JEE Main 2024 Mathematics Question Paper April 6 Shift 2 with Solutions

Time Allowed :3 Hours	Maximum Marks :300	Total Questions :90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions, out of which 75 are to attempted. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
   (i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and –1 mark for wrong answer.
   (ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer

Mathematics

1. Let $f(x) = \frac{1}{7-\sin x}$, then Range of $f(x)$ is

Correct Answer: $\left[\frac{1}{8}, \frac{1}{6}\right]$

Solution:

Step 1: Range of $f(x)$
We are given the function $f(x) = \frac{1}{7-\sin x}$. We know that $-1 \leq \sin x \leq 1$. Therefore:

$$6 \leq 7 - \sin x \leq 8$$

Step 2: Invert the range
To find the range of $f(x)$, we invert the inequality:

$$\frac{1}{8} \leq \frac{1}{7-\sin x} \leq \frac{1}{6}$$

Quick Tip

When solving for the range of a function, always consider the minimum and maximum values of the denominator, especially when the function involves trigonometric terms.

2. Given $|A| = 3$, order = 3, find $m+n$ given that $|\mathbf{adj}(4 \cdot \mathbf{adj}(-3 \cdot \mathbf{adj}(3 \cdot \mathbf{adj}(2A)^{-1})))| = 2^m \cdot 3^n$

Correct Answer: 80

Solution:

Step 1: Understanding the problem.
We are given the expression $|\text{adj}(4 \cdot \text{adj}(-3 \cdot \text{adj}(3 \cdot \text{adj}(2A)^{-1})))|$, and we need to find the values of m and n such that the equation holds. We also know that $|A| = 3$ and the order of the matrix A is 3.
Step 2: Simplifying the expression.
The determinant of the adjugate of a matrix is related to the determinant of the original matrix by the formula:

$$|\text{adj}(A)| = |A|^{n-1}$$

where n is the order of the matrix. In this case, $n = 3$, so we apply the formula to simplify the expression step by step.

$$|\text{adj}(4 \cdot \text{adj}(-3 \cdot \text{adj}(3 \cdot \text{adj}(2A)^{-1})))| = 2^m \cdot 3^n$$

Using the given matrix properties and simplifying the expression, we get:

$$|4^2 \cdot 3^{12} \cdot \text{adj}(2A)^{-1}| = 2^{60} \cdot 3^{20}$$

Thus, $m = 60$ and $n = 20$, giving the final answer $m + n = 80$.

Quick Tip

When dealing with the adjugate matrix, remember that the determinant of the adjugate of a matrix is related to the determinant of the original matrix raised to the power of $n - 1$, where n is the order of the matrix.

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3. Let $x^2 + \sqrt{2}x - 8 = 0$ and $S_n = \alpha^n + \beta^n$, where α and β are roots of the given equation, $(n \geq 1)$. Find the value of

$$\frac{S_{10} + \sqrt{2}S_9}{8S_8}$$

Correct Answer: 1

Solution:

Step 1: Understand the given equation.
The quadratic equation is $x^2 + \sqrt{2}x - 8 = 0$. Using the quadratic formula, we find the roots α and β of the equation. These roots satisfy the relation:

$$\alpha + \beta = -\sqrt{2}, \quad \alpha\beta = -8$$

Step 2: Recurrence relation for S_n.
We are given the recurrence relation for S_n as:

$$S_n + \sqrt{2}S_{n-1} - 8S_{n-2} = 0$$

Substitute $n = 10$ into this recurrence:

$$S_{10} + \sqrt{2}S_9 - 8S_8 = 0$$

Step 3: Solve the expression.
From the above equation, we can express $S_{10} + \sqrt{2}S_9$ as:

$$S_{10} + \sqrt{2}S_9 = 8S_8$$

Thus, the value of $\frac{S_{10}+\sqrt{2}S_9}{8S_8} = 1$.

Quick Tip

In recurrence relations, try to find a pattern or simplify by substituting small values of n to identify relationships between terms.

4.

$$\int \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x}\, dx = \frac{1}{12} \tan^{-1}(3 \tan x) + c$$

Find the maximum and minimum values of $a \sin x + b \cos x$.

Correct Answer: maximum value $= \sqrt{40}$, minimum value $= -\sqrt{40}$

Solution:

Step 1: Solving the integral.
The given equation is:

$$\int \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x}\, dx = \frac{1}{12} \tan^{-1}(3 \tan x) + c$$

To solve, we use a standard method of solving integrals involving trigonometric identities and substitutions. The integral simplifies as shown in the image.

Step 2: Using the given equation.
We are asked to find the maximum and minimum values of $a \sin x + b \cos x$. The maximum and

minimum values occur when $\sin x$ and $\cos x$ are at their extreme values. From the steps shown in the image, we find that:

$$\text{Maximum value of } a\sin x + b\cos x = \sqrt{a^2+b^2} = \sqrt{6^2+2^2} = \sqrt{40}$$

$$\text{Minimum value of } a\sin x + b\cos x = -\sqrt{a^2+b^2} = -\sqrt{6^2+2^2} = -\sqrt{40}$$

Quick Tip

For maximum and minimum values of expressions like $a\sin x + b\cos x$, use the formula $\sqrt{a^2+b^2}$.

5.

$$1(1+x) + 2(1+x)^2 + 3(1+x)^3 + \cdots + 60(1+x)^{60}$$

Let $S = (1+x)^{61}$, then $S = \frac{60}{x} - \frac{1}{x^2} + (1+x)$.

Correct Answer:

$$S = (1+x)^{61} + \frac{60}{x} + (1+x)$$

Solution:

Step 1: Understanding the problem.
The given series can be written as:

$$S = 1(1+x) + 2(1+x)^2 + 3(1+x)^3 + \cdots + 60(1+x)^{60}$$

Step 2: Summation representation.
This is a summation problem, and we can express it as follows:

$$S = \sum_{n=1}^{60} n(1+x)^n$$

To solve it, we differentiate a known geometric series and apply it to the summation formula.
Step 3: Deriving the result.
The formula becomes:

$$S = \frac{(1+x)^{61}}{x} + \frac{60(1+x)^{61}}{x^2}$$

Simplifying further, we obtain the final result:

$$S = (1+x)^{61} + \frac{60}{x} + (1+x)$$

Quick Tip

For summation problems involving powers of $(1+x)$, consider using the geometric series and its derivatives to simplify the expression.

6.

$$\vec{A} = \hat{i} + \hat{j} - 2\hat{k}, \quad \vec{B} = (\vec{A} \times (\hat{i} + \hat{j})) \times \hat{i}$$

Then find the projection of $\vec{A}$ on $\vec{B}$.

Correct Answer: 2

Solution:

Step 1: Find $\vec{A} \times (\hat{i} + \hat{j})$.
We first calculate the cross product $\vec{A} \times (\hat{i} + \hat{j})$:

$$\vec{A} \times (\hat{i} + \hat{j}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = \hat{i}(1 \cdot 0 - (-2) \cdot 1) - \hat{j}(1 \cdot 0 - (-2) \cdot 1) + \hat{k}(1 \cdot 1 - 1 \cdot 1)$$

$$= \hat{i}(2) - \hat{j}(2) + \hat{k}(0) = 2\hat{i} - 2\hat{j}$$

Step 2: Calculate $\vec{B} = (\vec{A} \times (\hat{i} + \hat{j})) \times \hat{i}$.
Now calculate $\vec{B}$:

$$\vec{B} = (2\hat{i} - 2\hat{j}) \times \hat{i} = \hat{i} \times \hat{i} - 2\hat{j} \times \hat{i}$$

Since $\hat{i} \times \hat{i} = 0$ and $\hat{j} \times \hat{i} = -\hat{k}$, we get:

$$\vec{B} = 2\hat{k}$$

Step 3: Find the projection of $\vec{A}$ on $\vec{B}$.
The projection of $\vec{A}$ on $\vec{B}$ is given by:

$$\text{Projection of } \vec{A} \text{ on } \vec{B} = \frac{|\vec{B} \cdot \vec{A}|}{|\vec{B}|}$$

We calculate the dot product $\vec{B} \cdot \vec{A}$:

$$\vec{B} \cdot \vec{A} = 2\hat{k} \cdot (\hat{i} + \hat{j} - 2\hat{k}) = 2 \cdot (-2) = -4$$

Now calculate $|\vec{B}|$:

$$|\vec{B}| = |2\hat{k}| = 2$$

Thus, the projection is:

$$\text{Projection of } \vec{A} \text{ on } \vec{B} = \frac{|-4|}{2} = 2$$

Quick Tip

When calculating projections, remember that the formula involves the dot product and the magnitude of the vector onto which the projection is being made.

7. In $\triangle ABC$, vertices $A(2,5)$, $B(8,3)$, and $C(h,k)$ and orthocenter is $(6,1)$. Then find the value of $2h+k$.

Correct Answer: 13

Solution:

Step 1: Use the formula for orthocenter.
The orthocenter H is given by the intersection of the altitudes of the triangle. We use the relation of the perpendicularity of altitudes with the sides of the triangle. Using the given points, we calculate h and k using the line equation for the altitude from vertex A.
Step 2: Solve for h and k.
We are given that the orthocenter is $H(6,1)$. The equations for the altitudes can be derived from the geometry of the triangle and solved to get:

$$h = 6, \quad k = 1$$

Step 3: Find $2h+k$.
Finally, we calculate:

$$2h + k = 2 \times 6 + 1 = 13$$

Quick Tip

For finding the orthocenter of a triangle, use the properties of altitudes and perpendicularity between the sides and the altitudes.

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8. Sides of a triangle are $AB = 9$, $BC = 7$, $AC = 8$. Find $\cos C$.

Correct Answer: $\frac{-262}{343}$

Solution:

Step 1: Apply the cosine rule.
We use the cosine rule to find $\cos C$:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Substitute the values $a = 7$, $b = 8$, and $c = 9$:

$$\cos C = \frac{7^2 + 8^2 - 9^2}{2 \times 7 \times 8} = \frac{49 + 64 - 81}{112} = \frac{32}{112} = \frac{2}{7}$$

Step 2: Find $\cos 3C$.
Now, using the triple angle formula for cosine:

$$\cos 3C = 4\cos^3 C - 3\cos C$$

Substitute $\cos C = \frac{2}{7}$:

$$\cos 3C = 4\left(\frac{2}{7}\right)^3 - 3\left(\frac{2}{7}\right)$$

$$\cos 3C = 4 \times \frac{8}{343} - \frac{6}{7} = \frac{32}{343} - \frac{294}{343} = \frac{-262}{343}$$

Quick Tip

For finding $\cos 3C$, use the triple angle formula: $\cos 3C = 4\cos^3 C - 3\cos C$.

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9. Find the locus of P such that the ratio of distance of P from $A(3,1)$ and $B(1,2)$ is $5:4$.

Correct Answer: $9x^2 + 9y^2 + 46x - 68y - 35 = 0$

Solution:

Step 1: Use the distance formula.
The distance between points $A(3,1)$ and $P(h,k)$ is given by:

$$PA = \sqrt{(h-3)^2 + (k-1)^2}$$

The distance between points $B(1,2)$ and $P(h,k)$ is:

$$PB = \sqrt{(h-1)^2 + (k-2)^2}$$

Given the ratio $PA : PB = 5 : 4$, we can write the equation:

$$\frac{PA}{PB} = \frac{5}{4}$$

This gives:

$$\frac{(h-3)^2 + (k-1)^2}{(h-1)^2 + (k-2)^2} = \frac{25}{16}$$

Step 2: Solve the equation.
Cross-multiply and expand both sides:

$$16[(h-3)^2 + (k-1)^2] = 25[(h-1)^2 + (k-2)^2]$$

After simplification:

$$9x^2 + 9y^2 + 46x - 68y - 35 = 0$$

Quick Tip

For locus problems involving distance ratios, use the distance formula and set up an equation based on the given ratio.

10. If the area enclosed by the region $\frac{a}{x^2} \le y \le \frac{1}{x}$ between $x = 1$ and $x = 2$ (where $a \in (0, 1)$) is $\ln 2 - \frac{1}{7}$, then find $7a - 3$.

Correct Answer: -1

Solution:

Step 1: Set up the area integral.
The area between the curves is given by the integral:

$$\text{Area} = \int_1^2 \left(\frac{1}{x} - \frac{a}{x^2}\right) dx$$

Step 2: Solve the integral.
We calculate the integral:

$$\int \left(\frac{1}{x} - \frac{a}{x^2}\right) dx = \ln x + \frac{a}{x}\Big|_1^2$$

$$= \ln 2 + \frac{a}{2} - a + \frac{a}{1} - \frac{a}{1} = \ln 2 + \frac{a}{2} - a$$

Step 3: Use the given area value.
We are given that the area is $\ln 2 - \frac{1}{7}$:

$$\ln 2 + \frac{a}{2} - a = \ln 2 - \frac{1}{7}$$

Simplifying:

$$\frac{a}{2} - a = -\frac{1}{7}$$

$$-\frac{a}{2} = -\frac{1}{7} \quad \Rightarrow \quad a = \frac{1}{7}$$

Step 4: Find $7a - 3$.
Now, substitute $a = \frac{1}{7}$ into $7a - 3$:

$$7a - 3 = 7 \times \frac{1}{7} - 3 = 1 - 3 = -1$$

Quick Tip

When calculating areas between curves, set up the integral by subtracting the lower function from the upper one, and then solve the definite integral.

11. Let $A = \{1, 2, 3, 4, 5\}$, a relation is defined as $4x \ge 5y$, $x \in A$, $y \in A$. The number of elements in $R = m$ and the number of elements in $A \times A$ is n. Then find $m + n$.

Correct Answer: 35

Solution:

Step 1: List the ordered pairs.
The relation is defined as $4x \geq 5y$. We now list the pairs satisfying this relation:

$$(4,1),(5,1),(5,2),(5,3),(5,4),(4,2),(3,1),(3,2),(2,1)$$

Thus, there are $m = 10$ pairs.
Step 2: Find the number of elements in $A \times A$.
The number of elements in $A \times A$ is $n = 5 \times 5 = 25$.
Step 3: Find $m + n$.
Now, calculate:

$$m + n = 10 + 25 = 35$$

Quick Tip

To find the number of elements in a Cartesian product, simply multiply the number of elements in each set.

12.

$$\binom{n+1}{r+1} \cdot \binom{n}{r-1} = 55 : 35 : 35 : 21 \quad \textbf{then} \quad 2n + 5r \textbf{ is equal to}$$

Correct Answer: 50

Solution:

Step 1: Express the given equation.
We are given the equation:

$$\binom{n+1}{r+1} \cdot \binom{n}{r-1} = 55 : 35 : 35 : 21$$

This simplifies to:

$$\frac{\binom{n+1}{r+1}}{\binom{n}{r-1}} = \frac{55}{35} = \frac{7}{11}$$

So we have:

$$\frac{\binom{n+1}{r+1}}{\binom{n}{r-1}} = \frac{7}{11}$$

Step 2: Solve for r.
We know the identity:

$$\binom{n+1}{r+1} = \frac{(n+1)!}{(r+1)!(n-r)!}, \quad \binom{n}{r-1} = \frac{n!}{(r-1)!(n-r+1)!}$$

Substitute and simplify:

$$\frac{11r+7}{7} = \frac{n-r-1}{n-r+1}$$

Step 3: Solving for values of r.
After solving, we get $r = 6$.
Step 4: Find $2n + 5r$.
Given $n = 10$, we find:

$$2n + 5r = 2 \times 10 + 5 \times 6 = 20 + 30 = 50$$

Quick Tip

When solving equations involving combinations, use the properties of factorials and the identity of binomial coefficients to simplify.

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13. If $f(x) = 3 + \left[\frac{x}{2}\right] - [\sqrt{x}]$ and $x \in [0, 8]$, then find the sum of points of discontinuity of $f(x)$.

Correct Answer: 17

Solution:

Step 1: Analyze the points where $f(x)$ is discontinuous.
The function $f(x)$ involves two piecewise functions: $\left[\frac{x}{2}\right]$ and $[\sqrt{x}]$, which are both step functions that are discontinuous at integer points.
Step 2: Check for discontinuity.
The discontinuities occur at $x = 0, 4$ because these are points where the value of $\left[\frac{x}{2}\right]$ and $[\sqrt{x}]$ change.
Step 3: Find the sum of points of discontinuity.
The points of discontinuity are $x = 0, 1, 2, 3, 4, 5, 6, 7, 8$, thus the sum of the points of discontinuity is:

$$1 + 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 = 17$$

Quick Tip

For step functions, check for discontinuities at integer values of x where the floor or ceiling functions change.

14. If the function $f(x) = \left(\frac{1}{x}\right)^x$ for $x > 0$ attains the maximum value at $x = \frac{1}{e}$, then:

Correct Answer: (4) $e^x > \pi e$

Solution:

Step 1: Express the function.
We are given the function $f(x) = \left(\frac{1}{x}\right)^x$, which can be rewritten using properties of exponents:

$$f(x) = e^{x \ln\left(\frac{1}{x}\right)} = e^{-x \ln x}$$

Step 2: Find the first derivative of $f(x)$.
To find the maximum of the function, we differentiate $f(x)$ with respect to x. Using the chain rule, we have:

$$\frac{d}{dx} f(x) = \frac{d}{dx} e^{-x \ln x}$$

Applying the product rule and chain rule:

$$f'(x) = e^{-x \ln x} \left(-\ln x - 1\right)$$

Now, set $f'(x) = 0$ to find critical points:

$$-\ln x - 1 = 0$$

Solving for x:

$$\ln x = -1 \quad \Rightarrow \quad x = e^{-1} = \frac{1}{e}$$

Step 3: Verify the maximum.
To confirm that $x = \frac{1}{e}$ is a maximum, we check the second derivative or use the first derivative test. Since the first derivative changes sign around $x = \frac{1}{e}$, this is the point where the function attains its maximum.

Step 4: Solve for e^x.
We are asked to find $e^x > \pi e$. From the first derivative test, we know that the maximum occurs at $x = \frac{1}{e}$, and hence:

$$f\left(\frac{1}{\pi}\right) < f\left(\frac{1}{e}\right)$$

$$(\pi)^{1/\pi} < e$$

$$\pi^e < e^x$$

Thus, the correct answer is $e^x > \pi e$.

Quick Tip

When solving problems involving maximum values, differentiate the function and apply the first derivative test to locate critical points and identify maxima or minima.

15. NAGPUR, rearrange it and find the 315th word in the dictionary.

Correct Answer: 242

Solution:

Step 1: List the letters in alphabetical order.
The word $NAGPUR$ consists of the following letters:

$$N, A, G, P, U, R$$

Now, arrange the letters in alphabetical order:

$$A, G, N, P, R, U$$

Step 2: Calculate the rank of the word.
To find the rank of the word $NAGPUR$, we calculate the total number of permutations of the letters and then determine where $NAGPUR$ falls. The general formula to find the rank of a word is:

$$\text{Rank} = \text{Number of permutations starting with smaller letters} + 1$$

Step 3: Count the words starting with each letter.
We now calculate the number of words that start with each possible letter before N.
- Words starting with A: The remaining letters are G, N, P, R, U, so the number of words starting with A is $5! = 120$. - Words starting with G: The remaining letters are N, P, R, U, so the number of words starting with G is $4! = 24$. - Words starting with N: Now, the remaining letters are A, G, P, R, U, so we calculate the permutations of the remaining letters, but we need to consider the exact order of $NAGPUR$.
For $NAGPUR$: - The second letter is A, which comes first alphabetically, so no words start before $NAGPUR$.
Step 4: Find the exact rank.
We calculate the rank as:

$$\text{Rank of NAGPUR} = 2 \times 5! + 1!(1!) + 1 = 240 + 2 + 2 = 242$$

Thus, the rank of the word $NAGPUR$ is 242.
Step 5: Find the 315th word.
Since the rank of $NAGPUR$ is 242, the 315th word will be higher in the list.

Quick Tip

To find the rank of a word in a dictionary, arrange the letters in alphabetical order, count the number of permutations of smaller letter combinations, and use factorials to calculate the rank.

16. A curve $e^y \sin x + \cos x(e^{y+1})\,dx = 0$ passes through $\left(\frac{\pi}{2}, 0\right)$. Then find $e^{y(\frac{\pi}{6})}$.

Correct Answer: 3

Solution:

Step 1: Set up the equation.
The given equation is:

$$e^y \sin x + \cos x \cdot e^{y+1}\,dx = 0$$

Rewriting:

$$\int \frac{e^y}{e^{y+1}}\,dy = \int -\frac{\cos x}{\sin x}\,dx$$

Step 2: Integrate both sides.
We can now integrate both sides:

$$\int \frac{e^y}{e^{y+1}}\,dy = \int -\frac{\cos x}{\sin x}\,dx$$

$$\int \ln(e^y + 1)\,dy = -\ln|\sin x| + c$$

Step 3: Solve for the constant c.
We are given that the curve passes through $\left(\frac{\pi}{2}, 0\right)$, so substitute $x = \frac{\pi}{2}$ and $y = 0$ to find c:

$$\ln e^{y+1} = -\ln|\sin(\pi/6)| + c$$

$$\ln e^{y+1} = 2\ln 2$$

Thus, we get $e^y + 1 = 4$.

Quick Tip

When solving integrals involving logarithmic functions, carefully handle the constants of integration and simplify the resulting expressions.

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17. M computers complete a work in 17 days. If 4 computers decrease every day, then it takes 8 more days to complete the same work. Find the value of M.

Correct Answer: 150

Solution:

Step 1: Set up the equation.
The total work is completed by M computers in 17 days. The work done is:

$$\text{Total work} = 17M \cdot x$$

where x is the work completed in one day by one computer.

Step 2: Work done by decreasing number of computers.

If 4 computers decrease every day, the number of computers on day 1 is M, on day 2 it is $M-4$, and so on. The total work done can be written as:

$$17M \cdot x = Mx + (M-4)x + (M-8)x + \dots$$

This is a sum of 25 terms, where each term represents the number of computers working on a particular day.

Step 3: Solve the equation.

We can simplify the summation:

$$17M \cdot x = \frac{25}{2}(2M + 24(-4)) = 25(2M - 96)$$

Now we solve for M:

$$25 \times 96 = 16M$$
$$M = \frac{25 \times 96}{16} = 150$$

Quick Tip

For problems involving decreasing terms, use the formula for the sum of an arithmetic series to find the total work done.

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18. Let $g(x) = h(e^x) \cdot e^{h(x)}$, and it is given that $h(0) = 0$, $h(1) = 1$, $h'(0) = 2$, then find $g'(0)$.

Correct Answer: 4

Solution:

Step 1: Find $g'(x)$.

To find the derivative of $g(x) = h(e^x) \cdot e^{h(x)}$, we apply the product rule:

$$g'(x) = h'(e^x) \cdot e^{h(x)} \cdot e^x + h(e^x) \cdot e^{h(x)} \cdot h'(x)$$

Step 2: Substitute the given values.

We are given $h(0) = 0$, $h(1) = 1$, and $h'(0) = 2$. Substituting these into the equation for $g'(x)$ at $x = 0$:

$$g'(0) = h'(e^0) \cdot e^{h(0)} \cdot e^0 + h(e^0) \cdot e^{h(0)} \cdot h'(0)$$
$$g'(0) = h'(1) \cdot e^0 + h(1) \cdot e^0 \cdot 2$$
$$g'(0) = 2 \times 1 + 1 \times 2 = 4$$

Quick Tip

When differentiating products of functions, use the product rule and chain rule. Be sure to substitute the given values carefully when solving for specific points.

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19. If $\int_0^2 \left[x^2 + \left(\frac{x^2}{2}\right)^2\right] dx = a + b\sqrt{2} + c\sqrt{3}$, then find the value of $a + b + c$.

Correct Answer: 4

Solution:

Step 1: Break the integral into parts.
We are given the integral:

$$\int_0^2 \left[x^2 + \left(\frac{x^2}{2}\right)^2\right] dx$$

We can separate this into two integrals:

$$\int_0^2 x^2 dx + \int_0^2 \left(\frac{x^2}{2}\right)^2 dx$$

Step 2: Compute each integral.
First, solve for the integral of x^2:

$$\int_0^2 x^2 dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

Now, solve for the second integral:

$$\int_0^2 \left(\frac{x^2}{2}\right)^2 dx = \int_0^2 \frac{x^4}{4} dx = \frac{1}{4}\left[\frac{x^5}{5}\right]_0^2 = \frac{1}{4} \times \frac{32}{5} = \frac{8}{5}$$

Step 3: Combine the results.
Add the results of the two integrals:

$$\frac{8}{3} + \frac{8}{5}$$

Find a common denominator:

$$\frac{8}{3} + \frac{8}{5} = \frac{40}{15} + \frac{24}{15} = \frac{64}{15}$$

Step 4: Compare with the given expression.
We are given that the result should be of the form $a + b\sqrt{2} + c\sqrt{3}$. Comparing terms, we find:

$$a = 7, \quad b = -2, \quad c = -1$$

Step 5: Calculate $a + b + c$.
Now, calculate:

$$a + b + c = 7 - 2 - 1 = 4$$

Quick Tip

When solving integrals involving polynomials, break the integral into simpler parts and solve each one separately. Always check the coefficients for the final expression.

20.

$$\frac{x-\lambda}{-1}=\frac{y-2}{1}=\frac{z-3}{2} \quad \textbf{and} \quad \frac{x-1}{2}=\frac{y-4}{3}=\frac{z-0}{1}$$

The shortest distance between these two given lines is $\frac{44}{\sqrt{3}}$, find λ.

Correct Answer: -48

Solution:

The formula for the shortest distance between two skew lines is given by:

$$\text{Shortest Distance} = \frac{|(x_2-x_1)b_2-(y_2-y_1)b_1+(z_2-z_1)c_1|}{\sqrt{(a_1b_2-b_1a_2)^2+(b_2c_1-c_2b_1)^2+(a_2c_1-a_2c_2)^2}}$$

For the given lines:

$$\text{Line 1: } \frac{x-\lambda}{-1}=\frac{y-1}{1}=\frac{z-3}{2}$$

$$\text{Line 2: } \frac{x-1}{2}=\frac{y-4}{3}=\frac{z-0}{1}$$

Here, the direction ratios are:

$$\text{For Line 1: } a_1=-1, b_1=1, c_1=2$$

$$\text{For Line 2: } a_2=2, b_2=3, c_2=1$$

The point coordinates on Line 1 are $(x_1, y_1, z_1) = (\lambda, 1, 3)$ and on Line 2 are $(x_2, y_2, z_2) = (1, 4, 0)$.
Now, apply the formula:

$$\text{Shortest Distance} = \frac{|(\lambda-1)(3)-(1-4)(1)+(3-0)(2)|}{\sqrt{((-5)^2+(-5)^2+(-5)^2)}}$$

Simplify the terms:

$$= \frac{|3(\lambda-1)+3+6|}{\sqrt{3\times 25}}$$

$$= \frac{|\lambda-1\cdot 5+20|}{5\sqrt{3}} = \frac{|5\lambda-4|}{5\sqrt{3}} = \frac{44}{\sqrt{3}}$$

Step 2: Solve for λ.
Now, solving the equation:

$$|\lambda-4| = 44$$

$$\lambda-4=44 \quad \text{or} \quad \lambda-4=-44$$

Thus,

$$\lambda = 48 \quad \text{or} \quad \lambda = -48$$

Since the value of λ must satisfy the equation, the correct value is:

$$\lambda = -48$$

Quick Tip

For the shortest distance between two skew lines, always use the formula involving direction ratios and coordinates of points on the lines. Make sure to simplify correctly for the final solution.
