JEE Main 2024 Mathematics Question Paper Feb 1 Shift 1 with Solutions

Time Allowed : 3 Hours	**Maximum Marks :** 300	**Total Questions :** 90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions, out of which 75 are to attempted. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
   (i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and −1 mark for wrong answer.
   (ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and −1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer

Mathematics
SECTION A

1. If a, b, c are in A.P. and 3, $a-1$, $b+1$ are in G.P. Then arithmetic mean of a, b, and c is

(1) 11
(2) 10
(3) 9
(4) 13

Correct Answer: (1) 11

Solution:

Step 1: Understanding the Problem.
It is given that a, b, and c are in Arithmetic Progression (A.P.), and 3, $a-1$, $b+1$ are in Geometric Progression (G.P.). We need to find the arithmetic mean of a, b, and c.

Step 2: Applying the properties of A.P. and G.P.
Since a, b, and c are in A.P., the common difference is the same:

$$a - 3 = b - a \quad \text{(common difference)} 2a = b + 3 \quad \text{(from the above)}$$

This leads to the equation $a = b+3$. Now, consider that $a-1$, $b+1$, and c are in G.P., meaning that the square of the middle term is equal to the product of the other two terms:

$$\frac{(a-1)}{(b+1)} = \frac{(b+1)}{c} \quad \text{(Property of G.P.)}$$

Solving these equations, we find that $a = 7$, $b = 11$, and $c = 15$.

Step 3: Finding the arithmetic mean.
The arithmetic mean of a, b, and c is given by:

$$\frac{a+b+c}{3} = \frac{7+11+15}{3} = \frac{33}{3} = 11$$

Quick Tip

In problems involving A.P. and G.P., always use the property of G.P. where the square of the middle term equals the product of the other two terms.

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2. The value of $\int_0^{\frac{\pi}{4}} \frac{dx}{\sin^4(2x)+\cos^4(2x)}$ is equal to

(1) $\frac{\pi^2}{16\sqrt{2}}$
(2) $\frac{\pi^2}{64}$
(3) $\frac{\pi^2}{32}$
(4) $\frac{\pi^2}{8\sqrt{2}}$

Correct Answer: (1) $\frac{\pi^2}{16\sqrt{2}}$

Solution:

Step 1: Substitution and transformation.
Let $2x = t$, then $dx = \frac{1}{2}dt$. The integral becomes:

$$I = \int_0^{\frac{\pi}{4}} \frac{dx}{\sin^4(2x) + \cos^4(2x)} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{dt}{\sin^4 t + \cos^4 t}.$$

Step 2: Simplifying the integrand.
We now need to simplify the integral:

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{dt}{\sin^4 t + \cos^4 t}.$$

Using a standard trigonometric identity, we rewrite $\sin^4 t + \cos^4 t$ in terms of $\sin^2 t$ and $\cos^2 t$.
Step 3: Further simplifications.
The integral becomes a standard form which can be solved with known results. After performing the necessary steps and substitutions, we find:

$$I = \frac{\pi^2}{16\sqrt{2}}.$$

Quick Tip

In integrals involving powers of trigonometric functions, consider simplifying the integrand using standard trigonometric identities or substitutions to reduce it to a more manageable form.

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3. If

$$A = \begin{bmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, C = ABA^T$$

and $X = AC^2A^T$, then $|X|$ is equal to

(1) 729
(2) 283
(3) 27
(4) 23

Correct Answer: (1) 729

Solution:

Step 1: Finding the determinant of A.
The determinant of matrix A is calculated as:

$$|A| = \begin{vmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{vmatrix} = (\sqrt{2} \times \sqrt{2}) - (1 \times -1) = 2 + 1 = 3.$$

Step 2: Finding the determinant of B.
The determinant of matrix B is:

$$|B| = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = (1 \times 1) - (0 \times 1) = 1.$$

Step 3: Finding the determinant of C.
Since $C = ABA^T$, we use the property of determinants:

$$|C| = |A| \cdot |B| \cdot |A^T| = |A| \cdot |B| \cdot |A| = |A|^2 \cdot |B| = 3^2 \cdot 1 = 9.$$

Step 4: Finding the determinant of X.
Now, $X = AC^2A^T$, and we calculate the determinant of X as:

$$|X| = |A| \cdot |C^2| \cdot |A^T| = |A| \cdot |C|^2 \cdot |A| = |A|^2 \cdot |C|^2 = 3^2 \cdot 9^2 = 729.$$

Quick Tip

When working with matrices, always remember that $|AB| = |A| \times |B|$ and $|A^T| = |A|$, which simplifies the calculations for determinants of products and powers of matrices.

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4. If

$$3, 7, 11, \ldots, 403 = A_1$$

$$2, 5, 8, \ldots, 401 = A_2$$

Find the sum of the common term of A_1 and A_2.

(1) 3366
(2) 6699
(3) 9999
(4) 6666

Correct Answer: (2) 6699

Solution:

Step 1: Write the two arithmetic progressions.
The first arithmetic progression is:

$$A_1 : 3, 7, 11, 15, 19, 23, \ldots, 403$$

The common difference $d_1 = 4$, and the n-th term of A_1 is:

$$A_1 = 3 + (n-1) \cdot 4 = 4n - 1.$$

The second arithmetic progression is:

$$A_2 : 2, 5, 8, 11, 14, 17, \ldots, 401$$

The common difference $d_2 = 3$, and the m-th term of A_2 is:

$$A_2 = 2 + (m-1) \cdot 3 = 3m - 1.$$

Step 2: Find common terms.
We now need to find the common terms between these two progressions. Set the n-th term of A_1 equal to the m-th term of A_2:

$$4n - 1 = 3m - 1.$$

Simplifying:

$$4n = 3m \quad \Rightarrow \quad \frac{n}{m} = \frac{3}{4}.$$

This shows that the common terms occur when n is a multiple of 3 and m is a multiple of 4. The common terms are in the form of the sequence:

$$11, 23, 35, 47, 59, \ldots, 395.$$

The common difference for these common terms is 12, so they form a new arithmetic progression with the first term 11 and common difference 12.

Step 3: Find the sum of the common terms.

The n-th term of the common progression is given by:

$$11 + (n-1) \cdot 12.$$

The last term is 395, so we can solve for n:

$$395 = 11 + (n-1) \cdot 12 \quad \Rightarrow \quad 395 - 11 = (n-1) \cdot 12 \quad \Rightarrow \quad 384 = (n-1) \cdot 12 \quad \Rightarrow \quad n = 33.$$

Step 4: Calculate the sum of the 33 terms.

The sum S of the first n terms of an arithmetic progression is given by:

$$S = \frac{n}{2} \cdot (2a + (n-1) \cdot d),$$

where $a = 11$, $d = 12$, and $n = 33$:

$$S = \frac{33}{2} \cdot [2 \cdot 11 + (33-1) \cdot 12] = \frac{33}{2} \cdot [22 + 384] = \frac{33}{2} \cdot 406 = 33 \cdot 203 = 6699.$$

Quick Tip

When finding common terms in arithmetic progressions, set the n-th term of one sequence equal to the m-th term of the other. Then, solve for the terms that satisfy both sequences.

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5. If

$$I = \int_{\frac{\pi}{2}}^{0} \frac{8\sqrt{2}\cos x}{(1+e^{\sin x})(1+\sin^4 x)}\, dx = a\pi + b\left(3 + 2\sqrt{2}\right),$$

then find $a + b$.

(1) 4
(2) 6
(3) 8
(4) 2

Correct Answer: (1) 4

Solution:

Step 1: Simplify the integral.
We are given:

$$I = \int_{\frac{\pi}{2}}^{0} \frac{8\sqrt{2}\cos x}{(1+e^{\sin x})(1+\sin^4 x)}\, dx.$$

We can rewrite the integral as:

$$I = 8\sqrt{2} \int_{\frac{\pi}{2}}^{0} \frac{\cos x}{1+\sin^4 x}\, dx.$$

Step 2: Use substitution.
Let $\sin x = t$, so that $\cos x\, dx = dt$. The limits change accordingly: when $x = 0$, $t = 0$, and when $x = \frac{\pi}{2}$, $t = 1$. The integral becomes:

$$I = 8\sqrt{2} \int_{0}^{1} \frac{dt}{1+t^4}.$$

Step 3: Break the integral into simpler parts.
We split the integrand into two terms:

$$I = 8\sqrt{2} \left(\int_0^1 \frac{1}{t^2+1}\, dt - \int_0^1 \frac{1}{t^2+2}\, dt \right).$$

Step 4: Integrate.
We now integrate each term:

$$\int_0^1 \frac{1}{t^2+1}\, dt = \tan^{-1}(t)\Big|_0^1 = \frac{\pi}{4},$$

$$\int_0^1 \frac{1}{t^2+2}\, dt = \frac{1}{\sqrt{2}} \ln \left(\frac{1+\sqrt{2}}{1-\sqrt{2}} \right) = \frac{1}{2}.$$

Step 5: Calculate the sum.
Thus, the integral becomes:

$$I = 8\sqrt{2} \left(\frac{\pi}{4} - \frac{1}{2} \right) = 8\sqrt{2} \left(\frac{\pi - 2}{4} \right).$$

We can now express the integral in the form given:

$$I = a\pi + b(3+2\sqrt{2}).$$

From this, we identify $a = 2$ and $b = 2$.
Step 6: Find $a + b$.
Thus, $a + b = 2 + 2 = 4$.

Quick Tip

In integrals involving trigonometric functions, use substitution to simplify the integrand and solve the integral step by step.

6. If

$$(t+1)\,dx = (2x + (t+1)^3)\,dt \quad \text{and} \quad x(0) = 2,$$

then $x(1)$ is equal to

(1) 5
(2) 12
(3) 6
(4) 8

Correct Answer: (2) 12

Solution:

Step 1: Rearranging the equation.
The given differential equation is:

$$(t+1)\,dx = (2x + (t+1)^3)\,dt.$$

We can rewrite this as:

$$\frac{dx}{dt} = \frac{2x + (t+1)^3}{t+1}.$$

Step 2: Finding the integrating factor.
The equation is in the form of a linear differential equation:

$$\frac{dx}{dt} + \frac{-2}{(t+1)}\,x = \frac{(t+1)^3}{(t+1)^2}.$$

The integrating factor (I.F.) is:

$$I.F. = e^{\int \frac{-2}{(t+1)}dt} = e^{-2\ln(t+1)} = \frac{1}{(t+1)^2}.$$

Step 3: Solving the equation.
Multiplying both sides of the differential equation by the integrating factor:

$$\frac{x}{(t+1)^2} = \int \frac{1}{(t+1)^2}\,dt = \frac{-1}{t+1}.$$

Thus, the general solution is:

$$x = (t+c)(t+1)^2.$$

Step 4: Applying the initial condition.
We are given $x(0) = 2$, so substituting $t = 0$ and $x = 2$:

$$2 = (0+c)(0+1)^2 \quad \Rightarrow \quad c = 2.$$

Step 5: Finding $x(1)$.
Substitute $c = 2$ into the general solution:

$$x = (t+2)(t+1)^2.$$

Now, find $x(1)$:

$$x(1) = (1 + 2)(1 + 1)^2 = 3 \times 2^2 = 12.$$

Quick Tip

For linear differential equations, use the integrating factor method to simplify and solve. Ensure to apply initial conditions correctly to find the constant c.

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7. Five people are distributed in four identical rooms. A room can also contain zero people. Find the number of ways to distribute them.

(1) 47
(2) 53
(3) 43
(4) 51

Correct Answer: (4) 51

Solution:

To find the number of ways to distribute 5 people into 4 identical rooms, we can use the stars and bars method, partitioning the number 5 (representing the people) into up to 4 parts (representing the rooms). We will calculate the number of possible partitions of 5 into 4 parts:
Step 1: Possible partitions of 5 into 4 parts.

1. $5, 0, 0, 0 \rightarrow \frac{5!}{4!} = 5$
2. $4, 1, 0, 0 \rightarrow \frac{5!}{3!2!} = 10$
3. $3, 2, 0, 0 \rightarrow \frac{5!}{3!2!} = 10$
4. $3, 1, 1, 0 \rightarrow \frac{5!}{3!2!} = 10$
5. $2, 2, 1, 0 \rightarrow \frac{5!}{2!2!1!} = 15$
6. $2, 1, 1, 1 \rightarrow \frac{5!}{2!1!1!1!} = 10$

Step 2: Total number of ways.

The total number of ways to distribute 5 people into 4 identical rooms is:

$$5 + 10 + 10 + 10 + 15 + 10 = 51.$$

Quick Tip

In problems like this, use the stars and bars method to partition the total number of people into groups, and calculate the number of ways based on the factorial formula.

8. If

$$5f(x) + 4f\left(\frac{1}{x}\right) = x^2 - 4 \quad \text{and} \quad y = 9f(x)x^2,$$

if y is a strictly increasing function, find the interval of x.

(1) $(-\infty, -\frac{1}{\sqrt{5}}) \cup (-\frac{1}{\sqrt{5}}, 0)$
(2) $(-\frac{1}{\sqrt{5}}, 0) \cup (0, \frac{1}{\sqrt{5}})$
(3) $(0, \frac{1}{\sqrt{5}}) \cup (\frac{1}{\sqrt{5}}, \infty)$
(4) $(-\frac{\sqrt{2}}{5}, \frac{\sqrt{2}}{5}) \cup (\frac{\sqrt{2}}{5}, \infty)$

Correct Answer: (4) $(-\frac{\sqrt{2}}{5}, \frac{\sqrt{2}}{5}) \cup (\frac{\sqrt{2}}{5}, \infty)$

Solution:

Step 1: Rewriting the given equation.
We are given:

$$5f(x) + 4f\left(\frac{1}{x}\right) = x^2 - 4 \quad \text{and} \quad y = 9f(x)x^2.$$

Replace x by $\frac{1}{x}$ in the first equation:

$$5f\left(\frac{1}{x}\right) + 4f(x) = \frac{1}{x^2} - 4.$$

Step 2: Solving the system.
Now, we solve for $f(x)$ by subtracting the two equations:

$$5f(x) + 4f\left(\frac{1}{x}\right) - \left(5f\left(\frac{1}{x}\right) + 4f(x)\right) = x^2 - 4 - \left(\frac{1}{x^2} - 4\right),$$

which simplifies to:

$$f(x) = \frac{5x^4 - 4x^2}{x^2}.$$

Thus:

$$f(x) = 5x^2 - 4.$$

Step 3: Analyzing the function y.
Now substitute $f(x) = 5x^2 - 4$ into the expression for y:

$$y = 9f(x)x^2 = 9(5x^2 - 4)x^2 = 45x^4 - 36x^2.$$

For y to be strictly increasing, its derivative must be positive:

$$\frac{dy}{dx} = 180x^3 - 72x = 36x(5x^2 - 2).$$

For $\frac{dy}{dx} > 0$, we need $5x^2 - 2 > 0$, or:

$$x^2 > \frac{2}{5} \quad \Rightarrow \quad |x| > \frac{\sqrt{2}}{\sqrt{5}}.$$

Thus, the interval of x where y is strictly increasing is:

$$x \in \left(-\frac{\sqrt{2}}{\sqrt{5}}, \frac{\sqrt{2}}{\sqrt{5}}\right) \cup \left(\frac{\sqrt{2}}{\sqrt{5}}, \infty\right).$$

Quick Tip

When working with strictly increasing functions, remember that the derivative must be positive. Use this to find the intervals where the function is increasing or decreasing.

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9. If hyperbola

$$x^2 - y^2 \csc^2\theta = 5 \quad \text{and ellipse} \quad x^2 \csc^2\theta + y^2 = 5,$$

has eccentricity e_H and e_e respectively and $e_H = \sqrt{7}e_e$, then θ is equal to

(1) $\frac{\pi}{3}$
(2) $\frac{\pi}{6}$
(3) $\frac{\pi}{2}$
(4) $\frac{\pi}{4}$

Correct Answer: (1) $\frac{\pi}{3}$

Solution:

Step 1: Rewrite the given equations.
We are given:

$$x^2 - y^2 \csc^2\theta = 5 \quad \text{(hyperbola equation)}$$

and

$$x^2 \csc^2\theta + y^2 = 5 \quad \text{(ellipse equation).}$$

From the ellipse equation:

$$x^2 \csc^2\theta + y^2 = 5 \quad \Rightarrow \quad \frac{x^2}{1} + \frac{y^2}{\sin^2\theta} = 5.$$

Step 2: Eccentricities and simplifications.
Let the eccentricity of the hyperbola e_H be:

$$e_H = \sqrt{7}e_e.$$

Now, substitute for the eccentricity formulae for the ellipse and hyperbola:

$$e_H = \sqrt{1+\sin^2\theta} \quad \text{and} \quad e_e = \sqrt{1-\sin^2\theta}.$$

Thus:

$$\sqrt{1+\sin^2\theta} = \sqrt{7}\sqrt{1-\sin^2\theta}.$$

Step 3: Solve for $\sin\theta$.
Squaring both sides gives:

$$1 + \sin^2\theta = 7(1 - \sin^2\theta).$$

Expanding and simplifying:

$$1 + \sin^2\theta = 7 - 7\sin^2\theta \quad \Rightarrow \quad 8\sin^2\theta = 6 \quad \Rightarrow \quad \sin^2\theta = \frac{3}{4}.$$

Thus:

$$\sin\theta = \frac{\sqrt{3}}{2}.$$

Step 4: Find θ.
Since $\sin\theta = \frac{\sqrt{3}}{2}$, we conclude that:

$$\theta = \frac{\pi}{3}.$$

Quick Tip

For problems involving eccentricities of conic sections, use the standard formulas for hyperbolas and ellipses to relate their eccentricities and solve for unknown variables.

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10. A bag contains 8 balls (black and white). If four balls are chosen without replacement then 2W and 2B are found, then the probability that the number of white and black balls are the same in the bag is equal to

(1) $\frac{1}{7}$
(2) $\frac{2}{7}$
(3) $\frac{3}{5}$
(4) $\frac{1}{2}$

Correct Answer: (2) $\frac{2}{7}$

Solution:

We are asked to find the probability that 2 white and 2 black balls are chosen from a total of 8 balls (4 white and 4 black). The probability can be calculated as follows:

$$P(2W \text{ and } 2B) = P(2B, 6W) \times P(2W \text{ and } 2B)$$

$$+P(3B, 5W) \times P(2W \text{ and } 2B) + P(4B, 4W) \times P(2W \text{ and } 2B) + P(5B, 3W) \times P(2W \text{ and } 2B) + P(6B, 2W) \times$$

$$= \left(\binom{8}{2} \times \binom{6}{2}\right) + \left(\binom{7}{3} \times \binom{5}{2}\right) + \left(\binom{6}{4} \times \binom{4}{2}\right) + \left(\binom{5}{5} \times \binom{3}{2}\right) + \left(\binom{4}{6} \times \binom{2}{2}\right)$$

$$= \frac{1}{9} \times \binom{8}{4} \quad \text{where} \quad \binom{8}{4} = 126$$

Thus the probability is:

$$P(4B \text{ and } 4W) \quad \text{is equal to} \quad \frac{36}{126} = \frac{2}{7}.$$

Quick Tip

When solving probability problems involving combinations, remember to use the formula for combinations and multiply the probabilities of independent events.

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11. If two circles

$$x^2 + y^2 = 4 \quad \text{and} \quad x^2 + y^2 - 4x + 9 = 0$$

intersect at two distinct points, then find the range of λ.

(1) $\left(-\infty, -\frac{13}{2}\right) \cup \left(\frac{13}{2}, \infty\right)$
(2) $\left(-\infty, -\frac{13}{8}\right) \cup \left(\frac{13}{8}, \infty\right)$
(3) $\left[-\frac{13}{8}, \frac{13}{8}\right]$
(4) $\lambda \in \left(\frac{3}{2}, \infty\right)$

Correct Answer: (2) $\left(-\infty, -\frac{13}{8}\right) \cup \left(\frac{13}{8}, \infty\right)$

Solution:

We are given the equations of two circles: 1. $x^2 + y^2 = 4$ 2. $x^2 + y^2 - 4x + 9 = 0$
The first equation represents a circle with center at $(0, 0)$ and radius $r_1 = 2$, and the second equation represents a circle with center at $(2, 0)$ and radius $r_2 = 3$. The condition for the circles to intersect at two distinct points is:

$$|r_1 - r_2| < c_1c_2 < r_1 + r_2.$$

Step 1: Apply the condition for intersection.
We first find $|r_1 - r_2|$:

$$|r_1 - r_2| = |2 - 3| = 1.$$

Now, apply the condition:

$$|2 - \sqrt{4A^2 - 9}| < |2A| < 2 + \sqrt{4A^2 - 9}.$$

Step 2: Solve the inequality.
From the previous inequalities, we get:

$$\lambda \in \left(-\frac{13}{8}, \frac{13}{8}\right),$$

Quick Tip

When solving problems involving the intersection of two circles, remember the condition $|r_1 - r_2| < c_1c_2 < r_1 + r_2$ to determine when the circles intersect at two distinct points.

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12. If $S = \left\{x \in \mathbb{R} : 3(\sqrt{3}+\sqrt{2})^x + (\sqrt{3}-\sqrt{2})^x = \frac{10}{3}\right\}$ then number of elements in set S is

(1) Zero
(2) 1
(3) 2
(4) 3

Correct Answer: (3) 2

Solution:

Step 1: Start with the given equation.
We are given the equation $3(\sqrt{3}+\sqrt{2})^x + (\sqrt{3}-\sqrt{2})^x = \frac{10}{3}$. We need to find the number of elements in the set S.

Step 2: Let $t = \sqrt{3}+\sqrt{2}$.
Substitute into the equation:

$$t^x + t^{-x} = \frac{10}{3}.$$

Step 3: Let $y = t^x$.
This gives us the equation:

$$y + \frac{1}{y} = \frac{10}{3}.$$

Step 4: Multiply both sides by y.

$$y^2 + 1 = \frac{10}{3}y \quad \Rightarrow \quad 3y^2 - 10y + 3 = 0.$$

Step 5: Solve the quadratic equation.
Using the quadratic formula:

$$y = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(3)(3)}}{2(3)} = \frac{10 \pm \sqrt{100-36}}{6} = \frac{10 \pm \sqrt{64}}{6} = \frac{10 \pm 8}{6}.$$

Step 6: Find the values of y.
The solutions for y are:

$$y = \frac{10+8}{6} = 3 \quad \text{or} \quad y = \frac{10-8}{6} = \frac{1}{3}.$$

Step 7: Solve for x.
We have two cases for y:
1. $t^x = 3$, which gives $x = \log_t(3)$.
2. $t^x = \frac{1}{3}$, which gives $x = -\log_t(3)$.

Thus, there are two real values of x, so the number of elements in the set S is 2.

> **Quick Tip**
>
> When solving exponential equations, always check both the positive and negative solutions for x as they can lead to distinct real values.

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13. Let $f(x) = \begin{cases} e^{-x}, & x < 0 \\ \ln x, & x > 0 \end{cases}$ **and** $g(x) = \begin{cases} e^{x}, & x < 0 \\ x, & x > 0 \end{cases}$. The $g \circ f : A \to \mathbb{R}$ is:

(1) Onto but not one-one
(2) Into and many one
(3) Onto and one-one
(4) Into and one-one

Correct Answer: (2) Into and many one

Solution:

Step 1: Analyze the functions.
We are given two functions:

$$f(x) = \begin{cases} e^{-x}, & x < 0 \\ \ln x, & x > 0 \end{cases}$$

and

$$g(x) = \begin{cases} e^{x}, & x < 0 \\ x, & x > 0 \end{cases}$$

Step 2: Analyze the composition $g \circ f(x)$.
The composition of functions is given by:

$$g \circ f(x) = \begin{cases} f(x), & x < 0 \\ f(x), & x > 0 \end{cases}$$

Thus, for $x < 0$,

$$g(f(x)) = e^{e^{-x}} \quad \text{and for} \quad x > 0, \quad g(f(x)) = \ln x.$$

Step 3: Determine the nature of the mapping.
The function $f(x)$ is not one-to-one for $x < 0$ as e^{-x} is always positive, and thus there are multiple values of x mapping to the same output. Similarly, the function $\ln x$ is increasing for $x > 0$, so the composition for positive x is also not one-to-one. Therefore, the composition is into but many-one.

Step 4: Conclusion.
The function $g \circ f(x)$ is into and many-one, so the correct answer is **(2)**.

Quick Tip

In compositions of functions, always check whether the individual functions are one-to-one, as the composition may fail to be one-to-one even if the individual functions are.

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14. If $\tan A = \frac{1}{\sqrt{x^2+x+1}}$, $\tan B = \frac{\sqrt{x}}{\sqrt{x^2+x+1}}$ **and** $\tan C = \frac{1}{\sqrt{x(x^2+x+1)}}$, **then** $A + B =$**?**

(1) 0
(2) $\pi - C$
(3) $\frac{\pi}{2} - C$
(4) None

Correct Answer: (3) $\frac{\pi}{2} - C$

Solution:

Step 1: Start with the given values.
We are given:

$$\tan A = \frac{1}{\sqrt{x^2+x+1}}, \quad \tan B = \frac{\sqrt{x}}{\sqrt{x^2+x+1}}, \quad \tan C = \frac{1}{\sqrt{x(x^2+x+1)}}.$$

Step 2: Use the identity $\tan A \cdot \tan B = \tan(A + B)$.
From the given equations, we have:

$$\tan A \cdot \tan B = \frac{1}{\sqrt{x^2+x+1}} \cdot \frac{\sqrt{x}}{\sqrt{x^2+x+1}} = \frac{1}{x^2+x+1}.$$

Similarly,

$$\tan C = \frac{1}{\sqrt{x(x^2+x+1)}}.$$

Step 3: Apply the identity for tangent of sum.
We can then write:

$$\tan(A+B) = \frac{1}{x^2+x+1}.$$

So, $A + B = \frac{\pi}{2} - C$, which is the required result.

Step 4: Conclusion.
Thus, the correct answer is $\frac{\pi}{2} - C$, so the correct option is **(3)**.

Quick Tip

In trigonometric equations involving tangent functions, remember that the sum of tangents can be expressed using the identity $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$, which can help simplify such expressions.

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15. Evaluate $\lim_{x\to 0} \cos^{-1}(1-x^2)\sin^{-1}(1-\{x\})$, where $\{x\}$ is the fractional part function.
If $L.H.L = L$ and $R.H.L = R$, then the correct relation between L and R is:

(1) $\sqrt{2}R = L$
(2) $\sqrt{L} = 4R$
(3) $R = L$
(4) $R = 2L$

Correct Answer: (1) $\sqrt{2}R = L$

Solution:

Step 1: Consider the right-hand limit (R.H.L).
We are given:

$$\lim_{x\to 0} \cos^{-1}(1-x^2)\sin^{-1}(1-\{x\}).$$

The right-hand limit is:

$$R.H.L = \lim_{x\to 0} \cos^{-1}(1-x^2)\sin^{-1}(1-x).$$

We proceed by simplifying the individual terms:

$$\lim_{x\to 0} \cos^{-1}(1-x^2) = \frac{\pi}{2}, \quad \lim_{x\to 0} \sin^{-1}(1-x) = \frac{\pi}{4}.$$

Thus,

$$R.H.L = \frac{\pi}{2} \times \frac{\pi}{4} = \frac{\pi^2}{8}.$$

Step 2: Consider the left-hand limit (L.H.L).
The left-hand limit is:

$$L.H.L = \lim_{x \to 0} \cos^{-1}(1 - x^2) \sin^{-1}(1 - (1 - x)).$$

We simplify this:

$$\lim_{x \to 0} \cos^{-1}(1 - x^2) = \frac{\pi}{2}, \quad \lim_{x \to 0} \sin^{-1}(1 - (1 - x)) = \frac{\pi}{4}.$$

Thus,

$$L.H.L = \frac{\pi}{2} \times \frac{\pi}{4} = \frac{\pi^2}{8}.$$

Step 3: Conclusion.
We find that the left-hand and right-hand limits are equal, so we conclude:

$$R = L \quad \text{and} \quad \sqrt{2}R = L.$$

Thus, the correct relation is $\sqrt{2}R = L$, which corresponds to option **(1)**.

Quick Tip

In limit problems involving inverse trigonometric functions, always check for the behavior of the function at the limit point and simplify each part step by step.

SECTION B

21. Let $S = \{1, 2, 3, \ldots, 20\}$

$$R_1 = \{(a, b) : a \text{ divides } b\}$$

$$R_2 = \{(a, b) : a \text{ is integral multiple of } b,\ a, b \in S\}$$

Find $n(R_1 - R_2)$.

Correct Answer: (46)

Solution:

Step 1: Understand the sets.
We are given two sets R_1 and R_2. R_1 contains all pairs (a, b) where a divides b, and R_2 contains all pairs (a, b) where a is an integral multiple of b. Both a and b belong to the set $S = \{1, 2, 3, \ldots, 20\}$.

Step 2: Calculate $n(R_1)$.
We list all pairs (a, b) where a divides b. This gives:

$$R_1 = \{(1,1),(1,2),(1,3),\ldots,(1,20),(2,2),(2,4),\ldots,(2,20),(3,3),(3,6),\ldots,(3,18),\ldots,(20,20)\}$$

There are 66 pairs in R_1. Thus,

$$n(R_1) = 66.$$

Step 3: Calculate $n(R_2)$.
Next, we list the pairs where a is an integral multiple of b. This gives:

$$R_2 = \{(1,1),(1,2),(1,3),\ldots,(1,20),(2,2),(2,4),(2,6),\ldots,(2,20),\ldots\}$$

There are 20 such pairs. Thus,

$$n(R_2) = 20.$$

Step 4: Find the number of elements in $R_1 - R_2$.
To find $n(R_1 - R_2)$, we subtract the number of elements in R_2 from R_1:

$$n(R_1 - R_2) = n(R_1) - n(R_2) = 66 - 20 = 46.$$

Step 5: Conclusion.
The correct answer is 46, corresponding to option **(46)**.

Quick Tip

When calculating set differences, subtract the number of elements in the second set from the first set to get the result.

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22. The number of solutions of the equation $x + 2y + 3z = 42$, where $x, y, z \in \mathbb{Z}$ and $x, y, z \geq 0$, is:

Correct Answer: (168)

Solution:

Step 1: Analyze the equation.
We are given the equation:

$$x + 2y + 3z = 42,$$

where $x, y, z \in \mathbb{Z}$ and $x, y, z \geq 0$. We need to determine the number of solutions for this equation under these conditions.
Step 2: Consider different values of z.
We will substitute different values of z into the equation and find the corresponding number of solutions for x and y. The number of solutions for each case is determined by how many pairs of (x, y) satisfy the equation.

- For $z = 0$, $x + 2y = 42$ gives 22 solutions.
- For $z = 1$, $x + 2y = 39$ gives 19 solutions.
- For $z = 2$, $x + 2y = 36$ gives 19 solutions.
- For $z = 3$, $x + 2y = 33$ gives 17 solutions.
- For $z = 4$, $x + 2y = 30$ gives 16 solutions.
- For $z = 5$, $x + 2y = 27$ gives 14 solutions.
- For $z = 6$, $x + 2y = 24$ gives 13 solutions.
- For $z = 7$, $x + 2y = 21$ gives 11 solutions.
- For $z = 8$, $x + 2y = 18$ gives 10 solutions.
- For $z = 9$, $x + 2y = 15$ gives 8 solutions.
- For $z = 10$, $x + 2y = 12$ gives 7 solutions.
- For $z = 11$, $x + 2y = 9$ gives 5 solutions.
- For $z = 12$, $x + 2y = 6$ gives 4 solutions.
- For $z = 13$, $x + 2y = 3$ gives 2 solutions.
- For $z = 14$, $x + 2y = 0$ gives 1 solution.

Step 3: Calculate the total number of solutions.
Now, we sum all the solutions for each case:

$$22 + 19 + 19 + 17 + 16 + 14 + 13 + 11 + 10 + 8 + 7 + 5 + 4 + 2 + 1 = 168.$$

Step 4: Conclusion.
Thus, the total number of solutions is 168, corresponding to option **(168)**.

Quick Tip

When solving Diophantine equations with constraints, break the problem down into smaller cases and consider different values of one variable to simplify the calculation.