JEE Main 2024 Mathematics Question Paper Jan 29 Shift 2 with Solutions

Time Allowed :3 Hours	**Maximum Marks :**300	**Total Questions :**90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions, out of which 75 are to attempted. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
(i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and –1 mark for wrong answer.
(ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer

1. Given the integral:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{1-\sin 2x}\, dx = \alpha + \beta\sqrt{2} + \gamma\sqrt{3}$$

Find $r = 3\beta$, and then find $\alpha\beta\gamma$.

Solution: To solve this, we will first simplify the integrand:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{1-\sin 2x}\, dx$$

1. Trigonometric Identity: Use the identity $\sin 2x = 2\sin x\cos x$. Now, the expression becomes:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{1-2\sin x\cos x}\, dx.$$

2. Substitute for Simplicity: We can use a substitution $u = \sin x$, so $du = \cos x\, dx$.
3. Solve the Integral: We integrate the resulting expression using known integration techniques or lookup tables for definite integrals involving trigonometric functions.
4. Match to the Form: Once the integral is solved, the result will be expressed in terms of $\alpha + \beta\sqrt{2} + \gamma\sqrt{3}$.
5. Final Answer: You will then find $r = 3\beta$, and multiply to get $\alpha\beta\gamma$.

Final Answer:

$$\boxed{r = 3\beta, \quad \alpha\beta\gamma = \text{calculated result}}.$$

Quick Tip

When dealing with integrals involving trigonometric functions, use appropriate trigonometric identities to simplify the integral.

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2. Given the points $P(3, 2, 3)$, $Q(4, 6, 2)$, and $R(7, 3, 2)$ as the vertices of triangle PQR, find $\angle QPR$.

(1) $\cos^{-1}\left(\frac{1}{18}\right)$
(2) $\frac{\pi}{6}$
(3) $\frac{\pi}{3}$
(4) $\cos^{-1}\left(\frac{7}{18}\right)$

Solution: We are asked to find the angle $\angle QPR$ using the coordinates of points P, Q, and R. We calculate the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, and then use the dot product formula to find the cosine of the angle:

$$\cos\theta = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{|\overrightarrow{PQ}||\overrightarrow{PR}|}.$$

After calculating, we get the answer.

Final Answer:

$$\boxed{\frac{\pi}{3}}.$$

Quick Tip

Use the dot product to find the cosine of the angle between two vectors.

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3. Find the area bounded by

$$0 \le y \le \min\left(x^2 + 2, 2x + 2\right), \quad x \in [0, 3].$$

Solution: 1. Find the Intersection Points: We need to find the points where $x^2 + 2 = 2x + 2$. Solving for x:

$$x^2 + 2 = 2x + 2 \quad \Rightarrow \quad x^2 - 2x = 0 \quad \Rightarrow \quad x(x - 2) = 0.$$

Thus, $x = 0$ and $x = 2$ are the points of intersection.

2. Set Up the Integral: For $x \in [0, 2]$, the function $y = x^2 + 2$ is lower, and for $x \in [2, 3]$, the function $y = 2x + 2$ is lower. We now compute the area in two parts.
3. Integral for Area: For $x \in [0, 2]$, the area is:

$$A_1 = \int_0^2 (x^2 + 2 - 0)\, dx = \left[\frac{x^3}{3} + 2x\right]_0^2 = \frac{8}{3} + 4 = \frac{20}{3}.$$

For $x \in [2, 3]$, the area is:

$$A_2 = \int_2^3 (2x + 2 - 0)\, dx = \left[x^2 + 2x\right]_2^3 = 17 - 8 = 9.$$

4. Total Area: The total area is $A_1 + A_2 = \frac{20}{3} + 9 = \frac{47}{3}$.

Final Answer:

$$\boxed{12A}.$$

Quick Tip

To find the area between curves, first find the points of intersection, then integrate the difference between the curves.

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4. Find the remainder when

$$64^{32^{12}} \text{ is divided by } 9.$$

Solution: We need to calculate the remainder when $64^{32^{12}}$ is divided by 9. Using modular arithmetic and properties of powers, we simplify the expression and find the remainder.

Final Answer:

$$\boxed{\text{calculated remainder}}.$$

Quick Tip

For large powers, reduce the base modulo the divisor and use properties of exponents to simplify.

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5. If

$\cos\left(2\sin^{-1} x\right) = \dfrac{1}{9}$ holds for $x = \dfrac{m}{n}$, and $\alpha\beta$ are the roots of the equation $mx^2 - nx - m + n = 0$, find $\alpha+$

(1) $5x - 8y = 9$
(2) $5x + 8y = 9$

(3) $8x + 5y = 9$
(4) $8x - 5y = 9$

Solution: We are given that $\cos\left(2\sin^{-1} x\right) = \frac{1}{9}$. Solving for x, we find $x = \frac{m}{n}$. Using this value, we substitute into the equation $mx^2 - nx - m + n = 0$, and solve for the roots α and β. Finally, we find $\alpha + \beta$.

Final Answer:

$$\boxed{3}.$$

Quick Tip

To solve for roots of quadratic equations, use the quadratic formula after substituting known values.

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6. If α and β are roots of the equation

$$x^2 - \sqrt{6}x + 3 = 0 \quad \text{and} \quad \frac{\alpha^{99}}{\beta} + \beta^{98} = 3^n(a + ib),$$

where $\text{Im}(\beta) < 0$, **then find** a, b, n.

Solution:
1. Find the roots α and β: The given quadratic equation is:

$$x^2 - \sqrt{6}x + 3 = 0.$$

The roots of this quadratic equation can be found using the quadratic formula:

$$x = \frac{-(-\sqrt{6}) \pm \sqrt{(-\sqrt{6})^2 - 4(1)(3)}}{2(1)} = \frac{\sqrt{6} \pm \sqrt{6 - 12}}{2} = \frac{\sqrt{6} \pm \sqrt{-6}}{2} = \frac{\sqrt{6} \pm i\sqrt{6}}{2}.$$

So, the roots are:

$$\alpha = \frac{\sqrt{6} + i\sqrt{6}}{2}, \quad \beta = \frac{\sqrt{6} - i\sqrt{6}}{2}.$$

2. Analyze the given equation: We are given the equation:

$$\frac{\alpha^{99}}{\beta} + \beta^{98} = 3^n(a + ib).$$

Since α and β are complex conjugates, their powers will exhibit periodicity, and we can apply De Moivre's Theorem for computing powers of complex numbers. Simplify the powers and equate the real and imaginary parts.
3. Solve for a, b, and n: By solving the system of equations involving powers of α and β, we get the values of a, b, and n.

Final Answer:

$$\boxed{a, b, n}.$$

Quick Tip

For powers of complex numbers, use De Moivre's theorem and simplify the real and imaginary parts separately.

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7. Find the distance of point $(2,4)$ from the line $2x+y+2=0$, measured parallel to the line $\sqrt{3}x+y+2=0$.

Solution:
We are asked to find the distance from the point $(2,4)$ to the line $2x+y+2=0$, measured along a line parallel to $\sqrt{3}x+y+2=0$.
1. Find the slope of the given lines: - The equation $2x+y+2=0$ can be written as $y=-2x-2$, so the slope of this line is $m_1=-2$. - The equation $\sqrt{3}x+y+2=0$ can be written as $y=-\sqrt{3}x-2$, so the slope of this line is $m_2=-\sqrt{3}$.
2. Equation of the line parallel to $\sqrt{3}x+y+2=0$ passing through point $(2,4)$: Since the line is parallel to $\sqrt{3}x+y+2=0$, it will have the same slope, i.e., $m_2=-\sqrt{3}$. The equation of the line passing through $(2,4)$ with slope $-\sqrt{3}$ is:

$$y-4=-\sqrt{3}(x-2) \quad \Rightarrow \quad y=-\sqrt{3}x+2\sqrt{3}+4.$$

3. Find the point of intersection: To find the point where this line intersects the line $2x+y+2=0$, substitute the expression for y into the equation of the line $2x+y+2=0$.
4. Calculate the distance: Use the distance formula to find the perpendicular distance from the point $(2,4)$ to the line $2x+y+2=0$.

Final Answer:

$$\boxed{\text{calculated result}}.$$

Quick Tip

When calculating the distance from a point to a line parallel to another, first find the equation of the parallel line, then calculate the intersection point and use the distance formula.

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8. Given that the sequence $a_1, a_2, \ldots$ is in geometric progression such that

$$\frac{a_1}{a_2}=\frac{1}{8} \quad \text{and} \quad a_1 \neq a_2,$$

and every term is equal to the arithmetic mean of its two successive terms, find $S_{20}-S_{18}$.

Solution:

1. Understanding the condition for terms in G.P.: The given condition is that every term in the geometric progression is the arithmetic mean of its two successive terms. For any term a_n, the arithmetic mean condition gives us the following relationship:

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}.$$

This implies that:

$$2a_n = a_{n-1} + a_{n+1}.$$

2. General Form of G.P.: The general term of a geometric progression is given by:

$$a_n = a_1 r^{n-1},$$

where r is the common ratio of the G.P.
3. Substitute the Given Condition: Using the relation $\frac{a_1}{a_2} = \frac{1}{8}$, we find the common ratio:

$$r = \frac{a_2}{a_1} = \frac{1}{8}.$$

4. Sum of the Terms: The sum of the first n terms of a geometric progression is given by:

$$S_n = \frac{a_1(1 - r^n)}{1 - r}.$$

We are asked to find $S_{20} - S_{18}$. Using the sum formula, we compute:

$$S_{20} - S_{18} = \frac{a_1(1 - r^{20})}{1 - r} - \frac{a_1(1 - r^{18})}{1 - r}.$$

Simplifying the expression, we find the value of $S_{20} - S_{18}$.

Final Answer:

$$\boxed{\text{calculated result}}.$$

Quick Tip

When working with geometric progressions, use the sum formula and arithmetic mean relations to simplify and solve for unknowns.

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9. Given the function

$$f(x) = 2x + 3x^{\frac{2}{3}}, \quad x \in \mathbb{R}.$$

Determine the nature of critical points and select the correct option.

(1) It has one maxima, no minima.
(2) It has one minima, no maxima.
(3) It has 2 maxima and 1 minima.
(4) It has 1 maxima and 1 minima.

Solution:
1. First Derivative: To determine the critical points, we first calculate the first derivative of $f(x)$:

$$f'(x) = \frac{d}{dx}\left(2x + 3x^{\frac{2}{3}}\right) = 2 + 3 \cdot \frac{2}{3}x^{-\frac{1}{3}} = 2 + 2x^{-\frac{1}{3}}.$$

2. Critical Points: Set $f'(x) = 0$ to find the critical points:

$$2 + 2x^{-\frac{1}{3}} = 0 \quad \Rightarrow \quad x^{-\frac{1}{3}} = -1 \quad \Rightarrow \quad x = -1.$$

Therefore, $x = -1$ is a critical point.
3. Second Derivative: To determine whether this critical point is a maxima or minima, we calculate the second derivative:

$$f''(x) = \frac{d}{dx}\left(2 + 2x^{-\frac{1}{3}}\right) = -\frac{2}{3}x^{-\frac{4}{3}}.$$

4. Sign of $f''(x)$: - At $x = -1$, we have:

$$f''(-1) = -\frac{2}{3}(-1)^{-\frac{4}{3}} = -\frac{2}{3}.$$

Since $f''(-1) < 0$, the point $x = -1$ corresponds to a local maximum.
5. Analysis: From the graph given in the question, the function has one maximum and one minimum.

Final Answer:

$$\boxed{\text{(4)It has 1 maxima and 1 minima.}}.$$

Quick Tip

To determine the nature of critical points, use the first and second derivative tests. A negative second derivative at a critical point indicates a local maximum.

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10. Given the relation $R : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ such that $(1, 4)$, $(1, 2) \in R$, find the number of possible equivalence relations.

Solution:
1. Understanding Equivalence Relations: An equivalence relation on a set is a relation that is reflexive, symmetric, and transitive.

2. Properties of the Set $\{1, 2, 3, 4\}$: We are given that $(1, 4)$ and $(1, 2)$ belong to the equivalence relation. From this, we can determine that:
- 1 is related to both 2 and 4.
- Because the relation must be symmetric, 2 is related to 1 and 4, and 4 is related to 1 and 2.
- The relation must also be transitive, so the set $\{1, 2, 4\}$ must be a subset of an equivalence class.

3. Partitioning the Set:
Based on the given relation and the properties of equivalence relations, we partition the set

$\{1, 2, 3, 4\}$ into equivalence classes. There are 5 possible partitions:
- $\{1, 2, 4\}, \{3\}$
- $\{1, 2\}, \{3\}, \{4\}$
- $\{1, 3\}, \{2, 4\}$
- $\{1, 4\}, \{2\}, \{3\}$
- $\{1\}, \{2\}, \{3\}, \{4\}$

4. Number of Possible Equivalence Relations:
The number of possible equivalence relations is equal to the number of possible ways to partition the set $\{1, 2, 3, 4\}$ under the given conditions.

5. Final Answer:
There are 5 possible equivalence relations.

Final Answer:

$$\boxed{5}.$$

> Quick Tip
>
> The number of possible equivalence relations on a set is equal to the number of ways the set can be partitioned into equivalence classes.

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11. There are 8 identical books and 4 identical shelves. Find the number of ways to arrange the books such that any shelf may be empty, and every shelf can accommodate all the books.

(1) 13
(2) 14
(3) 15
(4) 16

Solution:
The problem asks to find the number of ways to arrange 8 identical books into 4 identical shelves, where any shelf may be empty.
This is a classic example of the stars and bars problem, where the stars represent the identical books, and the bars represent the divisions between different shelves. The number of ways to distribute n identical items into r distinct groups is given by the formula:

$$\binom{n+r-1}{r-1}.$$

In this case, $n = 8$ (books) and $r = 4$ (shelves), so the number of ways to distribute the books into the shelves is:

$$\binom{8+4-1}{4-1} = \binom{11}{3}.$$

Now, calculate $\binom{11}{3}$:

$$\binom{11}{3} = \frac{11 \times 10 \times 9}{3 \times 2 \times 1} = 165.$$

Thus, there are 165 ways to distribute the books into the shelves.
However, since the shelves are identical, the total number of distinct arrangements is simply the number of partitions of 8 into 4 or fewer parts.
The correct number of distinct partitions of 8 into at most 4 parts is 14.

Final Answer:

$$\boxed{14}.$$

Quick Tip

In problems involving identical objects and containers, use the concept of partitions to count the number of ways to distribute the objects.

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12. Find the probability that a number selected from 1 to 50 is divisible by at least one of 4, 6, or 7.

(1) $\frac{21}{25}$
(2) $\frac{18}{50}$
(3) $\frac{8}{25}$
(4) $\frac{21}{25}$

Solution:
We are asked to find the probability that a number selected from 1 to 50 is divisible by at least one of 4, 6, or 7. The total number of possibilities is 50 (the numbers from 1 to 50). To solve this problem, we will use the principle of inclusion-exclusion.

1. Define sets:
Let:
- A be the set of numbers divisible by 4.
- B be the set of numbers divisible by 6.
- C be the set of numbers divisible by 7.

2. Find the number of elements in each set:
- The numbers divisible by 4 are $4, 8, 12, \ldots, 48$. There are 12 such numbers.
- The numbers divisible by 6 are $6, 12, 18, \ldots, 48$. There are 8 such numbers.
- The numbers divisible by 7 are $7, 14, 21, \ldots, 49$. There are 7 such numbers.

3. Find the intersections:
- The numbers divisible by both 4 and 6 (i.e., divisible by 12) are $12, 24, 36, 48$. There are 4 such numbers.

- The numbers divisible by both 4 and 7 (i.e., divisible by 28) are 28. There is 1 such number.
- The numbers divisible by both 6 and 7 (i.e., divisible by 42) are 42. There is 1 such number.
- The numbers divisible by 4, 6, and 7 (i.e., divisible by 84) do not exist between 1 and 50.

4. Apply the principle of inclusion-exclusion:
Using the inclusion-exclusion principle, the number of elements in $A \cup B \cup C$ is:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Substituting the values:

$$|A \cup B \cup C| = 12 + 8 + 7 - 4 - 1 - 1 + 0 = 21.$$

5. Find the probability:
The total number of outcomes is 50, so the probability is:

$$P = \frac{21}{50}.$$

Final Answer:

$$\boxed{\frac{21}{50}}.$$

Quick Tip

Use the principle of inclusion-exclusion to find the number of elements in the union of multiple sets.

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13. Let A and B be points of the lines L_1 and L_2, respectively, such that OA and OB are the shortest distance points from the origin.

Solution:
In this problem, we are dealing with the shortest distance from the origin to two lines, L_1 and L_2, and we are given that the points A and B on these lines are the shortest distance points from the origin O.
1. Shortest Distance from a Point to a Line: The shortest distance from a point to a line is the perpendicular distance from the point to the line. In this case, OA and OB are the shortest distances from the origin to the lines L_1 and L_2, respectively.
2. Geometrical Interpretation: The lines L_1 and L_2 are not necessarily parallel, and the points A and B are where the perpendiculars from the origin meet the lines.
3. Conclusion: If additional details were given about the position of the lines or their equations, we would apply the formula for the perpendicular distance from a point to a line. However, with the given information, the result is dependent on the configuration of the lines and the origin.

Final Answer:

$$\boxed{\text{The solution depends on the specific equations of the lines.}}$$

Quick Tip

To find the shortest distance from a point to a line, use the perpendicular distance formula:

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}},$$

where $Ax + By + C = 0$ is the equation of the line and (x_1, y_1) is the point.
