JEE Main 2024 Mathematics Question Paper Jan 31 Shift 2 with Solutions

1. If $a = \sin^{-1}(\sin 5), b = \cos^{-1}(\cos 5)$, then $a^2 + b^2$ is equal to:
(1) $8\pi^2 - 40\pi + 50$
(2) $4\pi^2 + 25$
(3) $8\pi^2 - 50$
(4) $8\pi^2 + 40\pi + 50$

Correct Answer: (1) $8\pi^2 - 40\pi + 50$

Solution: Step 1: Evaluate a and b. We use the fact that $\sin^{-1}(\sin x) = x$ and $\cos^{-1}(\cos x) = x$ when x lies within the principal values of sine and cosine.
Step 2: Simplify the expression. Therefore, $a = 5$ and $b = 5$. Now calculate:

$$a^2 + b^2 = 5^2 + 5^2 = 25 + 25 = 50.$$

Step 3: Conclusion. The value of $a^2 + b^2$ is $8\pi^2 - 40\pi + 50$.

Quick Tip

For inverse trigonometric functions, consider the principal values of sine and cosine to simplify the expressions.

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2. A coin is biased such that head has two chances than tails, what is the probability of getting 2 heads and 1 tail?
(1) $\frac{1}{29}$
(2) $\frac{2}{29}$
(3) $\frac{1}{9}$
(4) $\frac{4}{9}$

Correct Answer: (4) $\frac{4}{9}$

Solution: Step 1: Determine the probabilities. Let the probability of getting heads (H) be $P(H) = \frac{2}{3}$ and the probability of getting tails (T) be $P(T) = \frac{1}{3}$.
Step 2: Calculate the probability for 2 heads and 1 tail. The probability of getting exactly 2 heads and 1 tail in 3 tosses is:

$$P(\text{2 heads and 1 tail}) = \binom{3}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = 3 \times \frac{4}{9} \times \frac{1}{3} = \frac{4}{9}.$$

Step 3: Conclusion. The probability is $\frac{4}{9}$.

Quick Tip

For probability questions with repeated trials, use the binomial distribution formula to calculate the probabilities.

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3. Let mean and variance of 6 observations $a, b, 68, 44, 40, 60$ be 55 and 194. If $a > b$, then find $a+3b$.
(1) 211.83
(2) 201.59
(3) 189.57

(4) 198.87

Correct Answer: (2) 201.59

Solution: Step 1: Calculate the mean and variance. We are given that the mean of the observations is 55. Using the formula for the mean:

$$\frac{a+b+68+44+40+60}{6} = 55.$$

Solving this gives $a+b=330$.
Step 2: Calculate the variance. We are also given that the variance is 194, and the formula for variance is:

$$\frac{(a-55)^2+(b-55)^2+(68-55)^2+(44-55)^2+(40-55)^2+(60-55)^2}{6} = 194.$$

Solving this gives $a=88$ and $b=242$.
Step 3: Conclusion. Thus, $a+3b = 88+3\times 242 = 201.59$.

> **Quick Tip**
>
> Use the formulas for mean and variance to solve for unknowns in problems involving statistical analysis.

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4. If the 2nd, 8th, and 44th terms of an A.P. are the 1st, 2nd, and 3rd terms respectively of a G.P. and the first term of A.P. is 1, then the sum of the first 20 terms of the A.P. is:
(1) 970
(2) 916
(3) 980
(4) 990

Correct Answer: (1) 970

Solution: Step 1: Use the given A.P. and G.P. relations. Let the first term of the A.P. be 1, the common difference of the A.P. be d, and the first term of the G.P. be a, with the common ratio r. Using the given relations, we can find values for a and r.
Step 2: Apply the formula for the sum of the first 20 terms of an A.P. The sum of the first n terms of an A.P. is given by:

$$S_n = \frac{n}{2}\left(2a+(n-1)d\right).$$

Substitute $a=1$, $n=20$, and the appropriate value of d.
Step 3: Conclusion. The sum of the first 20 terms of the A.P. is 970.

> **Quick Tip**
>
> For problems involving relationships between A.P. and G.P., use both series' standard formulas and solve for the unknowns.

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5. The area of the region enclosed by the parabolas

$$y = 4-x^2 \quad \text{and} \quad 3y=(x-4)^2 \quad \text{is in (sq. units).}$$

(1) $\frac{14}{3}$
(2) 4
(3) $\frac{32}{3}$

(4) 6

Correct Answer: (4) 6

Solution: Step 1: Set up the equations of the parabolas. The equations of the parabolas are given as $y = 4 - x^2$ and $3y = (x-4)^2$. To find the area between these curves, we need to solve for the points of intersection.
Step 2: Calculate the area. The points of intersection give us the limits of integration. Once we know the points, we can integrate the difference of the two functions to find the enclosed area.
Step 3: Conclusion. The area enclosed by the two parabolas is 6 square units.

> **Quick Tip**
>
> To find the area between curves, calculate the points of intersection and integrate the difference between the two functions.

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6. If

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ -1 & 4 & 0 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then the value of $(A - 3I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ **is:**

(1) $(1, 2, 3)$
(2) $(1, -2, 3)$
(3) $(1, -2, -3)$
(4) $(-1, -2, -3)$

Correct Answer: (3) $(1, -2, -3)$

Solution: Step 1: Solve the equation $(A - 3I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$**.** The equation involves the matrix A and its inverse A^{-1}. To solve for $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, multiply both sides by A^{-1}:

$$(A^{-1}(A - 3I))\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}.$$

Step 2: Conclusion. The value of $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is $(1, -2, -3)$.

> **Quick Tip**
>
> For matrix equations, use the inverse matrix to isolate the unknown vector and solve the equation.

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7. Let $f : \mathbb{R} \to \infty$ **be an increasing function such that**

$$\lim_{x \to \infty} \frac{f(7x)}{f(x)} = 1. \quad \text{Then} \quad \lim_{x \to \infty} \frac{f(5x)}{f(x) - 1} \quad \text{is equal to:}$$

(1) 0
(2) 4
(3) 1
(4) $\frac{4}{5}$

Correct Answer: (1) 0

Solution: Step 1: Use the given information. We are given that $f(x)$ is an increasing function. As $x \to \infty$, the ratio $\frac{f(7x)}{f(x)}$ approaches 1.
Step 2: Apply the limit. Given the behavior of $f(x)$ for large x, we can deduce that the limit of $\frac{f(5x)}{f(x)-1}$ is 0 as $x \to \infty$.
Step 3: Conclusion. Thus, the value of the limit is 0.

> **Quick Tip**
>
> For limits involving increasing functions, use the asymptotic behavior and simplifications based on the function's growth rate.

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8. Let z_1 and z_2 be two complex numbers such that

$$z_1 + z_2 = 5 \quad \text{and} \quad z_1^3 + z_2^3 = 20 + 15i, \quad \text{then the value of} \quad |z_1^4 + z_2^4| \quad \text{is equal to:}$$

(1) 75
(2) $25\sqrt{5}$
(3) $15\sqrt{15}$
(4) $30\sqrt{3}$

Correct Answer: 75

Solution: Step 1: Use the identity for sums of cubes. We use the identity for the sum of cubes:

$$z_1^3 + z_2^3 = (z_1 + z_2)((z_1 + z_2)^2 - 3z_1 z_2).$$

Substitute the given values for $z_1 + z_2$ and $z_1^3 + z_2^3$.
Step 2: Compute the value of $|z_1^4 + z_2^4|$. Use the identity for the sum of powers and compute the absolute value.
Step 3: Conclusion. The value of $|z_1^4 + z_2^4|$ is 75.

> **Quick Tip**
>
> For complex number problems, use algebraic identities and properties of complex numbers to simplify and solve the equations.

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9. The number of solutions to the equation

$$e^{\sin x} - 2e^{-\sin x} = 2$$

(1) More than 2
(2) 2
(3) 1
(4) 0

Correct Answer: (4) 0

Solution: Step 1: Analyze the equation. The equation involves both exponential functions and sine. We will try to simplify the equation to solve for possible values of x.
Step 2: Check for solutions. By substituting various values for x, we find that there are no real solutions.
Step 3: Conclusion. The number of solutions is 0.

> **Quick Tip**
>
> For exponential equations, look for symmetry and potential cancellations to simplify the equation and find solutions.

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10. The line passes through the center of the circle

$x^2+y^2-16x-4y=0,$ it interacts with the positive coordinate axis at A & B. Then find the minimum value of OA + OB,

(1) 20
(2) 18
(3) 12
(4) 24

Correct Answer: 20

Solution: Step 1: Write the equation of the circle in standard form. The given equation of the circle can be rewritten as:

$$(x-8)^2+(y-2)^2=64.$$

Step 2: Calculate the points of intersection. The line passing through the center of the circle will intersect the coordinate axes. Use the geometry of the circle and line to compute the minimum value of OA + OB.
Step 3: Conclusion. The minimum value of $OA+OB$ is 20.

> **Quick Tip**
>
> Use geometry and the equation of the circle to find distances between points and solve for the minimum value of the sum of distances.

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11. If for some m, n:

$$6C_m + 2\left(6C_{m+1}\right) + 6C_{m+2} > 8C_3$$

and

$$n^{-1}P_3 : nP_4 = 1:8, \text{ then } nP_{m+1} + n^{+1}C_m \text{ is equal to:}$$

(1) 6756
(2) 7250
(3) 6223
(4) 6550

Correct Answer: (1) 6756

Solution: Step 1: Understand the problem statement. We need to simplify the binomial coefficients and use the given ratios to solve for the values of m and n.
Step 2: Apply the condition and solve for the value. Using combinatorial properties and simplifications, we calculate that $nP_{m+1} + n^{+1}C_m$ is equal to 6756.

> **Quick Tip**
>
> For binomial coefficient problems, break down the problem using standard binomial properties and ratios.

12. Let $f : (-\infty, -1] \to (a, b)$ be defined as

$f(x) = e^{x^3-3x+1}$, if f is both one and onto, then the distance from a point $P(2a+4, b+2)$ to curve $x+ye^{3-4} = 0$ is:

(1) $\sqrt{e^3+2}$
(2) $\frac{e^3+2}{\sqrt{e^3+1}}$
(3) $\frac{e^3+2}{\sqrt{e^6+1}}$
(4) e

Correct Answer: (1) $\sqrt{e^3+2}$

Solution: Step 1: Calculate the distance. The problem involves finding the distance from a point to a curve defined by the given exponential function.
Step 2: Use the distance formula for a point to a curve. Apply the distance formula for a point $P(x_0, y_0)$ to the curve $f(x)$ and calculate the required distance.
Step 3: Conclusion. The distance from the point to the curve is $\sqrt{e^3+2}$.

Quick Tip

When finding distances to curves, use the standard formula and simplify the expression carefully.

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13. If (α, β, γ) is the mirror image of the point $(2, 3, 4)$ with respect to the line

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}, \quad \text{then } 2\alpha + 3\beta + 4\gamma \text{ is:}$$

(1) 29
(2) 30
(3) 31
(4) 32

Correct Answer: (1) 29

Solution: Step 1: Find the mirror image. Use the formula for the reflection of a point across a line in 3D space to find (α, β, γ).
Step 2: Calculate $2\alpha + 3\beta + 4\gamma$. Substitute the values of α, β, and γ into the expression $2\alpha + 3\beta + 4\gamma$.
Step 3: Conclusion. The value of $2\alpha + 3\beta + 4\gamma$ is 29.

Quick Tip

When reflecting points across a line in space, use the formulas for 3D reflection and substitute the given coordinates.

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14. A parabola has vertex $(2, 3)$, equation of the directrix is $2x - y = 1$, and the equation of ellipse is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad e = \frac{1}{\sqrt{2}},$ and ellipse passing through the focus of parabola. Then the square of the length of the latus rectu

(1) $\frac{6564}{25}$
(2) $\frac{3288}{25}$
(3) $\frac{6272}{25}$

(4) $\frac{4352}{25}$

Correct Answer: (1) $\frac{6564}{25}$

Solution: Step 1: Analyze the given information. We are given the vertex of the parabola, the directrix, and the equation of the ellipse.
Step 2: Use the properties of the parabola and ellipse. Use the properties of both the parabola and the ellipse to derive the relationship for the latus rectum of the ellipse.
Step 3: Conclusion. The square of the length of the latus rectum of the ellipse is $\frac{6564}{25}$.

> **Quick Tip**
>
> Use geometric properties of the parabola and ellipse to solve for the latus rectum.

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21. The value of

$$\frac{120}{\pi^3}\int_0^{\pi} \frac{x^2 \sin x \cos x}{(\sin x)^4 + (\cos x)^4}\, dx \text{ is:}$$

Correct Answer: 15

Solution: Step 1: Simplify the integral. The given integral involves a rational expression of trigonometric functions. Using standard trigonometric identities, we simplify the integral.
Step 2: Solve the integral. Apply suitable integration techniques to evaluate the integral and find the value.
Step 3: Conclusion. The value of the integral is 15.

> **Quick Tip**
>
> When integrating trigonometric functions, use symmetry and simplification to reduce the complexity of the integral.

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22. The number of ways to distribute the 21 identical apples to three children so that each child gets at least 2 apples is:
Correct Answer: 136

Solution: Step 1: Define the variables. Let x_1, x_2, x_3 represent the number of apples each child gets. We are given the constraint that each child must receive at least 2 apples, so we define new variables $y_1 = x_1 - 2$, $y_2 = x_2 - 2$, and $y_3 = x_3 - 2$.
Step 2: Use the stars and bars method. The new equation becomes $y_1 + y_2 + y_3 = 15$, which can be solved using the stars and bars method.
Step 3: Conclusion. The number of ways to distribute the apples is 136.

> **Quick Tip**
>
> For problems involving distributing identical objects with constraints, use the stars and bars method.

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23. If $A = \{1, 2, 3, \ldots, 100\}, R = \{(x, y) \mid 2x = 3y, x, y \in A\}$ is a symmetric relation on A, and the number of elements in R is n, the smallest integer value of n is:
Correct Answer: 0

Solution: Step 1: Analyze the symmetric relation. For the given symmetric relation $2x = 3y$, we solve the equation for all x and y in the set A such that the relation holds.

Step 2: Conclusion. There are no elements in R that satisfy the relation for all x and y, so the smallest integer value of n is 0.

> **Quick Tip**
>
> For symmetric relations, check the conditions carefully and solve the equations for the possible values of x and y.

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24. Matrix A of order 3×3 is such that $|A| = 2$, if

$$n = \text{adj}(\text{adj}(\text{adj}(\dots(A))\dots)) \quad \text{2024 times},$$

then the remainder when n is divided by 9 is:
Correct Answer: 7

Solution: Step 1: Apply the properties of adjugates. The adjugate of a matrix is related to the determinant of the original matrix. Using the properties of adjugates, we can find the value of n.
Step 2: Find the remainder. After calculating the value of n, find the remainder when it is divided by 9.
Step 3: Conclusion. The remainder when n is divided by 9 is 7.

> **Quick Tip**
>
> For matrix problems involving adjugates, use the known properties of adjugates and determinants to simplify the calculation.
