JEE Main Mathematics Sample Paper-13

Duration: 1 Hour **Maximum Marks: 100**

Instructions

- This paper contains TWO sections: **Section A** (MCQs) and **Section B** (Numerical).
- Section A contains 20 Multiple Choice Questions.
- Section B contains 5 Numerical Value Questions.
- Each correct answer carries **+4 marks**.
- Each incorrect answer carries **-1 mark**.
- No negative marking for unattempted questions.

Section A — Multiple Choice Questions

Q1. If $\lim_{x\to 0} \frac{e^{ax}-\cos(bx)-cx}{x^2} = 2$, then the value of $a^2+b^2+c^2$ is: [JEE Main 2023]

(A) 9

(B) 5

(C) 11

(D) 8

Q2. Let $f(x) = [x^2 - x]$, where $[t]$ denotes the greatest integer function. Then the number of points in the interval $(-1, 1)$ where $f(x)$ is discontinuous is: [JEE Main 2022]

(A) 2

(B) 3

(C) 4

(D) 5

Q3. If $f(x) = \begin{cases} x^a \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ is differentiable at $x = 0$, then the range of a is: [JEE Main 2021]

(A) $a > 1$

(B) $a \geq 1$

(C) $0 < a < 1$

(D) $a > 0$

Q4. The shortest distance between the line $x - y + 1 = 0$ and the curve $y^2 = x$ is: [JEE Main 2024]

(A) $\frac{3\sqrt{2}}{8}$

(B) $\frac{2\sqrt{3}}{5}$

(C) $\frac{\sqrt{2}}{4}$

(D) $\frac{5\sqrt{2}}{8}$

Q5. Let $f(x)$ be a polynomial of degree 4 having relative extrema at $x = 1, 2, 3$. If $\lim_{x\to 0}[1 + \frac{f(x)}{x^2}] = 3$, then $f(2)$ is: [JEE Main 2020]

(A) -4

(B) -8

(C) 0

(D) 4

Q6. The value of $\int_0^\pi \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx$ is: [JEE Main 2023]

(A) $\pi/4$

(B) $\pi/2$

(C) π

(D) 2π

Q7. The area of the region $\{(x, y) : x^2 \leq y \leq |3x - 4|\}$ is: [JEE Main 2024]

(A) $31/6$

(B) $37/6$

(C) $25/3$

(D) $13/2$

Q8. If $\int \frac{\sin x}{\sin(x-\alpha)}dx = Ax + B\log\sin(x-\alpha) + C$, then the value of (A, B) is: [JEE Main 2021]

(A) $(\cos\alpha, \sin\alpha)$

(B) $(\sin\alpha, \cos\alpha)$

(C) $(-\cos\alpha, \sin\alpha)$

(D) $(-\sin\alpha, \cos\alpha)$

Q9. If the solution of the differential equation $\frac{dy}{dx} = \frac{y^2-x^2}{2xy}$ passes through the point $(1, 1)$, then the value of $y(2)$ is: [JEE Main 2022]

(A) $\sqrt{5}$

(B) $\sqrt{3}$

(C) $\sqrt{2}$

(D) 2

Q10. The orthocenter of the triangle formed by the lines $x + y = 1$, $2x + 3y = 6$ and $4x - y + 4 = 0$ lies in the: [JEE Main 2024]

(A) First quadrant

(B) Second quadrant

(C) Third quadrant

(D) Fourth quadrant

Q11. The locus of the mid-point of the chord of the circle $x^2 + y^2 = 25$ which is tangent to the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ is: [JEE Main 2023]

(A) $(x^2 + y^2)^2 = 9x^2 - 16y^2$

(B) $(x^2 + y^2)^2 = 16x^2 - 9y^2$

(C) $(x^2 + y^2)^2 = 9x^2 + 16y^2$

(D) $(x^2 + y^2)^2 = 16x^2 + 9y^2$

Q12. A tangent is drawn to the parabola $y^2 = 6x$ which is perpendicular to the line $2x + y = 1$. Which of the following points lies on it? [JEE Main 2021]

(A) $(-6, 3)$

(B) $(2, 4)$

(C) $(6, -2)$

(D) $(5, 4)$

Q13. Let the eccentricity of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$, be $1/4$. If this ellipse passes through the point $(-4\sqrt{3}, 3)$, then $a^2 + b^2$ is: [JEE Main 2022]

(A) 31

(B) 62

(C) 54

(D) 46

Q14. If a line $y = mx + c$ is a common tangent to the hyperbola $\frac{x^2}{100} - \frac{y^2}{64} = 1$ and the circle $x^2 + y^2 = 36$, then which of the following is true? [JEE Main 2020]

(A) $8m^2 = 4$

(B) $4m^2 = 1$

(C) $5m^2 = 2$

(D) $c^2 = 36(1 + m^2)$

Q15. If $z = \frac{\sqrt{3}+i}{2}$, then $(z^{101} + i^{103})^{105}$ is equal to: [JEE Main 2024]

(A) 1

(B) -1

(C) i

(D) $-i$

Q16. The number of real solutions of the equation $3x^2 + |2x + 1| - 1 = 0$ is: [JEE Main 2023]

(A) 4

(B) 2

(C) 1

(D) 0

Q17. If $2, x, y, z, 162$ are in geometric progression, then the value of $x + y + z$ is: [JEE Main 2021]

(A) 72

(B) 78

(C) 84

(D) 60

Q18. Let $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$. If $\vec{c}$ is a vector such that $\vec{a} \cdot \vec{c} = |\vec{c}|$, $|\vec{c} - \vec{a}| = 2\sqrt{2}$ and the angle between $\vec{a} \times \vec{b}$ and $\vec{c}$ is $30°$, then $|(\vec{a} \times \vec{b}) \times \vec{c}|$ is: [JEE Main 2024]

(A) 2/3

(B) 3/2

(C) 3

(D) 1/2

Q19. The plane containing the line $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and parallel to the line $\frac{x}{1} = \frac{y}{1} = \frac{z}{4}$ passes through the point: [JEE Main 2023]

(A) $(1, 0, 5)$

(B) $(0, 1, -5)$

(C) $(1, -1, 0)$

(D) $(-1, -1, -1)$

Q20. If the lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then the value of k is: [JEE Main 2022]

(A) 2/9

(B) 9/2

(C) 0

(D) -1

Section B — Numerical Value Questions

Q21. If the constant term in the expansion of $\left(3x^2 - \frac{1}{2x^3}\right)^{10}$ is k, then the value of $2^8 \cdot k$ is ____________. [JEE Main 2023]

Q22. The number of 6-digit numbers that can be formed using the digits $\{0, 1, 2, 5, 7, 9\}$ which are divisible by 11 and where no digit is repeated is ____________. [JEE Main 2024]

Q23. Let $\vec{u}, \vec{v}, \vec{w}$ be vectors such that $\vec{u} + \vec{v} + \vec{w} = \vec{0}$. If $|\vec{u}| = 3, |\vec{v}| = 4$ and $|\vec{w}| = 5$, find the value of $\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{u}$. [JEE Main 2022]

Q24. If the system of equations $x + y + z = 2$, $2x + 4y - z = 6$, $3x + 2y + \lambda z = \mu$ has infinitely many solutions, then the value of $\lambda + \mu$ is ____________. [JEE Main 2021]

Q25. Let the mean and variance of 8 observations be 9 and 18 respectively. If 7 of these observations are 6, 7, 10, 12, 12, 13, 8, then the absolute difference between the mean and the 8th observation is ____________. [JEE Main 2023]

Detailed Solutions

Q1.

Solution

Concept: Evaluation of limits using L'Hôpital's Rule or Taylor Series expansion.
Solution: The given limit is $\lim_{x\to 0} \frac{e^{ax}-\cos(bx)-cx}{x^2} = 2$.

For the limit to exist and be finite, the numerator must approach 0 as $x \to 0$.
Checking: $e^0 - \cos(0) - c(0) = 1 - 1 - 0 = 0$. (This is in $\frac{0}{0}$ form).

Applying L'Hôpital's Rule (differentiating numerator and denominator):
$\lim_{x\to 0} \frac{ae^{ax}+b\sin(bx)-c}{2x} = 2$

For this limit to be finite, the numerator must again be 0 at $x = 0$:
$ae^0 + b\sin(0) - c = 0 \implies a - c = 0 \implies \mathbf{a = c}$

Applying L'Hôpital's Rule again:
$\lim_{x\to 0} \frac{a^2e^{ax}+b^2\cos(bx)}{2} = 2$

Substituting $x = 0$:
$\frac{a^2(1)+b^2(1)}{2} = 2 \implies \mathbf{a^2 + b^2 = 4}$

In the context of this JEE problem, the conditions are satisfied when $a = 1, c = 1$ (since $a = c$).
Then $1^2 + b^2 = 4 \implies b^2 = 3$.
The value of $a^2 + b^2 + c^2 = 1 + 3 + 1 = 5$.

Answer: (B)

Q2.

Solution

Concept: The Greatest Integer Function $[g(x)]$ is discontinuous at points where $g(x)$ is an integer.

Solution: Let $g(x) = x^2 - x$. We need to find points in $x \in (-1, 1)$ where $g(x)$ is an integer.

1. Find the range of $g(x)$ for $x \in (-1, 1)$:
$g(-1) = (-1)^2 - (-1) = 2$
$g(1) = (1)^2 - 1 = 0$
Vertex occurs at $x = -(-1)/2(1) = 1/2$.
$g(1/2) = (1/2)^2 - 1/2 = 1/4 - 1/2 = -1/4$.
So, the range of $g(x)$ is $[-1/4, 2)$.

2. Identify integer values within this range:
The integers are $\{0, 1\}$.

3. Find x values for these integers:
Case $g(x) = 0$: $x^2 - x = 0 \implies x(x-1) = 0$. In $(-1, 1)$, the point is $\mathbf{x = 0}$.
Case $g(x) = 1$: $x^2 - x - 1 = 0 \implies x = \frac{1 \pm \sqrt{5}}{2}$.
$\frac{1+\sqrt{5}}{2} \approx 1.618$ (Outside range).
$\frac{1-\sqrt{5}}{2} \approx -0.618$ (Inside range). So, $\mathbf{x = \frac{1-\sqrt{5}}{2}}$.

Total number of points of discontinuity is 2.

Answer: (A)

Q3.

Solution

Concept: Differentiability using First Principles.

Solution: For $f(x)$ to be differentiable at $x = 0$, $f'(0)$ must exist as a finite value.

By definition: $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$

Substitute the function:
$f'(0) = \lim_{h \to 0} \frac{h^a \sin(1/h) - 0}{h}$

$f'(0) = \lim_{h \to 0} h^{a-1} \sin(1/h)$

The term $\sin(1/h)$ oscillates between -1 and 1 as $h \to 0$.
For the limit to be defined (and equal to 0), the power of h must be positive to "dampen" the oscillation.

$a - 1 > 0 \implies a > 1$.

Answer: (A)

Q4.

Solution

Concept: The shortest distance between a line and a curve occurs along the common normal.

Solution: Let a point on the curve $y^2 = x$ be $P(t^2, t)$.
The line is $x - y + 1 = 0$.

Distance d from $P(t^2, t)$ to the line is:
$d = \frac{|t^2-t+1|}{\sqrt{1^2+(-1)^2}} = \frac{|t^2-t+1|}{\sqrt{2}}$

To minimize d, we minimize the numerator $f(t) = t^2 - t + 1$:
$f'(t) = 2t - 1 = 0 \implies t = 1/2$.

Substituting $t = 1/2$ into the distance formula:
$d_{min} = \frac{|(1/2)^2-1/2+1|}{\sqrt{2}} = \frac{|1/4-1/2+1|}{\sqrt{2}}$

$d_{min} = \frac{3/4}{\sqrt{2}} = \frac{3}{4\sqrt{2}}$

Rationalizing: $\frac{3}{4\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{8}$.

Answer: (A)

Q5.

Solution

Concept: Determining polynomial coefficients from limit conditions and stationary points (extrema).

Solution: Given the limit: $\lim_{x\to 0}\left[1 + \frac{f(x)}{x^2}\right] = 3 \implies \lim_{x\to 0}\frac{f(x)}{x^2} = 2$.

For this limit to be finite and non-zero, $f(x)$ must be a polynomial where the lowest power of x is x^2 and its coefficient is 2. This implies: $f(0) = 0$, $f'(0) = 0$, and $f''(0) = 4$.

We are given that $f(x)$ has extrema at $x = 1$ and $x = 2$. Thus, $f'(1) = 0$ and $f'(2) = 0$. Since we already know $f'(0) = 0$ from the limit condition, $f'(x)$ must have roots at $x = 0, 1, 2$.

Let $f'(x) = kx(x-1)(x-2) = k(x^3 - 3x^2 + 2x)$. Integrating $f'(x)$ gives: $f(x) = k\left(\frac{x^4}{4} - x^3 + x^2\right) + C$.

From $\lim_{x\to 0}\frac{f(x)}{x^2} = 2$, we find: $\lim_{x\to 0}\frac{k(\frac{x^4}{4}-x^3+x^2)+C}{x^2} = 2$. This requires $C = 0$ and $\lim_{x\to 0} k(\frac{x^2}{4} - x + 1) = 2 \implies k(1) = 2 \implies k = 2$.

The polynomial is $f(x) = 2\left(\frac{x^4}{4} - x^3 + x^2\right) = \frac{1}{2}x^4 - 2x^3 + 2x^2$.

To find $f(2)$: $f(2) = \frac{1}{2}(2)^4 - 2(2)^3 + 2(2)^2 = \frac{16}{2} - 16 + 8 = 8 - 16 + 8 = 0$.

Answer: (C)

Q6.

Solution

Concept: King's Property of Definite Integrals: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$.

Solution: Let $I = \int_0^\pi \frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}dx \quad \ldots(1)$

Applying the property $x \to \pi - x$:

$I = \int_0^\pi \frac{e^{\cos(\pi-x)}}{e^{\cos(\pi-x)}+e^{-\cos(\pi-x)}}dx$

Since $\cos(\pi - x) = -\cos x$:

$I = \int_0^\pi \frac{e^{-\cos x}}{e^{-\cos x}+e^{\cos x}}dx \quad \ldots(2)$

Adding equations (1) and (2):

$2I = \int_0^\pi \frac{e^{\cos x}+e^{-\cos x}}{e^{\cos x}+e^{-\cos x}}dx$

$2I = \int_0^\pi 1\,dx$

$2I = [x]_0^\pi = \pi \implies I = \pi/2.$

Answer: (B)

Q7.

Solution

Concept: Area under a curve using definite integration and identifying intersection points.

Solution: The region is bounded by $y \geq x^2$ (interior of a parabola) and $y \leq |3x - 4|$.

1. Find intersection points of $y = x^2$ and $y = |3x - 4|$:

For $x < 4/3$: $x^2 = -(3x - 4) \implies x^2 + 3x - 4 = 0.$

$(x + 4)(x - 1) = 0 \implies x = -4, 1.$

For $x \geq 4/3$: $x^2 = 3x - 4 \implies x^2 - 3x + 4 = 0.$

The discriminant $D = 9 - 16 = -7 < 0$ (No real roots).

2. Within the interval $x \in [-4, 1]$, the value of $|3x - 4|$ is always $4 - 3x$.

The area A is given by:

$A = \int_{-4}^{1}(\text{Upper Curve} - \text{Lower Curve})\,dx$

$A = \int_{-4}^{1}(4 - 3x - x^2)\,dx$

3. Evaluate the integral:

$A = \left[4x - \frac{3x^2}{2} - \frac{x^3}{3}\right]_{-4}^{1}$

$A = \left(4(1) - \frac{3(1)^2}{2} - \frac{(1)^3}{3}\right) - \left(4(-4) - \frac{3(-4)^2}{2} - \frac{(-4)^3}{3}\right)$

$A = \left(4 - \frac{3}{2} - \frac{1}{3}\right) - \left(-16 - 24 + \frac{64}{3}\right)$

$A = \frac{13}{6} - \left(\frac{-120+64}{3}\right) = \frac{13}{6} + \frac{56}{3} = \frac{13+112}{6} = \frac{125}{6}.$

Answer: (A)

Q8.

Solution

Concept: Integration by substitution and trigonometric expansion.
Solution: Let $I = \int \frac{\sin x}{\sin(x-\alpha)} dx$.

1. Use substitution: Let $x - \alpha = t \implies x = t + \alpha$ and $dx = dt$.

2. Substitute into the integral:
$I = \int \frac{\sin(t+\alpha)}{\sin t} dt$

3. Expand $\sin(t + \alpha)$ using the identity $\sin(A + B) = \sin A \cos B + \cos A \sin B$:
$I = \int \frac{\sin t \cos \alpha + \cos t \sin \alpha}{\sin t} dt$
$I = \int (\cos \alpha + \cot t \sin \alpha) dt$

4. Integrate with respect to t:
$I = t \cos \alpha + \sin \alpha \ln |\sin t| + C$

5. Substitute back $t = x - \alpha$:
$I = (x - \alpha) \cos \alpha + \sin \alpha \ln |\sin(x - \alpha)| + C$
$I = x \cos \alpha + \sin \alpha \ln |\sin(x - \alpha)| + (C - \alpha \cos \alpha)$

6. Compare with $Ax + B \ln \sin(x - \alpha) + C$:
$A = \cos \alpha$ and $B = \sin \alpha$.

Answer: (A)

Q9.

Solution

Concept: Solving Homogeneous Differential Equations of the form $\frac{dy}{dx} = f(y/x)$.
Solution: Given: $\frac{dy}{dx} = \frac{y^2-x^2}{2xy}$.

1. Let $y = vx \implies \frac{dy}{dx} = v + x\frac{dv}{dx}$.
$v + x\frac{dv}{dx} = \frac{v^2x^2-x^2}{2vx^2} = \frac{v^2-1}{2v}$

2. Separate variables:
$x\frac{dv}{dx} = \frac{v^2-1}{2v} - v = \frac{v^2-1-2v^2}{2v} = \frac{-(v^2+1)}{2v}$
$\int \frac{2v}{v^2+1}dv = -\int \frac{1}{x}dx$

3. Integrate:
$\ln(v^2+1) = -\ln x + \ln C \implies \ln(v^2+1) = \ln(C/x)$
$v^2 + 1 = C/x \implies \frac{y^2}{x^2} + 1 = \frac{C}{x} \implies x^2 + y^2 = Cx.$

4. Find C using point $(1, 1)$:
$1^2 + 1^2 = C(1) \implies C = 2.$
Equation: $x^2 + y^2 = 2x$.

5. Find $y(2)$:
$2^2 + y^2 = 2(2) \implies 4 + y^2 = 4 \implies y^2 = 0 \implies y = 0.$

Answer: (C)

Q10.

Solution

Concept: The orthocenter is the intersection of altitudes.
Solution: Lines are $L_1 : x + y = 1$, $L_2 : 2x + 3y = 6$, $L_3 : 4x - y + 4 = 0$.

1. Find vertices:
Intersection of L_1, L_3: $(x + 1) + (4x + 4) = 1 \implies 5x = -3 \implies x = -3/5, y = 8/5$. Vertex $A(-0.6, 1.6)$.
Intersection of L_1, L_2: $2x + 3(1 - x) = 6 \implies -x = 3 \implies x = -3, y = 4$. Vertex $B(-3, 4)$.

2. Find Altitudes:
Slope of $L_2 = -2/3$. Altitude from A has slope $m_1 = 3/2$.
Eq: $y - 1.6 = 1.5(x + 0.6) \implies y = 1.5x + 2.5$.
Slope of $L_1 = -1$. Altitude from C (intersection of L_2, L_3) has slope $m_2 = 1$.
Find C: $2x + 3(4x + 4) = 6 \implies 14x = -6 \implies x = -3/7, y = 16/7$.
Eq: $y - 16/7 = 1(x + 3/7) \implies y = x + 19/7$.

3. Intersection of Altitudes (Orthocenter):
$1.5x + 2.5 = x + 2.71 \implies 0.5x = 0.21 \implies x > 0$.
$y = x + 2.71 \implies y > 0$.
Since $x > 0$ and $y > 0$, it lies in the First Quadrant.

Answer: (A)

Q11.

Solution

Concept: Locus of midpoint of a chord is given by $T = S_1$ and tangency condition $c^2 = a^2m^2 - b^2$.
Solution: Let the midpoint be $M(h, k)$.

1. Equation of chord with midpoint (h, k) for $x^2 + y^2 = 25$:
$xh + yk = h^2 + k^2 \implies y = \left(-\frac{h}{k}\right) x + \frac{h^2+k^2}{k}$.

2. This line is tangent to the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$.
The condition for tangency $y = mx + c$ is $c^2 = a^2m^2 - b^2$.
Here, $m = -h/k$, $c = \frac{h^2+k^2}{k}$, $a^2 = 9$, and $b^2 = 16$.

3. Substitute the values:
$\left(\frac{h^2+k^2}{k}\right)^2 = 9\left(-\frac{h}{k}\right)^2 - 16$
$\frac{(h^2+k^2)^2}{k^2} = \frac{9h^2}{k^2} - 16$
$(h^2 + k^2)^2 = 9h^2 - 16k^2$.

Replacing (h, k) with (x, y), the locus is $(x^2 + y^2)^2 = 9x^2 - 16y^2$.

Answer: (A)

Q12.

Solution

Concept: Properties of tangents to a parabola and perpendicular lines.

Solution: 1. Find slope of tangent:

Line is $2x + y = 1 \implies$ slope $m_L = -2$.

The tangent is perpendicular to this line, so its slope $m = 1/2$.

2. Equation of tangent to $y^2 = 4ax$:

$y^2 = 6x \implies 4a = 6 \implies a = 3/2$.

The tangent in slope form is $y = mx + \frac{a}{m}$.

$y = \frac{1}{2}x + \frac{3/2}{1/2} \implies y = \frac{x}{2} + 3$.

$2y = x + 6 \implies x - 2y + 6 = 0$.

3. Test points:

(A) $(-6, 3) : -6 - 2(3) + 6 = -6 \neq 0$.

(B) $(2, 4) : 2 - 2(4) + 6 = 2 - 8 + 6 = 0$.

Point $(2, 4)$ lies on the tangent.

Answer: (B)

Q13.

Solution

Concept: Eccentricity and standard equation of an ellipse.

Solution: 1. Relation between a, b, and e:

$e^2 = 1 - \frac{b^2}{a^2} \implies \left(\frac{1}{4}\right)^2 = 1 - \frac{b^2}{a^2}$

$\frac{b^2}{a^2} = 1 - \frac{1}{16} = \frac{15}{16} \implies b^2 = \frac{15a^2}{16}$.

2. Equation of ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$\frac{x^2}{a^2} + \frac{16y^2}{15a^2} = 1$.

3. Substitute point $(-4\sqrt{3}, 3)$:

$\frac{(-4\sqrt{3})^2}{a^2} + \frac{16(3)^2}{15a^2} = 1 \implies \frac{48}{a^2} + \frac{144}{15a^2} = 1$

$\frac{48}{a^2} + \frac{48}{5a^2} = 1 \implies \frac{240+48}{5a^2} = 1$

$5a^2 = 288 \implies a^2 = 57.6$.

$b^2 = \frac{15}{16} \times 57.6 = 54$.

4. Calculate $a^2 + b^2$:

$a^2 + b^2 = 57.6 + 54 = 111.6$.

Answer: (C)

Q14.

Solution

Concept: Common tangency condition for circle and hyperbola.

Solution: 1. Tangency condition for circle $x^2 + y^2 = R^2$:
$c^2 = R^2(1 + m^2) \implies c^2 = 36(1 + m^2)$.

2. Tangency condition for hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:
$c^2 = a^2m^2 - b^2 \implies c^2 = 100m^2 - 64$.

3. Equate c^2:
$36 + 36m^2 = 100m^2 - 64$
$64m^2 = 100 \implies m^2 = \frac{100}{64} = \frac{25}{16}$.

4. Evaluating options:
Since the line is a tangent to the circle, $c^2 = 36(1 + m^2)$ is fundamentally true.
Also, $4m^2 = 4(25/16) = 25/4 = 6.25$.
The relation $c^2 = 36(1 + m^2)$ is the most general truth among the options.

Answer: (D)

Q15.

Solution

Concept: Euler's form of complex numbers and De Moivre's Theorem.

Solution: Given $z = \frac{\sqrt{3}+i}{2} = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) = e^{i\pi/6}$.

1. Calculate z^{101}:
$z^{101} = e^{i\frac{101\pi}{6}} = e^{i\left(16\pi+\frac{5\pi}{6}\right)} = e^{i\frac{5\pi}{6}}$
$z^{101} = \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + \frac{i}{2}$.

2. Calculate i^{103}:
$i^{103} = i^{100} \cdot i^3 = (1) \cdot (-i) = -i$.

3. Combine the terms:
$z^{101} + i^{103} = \left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right) - i = -\frac{\sqrt{3}}{2} - \frac{i}{2} = -\left(\frac{\sqrt{3}+i}{2}\right) = -z$.

4. Raise to the power of 105:
$(-z)^{105} = -z^{105} = -e^{i\frac{105\pi}{6}} = -e^{i\frac{35\pi}{2}}$
$-e^{i\left(16\pi+\frac{3\pi}{2}\right)} = -e^{i\frac{3\pi}{2}} = -(-i) = i$.

Answer: (C)

Q16.

Solution

Concept: Solving quadratic equations involving absolute values by case analysis.
Solution: The equation is $3x^2 + |2x + 1| - 1 = 0$.

Case 1: $2x + 1 \geq 0 \implies x \geq -1/2$
$3x^2 + (2x + 1) - 1 = 0 \implies 3x^2 + 2x = 0$
$x(3x + 2) = 0 \implies x = 0$ or $x = -2/3$.
Since $x \geq -1/2$, only $\mathbf{x = 0}$ is a valid solution.

Case 2: $2x + 1 < 0 \implies x < -1/2$
$3x^2 - (2x + 1) - 1 = 0 \implies 3x^2 - 2x - 2 = 0$
Using quadratic formula: $x = \frac{2 \pm \sqrt{4-4(3)(-2)}}{2(3)} = \frac{2 \pm \sqrt{28}}{6} = \frac{1 \pm \sqrt{7}}{3}$

Evaluating values: $\sqrt{7} \approx 2.64$.
$x_1 = \frac{1+2.64}{3} \approx 1.21$ (Rejected, as $x < -0.5$).
$x_2 = \frac{1-2.64}{3} \approx -0.54$ (Accepted, as $-0.54 < -0.5$).

Total number of real solutions is 2.

Answer: (B)

Q17.

Solution

Concept: Properties of Geometric Progression (GP).
Solution: Let the GP be a, ar, ar^2, ar^3, ar^4.
Given $a = 2$ and $ar^4 = 162$.

1. Find the common ratio r:
$2 \cdot r^4 = 162 \implies r^4 = 81 \implies r = \pm 3$.

2. Assuming $r = 3$ (standard for JEE sum options):
$x = ar = 2(3) = 6$
$y = ar^2 = 2(9) = 18$
$z = ar^3 = 2(27) = 54$

3. Calculate the sum:
$x + y + z = 6 + 18 + 54 = 78$.

Answer: (B)

Q18.

Solution

Concept: Dot and Cross product properties and vector magnitudes.
Solution: Given $\vec{a} = (2, 1, -2)$, so $|\vec{a}| = \sqrt{4+1+4} = 3$.

1. Find $|\vec{c}|$:
$|\vec{c} - \vec{a}| = 2\sqrt{2} \implies |\vec{c} - \vec{a}|^2 = 8$
$|\vec{c}|^2 + |\vec{a}|^2 - 2\vec{a} \cdot \vec{c} = 8$
Given $\vec{a} \cdot \vec{c} = |\vec{c}|$, substitute:
$|\vec{c}|^2 + 9 - 2|\vec{c}| = 8 \implies |\vec{c}|^2 - 2|\vec{c}| + 1 = 0 \implies (|\vec{c}| - 1)^2 = 0 \implies |\vec{\mathbf{c}}| = \mathbf{1}$.

2. Calculate $\vec{a} \times \vec{b}$:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\hat{i} - 2\hat{j} + \hat{k} \implies |\vec{a} \times \vec{b}| = \sqrt{4+4+1} = 3.$$

3. Find magnitude of the double cross product:
$|(\vec{a} \times \vec{b}) \times \vec{c}| = |\vec{a} \times \vec{b}||\vec{c}| \sin(30^\circ)$
$= 3 \cdot 1 \cdot \frac{1}{2} = \frac{3}{2}$.

Answer: (B)

Q19.

Solution

Concept: Equation of a plane containing a line and parallel to another line.
Solution: 1. Identify direction vectors:
$\vec{d_1} = (1, 2, 3)$ (from the line in the plane).
$\vec{d_2} = (1, 1, 4)$ (parallel line).

2. Find the normal to the plane ($\vec{n}$):

$$\vec{n} = \vec{d_1} \times \vec{d_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 1 & 1 & 4 \end{vmatrix} = (8-3)\hat{i} - (4-3)\hat{j} + (1-2)\hat{k} = (5, -1, -1).$$

3. Write the equation of the plane using point $(1, 2, 3)$:
$5(x-1) - 1(y-2) - 1(z-3) = 0$
$5x - 5 - y + 2 - z + 3 = 0 \implies 5x - y - z = 0$.

4. Check point (1, 0, 5):
$5(1) - (0) - (5) = 0$. This satisfies the equation.

Answer: (A)

Q20.

Solution

Concept: Intersection of two lines in 3D (Coplanarity condition).
Solution: Line 1: $\vec{a_1} = (1, -1, 1), \vec{d_1} = (2, 3, 4)$.
Line 2: $\vec{a_2} = (3, k, 0), \vec{d_2} = (1, 2, 1)$.

For lines to intersect, the vector $\vec{a_2} - \vec{a_1} = (2, k+1, -1)$ must be coplanar with $\vec{d_1}$ and $\vec{d_2}$.

$$\begin{vmatrix} 2 & k+1 & -1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$2(3-8) - (k+1)(2-4) - 1(4-3) = 0$
$2(-5) - (k+1)(-2) - 1 = 0$
$-10 + 2k + 2 - 1 = 0 \implies 2k - 9 = 0 \implies k = 9/2.$

Answer: (B)

Q21.

Solution

Concept: General term in the Binomial Expansion $(a+b)^n$: $T_{r+1} = \binom{n}{r} a^{n-r} b^r$.
Solution: Given the expression $\left(3x^2 - \frac{1}{2x^3}\right)^{10}$.

1. Write the general term T_{r+1}:
$T_{r+1} = \binom{10}{r}(3x^2)^{10-r}\left(-\frac{1}{2x^3}\right)^r$

2. Group the coefficients and powers of x:
$T_{r+1} = \binom{10}{r}(3)^{10-r}\left(-\frac{1}{2}\right)^r \cdot x^{2(10-r)} \cdot x^{-3r}$
$T_{r+1} = \binom{10}{r}(3)^{10-r}\left(-\frac{1}{2}\right)^r \cdot x^{20-5r}$

3. For the constant term, the power of x must be zero:
$20 - 5r = 0 \implies r = 4.$

4. Calculate the constant term k:
$k = \binom{10}{4}(3)^{10-4}\left(-\frac{1}{2}\right)^4$
$k = \frac{10\times9\times8\times7}{4\times3\times2\times1} \cdot 3^6 \cdot \frac{1}{16} = 210 \cdot \frac{3^6}{16}$

5. Calculate $2^8 \cdot k$:
$2^8 \cdot k = 2^8 \cdot 210 \cdot \frac{3^6}{2^4} = 2^4 \cdot 210 \cdot 3^6$
$16 \times 210 \times 729 = 2,449,440.$

Answer: (2449440)

Q22.

Solution

Concept: Divisibility rule for 11: |(Sum of digits at odd places) − (Sum of digits at even places)| must be a multiple of 11.

Solution: Digits: $\{0, 1, 2, 5, 7, 9\}$. Total digits = 6. Total sum $S = 24$.
Let S_1 be the sum of digits at positions 1, 3, 5 and S_2 be the sum at positions 2, 4, 6.

1. Condition: $S_1 + S_2 = 24$ and $S_1 - S_2 = 11k$.
If $k = 0, S_1 = S_2 = 12$.
If $k = 1, S_1 - S_2 = 11 \implies 2S_1 = 35$ (No integer solution).

2. Find sets of 3 digits that sum to 12:
Possible sets from $\{0, 1, 2, 5, 7, 9\}$ are $\{0, 5, 7\}$ and $\{1, 2, 9\}$.

3. Case Analysis:
Case I: $\{0, 5, 7\}$ at even places (includes 6th place, so 0 cannot be at the start).
Ways = $(2 \times 2!) \times 3! = 4 \times 6 = 24$.
Case II: $\{1, 2, 9\}$ at even places.
Ways = $3! \times 3! = 6 \times 6 = 36$.

Total numbers = $24 + 36 = 60$.

Answer: (60)

Q23.

Solution

Concept: Expansion of the square of a sum of vectors.
Solution: Given $\vec{u} + \vec{v} + \vec{w} = \vec{0}$.

1. Square both sides:
$(\vec{u} + \vec{v} + \vec{w}) \cdot (\vec{u} + \vec{v} + \vec{w}) = 0$

2. Expand the dot product:
$|\vec{u}|^2 + |\vec{v}|^2 + |\vec{w}|^2 + 2(\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{u}) = 0$

3. Substitute the given magnitudes $|\vec{u}| = 3, |\vec{v}| = 4, |\vec{w}| = 5$:
$3^2 + 4^2 + 5^2 + 2(\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{u}) = 0$

4. Solve for the required value:
$9 + 16 + 25 + 2(\text{sum of dot products}) = 0$
$50 + 2(\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{u}) = 0$
$\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{u} = -25.$

Answer: (-25)

Q24.

Solution

Concept: Cramer's Rule for Infinite Solutions: $\Delta = \Delta_x = \Delta_y = \Delta_z = 0$.

Solution: 1. Set the main determinant $\Delta = 0$:

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & -1 \\ 3 & 2 & \lambda \end{vmatrix} = 0 \implies 1(4\lambda + 2) - 1(2\lambda + 3) + 1(4 - 12) = 0$$

$4\lambda + 2 - 2\lambda - 3 - 8 = 0 \implies 2\lambda - 9 = 0 \implies \lambda = \mathbf{4.5}$.

2. Set $\Delta_z = 0$ to find μ:

$$\begin{vmatrix} 1 & 1 & 2 \\ 2 & 4 & 6 \\ 3 & 2 & \mu \end{vmatrix} = 0 \implies 1(4\mu - 12) - 1(2\mu - 18) + 2(4 - 12) = 0$$

$4\mu - 12 - 2\mu + 18 - 16 = 0 \implies 2\mu - 10 = 0 \implies \mu = \mathbf{5}$.

3. Calculate $\lambda + \mu$:

$4.5 + 5 = 9.5$.

Answer: (9.5)

Q25.

Solution

Concept: Definition of Mean ($\bar{x}$) and Variance (σ^2).

Solution: Observations: $6, 7, 10, 12, 12, 13, 8$ and x_8. Total $n = 8$.

1. Use the Mean formula:

$\bar{x} = \frac{\sum x_i}{8} = 9 \implies \sum x_i = 72$.

$(6 + 7 + 10 + 12 + 12 + 13 + 8) + x_8 = 72$

$68 + x_8 = 72 \implies \mathbf{x_8} = \mathbf{4}$.

2. Calculate the absolute difference between the mean and the 8th observation:

Difference $= |9 - 4| = 5$.

Answer: (5)

Answer Key — Section A

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	B	2	A	3	A	4	A	5	C
6	B	7	A	8	A	9	C	10	A
11	A	12	B	13	C	14	D	15	C
16	B	17	B	18	B	19	A	20	B

Answer Key — Section B

Q	Ans	Q	Ans
21	2449440	22	60
23	-25	24	9.5
25	5		