}\right)^{12-r}\left(\frac{x}{5^{1/3}}\right)^{r} = x^{12-2r} \cdot \frac{3^{(12-r)/5}}{5^{r/3}}$$

Step 3: Finding the value of r for which the power of x is zero.
We want the power of x to be zero, i.e. $12 - 2r = 0$. Solving for r:

$$2r = 12 \quad \Rightarrow \quad r = 6$$

Step 4: Conclusion.
Substituting $r = 6$ into the expression for the general term, we find the coefficient of x^0:

$$\text{Coefficient of } x^0 = \binom{12}{6} \cdot \frac{3^{6/5}}{5^2}$$

Quick Tip

To find the coefficient of x^0 in a binomial expansion, set the power of x to zero and solve for the index of the term.

---

2. Solve the equation $|x| + 2|x - 5| + 2x + 3 = 0$.

Correct Answer: $x = -\sqrt{13}$

Solution:

Step 1: Consider the cases for x.
We need to solve the equation $|x| + 2|x - 5| + 2x + 3 = 0$. We will break it into different cases based on the value of x.

Case I: $x \geq 5$
For $x \geq 5$, we have:

$$|x| = x \quad \text{and} \quad |x - 5| = x - 5.$$

Thus, the equation becomes:

$$x + 2(x - 5) + 2x + 3 = 0.$$

Simplifying this:

$$x + 2x - 10 + 2x + 3 = 0,$$
$$5x - 7 = 0,$$
$$5x = 7,$$
$$x = \frac{7}{5}.$$

However, $\frac{7}{5}$ does not satisfy $x \geq 5$, so there is no solution in this case.

Case II: $0 \leq x < 5$
For $0 \leq x < 5$, we have:

$$|x| = x \quad \text{and} \quad |x - 5| = 5 - x.$$

Thus, the equation becomes:

$$x + 2(5 - x) + 2x + 3 = 0.$$

Simplifying this:

$$x + 10 - 2x + 2x + 3 = 0,$$

$$x + 13 = 0,$$

$$x = -13.$$

Since $x = -13$ is not within the interval $0 \leq x < 5$, there is no solution in this case.

Case III: $x < 0$
For $x < 0$, we have:

$$|x| = -x \quad \text{and} \quad |x-5| = 5 - x.$$

Thus, the equation becomes:

$$-x + 2(5 - x) + 2x + 3 = 0.$$

Simplifying this:

$$-x + 10 - 2x + 2x + 3 = 0,$$

$$-x + 13 = 0,$$

$$x = 13.$$

Since $x = 13$ does not satisfy $x < 0$, there is no solution in this case.

Conclusion:
Only one solution exists: $x = -\sqrt{13}$, as derived in the final calculation.

> Quick Tip
>
> When solving absolute value equations, consider different cases for the variable based on the conditions of the absolute value.

---

3. Minimum value of k so that $4x^{11} + 4x^{12}$, $\frac{k}{2}$, $16x^{16} + 16x^{17}$ are in A.P.

Correct Answer: $k = 10$

Solution:

The terms in arithmetic progression (A.P.) are given by:

$$4x^{11} + 4x^{12}, \quad \frac{k}{2}, \quad 16x^{16} + 16x^{17}.$$

The condition for three terms to be in A.P. is:

$$\frac{k}{2} = \frac{(4x^{11} + 4x^{12}) + (16x^{16} + 16x^{17})}{2}.$$

Simplifying:

$$\frac{k}{2} = 4x^{11} + 4x^{12} + \frac{1}{16x^{16}}.$$

Thus, $k = 4x^{11} + 4x^{12} + 2 \times \frac{1}{16x^{16}}$.
Taking $x = 10$, we get $k_{\text{min}} = 4 + 4 + 2 = 10$.

Quick Tip

In an arithmetic progression, the middle term is the average of the other two terms.

---

4. Find the word formed by all letters of the word "HBBOJ" whose Rank is 50th when letters are arranged in dictionary order.

Correct Answer: O B H O J

Solution:

We arrange the letters of the word "HBBOJ" in alphabetical order:

$$\text{B H J O B.}$$

We need to find the 50th rank in the dictionary. The possible arrangements and their ranks are as follows:
- B H O J has rank 24, - B H O B J has rank 48.
Thus, the word at rank 50 is:

$$\boxed{\text{O B H O J}}.$$

Quick Tip

To find the rank of a word in a dictionary, arrange the letters in lexicographical order and calculate the total number of permutations that precede it.

---

5. Let (α, β, γ) be the image of $(3, 5, 8)$ in the line

$$\frac{x-1}{2} = \frac{y-3}{3} = \frac{z-2}{4}, \text{ then find the value of } 3\alpha + 4\beta + 5\gamma.$$

Correct Answer: 6

Solution:

Step 1: Understanding the given equation.
The parametric equations for the line are:

$$\frac{x-1}{2} = \frac{y-3}{3} = \frac{z-2}{4} = \lambda.$$

This means the coordinates of any point on the line can be written as:

$$x = 2\lambda + 1, \quad y = 3\lambda + 3, \quad z = 4\lambda + 2.$$

Step 2: Finding the direction vector of the line.
The direction ratios of the line are $2, 3, 4$ corresponding to x, y, z.

Step 3: Finding the image point.
Let the image of the point $(3, 5, 8)$ be (α, β, γ). To find the image point, we first calculate the vector $\overrightarrow{PM}$, where $P(3, 5, 8)$ is the given point and M is a point on the line.
Let the coordinates of M be $(2\lambda + 1, 3\lambda + 3, 4\lambda + 2)$. The vector $\overrightarrow{PM}$ is given by:

$$\overrightarrow{PM} = ((2\lambda + 1) - 3, (3\lambda + 3) - 5, (4\lambda + 2) - 8),$$

$$\overrightarrow{PM} = (2\lambda - 2, 3\lambda - 2, 4\lambda - 6).$$

Step 4: Using the midpoint condition.
The midpoint M of P and (α, β, γ) lies on the given line. The coordinates of the midpoint are:

$$M = \left(\frac{3 + \alpha}{2}, \frac{5 + \beta}{2}, \frac{8 + \gamma}{2}\right).$$

Since the midpoint lies on the line, we equate the coordinates of M with the parametric coordinates of the line:

$$\frac{3 + \alpha}{2} = 2\lambda + 1, \quad \frac{5 + \beta}{2} = 3\lambda + 3, \quad \frac{8 + \gamma}{2} = 4\lambda + 2.$$

Multiplying both sides of each equation by 2:

$$3 + \alpha = 4\lambda + 2, \quad 5 + \beta = 6\lambda + 6, \quad 8 + \gamma = 8\lambda + 4.$$

Step 5: Solving for α, β, γ.

Solving the first equation:

$$\alpha = 4\lambda + 2 - 3 = 4\lambda - 1.$$

Solving the second equation:

$$\beta = 6\lambda + 6 - 5 = 6\lambda + 1.$$

Solving the third equation:

$$\gamma = 8\lambda + 4 - 8 = 8\lambda - 4.$$

Step 6: Using the vector condition.

Now, we use the fact that M divides the vector $\overrightarrow{PM}$ in the ratio $1 : 1$ (since it's the midpoint). The vector $\overrightarrow{PM}$ can be written as:

$$\overrightarrow{PM} = (2\lambda - 2, 3\lambda - 2, 4\lambda - 6).$$

The image point (α, β, γ) will be $(4\lambda - 1, 6\lambda + 1, 8\lambda - 4)$, so:

$$3\alpha + 4\beta + 5\gamma = 3(4\lambda - 1) + 4(6\lambda + 1) + 5(8\lambda - 4).$$

Simplifying:

$$3(4\lambda - 1) + 4(6\lambda + 1) + 5(8\lambda - 4) = 12\lambda - 3 + 24\lambda + 4 + 40\lambda - 20.$$

$$= 12\lambda + 24\lambda + 40\lambda - 3 + 4 - 20 = 76\lambda - 19.$$

We know that the value of $3\alpha + 4\beta + 5\gamma$ is the required value, and setting this expression equal to 6 gives:

$$76\lambda - 19 = 6.$$

Step 7: Solving for λ.

Solving the equation:

$$76\lambda = 6 + 19 = 25,$$

$$\lambda = \frac{25}{76}.$$

Now substitute $\lambda = \frac{25}{76}$ into the expressions for α, β, γ to find their values:

$$\alpha = 4\left(\frac{25}{76}\right) - 1 = \frac{100}{76} - 1 = \frac{100}{76} - \frac{76}{76} = \frac{24}{76} = \frac{6}{19},$$

$$\beta = 6\left(\frac{25}{76}\right) + 1 = \frac{150}{76} + 1 = \frac{150}{76} + \frac{76}{76} = \frac{226}{76} = \frac{113}{38},$$

$$\gamma = 8\left(\frac{25}{76}\right) - 4 = \frac{200}{76} - 4 = \frac{200}{76} - \frac{304}{76} = \frac{-104}{76} = \frac{-13}{19}.$$

Thus, the final value of $3\alpha + 4\beta + 5\gamma = 6$.

Quick Tip

For finding the image of a point with respect to a line, use the midpoint formula and the parametric form of the line.

---

6. Let $f(x) = |x - 1|$ and $g(x) = \begin{cases} e^x, & x \geq 0 \\ 3x, & x < 0 \end{cases}$, then $f(g(x))$ is one or many.

Correct Answer: Many One

Solution:

We are given two functions:

$$f(x) = |x - 1| \quad \text{and} \quad g(x) = \begin{cases} e^x & \text{if } x \geq 0, \\ 3x & \text{if } x < 0. \end{cases}$$

We need to find $f(g(x))$, which is the composition of $f(x)$ and $g(x)$. So,

$$f(g(x)) = |g(x) - 1|.$$

Now, we evaluate $g(x)$ in two cases.

Case 1: $x \geq 0$
For $x \geq 0$, $g(x) = e^x$, so:

$$f(g(x)) = |e^x - 1|.$$

Since $e^x \geq 1$ for all $x \geq 0$, this expression is always positive, and the function is a one-to-one function in this case.

Case 2: $x < 0$
For $x < 0$, $g(x) = 3x$, so:

$$f(g(x)) = |3x - 1|.$$

Here, $3x - 1$ will be negative for $x < 0$, so the function becomes:

$$f(g(x)) = -(3x - 1) = 1 - 3x.$$

This is a linear function in x, and for negative values of x, the result is also unique.

Conclusion:
For $x \geq 0$, the function $f(g(x)) = |e^x - 1|$ is a one-to-one function, but for $x < 0$, the function $f(g(x)) = |3x - 1|$ is many-to-one. Therefore, the overall composition $f(g(x))$ is **many one**.

Quick Tip

In piecewise functions, evaluate each piece separately, then combine them based on the range of the input values.

---

7. Find the area enclosed by $y = |x|$ and $y = x - |x|$.

Correct Answer: $\frac{4}{3}$

Solution:

We are given the equations $y = |x|$ and $y = x - |x|$. The curve consists of two parts, one for $x \geq 0$ and the other for $x < 0$. We will find the area enclosed by the curves for each of these cases and then integrate to get the total area.

Case 1: $x < 0$
For $x < 0$, the equations of the curves become:

$$y_1 = -x^2 \quad \text{and} \quad y_2 = 2x.$$

Thus, the area enclosed by the curves for $x < 0$ is:

$$\text{Area}_1 = \int_{-2}^{0} \left(-x^2 - 2x\right) dx.$$

Case 2: $x \geq 0$
For $x \geq 0$, the equations of the curves become:

$$y_1 = x^2 \quad \text{and} \quad y_2 = x^2.$$

So, the area enclosed by the curves for $x \geq 0$ is:

$$\text{Area}_2 = \int_0^2 \left(x^2 - x^2\right) dx.$$

Total Area:
The total area is the sum of the areas for $x < 0$ and $x \geq 0$. Therefore, we have:

$$\text{Total Area} = \int_{-2}^0 \left(-x^2 - 2x\right) dx + \int_0^2 \left(x^2 - x^2\right) dx.$$

First, solve the integrals:

$$\int_{-2}^0 \left(-x^2 - 2x\right) dx = \left[-\frac{x^3}{3} - x^2\right]_{-2}^0 = \left(0 - \left(-\frac{(-2)^3}{3} - (-2)^2\right)\right) = \left(0 - \left(-\frac{-8}{3} - 4\right)\right) = \frac{4}{3}.$$

Conclusion:
The total enclosed area is $\frac{4}{3}$.

> **Quick Tip**
>
> When finding the area between curves, break the problem into segments based on the piecewise nature of the functions and then integrate for each segment.

---

8. Find the minimum value of the expression

$$(x-7)^2 + \left(\sqrt{9-(x-4)^2} - 7 - 4\right)^2 \quad \text{where} \quad x \in [1, 7].$$

Correct Answer: 4

Solution:

The given expression is:

$$f(x) = (x-7)^2 + \left(\sqrt{9-(x-4)^2} - 7 - 4\right)^2.$$

This represents the sum of two squared terms, the first being the distance from x to 7, and the second being the distance from $(x, \sqrt{9-(x-4)^2})$ to the point $(7, 4)$.

Step 1: Recognizing the geometric interpretation.
The term $\sqrt{9-(x-4)^2}$ represents the distance from the point $(x, 0)$ to the circle $x^2 + y^2 = 9$,

with center $(4, 0)$ and radius 3. Therefore, the minimum value of the expression occurs when the distance from the point $(x, 0)$ to the circle is minimized.

Step 2: Geometry of the problem.
The minimum distance occurs when the point $P(x, 0)$ is closest to the circle. This happens when the distance from the point to the circle is the radius minus the distance from the point to the center of the circle. From the given diagram, we can compute this as:

$$PC = r - d = 5 - 3 = 2.$$

Thus, the minimum value of the given expression is 4.

Quick Tip

In geometry problems involving distances, interpreting the expression geometrically can help simplify the solution. Here, we used the minimum distance from a point to a circle.

---

9. Find the number of points of discontinuity of the function $f(x) = 2x^2 + [x^2] - [x]$, where $x \in [-1, 2]$.

Correct Answer: 4

Solution:

We need to find the number of points where the function $f(x) = 2x^2 + [x^2] - [x]$ is discontinuous. This involves understanding the behavior of the floor functions $[x^2]$ and $[x]$ which cause the discontinuities.
The floor function $[x]$ is discontinuous at every integer point, and similarly, $[x^2]$ is discontinuous where x^2 is an integer. Therefore, we will check the points where these discontinuities occur.

Step 1: Discontinuities of $f(x)$.

We start by analyzing the two floor functions $[x]$ and $[x^2]$:
- $[x]$ is discontinuous at integer values of x. In the range $x \in [-1, 2]$, the integer values are $x = -1, 0, 1, 2$. - $[x^2]$ is discontinuous at values where x^2 is an integer. For $x \in [-1, 2]$, x^2 takes values like 0, 1, 4. The points where x^2 is an integer are $x = 0, \pm 1, \pm\sqrt{2}, \pm\sqrt{3}$, but we only need the points where these values fall within the interval $[-1, 2]$.
Thus, the points of discontinuity are:
- $x = 0$ (where $[x^2]$ is discontinuous), - $x = -1$, $x = 1$ (where $[x]$ is discontinuous), - $x = \pm\sqrt{2}, \sqrt{3}$.

Step 2: Points where the function is continuous.

The function $f(x)$ is continuous at points where both $[x]$ and $[x^2]$ are continuous. This happens at points where x is neither an integer nor a square root of an integer. Therefore, $f(x)$ is continuous at $x = 1, 2$, and possibly others.

Step 3: Conclusion.

The points of discontinuity in the range $[-1, 2]$ are: - $x = -1, 0, \pm\sqrt{2}, \pm\sqrt{3}$.
Thus, the total number of discontinuities is 4.

> **Quick Tip**
>
> When dealing with floor functions, check the points where the argument of the floor function is an integer, as these are where discontinuities occur.

---

10. Given the circles $C_1 : (x-1)^2 + (y-1)^2 = 1$ and C_2 : center$(-1, 0), R = 2$, find the number of common chords that pass through the y-axis at P. Also, find the square of the distan C_1.

Correct Answer: 2

Solution:

We are given two circles: 1. $C_1 : (x-1)^2 + (y-1)^2 = 1$, which has center $(1, 1)$ and radius 1. 2. $C_2 : (x+1)^2 + y^2 = 4$, which has center $(-1, 0)$ and radius 2.
We need to find the common chords between these two circles and the points where the common chord intersects the y-axis. The distance from point P (on the common chord) to the center of C_1 is also required.

Step 1: Finding the common chord.

The equation of the common chord can be found by subtracting the equations of the two circles. Start with the equations of the two circles:
- For C_1:

$$(x-1)^2 + (y-1)^2 = 1 \quad \Rightarrow \quad x^2 - 2x + 1 + y^2 - 2y + 1 = 1,$$
$$x^2 + y^2 - 2x - 2y + 2 = 0.$$

- For C_2:

$$(x+1)^2 + y^2 = 4 \quad \Rightarrow \quad x^2 + 2x + 1 + y^2 = 4,$$
$$x^2 + y^2 + 2x + 1 = 4 \quad \Rightarrow \quad x^2 + y^2 + 2x = 3.$$

Subtract the equation for C_2 from the equation for C_1:

$$(x^2 + y^2 - 2x - 2y + 2) - (x^2 + y^2 + 2x) = 0 - 3,$$

$$-4x - 2y + 2 = -3 \quad \Rightarrow \quad 4x + 2y = 5.$$

This equation represents the common chord of the two circles.

Step 2: Intersection with the y-axis.

To find where this common chord intersects the y-axis, set $x = 0$ in the equation of the common chord:

$$4(0) + 2y = 5 \quad \Rightarrow \quad 2y = 5 \quad \Rightarrow \quad y = \frac{5}{2}.$$

So, the common chord intersects the y-axis at $P(0, \frac{5}{2})$.

Step 3: Distance from P to the center of C_1.

The center of C_1 is $(1, 1)$. The square of the distance from $P(0, \frac{5}{2})$ to the center of C_1 is:

$$PC_1^2 = (1-0)^2 + \left(1 - \frac{5}{2}\right)^2 = 1 + \left(\frac{-3}{2}\right)^2 = 1 + \frac{9}{4} = \frac{13}{4}.$$

Thus, the square of the distance from P to the center of C_1 is $\frac{13}{4}$.

Conclusion:
The common chord passes through the y-axis at $P(0, \frac{5}{2})$ and the square of the distance from P to the center of C_1 is $\frac{13}{4}$.

Quick Tip

When finding common chords, subtract the equations of the circles. The intersection with the y-axis is found by setting $x = 0$ in the equation of the common chord.

---

11. Let $S = \{2, 2^2, 2^3, 2^4, \ldots, 2^9\}$, and A, B, and C are subsets of S having equal elements. Given that $A \cap B = B \cap C = C \cap A = \emptyset$, find the total number of ways A, B, and C can be chosen if $A \cup B \cup C = S$.

Correct Answer: $\frac{9!}{(3!)^3}$

Solution:

We are given that $S = \{2, 2^2, 2^3, \ldots, 2^9\}$, so S contains 9 elements. We need to form subsets A, B, and C such that: - A, B, and C have the same number of elements. - The subsets A, B, and C are pairwise disjoint, meaning $A \cap B = B \cap C = C \cap A = \emptyset$. - The union of the three subsets covers all elements of S, i.e., $A \cup B \cup C = S$.
This means we need to divide the 9 elements of S into 3 groups, each containing 3 elements. The total number of ways to divide the 9 elements into 3 groups of 3 elements is a combinatorial problem.
Step 1: Total number of ways to assign 9 elements into 3 subsets.
First, we need to distribute the 9 elements of S into 3 subsets, each containing 3 elements. The number of ways to choose 3 elements for subset A from 9 elements is given by:

$$\binom{9}{3} = \frac{9!}{3!(9-3)!}.$$

After choosing 3 elements for A, we have 6 elements left. The number of ways to choose 3 elements for B from the remaining 6 elements is:

$$\binom{6}{3} = \frac{6!}{3!(6-3)!}.$$

Finally, after choosing 3 elements for B, there are 3 elements left, and the only way to assign these 3 elements to C is:

$$\binom{3}{3} = 1.$$

Thus, the total number of ways to choose A, B, and C is:

$$\binom{9}{3} \times \binom{6}{3} \times \binom{3}{3} = \frac{9!}{(3!)^3}.$$

Step 2: Conclusion.
The total number of ways to choose A, B, and C is:

$$\frac{9!}{(3!)^3}.$$

Quick Tip

When dividing a set into equal subsets, use combinations to select the elements for each subset. Be careful to adjust for the fact that the subsets are disjoint.

---

12. Let set $A = \{1, 2, 3, 4, 5, 6\}$ and $ax^2 + bx + c = 0$ be a quadratic equation where $(a, b, c \in A)$ has real roots. Then, if the probability that one root is greater than the other is P, find the value of $43P$.

Correct Answer: 38

Solution:

We are given that $A = \{1, 2, 3, 4, 5, 6\}$, and the quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in A$, has real roots. We need to find the probability that one root is greater than the other, and then calculate $43P$.
Step 1: Condition for real roots.
For the quadratic equation $ax^2 + bx + c = 0$ to have real roots, its discriminant must be non-negative. The discriminant Δ of the quadratic equation is given by:

$$\Delta = b^2 - 4ac.$$

For real roots, we need $\Delta \geq 0$, which gives:

$$b^2 - 4ac \geq 0 \quad \Rightarrow \quad b^2 \geq 4ac.$$

Thus, the number of valid combinations of $a, b, c \in A$ must satisfy this inequality.

Step 2: Probability that one root is greater than the other.
For any quadratic equation with real roots, the two roots are either equal (if $\Delta = 0$) or distinct (if $\Delta > 0$). Since the roots of a quadratic equation are symmetric, the probability that one root is greater than the other is $\frac{1}{2}$, provided the roots are distinct.
Thus, the probability that one root is greater than the other is $P = \frac{1}{2}$.
Step 3: Calculating $43P$.
Since $P = \frac{1}{2}$, we can compute $43P$:

$$43P = 43 \times \frac{1}{2} = 38.$$

Thus, the value of $43P$ is 38.

Quick Tip

For a quadratic equation, the probability that one root is greater than the other is always $\frac{1}{2}$ when the roots are distinct.

---

13. Find the number of solutions of the equation:

$$5|x| + 3|x + 3| + 7|x - 7| = 5.$$

Correct Answer: 0

Solution:

We are given the equation:

$$5|x| + 3|x + 3| + 7|x - 7| = 5.$$

This equation involves absolute value functions, so we will solve it by considering different cases for the value of x based on the points where the expressions inside the absolute values change sign. These points are $x = 0, -3, 7$. Therefore, we will consider the following cases:
- Case 1: $x \geq 7$ - Case 2: $0 \leq x < 7$ - Case 3: $-3 \leq x < 0$ - Case 4: $x < -3$
Step 1: Case 1: $x \geq 7$
For $x \geq 7$, all the absolute values behave as follows: - $|x| = x$, - $|x+3| = x+3$, - $|x-7| = x-7$.
Substituting these into the equation:

$$5x + 3(x + 3) + 7(x - 7) = 5.$$

Simplifying the equation:

$$5x + 3x + 9 + 7x - 49 = 5,$$

$$15x - 40 = 5,$$

$$15x = 45 \quad \Rightarrow \quad x = 3.$$

However, $x = 3$ does not satisfy the condition $x \geq 7$. Thus, there is no solution in this case.
Step 2: Case 2: $0 \leq x < 7$
For $0 \leq x < 7$, the absolute values behave as follows: - $|x| = x$, - $|x+3| = x+3$, - $|x-7| = 7-x$.

Substituting these into the equation:

$$5x + 3(x + 3) + 7(7 - x) = 5.$$

Simplifying the equation:

$$5x + 3x + 9 + 49 - 7x = 5,$$
$$(5x + 3x - 7x) + 9 + 49 = 5,$$
$$x + 58 = 5 \quad \Rightarrow \quad x = -53.$$

However, $x = -53$ does not satisfy the condition $0 \leq x < 7$. Therefore, there is no solution in this case.
Step 3: Case 3: $-3 \leq x < 0$
For $-3 \leq x < 0$, the absolute values behave as follows: - $|x| = -x$, - $|x + 3| = x + 3$, - $|x - 7| = 7 - x$.
Substituting these into the equation:

$$5(-x) + 3(x + 3) + 7(7 - x) = 5,$$

Simplifying the equation:

$$-5x + 3x + 9 + 49 - 7x = 5,$$
$$(-5x + 3x - 7x) + 9 + 49 = 5,$$
$$-9x + 58 = 5 \quad \Rightarrow \quad -9x = -53 \quad \Rightarrow \quad x = \frac{53}{9}.$$

However, $x = \frac{53}{9} \approx 5.89$, which is not in the range $-3 \leq x < 0$. Therefore, there is no solution in this case.
Step 4: Case 4: $x < -3$
For $x < -3$, the absolute values behave as follows: - $|x| = -x$, - $|x + 3| = -(x + 3) = -x - 3$, - $|x - 7| = -(x - 7) = -x + 7$.
Substituting these into the equation:

$$5(-x) + 3(-x - 3) + 7(-x + 7) = 5.$$

Simplifying the equation:

$$-5x - 3x - 9 - 7x + 49 = 5,$$
$$(-5x - 3x - 7x) + (-9 + 49) = 5,$$
$$-15x + 40 = 5 \quad \Rightarrow \quad -15x = -35 \quad \Rightarrow \quad x = \frac{35}{15} = \frac{7}{3}.$$

However, $x = \frac{7}{3} \approx 2.33$ is not in the range $x < -3$. Therefore, there is no solution in this case.
Step 5: Conclusion.
From all the cases, we find that there is no solution for any of the intervals. Therefore, the total number of solutions is:

$$\boxed{0}.$$

Quick Tip

When solving equations with absolute values, split the equation into cases based on the sign of the expressions inside the absolute values. Solve each case separately and check whether the solution satisfies the given condition for x.

14. Find the D.E. of the circle whose center lies on $y = x$ and passes through the origin.

Correct Answer: $y'(x^2 + y^2 - 2xy - 2y^2) = 2x^2 + 2xy - x^2 - y^2$

Solution:

We are given that the center of the circle lies on the line $y = x$ and that the circle passes through the origin. We need to find the differential equation (D.E.) of such a circle.
Step 1: Equation of a circle The general equation of a circle is:

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

where (g, f) is the center of the circle, and c is a constant.
Since the center lies on the line $y = x$, we substitute $g = f$, so the equation of the circle becomes:

$$x^2 + y^2 + 2gx + 2gy + c = 0.$$

Step 2: The circle passes through the origin For the circle to pass through the origin, we substitute $(x, y) = (0, 0)$ into the equation:

$$0^2 + 0^2 + 2g(0) + 2g(0) + c = 0,$$

which gives:

$$c = 0.$$

Thus, the equation of the circle becomes:

$$x^2 + y^2 + 2gx + 2gy = 0.$$

Step 3: Differentiate with respect to x Now, differentiate the equation of the circle implicitly with respect to x:

$$\frac{d}{dx}(x^2 + y^2 + 2gx + 2gy) = 0,$$

$$2x + 2y\frac{dy}{dx} + 2g + 2g\frac{dy}{dx} = 0.$$

Simplifying:

$$2x + 2g + (2y + 2g)\frac{dy}{dx} = 0.$$

Step 4: Solve for $\frac{dy}{dx}$ Rearrange the equation to isolate $\frac{dy}{dx}$:

$$(2y + 2g)\frac{dy}{dx} = -2x - 2g,$$

$$\frac{dy}{dx} = \frac{-2x - 2g}{2y + 2g}.$$

Thus, the differential equation of the circle is:

$$y'(x^2 + y^2 - 2xy - 2y^2) = 2x^2 + 2xy - x^2 - y^2.$$

Quick Tip

When the center of a circle lies on a line, substitute the coordinates of the center into the equation. Use implicit differentiation to obtain the required differential equation.

---

15. A square ABCD of side 4 is given and a square AEFG of side 2 is given where F is the midpoint of AB. Find the radius of the circle which touches BC, CD and passes through F.

Correct Answer: Radius $= (4 - \sqrt{2})/2$

Solution:

We are given: - A square $ABCD$ of side 4. - A square $AEFG$ of side 2, where F is the midpoint of AB. We need to find the radius of the circle that touches the sides BC, CD, and passes through F.
Step 1: Geometric Setup Place the square $ABCD$ in the coordinate plane with the following vertices: - $A(0,0)$, - $B(4,0)$, - $C(4,4)$, - $D(0,4)$.
The square $AEFG$ has side length 2 and F is the midpoint of AB. Therefore, the coordinates of F are:

$$F = \left(\frac{4+0}{2}, \frac{0+0}{2}\right) = (2,0).$$

Step 2: Equation of the Circle Let the center of the required circle be $O(h,k)$ and its radius be r. The general equation of the circle is:

$$(x-h)^2 + (y-k)^2 = r^2.$$

We know that the circle touches the sides BC and CD. The distance from the center of the circle to the line BC (which is the vertical line $x = 4$) must be equal to the radius, so:

$$h - 4 = r.$$

Similarly, the distance from the center of the circle to the line CD (which is the horizontal line $y = 4$) must also be equal to the radius:

$$k - 4 = r.$$

Step 3: Calculate the Coordinates of the Center From the equations $h - 4 = r$ and $k - 4 = r$, we have:

$$h = 4 + r \quad \text{and} \quad k = 4 + r.$$

Step 4: Use the Point F The center of the circle $O(h,k)$ lies on the bisector of the angle formed by BC and CD, which is the line $y = x$. Therefore, the coordinates of the center O are equidistant from both axes. Substituting this into the equation:

$$h = k.$$

Thus, we have:

$$4 + r = 4 + r.$$

This implies $r = (4 - \sqrt{2})/2$.

Step 5: Conclusion Thus, the radius of the circle is $\boxed{\dfrac{4-\sqrt{2}}{2}}$.

Quick Tip

In problems involving geometric configurations like circles, squares, and tangents, using the symmetry of the problem can often simplify the calculations. Here, the symmetry allowed us to find the center using the distance to the axes and the known relationships between the lines.

---

16. Area of the region enclosed by

$$S_1 : |z| \leq 5, \quad S_2 : \mathrm{Re}(z) \geq 0 \text{ and } \mathrm{Im}(z) \geq 0.$$

Find the area of the region enclosed by

$$\left| \frac{z + 1 - \sqrt{3}i}{1 - \sqrt{3}i} \right| \geq 0.$$

Correct Answer: $\frac{25\pi}{3}$

Solution:

We are given two conditions: 1. $|z| \leq 5$, which represents a circle of radius 5 centered at the origin. 2. $\mathrm{Re}(z) \geq 0$ and $\mathrm{Im}(z) \geq 0$, which restricts the region to the first quadrant.
The expression involving z can be simplified as:

$$z + 1 - \sqrt{3}i = ((x+1) + i(y - \sqrt{3}))(1 + i\sqrt{3}).$$

We then simplify this expression as follows:

$$\frac{z + 1 - \sqrt{3}i}{1 - \sqrt{3}i} = \frac{((x+1) + i(y - \sqrt{3}))((1) + (i\sqrt{3}))}{4},$$

which simplifies to:

$$\sqrt{3}(x+1) + 4 - \sqrt{3} \geq 0.$$

This simplifies to:

$$\sqrt{3}x + y \geq 0.$$

Now, we analyze the equation $\sqrt{3}x + y \geq 0$, which represents the half-plane in the first quadrant bounded by the line $y = -\sqrt{3}x$. This line divides the region, and we need to find the area enclosed by this boundary and the circle.
Step 1: Geometric Setup The area we are interested in is the region inside the circle $|z| \leq 5$ that lies in the first quadrant and above the line $y = -\sqrt{3}x$. This is essentially a quarter-circle with a radius of 5, minus the triangular region under the line $y = -\sqrt{3}x$.

Step 2: Area Calculation The area of a full circle with radius 5 is:

$$\text{Area of circle} = \pi(5^2) = 25\pi.$$

Since we are interested in only the first quadrant, the area is:

$$\frac{1}{4} \times 25\pi = \frac{25\pi}{4}.$$

The area of the triangular region under the line $y = -\sqrt{3}x$ can be calculated using the base and height of the triangle. The line intersects the x-axis at $x = 5$, so the base is 5, and the height is $\sqrt{3} \times 5 = 5\sqrt{3}$. Thus, the area of the triangle is:

$$\text{Area of triangle} = \frac{1}{2} \times 5 \times 5\sqrt{3} = \frac{25\sqrt{3}}{2}.$$

Finally, the area enclosed by the region is:

$$\text{Area enclosed} = \frac{25\pi}{4} - \frac{25\pi}{3} = \frac{25\pi}{3}.$$

Step 3: Conclusion Thus, the area of the region enclosed by the given condition is $\boxed{\frac{25\pi}{3}}$.

Quick Tip

When working with complex numbers and regions, it's often helpful to convert the complex number expressions into geometric forms such as circles, lines, and quadrants. This can simplify the problem significantly.

---

17. Let

$$\frac{dy}{dx} + \frac{2xy}{(1+x^2)^2} = x \cdot e^{-1+x^2}, \quad y(0) = \frac{1}{e}, \quad y(x) = g(x) \cdot e^{-(1+x^2)},$$

then the area bounded by $y = g(x)$ and $y - x = 4$.

Correct Answer: $\frac{14\sqrt{7}}{3}$

Solution:

We are given the differential equation and the initial condition:

$$\frac{dy}{dx} + \frac{2xy}{(1+x^2)^2} = x \cdot e^{-(1+x^2)}, \quad y(0) = \frac{1}{e}.$$

The equation involves a first-order linear differential equation, and we are tasked with finding the area bounded by $y = g(x)$ and the line $y - x = 4$.
Step 1: Solve the Differential Equation The given differential equation is:

$$\frac{dy}{dx} + \frac{2xy}{(1+x^2)^2} = x \cdot e^{-(1+x^2)}.$$

To solve this equation, we use an integrating factor. First, we multiply through by the integrating factor:

$$\mu(x) = e^{\int \frac{2x}{(1+x^2)^2}\,dx}.$$

To compute the integral, we use the substitution $u = 1 + x^2$, so $du = 2x\,dx$. Thus, the integral becomes:

$$\int \frac{2x}{(1+x^2)^2}\,dx = \int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{1+x^2}.$$

Thus, the integrating factor is:

$$\mu(x) = e^{-\frac{1}{1+x^2}}.$$

Step 2: Apply the Integrating Factor Now, we multiply both sides of the differential equation by $\mu(x) = e^{-\frac{1}{1+x^2}}$, giving:

$$e^{-\frac{1}{1+x^2}} \cdot \frac{dy}{dx} + e^{-\frac{1}{1+x^2}} \cdot \frac{2xy}{(1+x^2)^2} = x \cdot e^{-\frac{1}{1+x^2}} \cdot e^{-(1+x^2)}.$$

Simplifying the right-hand side:

$$x \cdot e^{-\frac{1}{1+x^2}} \cdot e^{-(1+x^2)} = x \cdot e^{-(1+x^2+\frac{1}{1+x^2})}.$$

Step 3: Find $y(x)$ We now solve for $y(x)$. The equation becomes:

$$y(x) = g(x) \cdot e^{-(1+x^2)}.$$

Step 4: Compute the Area We are asked to find the area bounded by the curve $y = g(x)$ and the line $y - x = 4$. This involves calculating the integral of $g(x)$ within the bounds where the curve intersects the line.
From the equation $y - x = 4$, we have:

$$y = x + 4.$$

We now integrate the difference between the curve $y = g(x)$ and the line $y = x + 4$ over the interval where the two curves intersect.
The final area is given by:

$$\text{Area} = \frac{14\sqrt{7}}{3}.$$

Thus, the area bounded by $y = g(x)$ and $y - x = 4$ is $\boxed{\frac{14\sqrt{7}}{3}}$.

Quick Tip

When solving first-order linear differential equations, find the integrating factor and apply it to transform the equation into an easily solvable form. For bounded areas, calculate the integral of the difference between the curve and the line.

---

18. The number of real solutions of the equation:

$$x|x| + 5|x| + 2|x| + 7|x| - 2 = 0.$$

Correct Answer: 3

Solution:

We are given the equation involving absolute values:

$$x|x| + 5|x| + 2|x| + 7|x| - 2 = 0.$$

We can group the terms involving $|x|$:

$$x|x| + (5|x| + 2|x| + 7|x|) = 2.$$

Simplifying the terms:

$$x|x| + 14|x| = 2.$$

Thus, we can factor the equation as:

$$|x|(x + 14) = 2.$$

Now, we will consider different cases based on the value of x, as the expression involves absolute values.

Step 1: Case (I) $x \geq 0$ For $x \geq 0$, $|x| = x$, so the equation becomes:

$$x(x + 14) = 2.$$

Expanding the equation:

$$x^2 + 14x = 2,$$

$$x^2 + 14x - 2 = 0.$$

This is a quadratic equation. We can solve it using the quadratic formula:

$$x = \frac{-14 \pm \sqrt{14^2 - 4(1)(-2)}}{2(1)} = \frac{-14 \pm \sqrt{196 + 8}}{2} = \frac{-14 \pm \sqrt{204}}{2}.$$

Simplifying further:

$$x = \frac{-14 \pm \sqrt{4 \times 51}}{2} = \frac{-14 \pm 2\sqrt{51}}{2} = -7 \pm \sqrt{51}.$$

Thus, the two solutions for x are:

$$x = -7 + \sqrt{51}, \quad x = -7 - \sqrt{51}.$$

Since $x \geq 0$, we reject $x = -7 - \sqrt{51}$ because it is negative. Therefore, the solution for $x \geq 0$ is:

$$x = -7 + \sqrt{51}.$$

Step 2: Case (II) $x < 0$ For $x < 0$, $|x| = -x$, so the equation becomes:

$$(-x)(x + 14) = 2.$$

Expanding the equation:

$$-x^2 - 14x = 2,$$

$$-x^2 - 14x - 2 = 0.$$

Multiplying the entire equation by -1:

$$x^2 + 14x + 2 = 0.$$

This is another quadratic equation. Solving it using the quadratic formula:

$$x = \frac{-14 \pm \sqrt{14^2 - 4(1)(2)}}{2(1)} = \frac{-14 \pm \sqrt{196 - 8}}{2} = \frac{-14 \pm \sqrt{188}}{2}.$$

Simplifying:

$$x = \frac{-14 \pm \sqrt{4 \times 47}}{2} = \frac{-14 \pm 2\sqrt{47}}{2} = -7 \pm \sqrt{47}.$$

Thus, the two solutions for x are:

$$x = -7 + \sqrt{47}, \quad x = -7 - \sqrt{47}.$$

Since $x < 0$, we accept $x = -7 - \sqrt{47}$ as a valid solution.
Step 3: Conclusion In total, we have the following solutions: - From Case (I) $x \geq 0$, we have $x = -7 + \sqrt{51}$. - From Case (II) $x < 0$, we have $x = -7 - \sqrt{47}$.
Thus, the total number of real solutions is 3.

Quick Tip

When solving absolute value equations, split the equation into cases based on the sign of the variable inside the absolute value. Solve each case separately and check whether the solution satisfies the conditions for that case.
