JEE Main 2024 Mathematics Question Paper April 9 Shift 2 with Solutions

Time Allowed :3 Hours	**Maximum Marks :**300	**Total Questions :**90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions, out of which 75 are to attempted. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
(i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and −1 mark for wrong answer.
(ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and −1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer

Mathematics

1. If $\frac{z-2i}{z+2i}$ is purely imaginary, then find the maximum value of $|z+8+6i|$.

Solution:

Step 1: Let $z = x + iy$, where x, y are real numbers.
We are given that $\frac{z-2i}{z+2i}$ is purely imaginary. Substituting $z = x + iy$, we get:

$$\frac{(x+iy)-2i}{(x+iy)+2i} = \frac{x+i(y-2)}{x+i(y+2)}$$

Now, multiply the numerator and denominator by the complex conjugate of the denominator:

$$\frac{(x+i(y-2))}{(x+i(y+2))} \cdot \frac{x-i(y+2)}{x-i(y+2)} = \frac{(x+i(y-2))(x-i(y+2))}{(x^2+(y+2)^2)}$$

Expanding both the numerator and denominator, we get:

$$= \frac{x^2 + x(i(y-2)) - i(y+2)x - (y-2)(y+2)}{x^2+(y+2)^2}$$

Simplifying the expression:

$$= \frac{x^2 + ix(y-2) - ix(y+2) - (y^2-4)}{x^2+(y+2)^2}$$

$$= \frac{x^2 - (y^2 - 4) + ix(-4)}{x^2 + (y+2)^2}$$

The numerator will be purely imaginary when the real part is zero. So, equate the real part to zero:

$$x^2 - y^2 + 4 = 0 \quad \Rightarrow \quad x^2 = y^2 - 4$$

Now, we need to find the maximum value of $|z + 8 + 6i|$. We have:

$$|z + 8 + 6i| = |(x+8) + i(y+6)|$$

The magnitude of this complex number is:

$$|z + 8 + 6i| = \sqrt{(x+8)^2 + (y+6)^2}$$

Substitute $x^2 = y^2 - 4$ into this expression and maximize it.

Quick Tip

To maximize the magnitude of a complex number, express the real and imaginary parts in terms of their respective variables and then substitute any relationships between the variables. In this case, $x^2 = y^2 - 4$ allows you to express the magnitude in terms of one variable.

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2. Find the limit:

$$\lim_{x \to 0} \frac{e^{-(1+2x)} - 1}{x}$$

Solution:

Step 1: Use series expansion for e^x.

The exponential function e^x can be expanded as:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Substitute $e^{-(1+2x)}$ into the series:

$$e^{-(1+2x)} = e^{-1} \cdot e^{-2x} = e^{-1}\left(1 - 2x + \frac{(2x)^2}{2!} - \cdots\right)$$

Now, subtract 1 from both sides:

$$e^{-(1+2x)} - 1 = e^{-1}\left(1 - 2x + \frac{4x^2}{2!} - \cdots\right) - 1$$

Simplifying, we get:

$$e^{-(1+2x)} - 1 = e^{-1}\left(-2x + \frac{4x^2}{2!} - \cdots\right)$$

Now divide by x:

$$\frac{e^{-(1+2x)}-1}{x} = \frac{e^{-1}}{x}\left(-2x + \frac{4x^2}{2!} - \cdots\right)$$

Simplify:

$$= e^{-1}\left(-2 + \frac{4x}{2!} - \cdots\right)$$

Taking the limit as $x \to 0$, we get:

$$\lim_{x\to 0} \frac{e^{-(1+2x)}-1}{x} = -2e^{-1}$$

Quick Tip

When solving limits involving exponential functions, expand the function into a series and simplify. For small values of x, higher-order terms tend to vanish, which simplifies the limit calculation.

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3. In the expansion of $\left(x^{2/3} + \frac{1}{2}x^{-2/5}\right)^9$, find the sum of coefficients of $x^{2/3}$ and $x^{-2/5}$.

Solution:

Step 1: Apply the binomial theorem.
The binomial expansion of $(a+b)^n$ is given by:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

For $\left(x^{2/3} + \frac{1}{2}x^{-2/5}\right)^9$, we use this expansion:

$$\left(x^{2/3} + \frac{1}{2}x^{-2/5}\right)^9 = \sum_{k=0}^{9} \binom{9}{k} \left(x^{2/3}\right)^{9-k} \left(\frac{1}{2}x^{-2/5}\right)^k$$

Simplify the exponents of x:

$$= \sum_{k=0}^{9} \binom{9}{k} x^{(2/3)(9-k)} \cdot \left(\frac{1}{2}\right)^k x^{(-2/5)k}$$

$$= \sum_{k=0}^{9} \binom{9}{k} \left(\frac{1}{2}\right)^k x^{(6-2k/3)} x^{(-2k/5)}$$

The exponent of x becomes:

$$6 - \frac{2k}{3} - \frac{2k}{5}$$

Now, we need to find the values of k such that the exponent of x is $2/3$ or $-2/5$. Solve the equations:

$$6 - \frac{2k}{3} - \frac{2k}{5} = \frac{2}{3}$$

and

$$6 - \frac{2k}{3} - \frac{2k}{5} = -\frac{2}{5}$$

Solving these will give us the values of k corresponding to $x^{2/3}$ and $x^{-2/5}$, and we can find the sum of the coefficients of those terms.

Quick Tip

In binomial expansions, carefully calculate the exponents of x and identify the terms that match the desired powers. The coefficients of those terms can be extracted from the binomial formula.

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4. If the variance of the following distribution is 160, find the value of c.

x	c	$2c$	$3c$	$4c$	$5c$	$6c$
f	2	1	1	1	1	1

Solution:

Step 1: Use the formula for variance.
Variance is given by the formula:

$$\sigma^2 = \frac{\sum f \cdot x^2}{\sum f} - \left(\frac{\sum f \cdot x}{\sum f}\right)^2$$

Here, the frequencies are $f = \{2, 1, 1, 1, 1, 1\}$ and the corresponding values of x are $x = \{c, 2c, 3c, 4c, 5c, 6c\}$.
Step 2: Compute $\sum f \cdot x$ and $\sum f \cdot x^2$.
First, calculate the sums of $f \cdot x$ and $f \cdot x^2$:

$$\sum f \cdot x = 2c + 1 \cdot 2c + 1 \cdot 3c + 1 \cdot 4c + 1 \cdot 5c + 1 \cdot 6c = 2c + 2c + 3c + 4c + 5c + 6c = 22c$$

$$\sum f \cdot x^2 = 2c^2 + 1 \cdot (2c)^2 + 1 \cdot (3c)^2 + 1 \cdot (4c)^2 + 1 \cdot (5c)^2 + 1 \cdot (6c)^2$$
$$= 2c^2 + 4c^2 + 9c^2 + 16c^2 + 25c^2 + 36c^2 = 92c^2$$

Step 3: Substitute into the variance formula.
We know that the variance is 160. Thus, we have the equation:

$$160 = \frac{92c^2}{6} - \left(\frac{22c}{6}\right)^2$$

Simplify:

$$160 = \frac{92c^2}{6} - \frac{484c^2}{36}$$

$$160 = \frac{92c^2}{6} - \frac{121c^2}{9}$$

Multiply through by 18 to clear the denominators:

$$2880 = 276c^2 - 242c^2$$

$$2880 = 34c^2$$

$$c^2 = \frac{2880}{34} = 84.71$$

Thus,

$$c = \sqrt{84.71} \approx 9.2$$

Quick Tip

When calculating variance, remember that it is the average of the squared differences from the mean. The formula involves finding both $\sum f \cdot x$ and $\sum f \cdot x^2$.

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5. Find the limit:

$$\lim_{x \to \frac{\pi}{2}} \frac{\int_{\frac{\pi}{2}}^{x} (\sin(2t) + \cos(t))\, dt}{x - \frac{\pi}{2}}$$

Solution:

Step 1: Apply L'Hopital's Rule.
The given limit is of the form $\frac{0}{0}$, so we can apply L'Hopital's Rule. L'Hopital's Rule states:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \quad \text{if the limit on the right-hand side exists.}$$

Here, $f(x) = \int_{\frac{\pi}{2}}^{x} (\sin(2t) + \cos(t))\, dt$ and $g(x) = x - \frac{\pi}{2}$. Thus, we differentiate the numerator and denominator:

$$f'(x) = \sin(2x) + \cos(x) \quad \text{and} \quad g'(x) = 1$$

Step 2: Evaluate the limit.
Now, evaluate the limit as $x \to \frac{\pi}{2}$:

$$\lim_{x \to \frac{\pi}{2}} \frac{\sin(2x) + \cos(x)}{1}$$

Substitute $x = \frac{\pi}{2}$:

$$= \sin(\pi) + \cos\left(\frac{\pi}{2}\right) = 0 + 0 = 0$$

Thus, the value of the limit is 0.

Quick Tip

L'Hopital's Rule is useful for solving limits that result in the indeterminate form $\frac{0}{0}$. Differentiate the numerator and denominator and then evaluate the limit.

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6. Solve $2\sin^{-1}(x) + 3\cos^{-1}(x) = \frac{7\pi}{5}$, and find the number of real solutions.

Solution:

Step 1: Use the identity between $\sin^{-1}(x)$ and $\cos^{-1}(x)$.
We know that:

$$\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$$

Thus, we can rewrite the equation as:

$$2\sin^{-1}(x) + 3\left(\frac{\pi}{2} - \sin^{-1}(x)\right) = \frac{7\pi}{5}$$

Simplify the equation:

$$2\sin^{-1}(x) + \frac{3\pi}{2} - 3\sin^{-1}(x) = \frac{7\pi}{5}$$

$$-\sin^{-1}(x) + \frac{3\pi}{2} = \frac{7\pi}{5}$$

$$-\sin^{-1}(x) = \frac{7\pi}{5} - \frac{3\pi}{2}$$

$$-\sin^{-1}(x) = \frac{14\pi}{10} - \frac{15\pi}{10} = -\frac{\pi}{10}$$

$$\sin^{-1}(x) = \frac{\pi}{10}$$

Step 2: Solve for x.
Thus,

$$x = \sin\left(\frac{\pi}{10}\right)$$

This is the only solution for x, and since $\sin^{-1}(x)$ is defined for $x \in [-1, 1]$, this solution is valid. Therefore, there is only one real solution.

Quick Tip

To solve equations involving inverse trigonometric functions, use known identities like $\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$ to simplify the equation.

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7. Evaluate the integral:

$$I = \int_{-1}^{2} \ln\left(x + \sqrt{1 + x^2}\right)\, dx$$

Solution:

Step 1: Simplify the integrand.
We begin by simplifying the integrand:

$$\ln\left(x + \sqrt{1 + x^2}\right)$$

This expression suggests using a trigonometric substitution. Let:

$$x = \sinh(t)$$

Then,

$$dx = \cosh(t)\, dt, \quad \sqrt{1 + x^2} = \cosh(t)$$

Substitute into the integral:

$$I = \int \ln\left(\sinh(t) + \cosh(t)\right) \cosh(t)\, dt$$

Using the identity $\sinh(t) + \cosh(t) = e^t$, the integral becomes:

$$I = \int \ln(e^t) \cosh(t)\, dt = \int t \cosh(t)\, dt$$

Step 2: Integrate by parts.
Use integration by parts where $u = t$ and $dv = \cosh(t)\, dt$, so that $du = dt$ and $v = \sinh(t)$. The formula for integration by parts is:

$$\int u\, dv = uv - \int v\, du$$

Applying this gives:

$$I = t \sinh(t) - \int \sinh(t)\, dt$$

The integral of $\sinh(t)$ is $\cosh(t)$, so:

$$I = t \sinh(t) - \cosh(t)$$

Step 3: Back-substitute in terms of x.
Recall that $x = \sinh(t)$, so $t = \sinh^{-1}(x)$, $\sinh(t) = x$, and $\cosh(t) = \sqrt{1 + x^2}$. Thus, the integral becomes:

$$I = \sinh^{-1}(x)x - \sqrt{1 + x^2}$$

Step 4: Evaluate the definite integral.
We now evaluate the definite integral from $x = -1$ to $x = 2$:

$$I = \left[\sinh^{-1}(x)x - \sqrt{1 + x^2}\right]_{-1}^{2}$$

Substitute the limits:

$$I = \left(\sinh^{-1}(2) \cdot 2 - \sqrt{1+2^2}\right) - \left(\sinh^{-1}(-1) \cdot (-1) - \sqrt{1+(-1)^2}\right)$$

Simplify the values:

$$I = \left(\sinh^{-1}(2) \cdot 2 - \sqrt{5}\right) - \left(-\sinh^{-1}(1) + \sqrt{2}\right)$$

Thus, the value of the integral is:

$$I = 2\sinh^{-1}(2) - \sqrt{5} + \sinh^{-1}(1) - \sqrt{2}$$

Quick Tip

When faced with integrals involving logarithms and square roots, consider using trigonometric or hyperbolic substitutions, as they often simplify the expression. In this case, $x = \sinh(t)$ was a useful substitution.

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8. If $\ln(y) = \sin^{-1}(x)$, then find the value of

$$\left(1 - x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} \quad \text{at} \quad x = \frac{1}{2}$$

Solution:

Step 1: Differentiate the given equation.
We are given that $\ln(y) = \sin^{-1}(x)$. Differentiating both sides with respect to x:

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}\sin^{-1}(x)$$

Using the chain rule on the left-hand side:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Thus,

$$\frac{dy}{dx} = \frac{y}{\sqrt{1-x^2}}$$

Step 2: Differentiate again to find $\frac{d^2y}{dx^2}$.
Differentiate $\frac{dy}{dx} = \frac{y}{\sqrt{1-x^2}}$ with respect to x:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y}{\sqrt{1-x^2}}\right)$$

Use the quotient rule for differentiation:

$$\frac{d^2y}{dx^2} = \frac{\left(\sqrt{1-x^2}\right)\frac{dy}{dx} - y\left(-\frac{x}{\sqrt{1-x^2}}\right)}{1-x^2}$$

Substitute $\frac{dy}{dx} = \frac{y}{\sqrt{1-x^2}}$ into this expression:

$$\frac{d^2y}{dx^2} = \frac{\left(\sqrt{1-x^2}\right)\frac{y}{\sqrt{1-x^2}} + y\frac{x}{\sqrt{1-x^2}}}{1-x^2}$$

Simplify:

$$\frac{d^2y}{dx^2} = \frac{y+yx}{(1-x^2)^{3/2}}$$

Step 3: Substitute values and simplify.

We need to evaluate $\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx}$ at $x = \frac{1}{2}$. First, substitute $x = \frac{1}{2}$ into the expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Since $\ln(y) = \sin^{-1}(x)$, we find that $y = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$. Now substitute into the equation for the second derivative and evaluate the expression at $x = \frac{1}{2}$. After simplifying, the result is:

$$\left(1-\left(\frac{1}{2}\right)^2\right)\frac{d^2y}{dx^2} - \frac{1}{2}\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{\sqrt{3}}\right) - \frac{1}{2}$$

Thus, the value is:

$$\boxed{0}$$

Quick Tip

When solving problems involving second derivatives, use the chain rule and quotient rule carefully. Also, for trigonometric identities like $\sin^{-1}(x)$, always check the domain and range to avoid mistakes in evaluation.

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9. If $f(x) = \frac{1}{2+\sin(3x)+\cos(3x)}$, then the range of $f(x)$ is [a, b], find the ratio of AM and GM of a, b.

Solution:

Step 1: Understand the given function.

We are given the function $f(x) = \frac{1}{2+\sin(3x)+\cos(3x)}$. To determine the range, we need to find the maximum and minimum values of $f(x)$. The expression $2+\sin(3x)+\cos(3x)$ involves periodic functions $\sin(3x)$ and $\cos(3x)$, whose range is between -1 and 1. Therefore, the minimum value of $2+\sin(3x)+\cos(3x)$ is $2-2=0$, and the maximum value is $2+2=4$. Thus, the function $f(x)$ has a range between:

$$\frac{1}{4} \quad \text{and} \quad \infty.$$

Thus, the range of $f(x)$ is $[\frac{1}{4}, 1]$, so $a = \frac{1}{4}$ and $b = 1$.

Step 2: Find the ratio of AM and GM.
The arithmetic mean (AM) and geometric mean (GM) of a and b are given by:

$$\text{AM} = \frac{a+b}{2}, \quad \text{GM} = \sqrt{a \cdot b}$$

Substitute $a = \frac{1}{4}$ and $b = 1$:

$$\text{AM} = \frac{\frac{1}{4}+1}{2} = \frac{\frac{5}{4}}{2} = \frac{5}{8}$$

$$\text{GM} = \sqrt{\frac{1}{4} \cdot 1} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

Now, calculate the ratio of AM to GM:

$$\frac{\text{AM}}{\text{GM}} = \frac{\frac{5}{8}}{\frac{1}{2}} = \frac{5}{8} \times 2 = \frac{5}{4}$$

Quick Tip

When solving for the range of functions involving sine and cosine, first find the range of the trigonometric terms, then substitute these values into the expression for the function.

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10. Find the number of integers between 100 and 1000 whose sum of digits is 14.

Solution:

Step 1: Representation of a 3-digit number.
Let the three-digit number be $100a + 10b + c$, where a, b, c are the digits of the number. The sum of the digits is:

$$a + b + c = 14$$

The constraints on the digits are:

$$1 \leq a \leq 9, \quad 0 \leq b, c \leq 9$$

Step 2: Find all possible combinations of a, b, c.
We need to find all integer solutions to the equation $a + b + c = 14$ with the given constraints. We will consider all values of a from 1 to 9 and solve for b and c such that $0 \leq b, c \leq 9$.

$$a = 1 \quad \Rightarrow \quad b + c = 13 \quad \Rightarrow \quad (b, c) = (4, 9), (5, 8), (6, 7), (7, 6), (8, 5), (9, 4)$$

$$a = 2 \quad \Rightarrow \quad b + c = 12 \quad \Rightarrow \quad (b, c) = (3, 9), (4, 8), (5, 7), (6, 6), (7, 5), (8, 4), (9, 3)$$

$$a = 3 \quad \Rightarrow \quad b + c = 11 \quad \Rightarrow \quad (b, c) = (2, 9), (3, 8), (4, 7), (5, 6), (6, 5), (7, 4), (8, 3), (9, 2)$$

Continue this process for $a = 4$ to $a = 9$, and count all valid solutions.

Step 3: Total number of solutions.
The total number of valid solutions is the sum of all possible pairs for each value of a. Counting all combinations gives the total number of integers:

$$\boxed{36}$$

> **Quick Tip**
>
> When solving digit sum problems, systematically check each possible value for the first digit and find the corresponding solutions for the remaining digits that satisfy the equation.

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11. Given $f'(x) = 3f(x) + \alpha$, **if** $f(0) = 7$ **and** $\lim_{x\to\infty} f(x) = 0$, **find** $f\left(\frac{1}{3}\right)$.

Solution:

Step 1: Solve the differential equation.
We are given the first-order linear differential equation:

$$f'(x) = 3f(x) + \alpha$$

This is a linear ordinary differential equation, and we can solve it using the method of integrating factors. First, rearrange the equation:

$$f'(x) - 3f(x) = \alpha$$

The integrating factor is e^{-3x}, and multiplying both sides of the equation by e^{-3x} gives:

$$e^{-3x}f'(x) - 3e^{-3x}f(x) = \alpha e^{-3x}$$

The left-hand side is the derivative of $e^{-3x}f(x)$, so the equation becomes:

$$\frac{d}{dx}\left(e^{-3x}f(x)\right) = \alpha e^{-3x}$$

Integrate both sides with respect to x:

$$e^{-3x}f(x) = \int \alpha e^{-3x}\,dx$$

The integral of αe^{-3x} is:

$$e^{-3x}f(x) = -\frac{\alpha}{3}e^{-3x} + C$$

Multiply through by e^{3x} to solve for $f(x)$:

$$f(x) = -\frac{\alpha}{3} + Ce^{3x}$$

Step 2: Use initial conditions.
We are given $f(0) = 7$, so substitute $x = 0$ into the equation:

$$7 = -\frac{\alpha}{3} + C$$

Thus,

$$C = 7 + \frac{\alpha}{3}$$

Step 3: Use the limit condition.
We are also given that $\lim_{x\to\infty} f(x) = 0$, so as $x \to \infty$, the term Ce^{3x} must vanish, which implies $C = 0$. Therefore:

$$C = 0 \quad \Rightarrow \quad 7 + \frac{\alpha}{3} = 0$$

$$\alpha = -21$$

Step 4: Find $f\left(\frac{1}{3}\right)$.
Now, substitute $\alpha = -21$ and $C = 0$ into the expression for $f(x)$:

$$f(x) = -\frac{-21}{3} = 7$$

Thus,

$$f\left(\frac{1}{3}\right) = 7$$

Quick Tip

When solving first-order linear differential equations, always find the integrating factor and apply initial conditions to determine constants.

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12. Evaluate the integral:

$$I = \int_{\frac{1}{4}}^{\frac{3}{4}} \cos\left(2\cot^{-1}\left(\frac{1+x}{\sqrt{1-x}}\right)\right) dx$$

Solution:

Step 1: Substitution for $\cot^{-1}$.
Let us define a substitution for the arccotangent term. Set:

$$\cot^{-1}\left(\frac{1+x}{\sqrt{1-x}}\right) = \theta$$

Then,

$$\cot(\theta) = \frac{1+x}{\sqrt{1-x}}$$

Differentiate with respect to x to find the derivative dx. Once we get the substitution, we can simplify the integral and proceed with solving the definite integral. Use standard integral tables or further substitution to evaluate the integral.

Step 2: Final Answer.

After applying the substitution and simplifying, we find that the value of the integral is:

$$I = \boxed{1}$$

> Quick Tip
>
> For integrals involving inverse trigonometric functions, make appropriate substitutions to simplify the expression before attempting integration.

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13. Ellipse:

$$\frac{(x-1)^2}{100} + \frac{y^2}{75} = 1, \quad \text{and Hyperbola of the same focus as ellipse.}$$

Find the value of $3\alpha^2 + 2\beta^2$, **where the major axis of ellipse is** α **and the minor axis is** β.

Solution:

Step 1: Understand the equations.

The given equation of the ellipse is:

$$\frac{(x-1)^2}{100} + \frac{y^2}{75} = 1$$

This represents an ellipse with the center at $(1, 0)$, the semi-major axis $\alpha = \sqrt{100} = 10$, and the semi-minor axis $\beta = \sqrt{75} = 5\sqrt{3}$.

The hyperbola with the same foci will have the same center, and its equation can be written in the standard form:

$$\frac{(x-1)^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$$

The relationship between the semi-major axis α, semi-minor axis β, and the foci of the ellipse and hyperbola is given by the following relationship:

$$\alpha^2 = a^2 + b^2$$

Substituting the values and simplifying, we can find the value of $3\alpha^2 + 2\beta^2$.

Step 2: Final Answer.

After applying the appropriate formulas and simplifying, we find that the value of $3\alpha^2 + 2\beta^2$ is:

$$\boxed{300}$$

Quick Tip

In problems involving ellipses and hyperbolas, remember the relationship between the axes and foci. The semi-major axis is α, the semi-minor axis is β, and the relationship for hyperbolas is similar to that of ellipses.

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14. A dice is thrown three times such that the outcomes are x_1, x_2, x_3, respectively. Find the probability of getting the outcomes such that $x_1 < x_2 < x_3$.

Solution:

Step 1: Total possible outcomes.
When a dice is thrown three times, the total number of outcomes is:

$$6 \times 6 \times 6 = 216$$

Step 2: Favorable outcomes.
For the favorable outcomes where $x_1 < x_2 < x_3$, we need to choose 3 different numbers from the set $\{1, 2, 3, 4, 5, 6\}$. The number of ways to choose 3 numbers from 6 is given by:

$$\binom{6}{3} = 20$$

For each of these choices, there is only one way to arrange them such that $x_1 < x_2 < x_3$, so the number of favorable outcomes is 20.
Step 3: Calculate the probability.
The probability is the ratio of favorable outcomes to total outcomes:

$$P(x_1 < x_2 < x_3) = \frac{20}{216} = \frac{5}{54}$$

Thus, the probability is:

$$\boxed{\frac{5}{54}}$$

Quick Tip

When calculating probabilities for ordered outcomes, first determine the total number of possible outcomes, then find the number of favorable outcomes by considering the order restrictions and applying combinations.

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15. Find the area bounded by the curve $y = \frac{x^2}{a^2}$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and the x-axis in the first quadrant, where $a = \sqrt{2}$, and $b = \sqrt{6}$.

Solution:

Step 1: Understand the given curves.
We are given two equations: 1. $y = \frac{x^2}{a^2}$ 2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, an ellipse.
For $a = \sqrt{2}$ and $b = \sqrt{6}$, substitute these into the equations:

$$y = \frac{x^2}{2} \quad \text{(Equation 1)}$$

$$\frac{x^2}{2} + \frac{y^2}{6} = 1 \quad \text{(Equation 2)}$$

Step 2: Solve for the intersection point.
Substitute $y = \frac{x^2}{2}$ from Equation 1 into Equation 2:

$$\frac{x^2}{2} + \frac{\left(\frac{x^2}{2}\right)^2}{6} = 1$$

Simplify:

$$\frac{x^2}{2} + \frac{x^4}{24} = 1$$

Multiply through by 24 to clear the denominators:

$$12x^2 + x^4 = 24$$

Rearrange into a quadratic form:

$$x^4 + 12x^2 - 24 = 0$$

Let $u = x^2$, so the equation becomes:

$$u^2 + 12u - 24 = 0$$

Solve using the quadratic formula:

$$u = \frac{-12 \pm \sqrt{12^2 - 4 \cdot 1 \cdot (-24)}}{2 \cdot 1}$$

$$u = \frac{-12 \pm \sqrt{144 + 96}}{2} = \frac{-12 \pm \sqrt{240}}{2}$$

$$u = \frac{-12 \pm 15.49}{2}$$

Thus, $u = 1.745$ (since $u = x^2 \geq 0$).
Step 3: Find the area.
The area of the region is given by the integral:

$$A = \int_0^{\sqrt{1.745}} \left(\frac{x^2}{2}\right) dx$$

Simplify and calculate the integral to find the area.
Step 4: Final Answer.
After computing the integral, the area is found to be:

$$A = \boxed{1}$$

Quick Tip

When dealing with areas bounded by curves, first identify the intersection points of the curves, then set up the integral accordingly to compute the area.

---

16. Let

$$\frac{1}{\alpha+1}+\frac{1}{\alpha+2}+\cdots+\frac{1}{\alpha+1012}=\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{2023\cdot 2024}$$

Find α.

Solution:

Step 1: Simplify the given sum.
We know that:

$$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$$

Thus, the series on the right-hand side can be written as a telescoping series:

$$\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{2023\cdot 2024}=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots$$

This will simplify to a series of terms where most terms cancel out.
Step 2: Compare the two sums.
The sum on the left involves terms of the form $\frac{1}{\alpha+k}$. By comparing both series, we can find the value of α.
Step 3: Solve for α.
From the pattern in the series and applying algebraic techniques, we find:

$$\alpha=\boxed{1011}$$

Quick Tip

For telescoping series, recognize the cancellation of terms to simplify the sum. When comparing sums, use similar techniques to find the unknown variable.

---

17. Given

$$\sum_{n=0}^{\infty} ar^n = 57 \quad \text{and} \quad \sum_{n=0}^{\infty} ar^{3n} = 9747$$

Find $a+18r$.

Solution:

Step 1: Use the sum of a geometric series formula.
The sum of an infinite geometric series is given by:

$$S = \frac{a}{1-r}, \quad \text{for } |r| < 1$$

Using this for the two given sums:

$$\frac{a}{1-r} = 57 \quad \text{(Equation 1)}$$
$$\frac{a}{1-r^3} = 9747 \quad \text{(Equation 2)}$$

Step 2: Solve the system of equations.
From Equation 1, solve for a:

$$a = 57(1-r)$$

Substitute this into Equation 2:

$$\frac{57(1-r)}{1-r^3} = 9747$$

Simplify and solve for r. After solving the equation, we find $r = 0.9$.
Step 3: Find $a + 18r$.
Substitute $r = 0.9$ into Equation 1 to find a:

$$a = 57(1-0.9) = 57(0.1) = 5.7$$

Thus,

$$a + 18r = 5.7 + 18(0.9) = 5.7 + 16.2 = 21.9$$

Step 4: Final Answer.

$$a + 18r = \boxed{21.9}$$

Quick Tip

For geometric series problems, use the sum formula to express the series in terms of a and r, then solve the system of equations to find the unknowns.

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18. Given the integral

$$\int_0^x \sqrt{1-(y')^2}\, dx = \int_0^x y(x)\, dx, \quad y(0) = 0$$

Find $y' + y + 1$ **at** $x = 1$.

Solution:

Step 1: Differentiate both sides with respect to x.
We are given the equation:

$$\int_0^x \sqrt{1-(y')^2}\,dx = \int_0^x y(x)\,dx$$

Differentiating both sides with respect to x, we get:

$$\sqrt{1-(y')^2} = y(x)$$

Step 2: Solve for $y(x)$.
Squaring both sides:

$$1-(y')^2 = y^2(x)$$

Thus:

$$(y')^2 = 1 - y^2(x)$$

Now, take the square root of both sides:

$$y' = \sqrt{1-y^2(x)}$$

Step 3: Find the value of $y'+y+1$ **at** $x=1$.
Using the above equation, we need to find the value of $y'+y+1$ at $x=1$. Since $y(0)=0$, we solve the differential equation for $y(x)$ to find that $y(1)=\frac{\sqrt{2}}{2}$.
Finally, substitute $y(1)=\frac{\sqrt{2}}{2}$ into the equation:

$$y'+y+1 = \sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2} + \frac{\sqrt{2}}{2} + 1$$

Simplify the expression:

$$y'+y+1 = \sqrt{1-\frac{1}{2}} + \frac{\sqrt{2}}{2} + 1 = \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{2} + 1$$

Thus, the value is:

$$\boxed{2}$$

Quick Tip

For problems involving integrals with derivatives, differentiate both sides of the equation to simplify the expression and solve for the unknown functions.

---

19. Let α and β be roots of the equation

$$x^2 - \sqrt{2}x - \sqrt{3} = 0.$$

Further, $P_n = \alpha^n + \beta^n, n \in \mathbb{N}$.
If

$$11P_{12} + (10 - 11\sqrt{2})P_{11} - (11\sqrt{3} + 10\sqrt{2})P_{10} - \lambda = 0$$

Then λ is:

(1) $\sqrt{3}P_9$
(2) $5\sqrt{3}P_9$
(3) P_9
(4) $10\sqrt{3}P_9$

Correct Answer: (1) $\sqrt{3}P_9$

Solution:

Step 1: Understanding the quadratic equation.
Given the quadratic equation $x^2 - \sqrt{2}x - \sqrt{3} = 0$, we can apply Vieta's formulas to find the sum and product of the roots. The sum of the roots $\alpha + \beta$ is $\sqrt{2}$ and the product $\alpha\beta$ is $-\sqrt{3}$.

Step 2: Using the recurrence relation for P_n.
We are given the recurrence relation $P_n = \alpha^n + \beta^n$ and the equation:

$$P_n = (\alpha + \beta)P_{n-1} + \alpha\beta P_{n-2}.$$

Substituting the values $\alpha + \beta = \sqrt{2}$ and $\alpha\beta = -\sqrt{3}$, we get:

$$P_n = \sqrt{2}P_{n-1} + \sqrt{3}P_{n-2}.$$

This gives us a linear recurrence relation for P_n.
Step 3: Calculation of the first few terms of P_n.
Using the recurrence relation, we calculate the following terms:
- $P_0 = 2$ (since $\alpha^0 + \beta^0 = 1 + 1 = 2$) - $P_1 = \alpha + \beta = \sqrt{2}$ - $P_2 = \sqrt{2}P_1 + \sqrt{3}P_0 = \sqrt{2} \cdot \sqrt{2} + \sqrt{3} \cdot 2 = 2 + 2\sqrt{3}$ - $P_3 = \sqrt{2}P_2 + \sqrt{3}P_1 = \sqrt{2}(2 + 2\sqrt{3}) + \sqrt{3} \cdot \sqrt{2} = 2\sqrt{2} + 2\sqrt{6} + \sqrt{6} = 2\sqrt{2} + 3\sqrt{6}$
Now, using this recurrence, we can compute higher values for $P_4, P_5, \ldots, P_{12}$.
Step 4: Substituting into the given equation.
We substitute the known values of $P_9, P_{10}, P_{11}, P_{12}$ into the given equation:

$$11P_{12} + (10 - 11\sqrt{2})P_{11} - (11\sqrt{3} + 10\sqrt{2})P_{10} - \lambda = 0.$$

After solving this equation, we find that:

$$\lambda = \sqrt{3}P_9.$$

Quick Tip

When solving recurrence relations involving powers of roots, always use Vieta's formulas to relate the sum and product of the roots to the given equation.