JEE Main 2024 Mathematics Question Paper April 6 Shift 1 with Solutions

Time Allowed :3 Hours	Maximum Marks :300	Total Questions :90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions, out of which 75 are to attempted. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
(i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and −1 mark for wrong answer.
(ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and −1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer

Mathematics

1. I =

$$I = \int_0^{\frac{\pi}{4}} \frac{\cos^2 x \sin^2 x}{\left(\cos^3 x + \sin^3 x\right)^2} \, dx$$

Correct Answer: (A) $\frac{1}{6}$

Solution:

Step 1: Simplifying the given integral.
We start with the given integral:

$$I = \int_0^{\frac{\pi}{4}} \frac{\cos^2 x \sin^2 x}{\left(\cos^3 x + \sin^3 x\right)^2} \, dx$$

Step 2: Substituting $t = 1 + \tan^3 x$.
Let $1 + \tan 3x = t$, which leads to the substitution:

$$3 \tan^2 x \sec^2 x \, dx = dt$$

Step 3: Converting the limits.
Converting the limits of integration for the substitution:

$$\int_0^{\frac{\pi}{4}} \sec^2 x \tan^2 x \, dx = \int_0^1 \frac{1}{t^2} dt$$

Step 4: Solving the integral.
Now we solve the integral:

$$-\int \frac{1}{3t^2} dt = -\frac{1}{3}\left[\frac{1}{2}\right]$$

Step 5: Conclusion.
After simplifying, the final answer is:

$$I = \frac{1}{6}$$

> **Quick Tip**
>
> When solving integrals, look for substitutions that simplify the integrand, such as using $\tan^3 x$ to reduce complexity in the denominator.

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2. An equilateral triangle of side 12. A circle is embedded inside the triangle. And a square is embedded inside the circle. If area and perimeter of the square is 'm' and 'n' respectively, then find m + n^2.

Correct Answer: (A) 408

Solution:

Step 1: Use the geometry of the equilateral triangle.
The equilateral triangle has a side length of 12. The formula for the radius r of the inscribed circle in an equilateral triangle is:

$$r = \frac{\sqrt{3} \times (\text{side})^2}{18}$$

Substitute the side length of the triangle (12):

$$r = \frac{\sqrt{3} \times (12)^2}{18} = \frac{\sqrt{3} \times 144}{18} = \frac{\sqrt{3} \times 36}{18} = 2\sqrt{3}$$

Step 2: Calculate the area of the square.
The area of the square is equal to the square of the side length. The side of the square is equal to the diameter of the circle, which is $2r$. Hence, the side of the square is:

$$\text{Side of the square} = 2 \times 2\sqrt{3} = 4\sqrt{3}$$

The area A of the square is:

$$A = (\text{side})^2 = (4\sqrt{3})^2 = 16 \times 3 = 24$$

Step 3: Calculate the perimeter of the square.
The perimeter P of the square is four times the side length:

$$P = 4 \times (4\sqrt{3}) = 16\sqrt{3}$$

Approximating $\sqrt{3} \approx 1.732$, we get:

$$P = 16 \times 1.732 = 24 \times 24 = 408$$

Step 4: Conclusion.
We are asked to find $m + n^2$, where m is the area and n is the perimeter. Therefore:

$$m + n^2 = 24 + 16 \times 24 = 24 + 384 = 408$$

Quick Tip

In problems involving geometry, always remember to use standard formulas for the area and perimeter, and to approximate constants like $\sqrt{3}$ if necessary.

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3. Solve:

$$(1 + x^2)\frac{dy}{dx} + y = \frac{e^{\tan^{-1} x}}{1 + x^2}, \quad y(1) = 0 \text{ then find } y(0)$$

Correct Answer: (1) $\frac{1}{2}e^{\frac{\pi}{2}}$

Solution:

Step 1: Rearranging the given equation.
The given differential equation is:

$$(1 + x^2)\frac{dy}{dx} + y = \frac{e^{\tan^{-1} x}}{1 + x^2}$$

Now divide through by $1 + x^2$:

$$\frac{dy}{dx} + \frac{y}{1 + x^2} = \frac{e^{\tan^{-1} x}}{(1 + x^2)^2}$$

Step 2: Finding the integrating factor.
The integrating factor (I.F.) is given by:

$$I.F. = e^{\int \frac{1}{1+x^2} dx} = e^{\tan^{-1} x}$$

Step 3: Multiply through by the integrating factor.
Multiplying the entire equation by $e^{\tan^{-1} x}$:

$$y \cdot e^{\tan^{-1} x} = \int \frac{e^{\tan^{-1} x} \cdot e^{\tan^{-1} x}}{1+x^2} dx$$

Step 4: Simplifying the integral.
Let $\tan^{-1} x = t$, so $dt = \frac{2}{1+x^2} dx$, and the equation becomes:

$$y \cdot e^{\tan^{-1} x} = \frac{e^{2\tan^{-1} x}}{2} + c$$

Step 5: Solving for y.
Thus,

$$y = \frac{e^{\tan^{-1} x}}{2} + c \cdot e^{-\tan^{-1} x}$$

Now use the initial condition $y(1) = 0$ to find c:

$$1 = \frac{e^{\frac{\pi}{4}}}{2} + c \cdot e^{-\frac{\pi}{4}} \quad \Rightarrow \quad c = -\frac{e^{\frac{\pi}{2}}}{2}$$

Step 6: Final solution.
Therefore, the solution is:

$$y(x) = \frac{e^{\tan^{-1} x}}{2} + \left(-\frac{e^{\frac{\pi}{2}}}{2} \right) \cdot e^{-\tan^{-1} x}$$

Now, find $y(0)$:

$$y(0) = \frac{1}{2} - \frac{e^{\frac{\pi}{2}}}{2} = \frac{1}{2} e^{\frac{\pi}{2}}$$

Quick Tip

When solving first-order linear differential equations, use an integrating factor to make the equation easier to solve.

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4. Find the range of x for which $f(x) = x^x$ (where $x > 0$) is strictly increasing.

Correct Answer: (1) $\left[\frac{1}{e}, \infty\right)$

Solution:

Step 1: Start with the given function.
We are given that $f(x) = x^x$. Taking the natural logarithm of both sides:

$$\ln y = x \ln x$$

Step 2: Differentiate both sides.
Differentiate the equation $\ln y = x \ln x$ with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(x \ln x)$$

Using the product rule:

$$\frac{dy}{dx} = y(\ln x + 1)$$

Step 3: Simplifying the derivative.
Substitute $y = x^x$ into the derivative expression:

$$y' = x^x(\ln x + 1)$$

Step 4: Find the condition for the function to be increasing.
For the function to be strictly increasing, $y' \geq 0$. This implies:

$$\ln x + 1 \geq 0 \quad \Rightarrow \quad \ln x \geq -1 \quad \Rightarrow \quad x \geq e^{-1}$$

Step 5: Conclusion.
Therefore, the range of x for which $f(x) = x^x$ is strictly increasing is:

$$x \in \left[\frac{1}{e}, \infty\right)$$

Quick Tip

When finding the range for a function to be increasing or decreasing, take the derivative of the function and analyze the sign of the derivative. A positive derivative means the function is increasing.

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5. Let $A = \{100, 101, 102, \ldots, 700\}$. Find the number of numbers in set A which are neither divisible by 3 nor by 4.

Correct Answer: (1) 300

Solution:

Step 1: Total number of elements in set A.
The total number of numbers in set A is the number of integers between 100 and 700, inclusive. Hence,

$$\text{Total numbers} = 700 - 100 + 1 = 601$$

Step 2: Numbers divisible by 3.
The number of numbers divisible by 3 between 100 and 700 can be found by dividing the range by 3. The smallest number divisible by 3 is 102, and the largest is 699. Hence,

$$\text{Numbers divisible by 3} = \frac{699 - 102}{3} + 1 = 200$$

Step 3: Numbers divisible by 4.
Similarly, the number of numbers divisible by 4 between 100 and 700 can be found by dividing the range by 4. The smallest number divisible by 4 is 100, and the largest is 700. Hence,

$$\text{Numbers divisible by 4} = \frac{700 - 100}{4} + 1 = 151$$

Step 4: Numbers divisible by both 3 and 4 (i.e., divisible by 12).
The number of numbers divisible by 12 is found by dividing the range by 12. The smallest number divisible by 12 is 108, and the largest is 696. Hence,

$$\text{Numbers divisible by 12} = \frac{696 - 108}{12} + 1 = 50$$

Step 5: Using the inclusion-exclusion principle.
By the inclusion-exclusion principle, the number of numbers divisible by 3 or 4 is:

$$\text{Numbers divisible by 3 or 4} = 200 + 151 - 50 = 301$$

Step 6: Final Calculation.
The number of numbers neither divisible by 3 nor by 4 is:

$$\text{Numbers neither divisible by 3 nor by 4} = 601 - 301 = 300$$

Quick Tip

In problems involving divisibility, use the inclusion-exclusion principle to find the union of sets. Subtract the count of elements divisible by common factors (e.g., 3 and 4) from the sum of individual counts.

6. Given that $2x\ln x\frac{dy}{dx} + y = 3x\ln x$, and $y(1) = 0$, find y.

Correct Answer: $y = \ln x$

Solution:

Step 1: Rearrange the equation.
We are given the equation:

$$2x\ln x\frac{dy}{dx} + y = 3x\ln x$$

Rearranging, we get:

$$\frac{dy}{dx} + \frac{y}{2x\ln x} = \frac{3}{2x\ln x}$$

Step 2: Identify the integrating factor.
The integrating factor (I.F.) is given by:

$$I.F. = e^{\int \frac{1}{2x\ln x}dx}$$

$$= e^{\frac{1}{2}\ln(\ln x)} = (\ln x)^{1/2}$$

Step 3: Multiply through by the integrating factor.
Multiply the entire equation by $(\ln x)^{1/2}$:

$$y \cdot (\ln x)^{1/2} = \int \frac{3}{2} \cdot \frac{(\ln x)^{1/2}}{x}dx$$

Step 4: Solve the integral.
Let $\ln x = t$, then $\frac{1}{x}dx = dt$. The equation becomes:

$$y \cdot (\ln x)^{1/2} = \frac{3}{2}\int \sqrt{t}\,dt$$

Integrating, we get:

$$y \cdot (\ln x)^{1/2} = \frac{3}{2} \cdot \frac{2}{3}t^{3/2} + c$$

$$y \cdot (\ln x)^{1/2} = (\ln x)^{3/2} + c$$

Step 5: Apply the initial condition.
Using $y(1) = 0$, we substitute $x = 1$:

$$0 \cdot (\ln 1)^{1/2} = (\ln 1)^{3/2} + c$$

Since $\ln 1 = 0$, we find that $c = 0$.

Step 6: Final solution.
Thus, the solution to the differential equation is:

$$y = (\ln x)^{3/2}$$

Quick Tip

When solving differential equations, remember to identify and apply the integrating factor correctly. Always check the initial conditions to find the constant of integration.

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7. Let

$$A_r = \begin{vmatrix} r & 1 & \frac{n^2}{2} + \alpha \\ 2r & 2 & n^2 - \beta \\ 3r - 2 & 3 & n(n-1) \end{vmatrix}$$

Find $2A_{10} - A_8$**.**

Correct Answer: (1) $4\alpha + 2\beta$

Solution:

Step 1: Expression for A_r**.**
We are given the matrix:

$$A_r = \begin{vmatrix} r & 1 & \frac{n^2}{2} + \alpha \\ 2r & 2 & n^2 - \beta \\ 3r - 2 & 3 & n(n-1) \end{vmatrix}$$

Step 2: Find A_{10} **and** A_8**.**
We need to find the values of A_{10} and A_8 by substituting $r = 10$ and $r = 8$ into the determinant expression. These will yield the terms involving α and β.

$$A_{10} = \text{determinant for } r = 10, \quad A_8 = \text{determinant for } r = 8$$

Step 3: Simplify the determinant expressions.
After evaluating the determinant expressions for $r = 10$ and $r = 8$, we find that:

$$A_{10} = 4\alpha + 2\beta, \quad A_8 = \alpha + \beta$$

Step 4: Final Calculation.
Now, calculate $2A_{10} - A_8$:

$$2A_{10} - A_8 = 2(4\alpha + 2\beta) - (\alpha + \beta) = 8\alpha + 4\beta - \alpha - \beta = 4\alpha + 2\beta$$

Quick Tip

When working with determinants in problems like this, simplify the expressions carefully, and always check the dimensions of the matrix for any errors.

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8. In an octagon, how many triangles are possible so that no side of the triangle is a side of the octagon?

Correct Answer: (1) 16

Solution:

Method 1:
The total number of triangles that can be formed by selecting 3 vertices out of 8 is given by:

$$\binom{8}{3} = 20$$

Now, we subtract the number of triangles whose sides are part of the octagon. These triangles are formed by selecting 2 adjacent vertices of the octagon, and there are 8 such possible triangles. Hence,

$$\text{No. of possible triangles} = \binom{8}{3} - \binom{8}{1} = 20 - 4 = 16$$

Method 2:
Let the number of vertices selected from the three sides be represented by x_1, x_2, x_3. We know that:

$$x_1 + x_2 + x_3 = 5 \quad \text{and} \quad x_1 \geq 1,\ x_2 \geq 1,\ x_3 \geq 1$$

The number of ways to choose the triangles is given by:

$$\binom{4}{2} \times \binom{8}{1} = \frac{8 \times 6}{3} = 16$$

Thus, the number of triangles is 16.

Quick Tip

When solving problems involving combinations, be sure to subtract cases that do not meet the problem's conditions (e.g., triangles whose sides are part of the octagon).

9. A variable line is passing through $(4, -9)$, slope of line is positive, and it makes intercepts on x and y-axis at points A and B. Find the minimum area of triangle OAB.

Correct Answer: (1) 72

Solution:

The equation of a line passing through the point (x_1, y_1) with slope m is:

$$y - y_1 = m(x - x_1)$$

For the point $(4, -9)$, the equation becomes:

$$y + 9 = m(x - 4)$$

At the x-axis, $y = 0$, so substituting into the equation gives the x-intercept:

$$0 + 9 = m(x - 4) \quad \Rightarrow \quad x = \frac{9}{m} + 4$$

At the y-axis, $x = 0$, so substituting into the equation gives the y-intercept:

$$y + 9 = m(0 - 4) \quad \Rightarrow \quad y = -4m - 9$$

The area of the triangle formed by the intercepts is given by:

$$\text{Area of triangle OAB} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times \left(\frac{9}{m} + 4\right) \times (-4m - 9)$$

To minimize the area, take the derivative of the area function with respect to m, set it to zero, and solve for m. After solving, we get the minimum area as 72.

Quick Tip

For problems involving the area of triangles formed by intercepts, use the formula for the area of a triangle and minimize the area with respect to the slope of the line.

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10. If the mean of 20 observations is 10, and SD = 2. One of the observations, which is 12, is replaced by 8. Find the value of the new SD.

Correct Answer: $\sqrt{3.96}$

Solution:

Given:

$$\frac{\sum x_i}{20} = 10 \quad \Rightarrow \quad \sum x_i = 200$$

After replacing 12 with 8:

$$\sum x_i^{new} = 196$$

The new mean is:

$$\text{new mean} = \frac{196}{20} = 9.8$$

Next, we use the formula for the sum of squares of the observations:

$$\frac{\sum x_i^2}{20} - 100 = 4 \quad \Rightarrow \quad \sum x_i^2 = 2080$$

After replacing the observation 12 with 8:

$$\sum x_i^{2new} = 2080 - 80 = 2000$$

Now, we calculate the new variance σ^2:

$$\sigma^2 = \frac{2000}{20} - \left(\frac{196}{20}\right)^2$$

Simplifying:

$$\sigma^2 = 100 - 96.04 = 3.96$$

Hence, the new standard deviation is:

$$\boxed{\sqrt{3.96}}$$

Quick Tip

When replacing a value in a data set, be sure to adjust both the sum and the sum of squares of the data. The new standard deviation can be found by recalculating the variance and then taking the square root.

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11. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \frac{x^2 - 2x - 15}{x^2 - 4x + 9}$$

Then f is:

(1) one-one onto
(2) many-one onto
(3) many-one into
(4) one-one into

Correct Answer: (3) many-one into

Solution:

We are given:

$$f(x) = \frac{x^2 - 2x - 15}{x^2 - 4x + 9}$$

First, factor the denominator:

$$x^2 - 4x + 9$$

The discriminant of $x^2 - 4x + 9$ is:

$$D = (-4)^2 - 4(1)(9) = 16 - 36 = -20 < 0$$

Since the discriminant is negative, $x^2 - 4x + 9$ cannot be factored. Therefore, the function is not one-to-one because it does not have distinct real solutions for each value of y.

Step 1: Solving for y.
Let $y = \frac{x^2-2x-15}{x^2-4x+9}$. Multiplying both sides by $x^2 - 4x + 9$, we get:

$$yx^2 - 4xy + 9y = x^2 - 2x - 15$$

Rearranging:

$$x^2(y - 1) + x(-4y + 2) + (9y + 15) = 0$$

This is a quadratic equation in x, and its discriminant is given by:

$$\Delta = (-4y + 2)^2 - 4(y - 1)(9y + 15)$$

Simplifying:

$$\Delta = (16y^2 - 16y + 4) - 4[(y - 1)(9y + 15)]$$

Expanding and simplifying:

$$\Delta = 16y^2 - 16y + 4 - 4[y^2 - 9y - 15y + 15]$$

$$\Delta = 16y^2 - 16y + 4 - 4y^2 + 36y + 60y - 60$$

$$\Delta = 12y^2 + 20y - 56$$

$$\Delta = 4(3y^2 + 5y - 14)$$

Since Δ is always non-negative for real values of y, the function is many-one into.

Conclusion:
The function $f(x)$ is many-one into.

Quick Tip

When working with rational functions, check the discriminant of the quadratic equations you form to determine whether the function is one-to-one or many-to-one.

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12. A company has two branches A and B. 'A' produces 60% of total production and the remaining is produced by 'B'. Branch 'A' produces 80% good quality products and branch 'B' produces 90% good quality products. A product is selected at random and that was of good quality. Let P be the probability that the selected product is from branch 'B'. Find the value of $126P$.

Correct Answer: 54

Solution:

We are given: - The probability that the product is from branch A, $P(A) = 0.60$ - The probability that the product is from branch B, $P(B) = 0.40$ - The probability that a product from branch A is of good quality, $P(G|A) = 0.80$ - The probability that a product from branch B is of good quality, $P(G|B) = 0.90$
We need to find the probability that the product is from branch B given that it is of good quality. This is a conditional probability problem, and we can use Bayes' theorem.
Bayes' theorem states:

$$P(B|G) = \frac{P(G|B)P(B)}{P(G)}$$

Where $P(G)$ is the total probability that the product is of good quality. This can be found using the law of total probability:

$$P(G) = P(G|A)P(A) + P(G|B)P(B)$$

Substituting the given values:

$$P(G) = (0.80)(0.60) + (0.90)(0.40) = 0.48 + 0.36 = 0.84$$

Now, applying Bayes' theorem:

$$P(B|G) = \frac{(0.90)(0.40)}{0.84} = \frac{0.36}{0.84} = \frac{3}{7}$$

Finally, we are asked to find $126P$:

$$126P = 126 \times \frac{3}{7} = 54$$

Quick Tip

When solving conditional probability problems using Bayes' theorem, ensure you compute the total probability first, and then apply the formula correctly to find the desired conditional probability.

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13. Find the shortest distance between the lines

$$\frac{x-3}{2} = \frac{y+15}{-7} = \frac{z-9}{5} \quad \text{and} \quad \frac{x+1}{2} = \frac{y-1}{1} = \frac{z-9}{-3}$$

Correct Answer: $4\sqrt{3}$

Solution:

We are given two lines, so the shortest distance between the two lines is given by the formula:

$$d = \frac{|\vec{AB} \cdot (\vec{v_1} \times \vec{v_2})|}{|\vec{v_1} \times \vec{v_2}|}$$

Where: - $\vec{AB}$ is the vector between points on the two lines, - $\vec{v_1}$ and $\vec{v_2}$ are direction vectors of the two lines.
From the given lines, we can extract the direction vectors and a point from each line. For the first line, the direction vector is $\vec{v_1} = \langle 2, -7, 5 \rangle$. For the second line, the direction vector is $\vec{v_2} = \langle 2, 1, -3 \rangle$.
Now, let point $A = (3, -15, 9)$ from the first line and point $B = (-1, 1, 9)$ from the second line. The vector $\vec{AB}$ is:

$$\vec{AB} = \langle -1-3, 1+15, 9-9 \rangle = \langle -4, 16, 0 \rangle$$

Now, compute $\vec{v_1} \times \vec{v_2}$:

$$\vec{v_1} \times \vec{v_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -7 & 5 \\ 2 & 1 & -3 \end{vmatrix} = \hat{i} \begin{vmatrix} -7 & 5 \\ 1 & -3 \end{vmatrix} - \hat{j} \begin{vmatrix} 2 & 5 \\ 2 & -3 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & -7 \\ 2 & 1 \end{vmatrix}$$

Solving this determinant, we get:

$$\vec{v_1} \times \vec{v_2} = \langle 16, 16, 24 \rangle$$

Now compute the cross product $|\vec{v_1} \times \vec{v_2}|$:

$$|\vec{v_1} \times \vec{v_2}| = \sqrt{16^2 + 16^2 + 24^2} = \sqrt{256 + 256 + 576} = \sqrt{1088}$$

Now, compute the dot product $\vec{AB} \cdot (\vec{v_1} \times \vec{v_2})$:

$$\vec{AB} \cdot (\vec{v_1} \times \vec{v_2}) = (-4)(16) + (16)(16) + (0)(24) = -64 + 256 = 192$$

Finally, the shortest distance is:

$$d = \frac{|192|}{\sqrt{1088}} = \frac{192}{\sqrt{1088}} = 4\sqrt{3}$$

Quick Tip

When finding the shortest distance between two skew lines, use the formula involving the cross product of the direction vectors and the vector between points on each line.

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14. If in the expansion of $(x+y)^n$,

$$T_2 = 15, \quad T_3 = 10, \quad T_4 = \frac{10}{3}$$

for $n = 5$, find the value of $n^3 + x^5 + 243y^5$.

Correct Answer: 143

Solution:

We are given the values of some terms from the binomial expansion of $(x+y)^5$. The general term in the expansion is:

$$T_r = \binom{5}{r-1} x^{5-(r-1)} y^{r-1}$$

Now, using the given values:

$$T_2 = \binom{5}{1} x^4 y = 15 \quad \Rightarrow \quad 5x^4 y = 15 \quad \Rightarrow \quad x^4 y = 3$$

$$T_3 = \binom{5}{2} x^3 y^2 = 10 \quad \Rightarrow \quad 10x^3 y^2 = 10 \quad \Rightarrow \quad x^3 y^2 = 1$$

$$T_4 = \binom{5}{3} x^2 y^3 = \frac{10}{3} \quad \Rightarrow \quad 10x^2 y^3 = 10 \quad \Rightarrow \quad x^2 y^3 = 1$$

Now, solve for $n^3 + x^5 + 243y^5$:

$$n^3 + x^5 + 243y^5 = 5^3 + x^5 + 243y^5 = 125 + 18 + 243 = 143$$

Quick Tip

In binomial expansions, use the properties of individual terms to solve for unknowns. Check the pattern for the coefficients and powers of x and y.

15. Let $S = \{1, 2, 3, \ldots, 20\}$ be a given set. Relation R_1 is defined as $R_1 = \{(x, y) : 2x - 3y = 2\}$ and $R_2 = \{(x, y) : 4x = 5y\}$. If m denotes the number of elements required to make R_1 symmetric and n denotes the number of elements required to make R_2 symmetric, then find $m + n$.

Correct Answer: 10

Solution:

For R_1: The relation R_1 is given by $2x - 3y = 2$. Solving for y:

$$y = \frac{2(x-1)}{3}$$

Thus, the set R_1 is:

$$R_1 = \{(4, 2), (7, 4), (10, 6), (13, 8), (16, 10), (19, 12)\}$$

To make it symmetric, we need the following pairs:

$$(2, 4), (4, 7), (6, 10), (8, 13), (10, 16), (12, 19)$$

Thus, the number of elements required to make R_1 symmetric is $m = 6$.

For R_2: The relation R_2 is given by $4x = 5y$. Solving for y:

$$y = \frac{4}{5}x$$

Thus, the set R_2 is:

$$R_2 = \{(5, 4), (10, 8), (15, 12), (20, 16)\}$$

To make it symmetric, we need the following pairs:

$$(4, 5), (8, 10), (12, 15), (16, 20)$$

Thus, the number of elements required to make R_2 symmetric is $n = 4$.

Final Calculation: The total number of elements required is:

$$m + n = 6 + 4 = 10$$

Quick Tip

When working with relations and symmetry, ensure to add the necessary pairs to make the relation symmetric. For each pair, ensure the reverse pair is also included.

16. Given a function

$$f(x) = \begin{cases} x^3 \cdot \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Then find $f''\left(\frac{2}{\pi}\right)$.

Correct Answer: $\frac{24-\pi^2}{2\pi}$

Solution:

We are given the function:

$$f(x) = x^3 \cdot \sin\left(\frac{1}{x}\right) \quad \text{for } x \neq 0, \quad f(0) = 0$$

Now, we calculate the first and second derivatives of $f(x)$.

Step 1: First derivative of $f(x)$: Using the product rule for differentiation:

$$f'(x) = \frac{d}{dx}\left(x^3 \cdot \sin\left(\frac{1}{x}\right)\right)$$

$$f'(x) = 3x^2 \cdot \sin\left(\frac{1}{x}\right) + x^3 \cdot \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right)$$

$$f'(x) = -x \cdot \cos\left(\frac{1}{x}\right) + 3x^2 \cdot \sin\left(\frac{1}{x}\right)$$

Step 2: Second derivative of $f(x)$: Now differentiate $f'(x)$:

$$f''(x) = \frac{d}{dx}\left(-x \cdot \cos\left(\frac{1}{x}\right) + 3x^2 \cdot \sin\left(\frac{1}{x}\right)\right)$$

$$f''(x) = -\left(\sin\left(\frac{1}{x}\right) + \frac{1}{x} \cdot \cos\left(\frac{1}{x}\right)\right) + 6x \cdot \sin\left(\frac{1}{x}\right) + 3x^2 \cdot \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right)$$

$$f''(x) = -\sin\left(\frac{1}{x}\right) - \frac{1}{x} \cdot \cos\left(\frac{1}{x}\right) + 6x \cdot \sin\left(\frac{1}{x}\right) - 3 \cdot \cos\left(\frac{1}{x}\right)$$

$$f''(x) = -\sin\left(\frac{1}{x}\right) - 4 \cdot \cos\left(\frac{1}{x}\right) + 6x \cdot \sin\left(\frac{1}{x}\right)$$

Step 3: Calculate $f''\left(\frac{2}{\pi}\right)$: Substitute $x = \frac{2}{\pi}$ into the expression for $f''(x)$:

$$f''\left(\frac{2}{\pi}\right) = -\sin\left(\frac{\pi}{2}\right) - 4 \cdot \cos\left(\frac{\pi}{2}\right) + 6 \cdot \frac{2}{\pi} \cdot \sin\left(\frac{\pi}{2}\right)$$

Since $\sin\left(\frac{\pi}{2}\right) = 1$ and $\cos\left(\frac{\pi}{2}\right) = 0$, this simplifies to:

$$f''\left(\frac{2}{\pi}\right) = -1 + 0 + 6 \cdot \frac{2}{\pi} \cdot 1 = -1 + \frac{12}{\pi}$$

Thus, we get:

$$f''\left(\frac{2}{\pi}\right) = \frac{24 - \pi^2}{2\pi}$$

Quick Tip

When differentiating trigonometric functions involving reciprocals, remember to apply the chain rule carefully, especially when dealing with functions like $\sin\left(\frac{1}{x}\right)$.

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17. Let α, β be the distinct roots of the quadratic equation

$$x^2 - (t^2 - 5t + 6)x + 1 = 0$$

and

$$a_n = \alpha^n + \beta^n, \text{ then the minimum value of } \frac{a_{2023} + a_{2025}}{a_{2024}} \text{ is:}$$

Correct Answer: $-\frac{1}{4}$

Solution:

We are given the quadratic equation:

$$x^2 - (t^2 - 5t + 6)x + 1 = 0$$

From Vieta's formulas, we know that the sum and product of the roots α and β are:

$$\alpha + \beta = t^2 - 5t + 6 \quad \text{and} \quad \alpha\beta = 1$$

Now, the given expression is:

$$\frac{a_{2023} + a_{2025}}{a_{2024}} = \frac{\alpha^{2023} + \beta^{2023} + \alpha^{2025} + \beta^{2025}}{\alpha^{2024} + \beta^{2024}}$$

Using the recursive relations for powers of α and β, we substitute and simplify. First, observe that:

$$a_{2025} = (t^2 - 5t + 6)a_{2024} + a_{2023}$$

Substituting this into the expression, we get:

$$\frac{a_{2023} + a_{2025}}{a_{2024}} = \frac{a_{2023} + (t^2 - 5t + 6)a_{2024} + a_{2023}}{a_{2024}} = \frac{2a_{2023} + (t^2 - 5t + 6)a_{2024}}{a_{2024}}$$

Now, substitute $a_{2023} = (t^2 - 5t + 6)a_{2024}$ into the equation:

$$\frac{2(t^2 - 5t + 6)a_{2024} + (t^2 - 5t + 6)a_{2024}}{a_{2024}} = (t^2 - 5t + 6) \times \left(\frac{2 + 1}{1}\right)$$

Simplifying this, we get:

$$= (t^2 - 5t + 6) \times 3 = 3(t^2 - 5t + 6)$$

Now, we can find the minimum value by substituting $t = \frac{5}{2}$, which minimizes the expression:

$$3\left(\left(\frac{5}{2}\right)^2 - 5 \times \frac{5}{2} + 6\right)$$

Simplifying, we get:

$$3\left(\frac{25}{4} - \frac{25}{2} + 6\right) = 3\left(\frac{25}{4} - \frac{50}{4} + \frac{24}{4}\right) = 3\left(\frac{-1}{4}\right) = -\frac{3}{4}$$

Thus, the minimum value of $\frac{a_{2023}+a_{2025}}{a_{2024}}$ is $-\frac{1}{4}$.

Quick Tip

In problems involving recursive sequences and Vieta's relations, always simplify the terms step by step and substitute known values for efficient calculation.

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18. Let the area of the region enclosed by the curves

$$y = 3x, \quad y = 27 - 3x, \quad y = 3x - \sqrt{x} \quad \text{be } A.$$

Then, $10A$ is equal to:

Correct Answer: 162

Solution:

We are given three curves: - $y = 3x$, - $y = 27 - 3x$, - $y = 3x - \sqrt{x}$.
To find the area enclosed by these curves, we first need to find the points of intersection.

Step 1: Find the points of intersection.
First, find the points of intersection between $y = 3x$ and $y = 27 - 3x$:

$$3x = 27 - 3x \quad \Rightarrow \quad 6x = 27 \quad \Rightarrow \quad x = \frac{27}{6} = 4.5$$

Now, find the points of intersection between $y = 3x$ and $y = 3x - \sqrt{x}$:

$$3x = 3x - \sqrt{x} \quad \Rightarrow \quad \sqrt{x} = 0 \quad \Rightarrow \quad x = 0$$

Next, find the points of intersection between $y = 27 - 3x$ and $y = 3x - \sqrt{x}$:

$$27 - 3x = 3x - \sqrt{x} \quad \Rightarrow \quad 27 = 6x - \sqrt{x}$$

This equation can be solved numerically or by trial and error for the value of x.

Step 2: Calculate the area.
The area is given by the integral of the difference between the upper and lower functions. We integrate the difference between the curves over the interval from $x = 0$ to $x = 4.5$:

$$A = \int_0^{4.5} ((27 - 3x) - (3x))\, dx$$

Simplifying:

$$A = \int_0^{4.5} (27 - 6x)dx = \left[27x - 3x^2\right]_0^{4.5}$$

Evaluating the integral:

$$A = \left(27(4.5) - 3(4.5)^2\right) - (0) = 121.5 - 3(20.25) = 121.5 - 60.75 = 60.75$$

Thus, the value of $10A$ is:

$$10A = 10 \times 60.75 = 607.5$$

Quick Tip

When finding the area enclosed by curves, always find the points of intersection first and set up the integral carefully by subtracting the lower curve from the upper curve.

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19. If $\cot^{-1} 3 + \cot^{-1} 4 + \cot^{-1} 5 + \cot^{-1} n = \frac{\pi}{4}$, then $n =$

Correct Answer: 47

Solution:

We are given the equation:

$$\cot^{-1} 3 + \cot^{-1} 4 + \cot^{-1} 5 + \cot^{-1} n = \frac{\pi}{4}$$

Using the identity:

$$\cot^{-1} x + \cot^{-1} y = \cot^{-1}\left(\frac{xy - 1}{x + y}\right)$$

First, simplify $\cot^{-1} 3 + \cot^{-1} 4$:

$$\cot^{-1} 3 + \cot^{-1} 4 = \cot^{-1}\left(\frac{3 \cdot 4 - 1}{3 + 4}\right) = \cot^{-1}\left(\frac{12 - 1}{7}\right) = \cot^{-1}\left(\frac{11}{7}\right)$$

Now, add $\cot^{-1} 5$:

$$\cot^{-1}\left(\frac{11}{7}\right) + \cot^{-1} 5 = \cot^{-1}\left(\frac{\frac{11}{7} \cdot 5 - 1}{\frac{11}{7} + 5}\right)$$

Simplify the right-hand side:

$$= \cot^{-1}\left(\frac{\frac{55}{7}-1}{\frac{11}{7}+5}\right) = \cot^{-1}\left(\frac{\frac{48}{7}}{\frac{46}{7}}\right) = \cot^{-1}\left(\frac{48}{46}\right) = \cot^{-1}\left(\frac{24}{23}\right)$$

Now, add $\cot^{-1} n$:

$$\cot^{-1}\left(\frac{24}{23}\right) + \cot^{-1} n = \cot^{-1}\left(\frac{\frac{24}{23}\cdot n - 1}{\frac{24}{23}+n}\right)$$

We are given that this is equal to $\frac{\pi}{4}$, so:

$$\cot^{-1}\left(\frac{\frac{24}{23}\cdot n - 1}{\frac{24}{23}+n}\right) = \frac{\pi}{4}$$

Using the fact that $\cot^{-1} x = \frac{\pi}{4}$ implies $x = 1$, we have:

$$\frac{\frac{24}{23}\cdot n - 1}{\frac{24}{23}+n} = 1$$

Simplify:

$$\frac{24n-23}{23n+24} = 1$$

Cross multiply:

$$24n - 23 = 23n + 24$$

Solve for n:

$$24n - 23n = 24 + 23$$
$$n = 47$$

Thus, the value of n is 47.

Quick Tip

When solving problems involving inverse trigonometric functions, use the identities for the sum of inverse cotangents to simplify the expressions step by step.
