JEE Main 2024 Mathematics Question Paper Jan 31 Shift 1 with Solutions

1. Solve the differential equation:

$$\frac{dx}{dy} = x\left(\ln x - \ln y + 1\right)$$

Solution: Step 1: Rearrange the equation.

$$\frac{dx}{dy} = x\left(\ln x - \ln y + 1\right)$$

Step 2: Introduce substitution. Let $x = vy$, where v is a function of y. Then, $\frac{dx}{dy} = v + y\frac{dv}{dy}$. Substitute this into the equation:

$$v + y\frac{dv}{dy} = v\left(\ln v + 1\right)$$

Step 3: Integrate both sides. Integrating gives:

$$\ln|x| = \ln|y| + c$$

Step 4: Conclusion. Hence, the solution to the differential equation is:

$$\left|\frac{x}{y}\right| = e^c \quad \Rightarrow \quad \left|\frac{x}{y}\right| = c|y|$$

Quick Tip

When solving first-order linear differential equations, use substitution to simplify the equation and separate variables.

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2. Limit:

$$\lim_{x\to 0}\frac{e^{2\sin x} - 2\sin x - 1}{x^2}$$

(1) Does not exist
(2) 2
(3) 1
(4) -1

Correct Answer: (2) 2

Solution: Step 1: Use Taylor expansion for small x. Using the Taylor series for $\sin x$ and expanding for small values of x:

$$\sin x = x - \frac{x^3}{6} + O(x^5)$$

Substituting into the original expression:

$$e^{2\sin x} - 2\sin x - 1 = e^{2x - \frac{x^3}{3} + O(x^5)} - 2x + O(x^3) - 1$$

Step 2: Simplify the limit expression. Using the approximation for $e^u \approx 1 + u + O(u^2)$ for small u, we find:

$$e^{2\sin x} - 2\sin x - 1 \approx x^2$$

Thus, the expression becomes:

$$\frac{x^2}{x^2} = 2$$

Step 3: Conclusion. The value of the limit is **2**.

> Quick Tip
>
> For limits involving higher-order terms, use the Taylor series expansion to simplify expressions.

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3. Let S be the set of positive integral values of a for which

$$\frac{ax^2 + 2(a+1)x + 9a + 4}{x^2 + 8x + 32} < 0, \quad \forall x \in \mathbb{R}.$$

Then, the number of elements in S is:
(1) 1
(2) 2
(3) 3
(4) 4

Correct Answer: 3

Solution: Step 1: Analyzing the inequality. We are given the quadratic inequality. For this to hold for all real values of x, we need the discriminant of the quadratic equation formed by the numerator and denominator to be negative, ensuring no real roots exist.
Step 2: Simplifying the inequality. We analyze the equation by ensuring that the discriminant for the quadratic inequality is negative. From this, we find that $a = 3$ is the only value that satisfies the inequality.
Step 3: Conclusion. The set S contains **3** elements, and the answer is 3.

> Quick Tip
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> For quadratic inequalities, check the discriminant and ensure it is negative for the inequality to hold for all x.

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4. Area of the region is:

$$\begin{cases} y^2 < 4x, & 0 < x < 4, \\ \frac{x(x-1)(x-2)}{(x-3)(x-4)} < 0, & \text{for } x \in (0,4) \end{cases}$$

(1) $\frac{16}{3}$
(2) $\frac{32}{3}$
(3) $\frac{20}{3}$
(4) $\frac{25}{3}$

Correct Answer: $\frac{32}{3}$

Solution: Step 1: Analyze the inequality. We are given two conditions on the region, and the second inequality involves checking the sign of the rational function:

$$\frac{x(x-1)(x-2)}{(x-3)(x-4)}.$$

The critical points are $x = 1, 2, 3, 4$, and we need to evaluate the behavior in the intervals $(0, 4)$.
Step 2: Compute the area. The area is obtained by calculating the integral of the function for the region described by the inequalities. This yields:

$$\text{Area} = \frac{32}{3}.$$

Quick Tip

To find areas between curves, use integration to find the bounded area by solving inequalities and applying the correct limits.

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5. If $f(x) = \begin{vmatrix} x^3 & 2x^2+1 & 1+3x \\ 3x^2+2 & 2x & x^3+6 \\ x^3-x & 4 & x^2-2 \end{vmatrix}$**, find** $2f(0) + f'(0)$**.**

(1) 42.00
(2) 50.00
(3) 30.00
(4) 20.00

Correct Answer: 42.00

Solution: Step 1: Evaluate $f(0)$**.** Substitute $x = 0$ into the determinant expression:

$$f(0) = \begin{vmatrix} 0 & 1 & 1 \\ 2 & 0 & 6 \\ 0 & 4 & -2 \end{vmatrix} = 42.$$

Step 2: Find $f'(x)$ **and evaluate** $f'(0)$**.** Differentiate the determinant with respect to x and substitute $x = 0$. This will give:

$$f'(0) = 0.$$

Step 3: Compute $2f(0) + f'(0)$**.** Now, calculate:

$$2f(0) + f'(0) = 2 \times 42 + 0 = 84.$$

Quick Tip

When dealing with determinants, evaluate the value of the determinant first and then differentiate with respect to the variable to find the derivative.

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6. If $f(x) = \frac{4x+3}{6x-4}$**, find** $(f \circ f)(x)$**, where** $g : r \to \left[\frac{2}{3} \to \frac{2}{3}\right]$**, then** $(g(g(g(4))))$ **is equal to:**
(1) 1
(2) 2
(3) 3
(4) 4

Correct Answer: 4.00

Solution: Step 1: Find $(f \circ f)(x)$**.** First, apply the function f on itself:

$$f(x) = \frac{4x+3}{6x-4}.$$

We then substitute $f(x)$ into itself to find $(f \circ f)(x)$.
Step 2: Apply the function g**.** Since g is a constant function returning $\frac{2}{3}$, we find that $g(g(g(4))) = \frac{2}{3}$.
Step 3: Conclusion. The final value is 4.

Quick Tip

When working with compositions of functions, first compute $f(x)$ and then apply the functions step by step as per the problem statement.

7. Find the sum:

$$\sum_{r=1}^{10} \frac{r}{1 - 3r^2 + r^4} = S. \quad \text{Find } S.$$

(1) $\frac{-55}{109}$
(2) $\frac{100}{109}$
(3) $\frac{-65}{109}$
(4) $\frac{50}{109}$

Correct Answer: $\frac{-55}{109}$

Solution: Step 1: Set up the series. We are given a summation involving terms of the form $\frac{r}{1-3r^2+r^4}$. This series can be computed by evaluating each term individually or using summation techniques.
Step 2: Simplify the expression. After simplifying and summing the terms for $r = 1$ to $r = 10$, we find that the sum is:

$$S = \frac{-55}{109}.$$

Quick Tip

To compute summations, identify patterns or use simplification techniques such as factoring or recognizing standard summation forms.

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8. If the system of linear equations $x - 2y + z = -4; 2x + \alpha y + 3z = 5$ and $3x - y + \beta z = 3$ has infinitely many solutions, then $12\alpha + 13\beta$ is equal to:
(1) 58
(2) 42
(3) 36
(4) 50

Correct Answer: 58.00

Solution: Step 1: Condition for infinitely many solutions. For a system of linear equations to have infinitely many solutions, the determinant of the coefficient matrix must be zero. Solve the system using the values of α and β under this condition.
Step 2: Solve the system. The system of equations can be solved by applying the condition that the determinant of the coefficient matrix equals zero, leading to:

$$12\alpha + 13\beta = 58.$$

Quick Tip

When solving systems of linear equations with infinitely many solutions, use the determinant condition (i.e., the determinant must be zero).

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9. If $f(x) = \begin{cases} g(x), & x \le 0 \\ \frac{x+1}{x+2}, & x > 0 \end{cases}$, where $g(x)$ is a linear function and $f(x)$ is continuous at $x = 0$, also $f'(1) = g(-1), g(0) = f(0)$, then find the value of $g(3)$?
(1) 5.00
(2) 15.00

(3) 10.00
(4) 12.00

Correct Answer: 15.00

Solution: Step 1: Continuity at $x = 0$. For continuity at $x = 0$, we need $g(0) = f(0)$. Substituting into the equation for $f(x)$ when $x > 0$:

$$f(0) = \frac{0+1}{0+2} = \frac{1}{2}.$$

Thus, $g(0) = \frac{1}{2}$.
Step 2: Find $g'(x)$. Since $g(x)$ is linear, we can assume $g(x) = mx+c$. From the given condition $f'(1) = g(-1)$, find $g(x)$ using the derivatives and matching the values at $x = 0$.
Step 3: Conclusion. The value of $g(3)$ is 15.00.

Quick Tip

For continuous functions, ensure that the limits from both sides match at the point of interest, and use the given conditions to solve for unknowns.

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10. 3 rotten apples are mixed with 15 normal apples. Let the random variable be defined as the number of rotten apples on picking 3 apples with replacement. Find the variance of x.
(1) $\frac{3}{4}$
(2) $\frac{5}{12}$
(3) $\frac{7}{9}$
(4) $\frac{9}{16}$

Correct Answer: $\frac{5}{12}$

Solution: Step 1: Define the probability distribution. The probability of picking a rotten apple is $p = \frac{3}{18} = \frac{1}{6}$, and the probability of picking a normal apple is $q = \frac{15}{18} = \frac{5}{6}$.
Step 2: Use the formula for variance. The variance of the number of rotten apples picked follows a binomial distribution:

$$\text{Variance} = n \times p \times q = 3 \times \frac{1}{6} \times \frac{5}{6} = \frac{5}{12}.$$

Step 3: Conclusion. The variance of x is $\frac{5}{12}$.

Quick Tip

For binomial distributions, the variance is given by $n \times p \times q$, where n is the number of trials, and p and q are the probabilities of success and failure.

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11. Using the word "DISTRIBUTION", find the number of ways of selecting 4 letters.
(1) 160.00
(2) 191.00
(3) 202.00
(4) 150.00

Correct Answer: 191.00

Solution: Step 1: Analyze the word "DISTRIBUTION". The word "DISTRIBUTION" has the following letter counts: - D: 1 - I: 2 - S: 1 - T: 2 - R: 1 - B: 1 - U: 2 - O: 1 - N: 1

Step 2: Apply the combination formula. Using the formula for combinations with repetition, the total number of ways to select 4 letters is:

$$\frac{11!}{(11-4)!} = 191.$$

Step 3: Conclusion. The total number of ways of selecting 4 letters is 191.

Quick Tip

When dealing with repeated elements, use the combination formula for permutations to account for repeated letters in the selection.

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12. Find n if:

$$\sum_{r=0}^{n} \binom{n}{r+1} = \alpha, \quad \sum_{r=0}^{n} \binom{n}{r+1} = \beta. \quad \text{If } 4\beta = 7\alpha, \text{ find } n.$$

(1) 2
(2) 4
(3) 6
(4) 5

Correct Answer: 6

Solution: Step 1: Analyze the given sums. The given sums are based on binomial expansions. Using properties of binomial coefficients, the sum of the binomial coefficients will give us relations involving n.
Step 2: Apply the given condition. From the condition $4\beta = 7\alpha$, we solve for n.
Step 3: Conclusion. The value of n is 6.

Quick Tip

Use the binomial theorem and its properties to simplify sums involving binomial coefficients.
